國 立 交 通 大 學

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碩　士　論　文

可加性細胞自動機上的熵

On the Entropy of Additive Cellular Automata

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中 華 民 國 九 十 九 年 六 月

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摘　　　要

本篇論文討論任意有限狀況的可加性細胞自動機(cellular automata)，並利用拓樸共軛(topological conjugate)與測度同構(measure isomorphic)的特性給予拓樸熵(topological entropy)及在伯努利測度(Bernoulli measure)下測度熵(measure-theoretic entropy)的公式。藉由拓樸熵與測度熵的公式能找到滿足變分原理(Variational Principle)的最大測度(maximum measure)，可用於物理的應用上。

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ABSTRACT

In this paper we consider the additive cellular automata with any finite states, we take the properties of the topological conjugate and measure isomorphic to give a formula for topological entropy and measure-theoretic entropy with Bernoulli measure. After having the measure-theoretic entropy formula and the topological entropy formula, we can find the maximum measure such that satisfy the Variational Principle, and more apply in physics.

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1 Introduction

One-dimensional Cellular Automata(CA) consists of infinite lattice with finite states and a local rule. Although the additive CA and the ergodic properties of the additive CA have been investigated, there are still strong connection between the additive CA and the ergodic properties.

In this paper, the ergodic properties are measure-theoretic entropy, topological entropy. Let F be an additive CA, i.e., the local rule of f is a linear function, which is defined with finite state. Akin [1] compute the additive CA of measure-theoretic entropy with uniform Bernoulli measure. Ban et al. [2] calculate a measure-theoretic entropy formula to permutive CA with Bernoulli measure. Our investigation give a measure-theoretic entropy formula to any additive CA. Ward [6] gives a formula for the calculation of the additive CA with prime states' topological entropy. But additive CA with prime states is a proper subset of permutive CA, Ban et al. [2] extend Ward's formula to permutive CA. D'amico et al. [3] demonstrate an algorithm for the computation of topological entropy for any finite states' CA. Our investigation extends Ward's and Ban's formula but different method with D'amico's method for any additive CA.

After having the measure-theoretic entropy formula and the topological entropy formula, we can find the maximum measure such that satisfy the Variational Principle, and more apply in physics.

2 Basic definition

In this section, we introduce some basic definition and state some known results about CA.

2.1 Cellular automata

For $m \geq 2$, let $\mathcal{S} = \{0, 1, \cdots, m-1\}, m \in \mathbb{N}$ be a finite alphabet and let $X = \mathcal{S}^{\mathbb{Z}}$ be the space of infinite sequence $x = (x_n)_{n=-\infty}^{\infty}$, $x_n \in \mathcal{S}$. For one dimension cellular automata, we use a simplified notation. A local rule $f : \mathcal{S}^{r-l+1} \to \mathcal{S}$ is denoted by $f(x_l, \cdots, x_r)$. The associated global rule $F : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ is defined by

$$[F(x)]_i = f(x_{i+l}, \cdots, x_{i+r}) \ \ where \ x \in \mathcal{S}^{\mathbb{Z}}, i \in \mathbb{Z}$$

where $[F(x)]_i$ means the ith component of $F(x)$.

Denote by f_m be the local rule f taken modulo m; similar to the global rule F_m. To sequence, denote by $x_{m;i}$ be the ith component of the sequence x_n modulo m. For integers, denote by $\mathbf{x}_m$ be the integer x taken modulo m. In this paper, we consider the linear local rule $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i \pmod{m}$ denoted by $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$, and we called the CA of the linear local rule be additive CA. Define the *dipolynomial of* f is $\mathbb{T}(x) = \sum_{i=l}^{r} a_i x^{-i}$. And we use the Bernoulli measure.

Let $\mathcal{P}, \mathcal{Q}$ be collections, define

$$\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, \ Q \in \mathcal{Q}\}.$$

Definition 2.1. *The local rule $f : \mathcal{S}^{r-l+1} \to \mathcal{S}$ for a given CA is said to be leftmost (respectively rightmost) permutive if there exists an integer $i, l \leq i \leq -1$ (respectively $1 \leq i \leq r$) such that*

(i) f *is a permutation at* x_i*;*

(ii) f *does not depend on* x_j *for* $j < i$ *(respectively* $j > i$*).*

where $x = (x_j)_{j=l}^{r} \in \mathcal{S}^{r-l+1}$.

Proposition 2.2. *Let* $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$*,* f *is permutive in the jth variable if and only if* $\gcd(a_j, m) = 1$.

Definition 2.3. *The local rule* $f : \mathcal{S}^{r-l+1} \to \mathcal{S}$ *is called bipermutive if* f *is both leftmost and rightmost permutive.* f *is called permutive provided* f *is one of the following three cases:*

(i) f *is leftmost permutive and does not depend on* x_i *for* $i > 0$*;*

(ii) f *is rightmost permutive and does not depend on* x_i *for* $i < 0$*;*

(iii) f *is bipermutive.*

2.2 Measure-theoretic Entropy of Cellular Automata

Definition 2.4. *A partition of* $(\mathbf{X}, \mathcal{B}, \mu)$ *is a disjoint collection of elements of* $\mathcal{B}$ *whose union is* $\mathbf{X}$*. That is,* $\overset{\circ}{\bigcup_{A \in \mathcal{B}}} A = \mathbf{X}$.

Definition 2.5. *Let* $(\mathbf{X}, \mathcal{B}, \mu)$ *be a probability space and* α *be a partition of* $\mathbf{X}$*, define*

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$$

Definition 2.6. *Let* $(\mathbf{X}, \mathcal{B}, \mu)$ *be a probability space,* $F : \mathbf{X} \to \mathbf{X}$ *be a measure-preserving transformation and* α *be a partition of* $\mathbf{X}$*, define*

$$h_\mu(F, \alpha) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} F^{-i} \alpha \right).$$

The measure-theoretic entropy is

$$h_\mu(F) = \sup_\alpha h_\mu(F, \alpha).$$

Definition 2.7. *Define* $\mu = (p_0, p_1, \cdots, p_{m-1})$ *be a Bernoulli measure, if*

$$\mu({}_a[s_a, \cdots, s_b]_b) = p_{s_a} \cdots p_{s_b}, \quad \text{for } {}_a[s_a, \cdots, s_b]_b \subset \mathbf{X} \text{ and } p_i > 0 \ \forall i = s_a, \cdots, s_b.$$

