# 國 立 交 通 大 學

## 應用數學系

## 碩 士 論 文

線性橢圓偏微分方程之研究

Topics on Linear Elliptic Partial Differential Equations

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指導教授：李榮耀　博士

中 華 民 國 九 十 八 年 六 月

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## 摘要

本文的目標是探討一些常用於解決線性橢圓偏微分方程的古典方法及應用。首先，我們給一個有關於在靜電勢中Laplace方程的實際例子並利用有限元素法解之。再來介紹常用的古典解題技巧，像是在不同定義域中分離變數法的使用以及有限與無限空間的傅立葉轉換。最後我們介紹數值方法中的有限差分法並藉助軟體Mathematica去計算一個擁有Dirichlet邊界條件的Laplace方程問題 。

Topics on Linear Elliptic Partial Differential Equations

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# Abstract

The aim of this paper is to investigate several classical methods and applications of the linear elliptic partial differential equations. First, a practical example is given based on the Laplace's equation for the electrostatic potential, and is solved by Finite element method. Secondly, classical solving techniques are introduced, such as separation of variables in different domains, and Fourier transforms in both finite and infinite domains. At last, numerical Finite difference method is introduced to solve the Laplace's equation on a square with nonhomogeneous Dirichlet boundary condition, which is computed by Mathematica.

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# Contents

# Chapter 1

# Introduction

The linear second-order partial differential equation is often facilitated by a recognition of the type of the differential equation in question. For, depending on the type of the equation, it is frequently possible by means of a coordinate transformation to reduce the equation to one of three canonical forms. These canonical forms correspond to different simple forms in which the second-order derivative terms can appear in the equation. Moreover, the type of a partial differential equation plays a decisive role in determining the kind of auxiliary condition that can be considered with the equation so that the resulting problem has a unique solution.

For purposes of classification, it is not necessary to restrict consideration to linear equation. The classification is also valid for equations that are linear only in their second derivatives. Such equation, in two independent variables, are the form

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y) \qquad (x,y) \in \text{domain } \Omega \tag{1.1}$$

where f is any function of its arguments whatsoever. It is assumed that a, b and c have continuous first partial derivatives in $\Omega$ and that these coefficient functions do not all vanish simultaneously. We classify such PDEs into one of three types-**elliptic**, **parabolic** or **hyperbolic**, each type of PDE displays characteristics quite distinct from the others.

At a point (x,y), partial differential equation (1.1) is said to be

$$\text{hyperbolic} \quad \text{if} \quad b^2 - 4ac > 0; \tag{1.2}$$

$$\text{parabolic} \quad \text{if} \quad b^2 - 4ac = 0; \tag{1.3}$$

$$\text{elliptic} \quad \text{if} \quad b^2 - 4ac < 0. \tag{1.4}$$

Elliptic differential equations typically occur in problems that describe stationary situations; i.e., time has no explicit role. The *Laplacian* $\nabla^2 := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is called the Laplace operator in n-dimensional space. Sometimes we may write $\triangle$ instead. The simplest and most well know elliptic equation is the *Laplace's equation*, defined on a domain $\Omega \subset \mathbb{R}^d$(d=1,2,3), say:

$$\nabla^2 u = 0, \qquad \mathbf{x} \in \Omega. \tag{1.5}$$

In the inhomogeneous case we have the *Poisson's equation*

$$\nabla^2 u = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{1.6}$$

A further type that is often encountered is the *Helmholtz's equation*, which is actually related to the eigenvalue problem of (1.5):

$$\nabla^2 u - \lambda u = 0, \qquad \mathbf{x} \in \Omega,\ \lambda \in \mathbb{R}. \tag{1.7}$$

Any solution in $C^2(\Omega)$ of the equation (1.5) is called a *harmonic functin*. The operator also occurs quite often in time-dependent problems like the heat equation of the wave equation (see [6]).

In order to define a solution more precisely, we have to specify a boundary condition. Three common cases are distinguished for x∈ $\partial\Omega$

$$u(\mathbf{x}) = a(\mathbf{x}) \qquad (\textit{Dirichlet}), \tag{1.8}$$

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = b(\mathbf{x}) \qquad (\textit{Neumann}), \tag{1.9}$$

$$\alpha u(\mathbf{x}) + \beta \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = c(\mathbf{x}), \quad \alpha, \beta \neq 0 \qquad (\textit{Robin}). \tag{1.10}$$

The boundary condition of (1.8) is often called Dirichlet condition or boundary condition of the first kind. For example, the problem can be interpreted as finding the equilibrium temperature in $\Omega$ when a fixed temperature distribution is given on the $\partial\Omega$.

The boundary conditon of (1.9) is called the Neumann condition or boundary condition of the second kind. The $\frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u$ where $\mathbf{n}$ is the outward unit normal on $\partial\Omega$. Such a problem arises, for example, when we determine the equilibrium temperature in a uniform body $\Omega$ for which a given amount of heat is supplied to the $\partial\Omega$. And the (1.10) is called a Robin(mixed) boundary condition. Physically, we allow for radiation of heat from the boundary of the body into the surrounding medium. We can easily establish the uniqueness of a solution of (1.5) and either one of the two boundary conditions (1.9) or (1.10) (see [8]).

Elliptic partial equation has many applications in engineering, physics, medical and material science. For example, the measurement of the cortical thickness, the estimate of the electrostatic field around human head in front of Video display unit, state decomposition in microwave tube, *Navier-Stokes equation* in incompressible fluid and so on.

Next, we offer the math model being excerpted from [9], [10] and introduce the techniques of separation of variables; Fourier transforms in both finite and infinite domains; Green's functions and cosine transform in different domain. And last, we use numerical method (Finite difference method) to solve the Dirichlet problem (2.3) and compute it by **Mathematica**.

## 1.1 Laplace's equation as a mathematical model

Now, we offer the math model which is excerpted from [9], [10]. Video display units (VDU's) based on cathode ray tube (CRT) are sources of several types of radiation e.g. X ray radiation, optical radiation electromagnetic radiation and electrostatic field. Along with the expanding use of VDU's some concerns about the effect of these field on the human health have appeared. Over the years of work it has been proven that levels of X ray radiation, optical radiation, high(MHz) and low (kHz) frequency electromagnetic fields stay well below technical guidelines, hence, not considered to be harmful for human health. On the other hand, extremely low frequency (ELF) electromagnetic and electrostatic fields might be associated with some skin diseases, suppression of melatonin, or induction of phosphenes in the eyes. Nevertheless, there is no strong evidence of adverse health effects from domestic levels of ELF electromagnetic fields.

The realistic, three-dimensional, anatomically based model of the human head exposed to electrostatic field from VDU is proposed. The electrostatic field around the human head is assessed using the standard Finite Element Method (FEM). Special attention is focused to the correlation between field strength and distance between display and head. The motivation for studying this problem is lack of information of electrostatic field at the surface of human face, and its nature. That information then can be used for solving problems like particle deposition on human face and eyes.