Theorem 2.8. *([2])*
If f *be permutive and depends only on* $x_i, \cdots, x_j$*, where* $i \le j$*,* $i, j \in \mathbb{Z}$*. Denote by* $\widehat{i} = -\min\{i, 0\}$ *and* $\widehat{j} = \max\{j, 0\}$*, then* $h_\mu(F) = -(\widehat{i} + \widehat{j}) \sum_{k=0}^{m-1} p_k \log p_k$*. i.e.,*

(i) *If* f *is leftmost permutive and* $i < j \le 0$*, then* $h_\mu(F) = -\widehat{i} \sum_{k=0}^{m-1} p_k \log p_k$*;*

(ii) *If* f *is rightmost permutive and* $0 \le i < j$*, then* $h_\mu(F) = -\widehat{j} \sum_{k=0}^{m-1} p_k \log p_k$*;*

(iii) *If* f *is bipermutive, then* $h_\mu(F) = -(\widehat{i} + \widehat{j}) \sum_{k=0}^{m-1} p_k \log p_k$.

2.3 Topological Entropy of Cellular Automata

Definition 2.9. *Let* $(\mathbf{X}, F)$ *be a continuous function and* $\mathcal{P}$ *be an open cover of* $\mathbf{X}$*, define*

$$H(\mathcal{P}) = \inf\{\log card\widehat{\mathcal{P}}\}$$

where the infimum is taken over the set of finite subcover $\widehat{\mathcal{P}}$ *of* $\mathcal{P}$ *and card*$\widehat{\mathcal{P}}$ *is the cardinality of* $\widehat{\mathcal{P}}$*.*

Definition 2.10. *If* F *be a linear 1-dimension CA over* $\mathbf{X}$ *and* $\mathcal{P}$ *be the open cover of* $\mathbf{X}$*, define the topological entropy of* $\mathcal{P}$ *is*

$$h_{top}(F, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} F^{-i}\mathcal{P}\right).$$

The topological entropy of F *is*

$$h_{top}(F) = \sup_{\mathcal{P}} h_{top}(F, \mathcal{P}).$$

Theorem 2.11. *([2])*
If f *be permutive and depends only on* $x_i, \cdots, x_j$*, where* $i \le j$*,* $i, j \in \mathbb{Z}$*. Denote by* $\widehat{i} = -\min\{i, 0\}$ *and* $\widehat{j} = \max\{j, 0\}$*, then* $h_{top}(F) = (\widehat{i} + \widehat{j}) \log m$*. i.e.,*

(i) *If* f *is leftmost permutive and* $i < j \le 0$*, then* $h_{top}(F) = \widehat{i} \log m$*;*

(ii) *If* f *is rightmost permutive and* $0 \le i < j$*, then* $h_{top}(F) = \widehat{j} \log m$*;*

(iii) *If* f *is bipermutive, then* $h_{top}(F) = (\widehat{i} + \widehat{j}) \log m$*.*

2.4 Variational Principle

Theorem 2.12 (Boltzmann; Variational Principle)**.**

$$h_{top}(F) = \sup_{\mu} h_{\mu}(F)$$

In general, we know that $h_{top}(F) \ge h_{\mu}(F)$ for any probability measure μ. To consider the Variational Principle, we give the following definition:

Definition 2.13. *If* $(\mathbf{X}, \mathcal{B}, \mu)$ *be a probability space, and* $F : \mathbf{X} \to \mathbf{X}$ *be a measure preserving transformation. If there exist a probability measure* μ *such that* $h_{\mu}(F) = h_{top}(F)$*, then* μ *is the maximum measure.*

3 Entropy

In this subsection, we consider the entropy of the additive CA. By reprove some results from [1] with different method, we have the entropy formula of the additive CA. By [6] we have the following lemma:

Proposition 3.1. *If* $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$*, and the dipolynomial of* f *is* $\mathbb{T}(f) = \sum_{i=l}^{r} a_i x^{-i}$*, then* $\mathbb{T}$ *is bijective.*

Proof. Denote by $\mathfrak{L}$ be the collection of additive local rules. For each $f \in \mathfrak{L}$, define $\mathbb{T} : \mathfrak{L} \to \mathbb{Z}_m[[x, x^{-1}]]$ by $\mathbb{T}(f) = \sum_{i=l}^{r} a_i x^{-i}$, $i \in \mathbb{Z}$. We may check:

(i) linear:

$\forall f = \sum_{i=l}^{r} a_i x_i, g = \sum_{i=l}^{r} b_i x_i \in \mathfrak{L}$,

$$\begin{aligned}
\mathbb{T}(\alpha f + g) &= \mathbb{T}(\alpha \sum_{i=l}^{r} a_i x_i + \sum_{i=l}^{r} b_i x_i) \\
&= \mathbb{T}(\sum_{i=l}^{r} (\alpha a_i + b_i) x_i) \\
&= \sum_{i=l}^{r} (\alpha a_i + b_i) x^{-i} \\
&= \alpha \sum_{i=l}^{r} a_i x^{-i} + \sum_{i=l}^{r} b_i x^{-i} \\
&= \alpha \mathbb{T}(f) + \mathbb{T}(g).
\end{aligned}$$

(ii) one-to-one:

$\forall \mathbb{T}(f) \neq \mathbb{T}(g)$, we have $\sum_{i=l}^{r} a_i x^{-i} = f \neq g = \sum_{i=l}^{r} b_i x^{-i}$ and $a_i \neq b_i$ for some i, so $\sum_{i=l}^{r} a_i x_i \neq \sum_{i=l}^{r} b_i x_i$.

(iii) Onto:

$\forall \sum_{i=l}^{r} a_i x_{-i} \in \mathbb{Z}_m[[x, x^{-1}]]$, $\exists f = \sum_{i=l}^{r} a_i x_i$ such that $\mathbb{T}(f) = \sum_{i=l}^{r} a_i x^{-i}$.

Hence, $\mathbb{T}$ is bijective. □

The following proposition is easy to check.