Assuming the charge density to be negligible, the 3D electrostatic field between a VDU and human head is governed by the Laplace's equation for the electric potential $\varphi$ :

$$\nabla^2 \varphi = 0 \tag{1.11}$$

with the associated boundary conditions :

$$\varphi = \varphi_s \quad \text{on the display,} \tag{1.12}$$

where $\varphi_s$ denotes the electric potential on the display boundary,

$$\varphi = \varphi_h \quad \text{on the head,} \tag{1.13}$$

where $\varphi_h$ denotes the electric potential on the head boundary, and

$$\nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on the far field boundaries.} \tag{1.14}$$

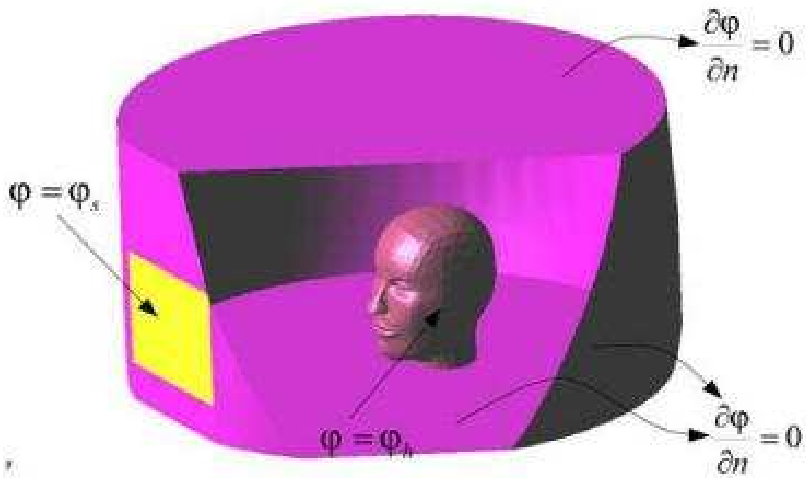


Figure 1.1: Geometry and boundary conditions for numerical 3D model of human seated in front of a VDU.

As shown in Figure(1.1), the Dirichlet boundary conditions (1.12) and (1.13) are applied on the face and the display, while the Neumann conditions (1.14) are imposed on all remaining exterior boundaries. It is worth noting that the head is considered to be the perfect conductor thus being itself an equipotential surface with potential $\varphi_h$ determinated by the boundary condition (1.13).

### 1.1.1 Model properties

Presented formulation requires input parameters $\ell_s,\ d_s,\ \varphi_s$ and $\varphi_h$. They represent distance between display and nose tip, size of display (diagonal display size given in inches), electrostatic potential on display and electrostatic potential on head, respectively. The mean electric potential on a CRT monitors is in the range of $1-15$kV. In this case the electric potential of the display is assumed to be very high (15 kV) and it is regarded as worst-case scenario. A parameter study is performed in accordance to a data given in the Table 1.1

It is important to note that, the eyebrows are considered to be at the same potential as the face. Other parameters such as temperature, humidity and conductivity of the surface of the screen glass are not considered in order to simplify the model. But, it has to be underlined that air humidity is very important factor, and in this calculations is considered very low (dry air).

### 1.1.2 Numerical solutions

Applying the weighting residual approach to equation (1.11) yields:

$$\iiint_\Omega \nabla^2\varphi \, W_j \, \mathrm{d}\Omega = 0 \quad \text{where } W_j \text{ is the weight function.} \tag{1.15}$$

By using the weak formulation of the problem:

$$\nabla^2\varphi W_j = \nabla(\nabla\varphi W_j) - \nabla\varphi\nabla W_j \tag{1.16}$$

and generalized Gauss theorem:

$$\iiint_\Omega \nabla\cdot\overrightarrow{F}\,\mathrm{d}\Omega = \oiint_\Gamma \overrightarrow{F}\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{1.17}$$

were $\overrightarrow{F}$ is an arbitrary vector function, then the integral formulation (variational equation) of the Laplace's equation is obtained:

$$\iiint_\Omega \nabla\varphi\nabla W_j\,\mathrm{d}\Omega = \oiint_\Gamma \frac{\partial\varphi}{\partial\mathbf{n}} W_j\,\mathrm{d}\Gamma \tag{1.18}$$

Then by using the Galerkin-Bubnov procedure $(W_j = N_j)$, equation (1.18) becomes:

$$\iiint_\Omega \nabla\varphi\nabla N_j\,\mathrm{d}\Omega = \oiint_\Gamma \frac{\partial\varphi}{\partial\mathbf{n}} N_j\,\mathrm{d}\Gamma \tag{1.19}$$

Furthermore, taking into account nonhomogenus Neumann condition on the part of the boundary $\frac{\partial\varphi}{\partial n} = 0$, equation (1.19) simplifies into:

$$\iiint_\Omega \nabla\varphi\nabla N_j\,\mathrm{d}\Omega = 0 \tag{1.20}$$

Using finite element algorithm, the unknown solution (potential) on the element is expressed by linear combination of shape functions:

$$\widetilde{\varphi}^e = \sum_{i=1}^{4} \alpha_i N_i \tag{1.21}$$

| | $\ell_s$ (m) | $d_s$ (in.) | $\varphi_s$ (kV) | $\varphi_h$ (kV) |
|---|---|---|---|---|
| Standard condition | 0.4 | 14 | 15 | 0 |

Table 1.1: Standard condition for a person seated in front of a VDU.

or using matrix notation:

$$\widetilde{\varphi}^e = \{\mathbf{N}\}^T\{\alpha\} = [N_1 \;\; N_2 \;\; N_3 \;\; N_4] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{1.22}$$

where vector $\{\alpha\}$ represents unknown coefficients of the solution, and shape functions are given by:

$$N_i(x,y,z) = \frac{1}{D}(V_i + a_i x + b_i y + c_i z) \quad \text{for i=1,2,3,4}, \;\; (x,y,z) \in e \tag{1.23}$$

Combining equations (1.20) to (1.23) results in the global matrix system:

$$[a]\{\alpha\} = \{Q\} \tag{1.24}$$

where $\{Q\}$ represents the flux vector. Global matrix $[a]$ is composed from local finite element matrices $[a]^e_{ij}$.