Proposition 3.2. *If* $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$, *define* $\widehat{\chi} : \mathcal{S}^{\mathbb{Z}} \to \mathbb{Z}_m[[x, x^{-1}]]$ *by* $\widehat{\chi}(b_n) = \sum_{i=-\infty}^{\infty} b_i x^i$, $\forall (b_n) \in \mathcal{S}^{\mathbb{Z}}$. *Then* $\widehat{\chi}$ *is a bijection.*

Proof. Similar to the proof of Proposition 3.1. □

Proposition 3.3. *If* $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$, *and*

$$\begin{array}{ccc}
\mathcal{S}^{\mathbb{Z}} & \xrightarrow{\;F\;} & \mathcal{S}^{\mathbb{Z}} \\
\downarrow{\scriptstyle \widehat{\chi}} & & \downarrow{\scriptstyle \widehat{\chi}} \\
\mathbb{Z}_m[[x, x^{-1}]] & \xrightarrow[\;\widetilde{\mathbb{T}}\;]{} & \mathbb{Z}_m[[x, x^{-1}]]
\end{array}$$

where $\mathbb{T}$ *and* $\widehat{\chi}$ *is defined as Proposition 3.1 and Proposition 3.2, and* $\widetilde{\mathbb{T}} = g \cdot \mathbb{T}$, $\forall g(x) = \sum_{i=l}^{r} a_i x^i \in \mathbb{Z}_m[[x, x^{-1}]]$. *Then the diagram commute.*

Proof. $\forall b_n \in \mathcal{S}^{\mathbb{Z}}$,
$\widehat{\chi}F(b_n) = \widehat{\chi}\left[(\sum_{i=l+n}^{r+n} a_{i-n}b_i)_n\right] = \sum_{n=-\infty}^{\infty}\sum_{i=l+n}^{r+n} a_{i-n}b_i x^n$
$\widetilde{\mathbb{T}}\widehat{\chi}(b_n) = \mathbb{T}\cdot\left(\sum_{n=-\infty}^{\infty} b_n x^n\right) = \sum_{i=l}^{r} a_i x^{-i}\left(\sum_{n=-\infty}^{\infty} b_n x^n\right)$
$\forall k \in \mathbb{N}$, for the component x^k of $\widehat{\chi}F(b_n)$ the coefficient is $\sum_{i=l+k}^{r+k} a_{i-k}b_i$, and the component x^k of $\widetilde{\mathbb{T}}\widehat{\chi}(b_n)$ the coefficient is $\sum_{i=l+k}^{r+k} a_{i-k}b_i$, so $\widehat{\chi}F(b_n) = \widetilde{\mathbb{T}}\widehat{\chi}(b_n)$. Hence, the diagram commute. $\square$

We give an example for Proposition 3.3

Example 3.4. *If $f(x_{-1}, x_0, x_1) = x_1$, $m = 2$, then $\mathbb{T}(f) = x^{-1}$, and we have the following diagram*

$$\begin{array}{ccc} \mathcal{S}^{\mathbb{Z}} & \xrightarrow{F} & \mathcal{S}^{\mathbb{Z}} \\ \widehat{\chi}\downarrow & & \downarrow\widehat{\chi} \\ \mathbb{Z}_m[[x, x^{-1}]] & \xrightarrow[\widetilde{\mathbb{T}}]{} & \mathbb{Z}_m[[x, x^{-1}]] \end{array}$$

$\forall (b_n) \in \mathcal{S}^{\mathbb{Z}}$, $\widehat{\chi}F(b_n) = \widehat{\chi}(b_{n+1}) = \sum_{i=-\infty}^{\infty} b_{i+1}x^i$ and $\widetilde{\mathbb{T}}\widehat{\chi}(b_n) = \mathbb{T}\cdot\left(\sum_{i=-\infty}^{\infty} b_i x^i\right) = \sum_{i=-\infty}^{\infty} b_i x^{i-1} = \widehat{\chi}F(b_n)$, so the diagram commute. And by [4] we have $\widehat{\chi}$ is bijective, hence F and $\widetilde{\mathbb{T}}$ have bijective relation.

3.1 Measure-theoretical Entropy

Theorem 3.5. *Let F be the 1-dimension CA over $\mathcal{S}^{\mathbb{Z}}$ with local rule $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$ and let $m = p_1^{k_1} \cdots p_n^{k_n}$ denote the prime factor decomposition of m. For $i = 1, \cdots, n$ define*

$$\widehat{p_i} = \{0\} \cup \{j : \gcd(a_j, p_i) = 1\},\ \widehat{l_i} = \min \widehat{p_i},\ \widehat{r_i} = \max \widehat{p_i}.$$

If μ is F-invariant and $\mu \in \mathfrak{F}$ where $\mathfrak{F} = \{\mu : (\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_{p_1^{k_1}} \times \cdots \times \mathbf{X}_{p_n^{k_n}}, \mathcal{B}_{p_1^{k_1}} \times \cdots \times \mathcal{B}_{p_n^{k_n}}, \mu_{p_1^{k_1}} \times \cdots \times \mu_{p_n^{k_n}}, F_{p_1^{k_1}} \times \cdots \times F_{p_n^{k_n}})\}$, then

$$h_\mu(F) = -\sum_{i=1}^{n}(\widehat{r_i} - \widehat{l_i}) \sum_{j=0}^{p_i^{k_i}-1} p_{i_j} \log p_{i_j}.$$

where $p_{i_j} = \sum_{l=0}^{m_i - 1} p_j + l p_i^{k_i}$, $m_i = \frac{m}{p_i^{k_i}}$.

To proof our main Theorem 3.5, we may prove the following lemma.

Lemma 3.6. *Let F be a 1-dimension CA over $\mathcal{S}^{\mathbb{Z}}$, where $\mathcal{S} = \{0, 1, \cdots, p^k - 1\}$ and p is a prime with local rule $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$, $m = p^k$. Let*

$$\widehat{p} = \{0\} \cup \{j : (a_j, p) = 1\},\ \widehat{l} = \min \widehat{p},\ \widehat{r} = \max \widehat{p}$$

Then,

$$h_\mu(F) = -(\widehat{r} - \widehat{l}) \sum_{i=0}^{p^k-1} p_i \log p_i$$

Proof. By Proposition 3.3, we associate to the dipolynomial of f is $\mathbb{T}(f) = \sum_{i=l}^{r} a_i x^{-i}$. Then the dipolynomial associated to $f^n(x)$ is $\mathbb{T}^n(f)$. Let $\mathbb{T}(f) = \mathbb{T}_1(f) + p\mathbb{T}_2(f)$, where $\mathbb{T}_1(f)$ contains all monomials whose coefficiets are coprime with p.
Claim. : $(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^i} \equiv (\mathbb{T}_1(f))^{p^i} \pmod{p^{i+1}}$
when $i = 1$, then

$$
\begin{aligned}
(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^p &= \sum_{j=0}^{p} \binom{p}{j} (\mathbb{T}_1(f))^j (p\mathbb{T}_2(f))^{p-j} \\
&\equiv \sum_{j=p-1}^{p} \binom{p}{j} (\mathbb{T}_1(f))^j (p\mathbb{T}_2(f))^{p-j} \pmod{p^2} \\
&= p(\mathbb{T}_1(f))^{p-1} p\mathbb{T}_2(f) + (\mathbb{T}_1(f))^p \\
&\equiv (\mathbb{T}_1(f))^p \pmod{p^2}
\end{aligned}
$$

Suppose when $i = k$,

$$
(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^k} \equiv (\mathbb{T}_1(f))^{p^k} \pmod{p^{k+1}} \tag{1}
$$

when $i = k + 1$,

$$
\begin{aligned}
(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^{k+1}} &= (\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^k \cdot p} \\
&= \left[(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^k}\right]^p \\
&= \left[qp^{k+1} + (\mathbb{T}_1(f))^{p^k}\right]^p \quad by(1) \\
&= \sum_{j=0}^{p} \binom{p}{j} (qp^{k+1})^j \left[(\mathbb{T}_1(f))^{p^k}\right]^{p-j} \\
&\equiv \sum_{j=0}^{1} \binom{p}{j} (qp^{k+1})^j \left[(\mathbb{T}_1(f))^{p^k}\right]^{p-j} \pmod{p^{k+2}} \\
&= \left[(\mathbb{T}_1(f))^{p^k}\right]^p + p(qp^{k+1}) \left[(\mathbb{T}_1(f))^{p^k}\right]^{p-1} \\
&\equiv (\mathbb{T}_1(f))^{p^{k+1}} \pmod{p^{k+2}}
\end{aligned}
$$

By Induction, we have

$$
(\mathbb{T}_1(f) + p\mathbb{T}_2(f))^{p^i} \equiv (\mathbb{T}_1(f))^{p^i} \pmod{p^{i+1}}, \ \forall i \in \mathbb{N}
$$

Take $n = p^{k-1}$, so that

$$
\mathbb{T}^n(f) \equiv \mathbb{T}_1^n(f) \pmod{p^k}
$$

It is easy to see that f^n is permutive. i.e., $f^n(x_{nl}, \cdots, x_{nr}) = \sum_{i=n\widehat{l}}^{n\widehat{r}} b_i x_i$ *with* $(b_{n\widehat{l}}, p) = 1 = (b_{n\widehat{r}}, p)$. By Theorem 2.8, we have $h_\mu(F^n) = -n(\widehat{r} - \widehat{l}) \sum_{i=0}^{p^k - 1} p_i \log p_i$. Hence,

$$
h_\mu(F) = \frac{h_\mu(F^n)}{n} = -(\widehat{r} - \widehat{l}) \sum_{i=0}^{p^k - 1} p_i \log p_i
$$

$\square$

In general, if f is a additive local rule CA then $\exists n \geq 1$ such that f^n is permutive.

From the proof of Lemma 3.6, we know that $n = p^{k-1}$. The following is a simple example of Lemma 3.6.

Example 3.7. *Let $\mathcal{S} = \{0,1,2,3\}$, μ be the Bernoulli measure and $f(x_{-1}, x_0, x_1, x_2) = 2x_{-1} + x_0 + 3x_1 + 2x_2$. If by Lemma 3.6, we have $\widehat{p} = \{0,1\}$, $\widehat{l} = \{0\}$, $\widehat{r} = \{1\}$, then $\exists n = 2$ such that $f^2(x_0, x_1, x_2) = b_0x_0 + b_1x_1 + b_2x_2$. But it is easy to see that $f^2(x_0, x_1, x_2) = x_0 + 2x_1 + x_2$. Then $h_\mu(F^2) = -2\sum_{i=0}^{3} p_i \log p_i$, we have $h_\mu(F) = -\sum_{i=0}^{3} p_i \log p_i$. Hence, that is the same result with Lemma 3.6. If we take μ is the uniform Bernoulli measure, then $h_\mu(F) = -\sum_{i=0}^{3} \frac{1}{4} \log \frac{1}{4} = 2\log 2$.*

Lemma 3.8.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{X} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathbf{X}_p \times \mathbf{X}_q & \xrightarrow[F_p \times F_q]{} & \mathbf{X}_p \times \mathbf{X}_q \end{array}$$

are diagram commute.

Proof.

$$\begin{array}{ccc} \mathcal{S}^{r-l+1} & \xrightarrow{f} & \mathcal{S} \\ \downarrow{\scriptstyle \phi^{r-l+1}} & & \downarrow{\scriptstyle \phi} \\ \mathcal{S}_p^{r-l+1} \times \mathcal{S}_q^{r-l+1} & \xrightarrow[f_p \times f_q]{} & \mathcal{S}_p \times \mathcal{S}_q \end{array}$$

Let $\mathcal{S}_p = \{0, 1, \cdots, p-1\}, \mathcal{S}_q = \{0, 1, \cdots, q-1\}$, $f_i : \mathcal{S}_i^{r-l+1} \to \mathcal{S}_i$, for $i = p, q$.
Take $\phi(x) = (\mathbf{x}_p, \mathbf{x}_q)$ for $x \in \mathcal{S}$, then $\forall (x_n)_{n=l}^{r} \in \mathcal{S}^{r-l+1}$

$$\begin{aligned} (f_1 \times f_2)\phi^{r-l+1}(x_n) &= (f_p \times f_q)(x_{p;n} \times x_{q;n}) \\ &= (f_p \times f_q)(x_{p;n} \times x_{q;n}) \\ &= \left(\sum_{i=l}^{r} a_i x_i \pmod p, \sum_{i=l}^{r} a_i x_i \pmod q \right) \end{aligned}$$

and $\phi f(x_n) = \phi\left[\sum_{i=l}^{r} a_i x_i\right] = \left(\sum_{i=l}^{r} a_i x_i \pmod p, \sum_{i=l}^{r} a_i x_i \pmod q\right)$, we have $(f_p \times f_q) \circ \phi^{r-l+1} = \phi \circ f$.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{X} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathbf{X}_p \times \mathbf{X}_q & \xrightarrow[F_p \times F_q]{} & \mathbf{X}_p \times \mathbf{X}_q \end{array}$$