Potential gradient is defined as:

$$\nabla\widetilde{\varphi} = \frac{\partial\widetilde{\varphi}}{\partial x}\overrightarrow{e}_x + \frac{\partial\widetilde{\varphi}}{\partial y}\overrightarrow{e}_y + \frac{\partial\widetilde{\varphi}}{\partial z}\overrightarrow{e}_z \tag{1.25}$$

and expressed in terms of shape functions is given by:

$$\nabla\widetilde{\varphi} = \begin{bmatrix} \frac{\partial\widetilde{\varphi}}{\partial x} \\ \frac{\partial\widetilde{\varphi}}{\partial y} \\ \frac{\partial\widetilde{\varphi}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_4}{\partial z} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}. \tag{1.26}$$

Combining equation (1.20) and (1.26) yields the finite element matrix :

$$[a]^e = \iiint_{\Omega^e} \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_4}{\partial z} \end{bmatrix} \mathrm{d}\Omega. \tag{1.27}$$

Having computed the scalar potential distribution one may determine the electrostatic field from expression:

$$\overrightarrow{E} = -\nabla\varphi. \tag{1.28}$$

# Chapter 2

# The methods of solving Elliptic PDEs

## 2.1 Separation of variables to construct solutions of system of Laplace's equation.

### 2.1.1 The domain is a rectangular (dimension $= 2$) in orthogonal coordinates.

Consider

$$u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < y < \pi. \tag{2.1}$$

We may apply separation of variables to the equation (2.1) with the form $u(x,y) = X(x)Y(y)$ and divide by u. The process gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y}. \tag{2.2}$$

The left-hand side of the equation (2.2) depends only upon x and the right-hand side is independent of x. If we take the partial derivative with respect to x,

we find that

$$\frac{d}{dx}[\frac{X''}{X}] = 0.$$

It follows that

$$\frac{X''}{X} = -\lambda \quad \text{where } \lambda \text{ is a constant.}$$

By the equation (2.2), we find

$$\frac{Y''}{Y} = \lambda.$$

Thus $u = X(x)Y(x)$ is a solution of Laplace's equation if and only if X(x) and Y(y) satisfy the two ordinary differential equations

$$\begin{cases} X'' + \lambda X = 0, \\ Y'' - \lambda Y = 0. \end{cases}$$

The homogeneous problem always has the trivial solution $X \equiv 0$, but this is no use to us. We are interested in case to find the non-trivial solution of X(x). Since we can see the relationship between $\lambda$ and X(x) in the previous paper [1], we focus only on the changing of the boundary condition.

<u>Case1</u>

$$u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < y < \pi, \tag{2.3}$$

$$u(0, y) = u(\pi, y) = u(x, \pi) = 0, \tag{2.4}$$

$$u(x, 0) = x^2(\pi - x). \tag{2.5}$$

If we consider those solutions of the equation which satisfy these conditions, we must have

$$X'' + \lambda X = 0 \qquad \text{for} \quad 0 < x < \pi, \tag{2.6}$$

$$X(0) = X(\pi) = 0. \tag{2.7}$$

and

$$Y'' - \lambda Y = 0 \qquad \text{for} \quad 0 < y < \pi, \tag{2.8}$$

$$Y(\pi) = 0. \tag{2.9}$$

Considering the equation (2.6), the general solution is $X(x) = a \sin \sqrt{\lambda}x + b \cos \sqrt{\lambda}x$ where a, b are determined to satisfy boundary conditions (2.7).

So we have

$$\begin{cases} X(0) = a \times 0 + b \times 1 = b = 0, \\ X(\pi) = a\ \sin \sqrt{\lambda}\pi = 0. \end{cases} \tag{2.10}$$

It implies either

$$a = 0 \quad \Longrightarrow X(x) = 0 \quad \text{(trivial solution)}$$

or

$$\sin\sqrt{\lambda}\pi = 0 \Longrightarrow \sqrt{\lambda}\pi = n\pi \Longrightarrow \lambda = n^2\text{,n=1,2,...}$$

We have $\lambda = n^2 = \lambda_n$. Thus $X_n(x) = \sin\sqrt{\lambda_n}x$ is the solution of X(x) system (2.6), (2.7) when $\lambda = n^2 = \lambda_n$.

In equation (2.8); for each $\lambda_n$, we have linear independent solutions $e^{\sqrt{\lambda_n}y}$ and $e^{-\sqrt{\lambda_n}y}$. Thus we get the general solution of $Y_n(y) = a\, e^{\sqrt{\lambda_n}y} + b\, e^{-\sqrt{\lambda_n}y}$ where a, b are constants. By boundary condition (2.9), we get $Y_n(\pi) = a\, e^{\sqrt{\lambda_n}\pi} + b\, e^{-\sqrt{\lambda_n}\pi} = 0$. It implies $a = -\frac{e^{-\sqrt{\lambda_n}\pi}}{2}$ and $b = \frac{e^{\sqrt{\lambda_n}\pi}}{2}$.

$$\Rightarrow Y_n(y) = \frac{1}{2}(e^{\sqrt{\lambda_n}\pi}\, e^{-\sqrt{\lambda_n}y} - e^{-\sqrt{\lambda_n}\pi}\, e^{\sqrt{\lambda_n}y}) = \sinh\ \sqrt{\lambda_n}(\pi - y).$$

For each $\lambda_n = n^2, X_n(x) = \sin nx$ and $Y_n(y) = \sinh\ n(\pi - y)$. We have constructed the particular solutions

$$u_n(x, y) = X_n(x)\, Y_n(y) = \sin nx\ \sinh\ n(\pi - y) \tag{2.11}$$

of the system (2.3), (2.4). Any finite linear combinations of $u_n(x, y)$ is also a solution of the system (2.3), (2.4). We attempt to represent the solution u as an infinite series in terms of $u_n$ :

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin nx\ \sinh\ n(\pi - y). \tag{2.12}$$

We need to determine the coefficients $c_n$ so as to satisfy the nonhomogeneous boundary condition (2.5). Thus, setting y=0 in (2.12), the coefficients

$$b_n = c_n \sinh\ n\pi,$$

must satisfy the relation

$$x^2(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx. \tag{2.13}$$

The expansion of an arbitrary function in a series of eigenfunctions is called a **Fourier series**. The particular case where the eigenfunctions are all sines is called a Fourier sine series. If we define

$$< f(x), g(x) > := \int_0^{\pi} f(x)g(x)\,\mathrm{d}x,$$

then

$$b_n = \frac{< x^2(\pi - x), \sin nx >}{< \sin nx, \sin nx >}. \tag{2.14}$$

Thus we find

$$u(x,y) = -4\sum_{n=1}^{\infty}[1+2(-1)^n]n^{-3}\frac{\sinh\ n(\pi-y)}{\sinh n\pi}\sin nx. \tag{2.15}$$

In order to verify the function u represented by the series (2.15), with $b_n$ given by (2.14) is the solution of the problem (2.3), (2.4) and (2.5), we should check whether the nonhomogeneous boundary condition $f(x) = x^2(\pi - x)$ is continuous and piecewise smooth on $[0, \pi]$ and that $f(0) = f(\pi) = 0$. Then the Fourier sine series (2.13) of $x^2(\pi - x)$ coverges absolutely and uniformly to the function on $[0, \pi]$.

Now, for $y \geq 0$,

$$\sum_{n=1}^{\infty}|c_n \sin nx\ \sinh\ n(\pi-y)| \leq \sum_{n=1}^{\infty}\frac{c\, e^{-ny}}{1-e^{-2\pi}} \qquad (c\ = \frac{2}{\pi}\int_0^{\pi}|f(x)|\,\mathrm{d}x = \frac{\pi^3}{6})$$

where the series on the right converges. Therefore, the series (2.15) converges absolutely and uniformly to u(x,y) for $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$. Since each term of the series is continuous and satisfies boundary conditions (2.4), (2.5), it follows that u, too (see [8]).