Take $[\Phi(x)]_i = \phi(x_i)$ where $x \in \mathcal{S}^{\mathbb{Z}}$ and $x_i \in \mathcal{S}$, then we have $(f_p \times f_q) \circ \phi^{r-l+1}(x_t) = \phi \circ f(x_t)$ for $x_t \in \mathcal{S}^{r-l+1}$, i.e., $[(F_p \times F_q)(\Phi(x))]_i = [\Phi(F(x))]_i$. Hence $(F_p \times F_q) \circ \Phi = \Phi \circ f$
Hence,

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{X} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathbf{X}_p \times \mathbf{X}_q & \xrightarrow[F_p \times F_q]{} & \mathbf{X}_p \times \mathbf{X}_q \end{array}$$

are diagram commute. $\square$

Definition 3.9. *Let* $(\mathbf{X}, \mathcal{B}_1, \mu_{\mathbf{X}})$ *and* $(Y, \mathcal{B}_2, \mu_Y)$ *be two probability space. Define* μ_Y *is a* $push-forward$ *measure of* $\mu_{\mathbf{X}}$,$\exists \Psi : \mathbf{X} \to Y$ *and* Ψ *is onto such that* $E \subseteq Y$*: measurable iff* $\Psi^{-1}(E) \subseteq \mathbf{X}$*:measurable and* $\mu_Y(E) = \mu_{\mathbf{X}}\left(\Psi^{-1}(E)\right)$*.*

$$\mathbf{X} \xrightarrow[onto]{\Psi} Y$$

Lemma 3.10. $\mathfrak{F} = \{\mu : (\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_p \times \mathbf{X}_q, \mathcal{B}_p \times \mathcal{B}_q, \mu_p \times \mu_q, F_p \times F_q)\}$ *is nonempty where* μ_p *and* μ_q *are push-forward measures of* μ*.*

Proof. By the proof of Lemma 3.8 we know that

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{X} \\ \Phi \downarrow & & \downarrow \Phi \\ \mathbf{X}_p \times \mathbf{X}_q & \xrightarrow[F_p \times F_q]{} & \mathbf{X}_p \times \mathbf{X}_q \end{array}$$

are diagram commute. And since μ_p and μ_q are push-forward measures of μ, we have the following relation:

$$\mathbf{X} \xrightarrow{\Psi_p} \mathbf{X}_p$$

$$\mathbf{X} \xrightarrow{\Psi_q} \mathbf{X}_q$$

where Ψ_p and Ψ_q are onto, and $\mu_p(E) = \mu\left(\Psi_p^{-1}(E)\right)$ and $\mu_q(F) = \mu\left(\Psi_q^{-1}(F)\right), \forall E \subseteq \mathbf{X}_p,\ F \subseteq \mathbf{X}_q$

<u>**Claim.**</u> :

$$\mathbf{X} \xrightarrow{\Psi} \mathbf{X}_p \times \mathbf{X}_q$$

i.e., $\exists \Psi : \mathbf{X} \to \mathbf{X}_p \times \mathbf{X}_q$ and Ψ is onto such that $E \subseteq \mathbf{X}_p \times \mathbf{X}_q$: measurable iff $\Psi^{-1}(E) \subseteq \mathbf{X}$:measurable and $\mu_{\mathbf{X}_p \times \mathbf{X}_q}(E) = \mu_{\mathbf{X}}\left(\Psi^{-1}(E)\right)$

If we take μ be the uniform Bernoulli measure, consider the cylinder set $\forall [a] \times [b] \in \mathbf{X}_p \times \mathbf{X}_q$ by the Chinese Remainder Theorem, $\exists [y] \in \mathbf{X}$ such that $\Psi([y]) = [a] \times [b]$. Since $\mu \circ \Psi^{-1}([a] \times [b]) = \mu([y]) = \frac{1}{m}$, and

$$\begin{aligned} \mu_p \times \mu_q([a] \times [b]) &= \mu_p([a]) \cdot \mu_q([b]) \\ &= \frac{1}{p} \cdot \frac{1}{q} = \frac{1}{m} \end{aligned}$$

we have $\mu \circ \Psi^{-1} = \mu_p \times \mu_q$, i.e., $\mu_p \times \mu_q$ is a push-forward measure of μ, Ψ is bijective and preserve complements, countable unions and intersection, by [6] $(\mathbf{X}, \mathcal{B}, \mu, F)$ and $(\mathbf{X}_p \times \mathbf{X}_q, \mathcal{B}_p \times \mathcal{B}_q, \mu_p \times \mu_q, F_p \times F_q)$ algebras are isomorphic. Hence, $(\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_p \times \mathbf{X}_q, \mathcal{B}_p \times \mathcal{B}_q, \mu_p \times \mu_q, F_p \times F_q)$ where μ is the Bernoulli measure. Furthermore, $\mathfrak{F}$ is nonempty. □

Lemma 3.11. *Let* $\mathcal{S} = \{0, 1, \cdots, m-1\}$ *be a finite alphabet with* $m = pq$ *and* $\gcd(p, q) = 1$*,* F *be a linear 1-dimension CA over* $\mathcal{S}^{\mathbb{Z}}$ *with local rule*
$f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$*. If* $\mu \in \mathfrak{F}$ *is* F*-invariant and*
$\mathfrak{F} = \{\mu : (\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_p \times \mathbf{X}_q, \mathcal{B}_p \times \mathcal{B}_q, \mu_p \times \mu_q, F_p \times F_q)\}$ *where* μ_p *and* μ_q *are push-forward measures of* μ*, then*

$$h_\mu(F) = h_{\mu_p}(F_p) + h_{\mu_p}(F_q)$$

Proof. By [5], and $(\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_p \times \mathbf{X}_q, \mathcal{B}_p \times \mathcal{B}_q, \mu_p \times \mu_q, F_p \times F_q)$. So,

$$h_\mu(F) = h_{\mu_p \times \mu_q}(F_p \times F_q) = h_{\mu_p}(F_p) + h_{\mu_p}(F_q)$$