There remains to be verified that the series (2.15) satisfies the Laplace's equation (2.3). Since the series for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ are all dominated by a constant times $\sum n^2 e^{-ny}$ which is converge uniformly for any $y > 0$, it follows that these derivative of u exist and may be obtained by term-by-term differentiation. We see that

$$u_{xx} + u_{yy} = \sum_{n=1}^{\infty} c_n \sinh\ n(\pi - y)\sin nx[-n^2 + n^2] = 0.$$

The completes the verification that (2.12) with (2.14) is a solution of the problem (2.3), (2.4), (2.5) under the conditions that nonhomogeneous boundary condition is continuous and piecewise smooth on $[0, \pi]$ and vanishes at the end points.

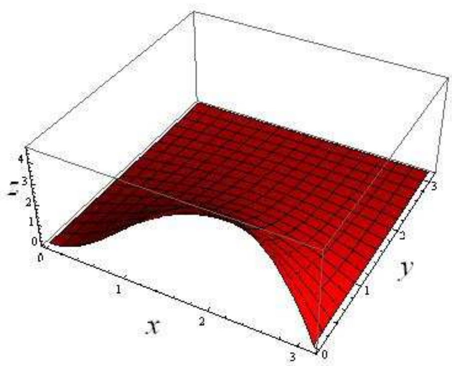


Figure 2.1: The graphic of the case1 where the boundary condition is $u(0, y) = u(\pi, y) = u(x, \pi) = 0.$

Case2

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < y < \pi, \\ u(0,y) = u(\pi,y) = u_y(x,\pi) = 0, \\ u(x,0) = x^2(\pi - x). \end{cases} \tag{2.16}$$

We have

$$\begin{cases} X'' + \lambda X = 0 \qquad \text{for} \quad 0 < x < \pi, \\ X(0) = X(\pi) = 0 \end{cases} \tag{2.17}$$

and

$$\begin{cases} Y'' - \lambda Y = 0 \qquad \text{for} \quad 0 < y < \pi, \\ Y'(\pi) = 0. \end{cases} \tag{2.18}$$

The systems (2.17), (2.18) imply

$$X_n(x) = \sin nx \quad \text{and} \quad Y_n(y) = \cosh\ n(\pi - y).$$

Thus we find

$$u(x,y) = -4\sum_{n=1}^{\infty}[1 + 2(-1)^n]n^{-3}\frac{\cosh\ n(\pi - y)}{\cosh n\pi}\sin nx. \tag{2.19}$$

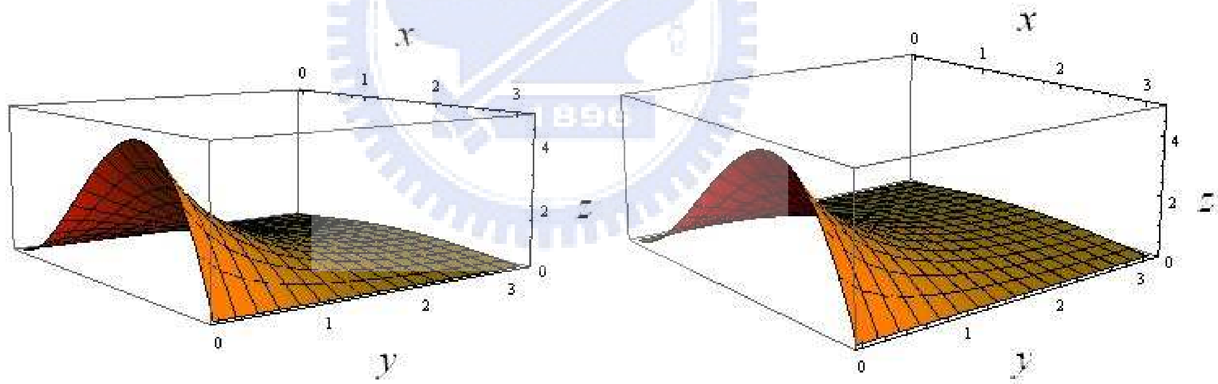


Figure 2.2: The graphic of the case2 where the boundary conditions are $u(0,y) = u(\pi,y) = u_y(x,\pi) = 0$ and the comparison between the graphic and the Figure 2.1.

Case3

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < y < \pi, \\ u(0,y) = u_x(\pi,y) = u(x,\pi) = 0, \\ u(x,0) = x^2(\pi - x). \end{cases} \tag{2.20}$$

We have

$$\begin{cases} X'' + \lambda X = 0 \qquad \text{for} \quad 0 < x < \pi, \\ X(0) = X'(\pi) = 0 \end{cases} \tag{2.21}$$

and

$$\begin{cases} Y'' - \lambda Y = 0 \qquad \text{for} \quad 0 < y < \pi, \\ Y(\pi) = 0. \end{cases} \tag{2.22}$$

The systems (2.21), (2.22) imply

$$X_n(x) = \sin(n - \frac{1}{2})x \quad \text{and} \quad Y_n(y) = \sinh\ (n - \frac{1}{2})(\pi - y).$$

Thus we find

$$u(x,y) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{[(-24 + \pi^2(1-2n)^2)(-1)^n + 4\pi(1-2n)]}{(1-2n)^4} \frac{\sinh(n-\frac{1}{2})(\pi - y)}{\sinh(n - \frac{1}{2})\pi} \sin(n - \frac{1}{2})x. \tag{2.23}$$

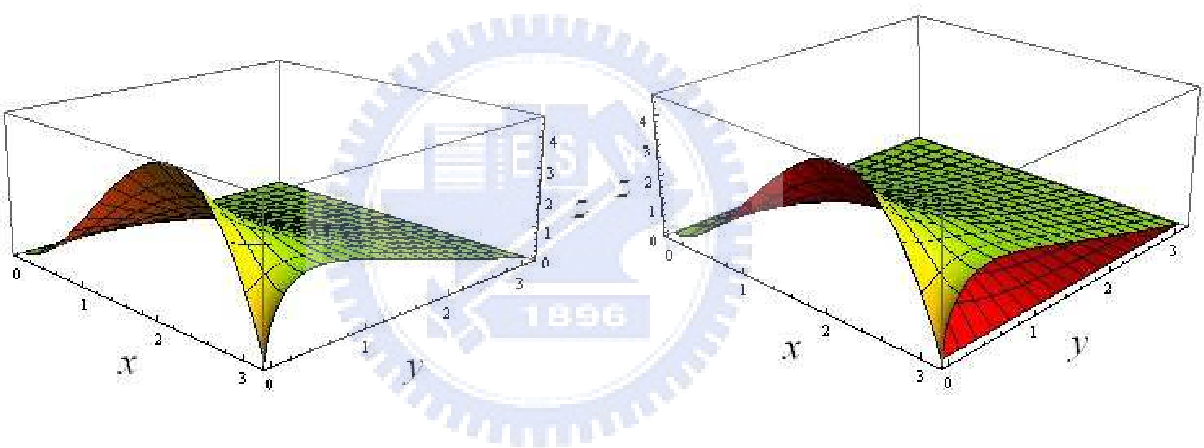


Figure 2.3: The graphic of the case3 where the boundary conditions are $u(0,y) = u_x(\pi, y) = u(x,\pi) = 0$ and the comparison between the graphic and the Figure 2.1.

Case4

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < y < \pi, \\ u_x(0,y) = u(\pi, y) = u(x,\pi) = 0, \\ u(x,0) = x^2(\pi - x). \end{cases} \tag{2.24}$$

We have

$$\begin{cases} X'' + \lambda X = 0 \qquad \text{for} \quad 0 < x < \pi, \\ X'(0) = X(\pi) = 0 \end{cases} \tag{2.25}$$

and

$$\begin{cases} Y'' - \lambda Y = 0 \qquad \text{for} \quad 0 < y < \pi, \\ Y(\pi) = 0. \end{cases} \tag{2.26}$$

The systems (2.25), (2.26) imply

$$X_n(x) = \cos(n - \frac{1}{2})x \quad \text{and} \quad Y_n(y) = \sinh\ (n - \frac{1}{2})(\pi - y).$$

Thus we find

$$u(x, y) = \frac{-64}{\pi} \sum_{n=1}^{\infty} \frac{[3 + \pi(1 - 2n)(-1)^{n+1}]}{(1 - 2n)^4} \frac{\sinh(n - \frac{1}{2})(\pi - y)}{\sinh(n - \frac{1}{2})\pi} \cos(n - \frac{1}{2})x. \tag{2.27}$$

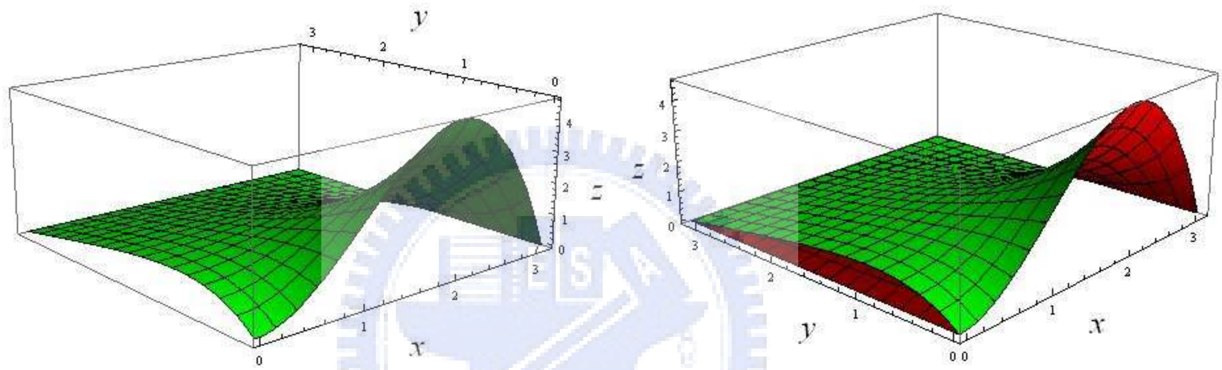


Figure 2.4: The graphic of the case4 where the boundary conditions are $u_x(0, y) = u(\pi, y) = u(x, \pi) = 0$ and the comparison between the graphic and the Figure 2.1.

Considering Figure 2.1, we see the boundary condition (2.5) decides the graphic in the boundary $y = 0$ and the figures in other sides ($y = \pi, x = 0, x = \pi$) are determined on the boundary equations (2.4). Moreover, the Maximum Principle tells that the maximum value of u(x,y) always appears on the boundary. Undering the different kind of the boundary conditions, the system (2.19) shows that even though we differentiate u(x,y) respect to y in the boundary condition. The distinct solutions of the case1 and case2 don't make any difference between Figure 2.1 and Figure 2.2. The Figure 2.3 and 2.4 show that the boundary conditions which are differentiated respect to x will change the shape in that side.

### 2.1.2 The domain is a disk (dimension $= 2$) in polar coordinates

We consider a solution u of Laplace's equation in the unit disk with values given on the boundary. It is natural to introduce the polar coordinates

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x). \tag{2.28}$$

A computation shows that Laplace's equation in these coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} = 0 \qquad \text{for} \quad r < 1, \tag{2.29}$$

$$u(1, \theta) = f(\theta). \tag{2.30}$$

The function $f(\theta)$ is a given continuously differentiable function which is periodic of period $2\pi$. The solution $u(r, \theta)$ must also be periodic of period $2\pi$ in $\theta$. We may apply separation of variables to (2.29), (2.30) with the form $u(r, \theta) = R(r)\Theta(\theta)$ as detail in [1]. There is another way to show the answer of the system (2.29), (2.30).

If $f(\theta)$ is absolutely integrable, we seek a solution of the problem (2.29), (2.30) with the form

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n r^n \cos n\theta + b_n r^n \sin n\theta). \tag{2.31}$$

Therefore $a_n$ and $b_n$ are the Fourier coefficients of $f(\theta)$,

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi \, \mathrm{d}\phi,$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi \, \mathrm{d}\phi.$$

Thus we find

$$u(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\theta - \phi)] \, \mathrm{d}\phi. \tag{2.32}$$

To evaluate the series,

$$\begin{aligned}
&[r^2 + 1 - 2r\cos\theta]\sum_{n=1}^{\infty} r^n \cos n\theta \\
&= \sum_{n=1}^{\infty}\{[r^{n+2} + r^n]\cos n\theta - r^{n+1}[\cos(n+1)\theta + \cos(n-1)\theta]\} \\
&= \sum_{n=1}^{\infty} r^{n+2}\cos n\theta + \sum_{n=1}^{\infty} r^n \cos n\theta - \sum_{n=2}^{\infty} r^n \cos n\theta - \sum_{n=0}^{\infty} r^{n+2}\cos n\theta \\
&= r\cos\theta - r^2.
\end{aligned}$$

Then

$$[r^2 + 1 - 2r\cos\theta][\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos \ n\theta] = \frac{1}{2}(1 - r^2)$$

and

$$\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos\, n\theta = \frac{1-r^2}{2[r^2+1-2r\cos\theta]}. \tag{2.33}$$

Replacing $\theta$ by $\theta-\phi$, we find that if $\int |f(\theta)|\,\mathrm{d}\theta$ is finite, the series solution (2.32) can be represented as the integral

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{1+r^2-2r\cos(\theta-\phi)}\,\mathrm{d}\phi \tag{2.34}$$

for $r<1$. This is called **Poisson's integral formula**. If boundary problem is on a circle of radius R, the series can also be written in the form as

$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{r^2+R^2-2rR\cos(\theta-\phi)}\,\mathrm{d}\phi \qquad \text{for } r<R. \tag{2.35}$$

This is the general form of Poisson's integral formula.

These formulas again represent a continuous solution when we define $u(R,\theta)=f(\theta)$ is continuous and periodic and $\int f'^2\,\mathrm{d}\theta$ is finite. Furthermore, the series (2.31) or the Poisson integral (2.35) satisfies Laplace's equation under $\int_{-\pi}^{\pi} |f|\,\mathrm{d}\theta$ is finite.