□

Example 3.12. *Let* $\mathcal{S} = \{0, 1, \cdots, 5\}$ *and* $f(x_{-2}, x_{-1}, x_0, x_1) = x_{-2} + 4x_{-1} + 2x_0 + 5x_1$, $\mu = (p_0, p_1, p_2, p_3, p_4, p_5)$ *be the Bernoulli measure, from Theorem 2.8* $h_\mu(F) = -3\sum_{i=0}^{6} p_i \log p_i$. *And* $\exists \mathbf{X}_2,\ \mathbf{X}_3$ *and* $\mu_2 = (p'_0, p'_1),\ \mu_3 = (p''_0, p''_1, p''_2)$ *are push-forward measures of* μ.
So, we have

$$\begin{cases} p'_0 = p_0 + p_2 + p_4 & \quad p''_0 = p_0 + p_3 \\ p'_1 = p_1 + p_3 + p_5 \quad \text{and} & \quad p''_1 = p_1 + p_4 \\ & \quad p''_2 = p_2 + p_5 \end{cases}$$

and $f_2(x_{-2}, x_1) = x_{-2} + x_1$, $f_3(x_{-2}, \cdots, x_1) = x_{-2} + x_{-1} + 2x_0 + 2x_1$ *are permutive. then* $h_{\mu_2}(F_2) = -3\sum_{i=0}^{1} p'_i \log p'_i$ *and* $h_{\mu_3}(F_3) = -3\sum_{i=0}^{2} p''_i \log p''_i$. *We have*

$$\begin{aligned} h_{\mu_2}(F_2) + h_{\mu_3}(F_3) &= -3\sum_{i=0}^{1} p'_i \log p'_i - 3\sum_{i=0}^{2} p''_i \log p''_i \\ &= -3[(p_0 + p_2 + p_4)\log(p_0 + p_2 + p_4) + (p_1 + p_3 + p_5)\log(p_1 + p_3 + p_5)] \\ &\quad - 3[(p_0 + p_3)\log(p_0 + p_3) + (p_1 + p_4)\log(p_1 + p_4) + (p_2 + p_5)\log(p_2 + p_5)] \end{aligned}$$

where $(\mathbf{X}, \mathcal{B}, \mu, F) \cong (\mathbf{X}_2 \times \mathbf{X}_3, \mathcal{B}_2 \times \mathcal{B}_3, \mu_2 \times \mu_3, F_2 \times F_3)$. *It is evidently Lemma 3.11.*

Example 3.13. *Let* $m = 6$, $f(x_{-2}, x_{-1}, x_0, x_1) = 2x_{-2} + 5x_{-1} + 4x_0 + 3x_1$, *and* $\mu = (p_0, p_1, p_2, p_3, p_4, p_5)$ *be Bernoulli measure, then* $\exists \mathbf{X}_2,\ \mathbf{X}_3$ *and* $\mu_2 = (p'_0, p'_1),\ \mu_3 = (p''_0, p''_1, p''_2)$ *are push-forward measures of* μ.
So, we have

$$\begin{cases} p'_0 = p_0 + p_2 + p_4 & \quad p''_0 = p_0 + p_3 \\ p'_1 = p_1 + p_3 + p_5 \quad \text{and} & \quad p''_1 = p_1 + p_4 \\ & \quad p''_2 = p_2 + p_5 \end{cases}$$

and $f_2(x_{-1}, x_1) = x_{-1} + x_1$, *and* $f_3(x_{-2}, x_{-1}, x_0) = 2x_{-2} + 2x_{-1} + x_0$ *are permutive. Then* $h_{\mu_2}(F_2) = -2\sum_{i=0}^{1} p'_i \log p'_i$ *and* $h_{\mu_3}(F_3) = -2\sum_{i=0}^{2} p''_i \log p''_i$. *Hence,*

$$\begin{aligned} h_\mu(F) &= h_{\mu_2}(F_2) + h_{\mu_3}(F_3) \\ &= -2\sum_{i=0}^{1} p'_i \log p'_i - 2\sum_{i=0}^{2} p''_i \log p''_i \\ &= -2[(p_0 + p_2 + p_4)\log(p_0 + p_2 + p_4) + (p_1 + p_3 + p_5)\log(p_1 + p_3 + p_5)] \\ &\quad - 2[(p_0 + p_3)\log(p_0 + p_3) + (p_1 + p_4)\log(p_1 + p_4) + (p_2 + p_5)\log(p_2 + p_5)] \end{aligned}$$

Proof of Theorem 3.5: The proof easily follows from Lemma 3.6 and Lemma 3.11.

Example 3.14. *Let* $m = 2^2 \times 3 \times 5 = 60$, μ *be the Bernoulli measure,* $\mu_4 = (p'_0, p'_1, p'_2, p'_3)$, $\mu_3 = (p''_0, p''_1, p''_2)$, *and* $\mu_5 = (p'''_0, \cdots, p'''_4)$ *be the push-forward measure of* μ. *Suppose* $f(x_{-2}, \cdots, x_4) = 32x_{-2} + 18x_{-1} + 7x_0 + 43x_1 + x_2 + 25x_3 + 57x_4$. *So,* $f_4(x_{-1}, \cdots, x_4) = 2x_{-1} + 3x_0 + 3x_1 + x_2 + x_3 + x_4$, $f_3(x_{-2}, \cdots, x_3) = 2x_{-2} + x_0 + x_1 + x_2 + x_3$, *and* $f_5(x_{-2}, \cdots, x_4) = 2x_{-2} + 3x_{-1} + 2x_0 + 3x_1 + x_2 + 2x_4$. *Then we have* $h_{\mu_4}(F_4) = -4\sum_{i=0}^{3} p'_i \log p'_i$, $h_{\mu_3}(F_3) = -5\sum_{i=0}^{2} p''_i \log p''_i$, *and* $h_{\mu_5}(F_5) = -6\sum_{i=0}^{4} p'''_i \log p'''_i$. *Hence,*

$$\begin{aligned} h_\mu(F) &= h_{\mu_4}(F_4) + h_{\mu_3}(F_3) + h_{\mu_5}(F_5) \\ &= -4\sum_{i=0}^{3} p'_i \log p'_i - 5\sum_{i=0}^{2} p''_i \log p''_i - 6\sum_{i=0}^{4} p'''_i \log p'''_i. \end{aligned}$$

If μ be the uniform Bernoulli measure, then μ_4, μ_3, and μ_5 are also uniform Bernoulli measure. Hence, $h_\mu(F) = 8\log 2 + 5\log 3 + 6\log 5$.

3.2 Topological Entropy

Theorem 3.15. *Let F be the 1-dimension CA over $\mathcal{S}^{\mathbb{Z}}$ with local rule $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$ and let $m = p_1^{k_1} \cdots p_n^{k_n}$ denote the prime factor decomposition of m. For $i = 1, \cdots, n$ define*

$$\widehat{p_i} = \{0\} \cup \{j : \gcd(a_j, p_i) = 1\}, \ \widehat{l_i} = \min \widehat{p_i}, \ \widehat{r_i} = \max \widehat{p_i}.$$

Then

$$h_{top}(F) = \sum_{i=1}^{n} k_i(\widehat{r_i} - \widehat{l_i}) \log(p_i).$$

To proof our main Theorem 3.15, we may prove the following lemma.