Now, we solve the system (2.36), (2.37) by the Green function. Assume the dimension is n $(n \geq 2)$, and $\overrightarrow{X}=(x_1,x_2,\ldots,x_n)$ and $\overrightarrow{Y}=(y_1,y_2,\ldots,y_n)$.
If $u(\overrightarrow{X})$ satisfies

$$\triangle u = 0 \quad \text{in} \quad B^0(0,r), \tag{2.36}$$

$$u = g \quad \text{on} \quad \partial B(0,r), \tag{2.37}$$

then

$$u(\overrightarrow{X}) = \frac{r^2-\left\|\overrightarrow{X}\right\|^2}{n\alpha(n)}\int_{\partial B(0,r)} \frac{g(\overrightarrow{Y})}{\left\|\overrightarrow{X}-\overrightarrow{Y}\right\|^n}\,\mathrm{d}S(\overrightarrow{Y}) \tag{2.38}$$

where $n\alpha(n)$ is the surface area of ball with radius r in $\mathbb{R}^n$ (see [4]).

Example I:

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad \text{for} \quad r<1, \tag{2.39}$$

$$u(1,\theta)=\sin^3\theta. \tag{2.40}$$

Answer :

since $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$, the system (2.39), (2.40) can be transformed into

$$u_{xx} + u_{yy} = 0 \quad \text{for} \quad x^2+y^2<1, \tag{2.41}$$

$$u(x,y) = y^3 \quad \text{for} \quad x^2+y^2=1. \tag{2.42}$$

We can solve the system (2.39), (2.40) by separation of variables. The answer is

$$\begin{aligned} u(r,\theta) &= \tfrac{1}{4}(3r\sin\theta - r^3\sin 3\theta). \\ (= u(x,y) &= \tfrac{1}{4}(3y - 3yx^2 + y^3)) \end{aligned} \tag{2.43}$$

If we solve the system (2.41), (2.42) by the Green function, we get the answer

$$u(x_1,x_2) = \frac{1-(x_1^2+x_2^2)}{2\pi}\int_{\partial B(0,1)} \frac{y_2^3}{(x_1-y_1)^2+(x_2-y_2)^2}\,\mathrm{d}S(\overrightarrow{Y}). \tag{2.44}$$

Claim : The answers (2.43) and (2.44) are identical

$$u(x_1,x_2) = \frac{1-(x_1^2+x_2^2)}{2\pi}\int_{\partial B(0,1)} \frac{y_2^3}{(x_1-y_1)^2+(x_2-y_2)^2}\,\mathrm{d}S(\overrightarrow{Y})$$

$(y_1 = \cos\theta, y_2 = \sin\theta, dS(\overrightarrow{Y}) = d\theta.)$

$$= \frac{1-(x_1^2+x_2^2)}{2\pi}\int_0^{2\pi} \frac{\sin^3\theta}{(x_1-\cos\theta)^2+(x_2-\sin\theta)^2}\,\mathrm{d}\theta$$

(By Residue Theorey, let $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$ and $\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$. Thus we find $d\theta = \frac{dz}{iz}$.)

$$= \frac{1-(x_1^2+x_2^2)}{2\pi}\int_c \frac{\frac{1}{8z}(z-\frac{1}{z})^3}{[x_1-\frac{1}{2}(z+\frac{1}{z})]^2+[x_2-\frac{1}{2i}(z-\frac{1}{z})]^2}\,\mathrm{d}z$$

$$= \frac{1-(x_1^2+x_2^2)}{16\pi(-x_1+ix_2)}\int_c \frac{(z^2-1)^3}{z^3(z-\frac{1}{x_1-ix_2})(z-(x_1+ix_2))}\,\mathrm{d}z.$$

Let $\alpha_1 = \frac{1}{x_1-ix_2}$, and $\alpha_2 = x_1+ix_2$. We find

$$f(z) = \frac{(z^2-1)^3}{z^3(z-\frac{1}{x_1-ix_2})(z-(x_1+ix_2))} = \frac{(z^2-1)^3}{z^3(z-\alpha_1)(z-\alpha_2)}.$$

Thus f(z) has simple poles at $z = \alpha_1, \alpha_2$ and pole of order 3 at origin. Since $\alpha_1$ may be outside of the unit ball, we only consider the Res(f;$\alpha_2$) and Res(f;0).

It implies

$$I = \frac{[1-(x_1^2+x_2^2)](-2\alpha_1\pi i)}{16\pi}[Res(f;\alpha_2)+Res(f;0)] \tag{2.45}$$

where

$$Res(f;\alpha_2) = \lim_{z\to\alpha_2}(z-\alpha_2)f(z) = \frac{(\alpha_2^2-1)^3}{\alpha_2^3(\alpha_2-\alpha_1)},$$
$$Res(f;0) = \lim_{z\to 0}\frac{d^2}{dz^2}[z^3 f(z)] = \frac{1}{\alpha_2-\alpha_1}(\frac{3}{\alpha_1}-\frac{3}{\alpha_2}-\frac{1}{\alpha_1^3}+\frac{1}{\alpha_2^3}).$$

Thus we find

$$I = \frac{1}{4}(x_2^3+3x_2-3x_1^2x_2) = u(x_1,x_2).$$

Example II :

$$\begin{cases} \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta} = 0 & \text{for} \quad r<1, 0<\theta<\frac{\pi}{2}, \\ u(r,0) = u_\theta(r,\frac{1}{2}\pi) = 0, & \\ u(1,\theta) = \theta. & \end{cases} \tag{2.46}$$

Answer :

Using separation of variables, we get

$$u(r,\theta) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)\theta\, r^{2n-1}. \tag{2.47}$$

If we transform the system (2.47) from Polar coordinate into Cartesian coordinate, we get

$$\begin{cases} u_{xx}+u_{yy} = 0 & \text{for} \quad x^2+y^2<1, 0<\tan^{-1}(\frac{y}{x})<\frac{\pi}{2}, \\ u(x,y) = 0 & \text{for} \quad y=0, \\ -u_x(x,y)y^2 = 0 & \text{for} \quad x=0, \\ u(x,y) = \tan^{-1}(\frac{y}{x}) & \text{for} \quad x^2+y^2=1. \end{cases} \tag{2.48}$$

The solution of the problem (2.47) will also be the solution of the problem (2.48)

$$u(r,\theta) = u(x,y) = \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{\pi(2n-1)^2}\sin[(2n-1)\tan^{-1}(\frac{y}{x})](x^2+y^2)^{\frac{2n-1}{2}}. \tag{2.49}$$

### 2.1.3 The domain is a cube (dimension $= 3$) in orthogonal coordinates

Consider

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{for} \quad 0 < x < \pi, 0 < y < \pi, 0 < z < \pi, \tag{2.50}$$

$$u = 0 \quad \text{for} \quad x = 0 \text{ and } \pi, y = 0 \text{ and } \pi, z = \pi, \tag{2.51}$$

$$u(x, y, 0) = g(x, y). \tag{2.52}$$

We apply the method of separation of variables to a product function $V(x, y, z) = X(x)Y(y)Z(z)$ which solves the system (2.50), (2.51) and (2.52). We have

$$\begin{aligned} &\tfrac{X''}{X} + \tfrac{Y''}{Y} = -\tfrac{Z''}{Z} = C_1 \text{ and} \\ &\tfrac{X''}{X} = C_1 - \tfrac{Y''}{Y} = C_2 \end{aligned} \tag{2.53}$$

where $C_1$ and $C_2$ are constants. The homogeneous boundary conditions (2.51) give

$$\begin{aligned} &X(0) = X(\pi) = 0, \\ &Y(0) = Y(\pi) = 0, \\ &Z(\pi) = 0. \end{aligned} \tag{2.54}$$

By the systems (2.53) and the boundary conditions (2.54), we get three ODE systems

$$\begin{cases} X'' - C_2 X = 0, \\ X(0) = X(\pi) = 0. \end{cases} \tag{2.55}$$

$$\begin{cases} Y'' - (C_1 - C_2)Y = 0, \\ Y(0) = Y(\pi) = 0. \end{cases} \tag{2.56}$$

$$\begin{cases} Z'' + C_1 Z = 0, \\ Z(\pi) = 0. \end{cases} \tag{2.57}$$

By the systems (2.55), (2.56) , we get $X_n(x) = \sin nx$ and $Y_m(y) = \sin my$. It implies $C_2 = -n^2$ and $C_1 - C_2 = -m^2$, then $C_1 = -m^2 - n^2$. So that the system (2.57) shows $Z(z) = \sinh \sqrt{m^2 + n^2}(\pi - z)$. We seek a solution of the form

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} \sinh \sqrt{m^2 + n^2}(\pi - z) \sin nx \sin my. \tag{2.58}$$

By the boundary condition (2.52), we obtain

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm} \sinh \sqrt{m^2+n^2}\pi \sin nx \sin my.$$

Therefore,

$$\alpha_{nm} \sinh \sqrt{m^2+n^2}\pi = \delta_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} g(x,y) \sin nx \sin my \, \mathrm{d}x \, \mathrm{d}y$$

and

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \delta_{nm} \frac{\sinh \sqrt{m^2+n^2}(\pi - z)}{\sinh \sqrt{m^2+n^2}\pi} \sin nx \sin my. \tag{2.59}$$

Since we must verify that the series (2.59) gives a solution to the system (2.50), (2.51) and (2.52), we should check whether $\int_0^{\pi}\int_0^{\pi} |g| \, \mathrm{d}x \, \mathrm{d}y$ is finite. (the $\delta_{nm}$ are uniformly bounded). Then the series (2.59) and all its derivatives converge absolutely and uniformly for $z_0 \le z \le \pi, 0 \le x \le \pi, 0 \le y \le \pi$ for any constant $z_0 > 0$. It follows that u(x, y, z) is infinitely differentiable for $z > 0$ and satisfies Laplace's equation.

In order to sure that u satisfies the boundary condition at z=0, we should check whether g(x,y) is continuous and continuously differentiable and the squares of its second partial derivatives have finite integrals, then the double Fourier series converges absolutely and uniformly to g(x,y) as a double series (see [2]).

Example:

$$\nabla^2 u - u = 0 \quad \text{for} \quad 0 < x < \pi, 0 < y < \frac{\pi}{2}, 0 < z < 1, \tag{2.60}$$

$$u = 0 \quad \text{for} \quad x = 0, y = 0, z = 1, \tag{2.61}$$

$$u_x = 0 \quad \text{for} \quad x = \pi, \tag{2.62}$$

$$u_y = 0 \quad \text{for} \quad y = \frac{\pi}{2}, \tag{2.63}$$

$$u_z(x,y,0) = 2x - \pi. \tag{2.64}$$

Answer :

Considering a product function $V(x,y,z) = X(x)Y(y)Z(z)$ which solves the system (2.60), (2.61), (2.62), (2.63) and (2.64), we have

$$
\begin{aligned}
&X''YZ + XY''Z + XYZ'' - XYZ = 0.\\
\Longrightarrow &\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} - 1 = 0\\
\Longrightarrow &\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} + 1 = C_1\\
\Longrightarrow &\frac{X''}{X} = C_1 - \frac{Y''}{Y} = C_2
\end{aligned}
$$

where $C_1$ and $C_2$ are constants. The homogenous boundary conditions (2.61), (2.62) and (2.63) give

$$
\begin{aligned}
&X(0) = X'(\pi) = 0,\\
&Y(0) = Y'(\frac{\pi}{2}) = 0,\\
&Z(1) = 0.
\end{aligned}
$$

Thus, we get three ODE systems

$$
\begin{cases} X'' - C_2 X = 0, \\ X(0) = X'(\pi) = 0. \end{cases} \tag{2.65}
$$

$$
\begin{cases} Y'' - (C_1 - C_2)Y = 0, \\ Y(0) = Y'(\frac{\pi}{2}) = 0. \end{cases} \tag{2.66}
$$

$$
\begin{cases} Z'' + (C_1 - 1)Z = 0, \\ Z(1) = 0. \end{cases} \tag{2.67}
$$

By the system (2.65), we have $C_2 = (n - \frac{1}{2})^2$ with correspond eigenfunction

$$
X_n(x) = \sin(n - \frac{1}{2})x
$$

and the system (2.66) implies

$$
Y_m(y) = \sin(2m - 1)y,
$$

hence $C_1 - C_2 = -(2m - 1)^2$ and $C_1 - 1 = -[(n - \frac{1}{2})^2 + (2m - 1)^2 - 1]$.

By the system (2.67), we get

$$
Z(z) = \sinh\sqrt{(n - \frac{1}{2})^2 + (2m - 1)^2 + 1}(1 - z).
$$

We seek the solution of form

$$
u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm} \sin(n - \frac{1}{2})x\ \sin(2m - 1)y\ \sinh\sqrt{(n - \frac{1}{2})^2 + (2m - 1)^2 + 1}(1 - z). \tag{2.68}
$$

By the boundary condition (2.64), we make

$$-\sqrt{(n-\tfrac{1}{2})^2+(2m-1)^2+1}\cosh(\sqrt{(n-\tfrac{1}{2})^2+(2m-1)^2+1})\alpha_{nm}=\delta_{nm}$$

$$=\frac{\int_0^{\frac{\pi}{2}}\int_0^{\pi}(2x-\pi)\sin(n-\dfrac{1}{2})x\sin(2m-1)y\,\mathrm{d}x\,\mathrm{d}y}{\int_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2(n-\dfrac{1}{2})x\ \sin^2(2m-1)y\,\mathrm{d}x\,\mathrm{d}y}$$

$$=-\frac{8}{\pi}\frac{(\frac{1}{n-\frac{1}{2}}+\frac{2(-1)^n}{\pi(n-\frac{1}{2})^2})}{2m-1}.$$

$$\Longrightarrow\alpha_{nm}=\frac{8(\frac{1}{n-\frac{1}{2}}+\frac{2(-1)^n}{\pi(n-\frac{1}{2})^2})}{\pi(2m-1)\sqrt{(n-\frac{1}{2})^2+(2m-1)^2+1}\cosh(\sqrt{(n-\frac{1}{2})^2+(2m-1)^2+1})}$$

and

$u(x,y,z)=$

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(\frac{1}{n-\frac{1}{2}}+\frac{2(-1)^n}{\pi(n-\frac{1}{2})^2})\frac{\sin(n-\frac{1}{2})x\ \sin(2m-1)y\ \sinh\sqrt{(n-\frac{1}{2})^2+(2m-1)^2+1}(z-1)}{(2m-1)\sqrt{(n-\frac{1}{2})^2+(2m-1)^2+1}\cosh(\sqrt{(n-\frac{1}{2})^2+(2m-1)^2+1})} \tag{2.69}$$