Lemma 3.16. *Let $\mathcal{S} = \{0, 1, \cdots, p^k - 1\}$ be a finite alphabet with p is a prime, and $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$ be any linear local rule defined over $\mathcal{S}^{\mathbb{Z}}$. Let F be the 1-dimension global transition map associated to f. Let $\widehat{p}, \widehat{r}, \widehat{l}$ are defined as Lemma 3.6. Then*

$$h_{top}(F) = k(\widehat{r} - \widehat{l}) \log(p).$$

Proof. By the proof of Lemma 3.6, we have $\exists n \geq 1$ such that the local rule f^n associated to F^n has the form

$$f^n(x_{nl}, \cdots, x_{nr}) = \sum_{i=n\widehat{l}}^{n\widehat{r}} b_i x_i \ \textit{with} \ (b_{n\widehat{l}}, p) = 1 = (b_{n\widehat{r}}, p).$$

That is, f^n is permutive. By Theorem 2.11., we have $h_{top}(F^n) = nk(\widehat{r} - \widehat{l}) \log p$. Hence, $h_{top}(F) = \frac{h_{top}(F^n)}{n} = k(\widehat{r} - \widehat{l}) \log p$ □

Example 3.17. *Let $\mathcal{S} = \{0, 1, 2, 3\}$ and $f(x_{-1}, x_0, x_1, x_2) = 2x_{-1} + x_0 + 3x_1 + 2x_2$. By Example 3.7 and Theorem 2.11, we have $h_{top}(F^2) = 2\log 4$. So $h_{top}(F) = \frac{2\log 4}{2} = \log 4 = 2\log 2$, the result is same as Lemma 3.16.*

Lemma 3.18. *F and $F_p \times F_q$ are topological conjugate.*

Proof.

$$\begin{array}{ccc} \mathcal{S}^{r-l+1} & \xrightarrow{\quad f \quad} & \mathcal{S} \\ \big\downarrow{\scriptstyle \phi^{r-l+1}} & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{S}_p^{r-l+1} \times \mathcal{S}_q^{r-l+1} & \xrightarrow[f_p \times f_q]{} & \mathcal{S}_p \times \mathcal{S}_q \end{array}$$

Let $\mathcal{S}_p = \{0, 1, \cdots, p-1\}, \mathcal{S}_q = \{0, 1, \cdots, q-1\}$, $f_i : \mathcal{S}_i^{r-l+1} \to \mathcal{S}_i$, for $i = p, q$.

(i) By Lemma 3.8, we have

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\quad F \quad} & \mathbf{X} \\ \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \Phi} \\ \mathbf{X}_p \times \mathbf{X}_q & \xrightarrow[F_p \times F_q]{} & \mathbf{X}_p \times \mathbf{X}_q \end{array}$$

the diagram commute.

(ii) $\forall (x_n),\ (y_n) \in \mathcal{S}^{\mathbb{Z}},\ x_n \neq y_n$, then $x_i \neq y_i$ for some $i \in \mathbb{Z}$ we have $[\Phi(x)]_i = \phi(x_i) \neq \phi(y_i) = [\Phi(y)]_i$, so Φ is one-to-one. And $\forall a, b \in \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$ *where* $a = (a_n) \in \mathcal{S}_p^{\mathbb{Z}},\ b = (b_n) \in \mathcal{S}_q^{\mathbb{Z}},\ (a_i)_{i=l}^{r} \times (b_i)_{i=l}^{r} \in \mathcal{S}_p^{r-l+1} \times \mathcal{S}_q^{r-l+1}$, by the Chinese Remainder Theorem, $\exists c = (c_n) \in \mathcal{S}^{\mathbb{Z}},\ (c_i)_{i=l}^{r} \in \mathcal{S}^{r-l+1}$ *such that* $[\Phi(c)]_i = \phi[(c_i)_{i=l}^{r}] = (a_i)_{i=l}^{r} \times (b_i)_{i=l}^{r}$, so Φ is onto. Therefore, Φ is bijective.

(iii) Define the metric on $\mathcal{S}^{\mathbb{Z}}$:

$$d(x,y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{m^{|i|}}, \quad \forall x,\ y \in \mathcal{S}^{\mathbb{Z}}$$

and the metric on $\mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}},\ \forall (x_{p;n}, x_{q;n}), (y_{p;n}, y_{q;n}) \in \mathcal{S}_1^{\mathbb{Z}} \times \mathcal{S}_2^{\mathbb{Z}}$

$$d_p \times d_q\left((x_{p;n}, x_{q;n}), (y_{p;n}, y_{q;n})\right) = \sum_{i=-\infty}^{\infty} \frac{|x_{p;i} - y_{p;i}|}{m^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{|x_{q;i} - y_{q;i}|}{m^{|i|}}$$

Given $x,\ y \in \mathcal{S}^{\mathbb{Z}}$, $\forall \epsilon > 0,\ \exists \delta = \frac{\epsilon}{p+q}$, if $d(x,y) = \sum\limits_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{m^{|i|}} < \delta$, then

$$\begin{aligned}
d_p \times d_q(\Phi(x), \Phi(y)) &= d_p \times d_q\left((x_{p;n}, x_{q;n}), (y_{p;n}, y_{q;n})\right) \\
&= \sum_{i=-\infty}^{\infty} \frac{|x_{p;i} - y_{p;i}|}{m^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{|x_{q;i} - y_{q;i}|}{m^{|i|}} \\
&\leq \sum_{i=-\infty}^{\infty} \frac{p|x_i - y_i|}{m^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{q|x_i - y_i|}{m^{|i|}} \\
&(\mathit{since}\ |x_{p;i} - y_{p;i}| \leq p|x_i - y_i|\ \mathit{and}\ |x_{q;i} - y_{q;i}| \leq q|x_i - y_i|) \\
&= (p+q) \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{m^{|i|}} \\
&< (p+q)\delta = \epsilon
\end{aligned}$$

so Φ is continuous. Similarly, $x,\ y \in \mathcal{S}^{\mathbb{Z}}$, $\forall \epsilon > 0,\ \exists \delta = \frac{\epsilon}{m}$, if

$$d_p \times d_q\left((x_{p;n}, x_{q;n}), (y_{p;n}, y_{q;n})\right) = \sum_{i=-\infty}^{\infty} \frac{|x_{p;i} - y_{p;i}|}{m^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{|x_{q;i} - y_{q;i}|}{m^{|i|}} < \delta$$

then

$$\begin{aligned}
d(\Phi^{-1}(x_{p;n}, x_{q;n}), \Phi^{-1}(y_{p;n}, y_{q;n})) &\leq \sum_{i=-\infty}^{\infty} \frac{m\{|x_{p;i} - y_{p;i}| + |x_{q;i} - y_{q;i}|\}}{m^{|i|}} \\
&= m\left[\sum_{i=-\infty}^{\infty} \frac{|x_{p;i} - y_{p;i}|}{m^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{|x_{q;i} - y_{q;i}|}{m^{|i|}}\right] \\
&< m\delta = \epsilon
\end{aligned}$$

so Φ^{-1} is continuous.

By (i)(ii)(iii) F and $F_p \times F_q$ are topological conjugate. □

Lemma 3.19. *Let $\mathcal{S} = \{0, 1, \cdots, m-1\}$ be a finite alphabet with $m = pq$ and $\gcd(p, q) = 1$, F be a linear 1-dimension CA over $\mathcal{S}^{\mathbb{Z}}$ with local rule* $f(x_l, \cdots, x_r) = \sum_{i=l}^{r} a_i x_i$. *Then*

$$h_{top}(F) = h_{top}(F_p) + h_{top}(F_q)$$

Proof. From Lemma 3.18, F and $F_p \times F_q$ are topological conjugate. Hence,

$$\begin{aligned} h_{top}(F) &= h_{top}(F_p \times F_q) \\ &= h_{top}(F_p) + h_{top}(F_q) \end{aligned}$$

□

Example 3.20. *Let $\mathcal{S} = \{0, 1, \cdots, 5\}$ and $f(x_{-2}, x_{-1}, x_0, x_1) = x_{-2} + 4x_{-1} + 2x_0 + 5x_1$, then we have $f_2(x_{-2}, x_{-1}, x_0, x_1) = x_{-2} + x_1 \pmod 2$ and $f_3(x_{-2}, x_{-1}, x_0, x_1) = x_{-2} + x_{-1} + 2x_0 + 2x_1 \pmod 3$. Since f, f_2 and f_3 are permutive, by Theorem 2.11 we have $h_{top}(F) = 3\log 6$, $h_{top}(F_2) = 3\log 2$ and $h_{top}(F_3) = 3\log 3$. So $h_{top}(F) = 3\log 6 = 3\log 2 + 3\log 3 = h_{top}(F_2) + h_{top}(F_3)$. It is evidently Lemma 3.19.*

Example 3.21. *Let $m = 6$, $f(x_{-2}, x_{-1}, x_0, x_1) = 2x_{-2} + 5x_{-1} + 4x_0 + 3x_1$. Then, we have $f_2(x_{-1}, x_1) = x_{-1} + x_1$, and $f_3(x_{-2}, x_{-1}, x_0) = 2x_{-2} + 2x_{-1} + x_0$ are permutive. So, $h_{top}(F_2) = 2\log 2$ and $h_{top}(F_3) = 2\log 3$. $h_{top}(F) = h_{top}(F_2) + h_{top}(F_3) = 2\log 2 + 2\log 3$.*

Compare with Example 3.13, we have

$$\begin{aligned} h_\mu(F) = &-2[(p_0 + p_2 + p_4)\log(p_0 + p_2 + p_4) + (p_1 + p_3 + p_5)\log(p_1 + p_3 + p_5)] \\ &-2[(p_0 + p_3)\log(p_0 + p_3) + (p_1 + p_4)\log(p_1 + p_4) + (p_2 + p_5)\log(p_2 + p_5)] \end{aligned}$$

and we have the probability measure constraint $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$. We use the method of Lagrange Multipliers, we have

$$(p_0,\, p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (p_0,\, p_1,\, \frac{1}{6} - p_0 + p_1,\, \frac{1}{3} - p_0,\, \frac{1}{3} - p_1,\, \frac{1}{6} + p_0 - p_1).$$

then $h_\mu(F) = h_{top}(F)$. Since μ is isomorphic to $\mu_2 \times \mu_3$, $\mu([0]) = \mu_2([0]) \times \mu_3([0])$ and $\mu([1]) = \mu_2([1]) \times \mu_3([1])$.
We have

$$\begin{cases} p_0 = (p_0 + p_2 + p_4) \times (p_0 + p_3) \\ p_1 = (p_1 + p_3 + p_5) \times (p_1 + p_4) \end{cases}$$

So, $p_0 = p_1 = p_2 = p_3 = p_4 = p_5 = \frac{1}{6}$. Hence, the uniform Bernoulli measure is a maximum measure.

Proof of Theorem 3.15: The proof easily follows from Lemma 3.19 and Lemma 3.16.

Example 3.22. *Let $m = 2^2 \times 3 \times 5 = 60$, consider the local rule $f(x_{-2}, \cdots, x_4) = 32x_{-2} + 18x_{-1} + 7x_0 + 43x_1 + x_2 + 25x_3 + 57x_4$. From Theorem 3.15. we have $\widehat{p}_4 = \{0, 1, 2, 3, 4\}$, $\widehat{p}_3 = \{-2, 0, 1, 2, 3\}$, $\widehat{p}_5 = \{-2, -1, 0, 1, 2, 4\}$, $\widehat{l}_4 = 0$, $\widehat{l}_3 = -2$, $\widehat{l}_5 = -2$, $\widehat{r}_4 = 4$, $\widehat{r}_3 = 3$, $\widehat{r}_5 = 4$ So, $h_{top}(F_4) = 8\log 2$, $h_{top}(F_3) = 5\log 3$, $h_{top}(F_5) = 6\log 5$, and then $h_{top}(F) = 9\log 2 + 10\log 3$.*

This example compared with Example 3.14, if we take the uniform Bernoulli measure we have $h_{top}(F) = h_\mu(F)$. The uniform Bernoulli measure is the maximum measure.

Now, we know that the uniform is the maximum measure. And by the above discussion, we give a conjecture that the maximum measure is unique.

4 Conclusion

- We give a formula for the measure-theoretic entropy and the topological entropy of the additive CA. After having the formula, there is more question to consider about the additive CA and the ergodic properties.
- The uniform Bernoulli measure is a maximum measure. We guess that the unique maximum measure is uniform Bernoulli measure.

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