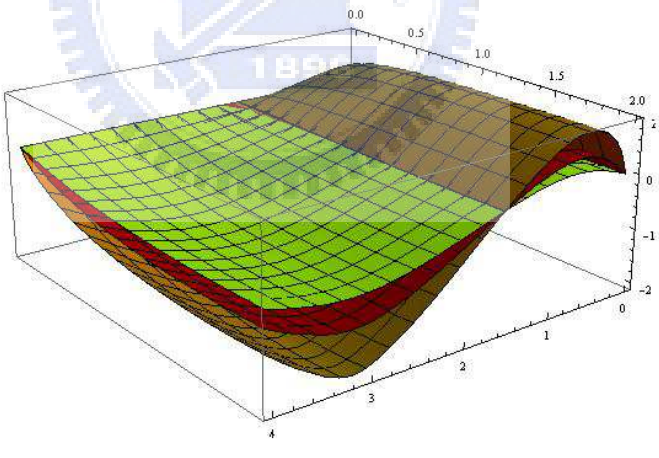


Figure 2.5: The projection of the solution (2.69) on the $z=0,\ \frac{1}{4},\ \frac{1}{2}$ (brown, red, green) .

Because we doubt if the graphics of $u(x,y,0),\ u(x,y,\frac{1}{4})$ and $u(x,y,\frac{1}{2})$ would intersect at the same line in the region $[1.4,1.6]\times[0,2]$ on the xy-plane, we use the numerical method to show the values in Appendix B.1, and the answer is no.

### 2.1.4 The domain is a cylinder (dimension $= 3$) in cylindrical coordinates

If we introduce cylindrical coordinates r, $\theta$, z such that

$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$z = z.$$

Laplace's equation in cylindrical coordinates becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{for} \quad 0 < r < R_1, 0 < z < \pi, \tag{2.70}$$

$$u(r,\theta,0) = u(r,\theta,\pi) = 0, \tag{2.71}$$

$$u(R_1,\theta,z) = g(\theta,z). \tag{2.72}$$

Consider the product function $V(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ such that $V(r,\theta,z)$ solves the system (2.70), (2.71) and (2.72). Then

$$\frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = C_1 \tag{2.73}$$

$$\text{and } \frac{r^2R'' + rR'}{R} - r^2C_1 = -\frac{\Theta''}{\Theta} = C_2 \tag{2.74}$$

where $C_1$ and $C_2$ are constants.

By the boundary conditions (2.71) and the equation (2.73), we have

$$\begin{cases} Z'' + C_1 Z = 0, \\ Z(0) = Z(\pi) = 0. \end{cases}$$

It implies that

$$Z_n(z) = \sin nz \text{ with } C_1 = n^2, n = 1, 2 \ldots.$$

Considering that $\Theta$ is periodic of period $2\pi$, then

$$\begin{cases} \Theta'' + C_2\Theta = 0, \\ \Theta(0) = \Theta(2\pi), \\ \Theta'(0) = \Theta'(2\pi). \end{cases}$$

Then $C_2 = m^2, m = 0, 1, 2, \ldots$, and the corresponding eigenfunctions are

$$\Theta_0 = 1,\ \Theta_m = \sin m\theta, \cos m\theta. \qquad (m = 1, 2, \ldots \text{ are double eigenvalues})$$

Finally, we obtain the differential equation

$$R''_{mn} + \frac{1}{r}R'_{mn} - (\frac{m^2}{r^2} + n^2)R_{mn} = 0, \tag{2.75}$$

$$R_{mn}(R_1) = 1. \tag{2.76}$$

This equation is singular at r=0. In place of a boundary condition we impose the condition that R remain finite at r=0.

(Frobenius method) We seek a solution of the form

$$R_{mn}(r) = \gamma \sum_{k=0}^{\infty} c_k r^{\alpha+k} \qquad c_0 \neq 0 \text{ and } \alpha, \gamma \text{ are constants.}$$

By the equation (2.75), we have

$$\begin{aligned}
& (\alpha+k)^2 \sum_{k=0}^{\infty} c_k r^{\alpha+k-1} - m^2 \sum_{k=0}^{\infty} c_k r^{\alpha+k-1} - n^2 \sum_{k=0}^{\infty} c_k r^{\alpha+k+1} \\
= \; & \alpha^2 c_0 r^{\alpha-1} + (\alpha+1)^2 c_1 r^{\alpha} - m^2 c_0 r^{\alpha-1} - m^2 c_1 r^{\alpha} + \sum_{k=2}^{\infty} ([(\alpha+k)^2 - m^2] c_k - n^2 c_{k-2}) r^{\alpha+k-1} \\
= \; & \{(\alpha^2 - m^2) c_0 + [(\alpha+1)^2 - m^2] c_1 r\} r^{\alpha-1} + \sum_{k=2}^{\infty} ([(\alpha+k)^2 - m^2] c_k - n^2 c_{k-2}) r^{\alpha+k-1} = 0.
\end{aligned}$$

The power series is identically zero only if all its coefficients vanish. It implies that

$$\alpha = \pm m, \quad c_1 = 0 \quad \text{and} \quad c_\iota = \frac{n^2 c_{\iota-2}}{(\alpha+\iota)^2 - m^2}.$$

Hence $c_1 = c_3 = c_5 = \ldots = 0$ and

$$\begin{aligned}
c_2 &= \frac{n^2 c_0}{[(\alpha+2)^2 - m^2]} \\
c_4 &= \frac{n^2 c_2}{(\alpha+4)^2 - m^2} = \frac{n^4 c_0}{[(\alpha+4)^2 - m^2][(\alpha+2)^2 - m^2]} \\
&\vdots \\
c_{2\iota} &= \frac{n^{2\iota} c_0}{[(\alpha+2\iota)^2 - m^2][(\alpha+2(\iota-1))^2 - m^2]\ldots[(\alpha+2)^2 - m^2]}.
\end{aligned}$$

Thus we find

$$\begin{aligned}
& \sum_{k=0}^{\infty} c_k r^{\alpha+k} \\
&= c_0 r^{\alpha} + \sum_{k=1}^{\infty} c_k r^{\alpha+k} \\
&= c_0 r^{\alpha} + \sum_{\iota=1}^{\infty} c_{2\iota} r^{\alpha+2\iota} \\
&= c_0 r^{\alpha} [1 + \sum_{\iota=1}^{\infty} \frac{(rn)^{2\iota}}{[(\alpha+2\iota)^2 - m^2][(\alpha+2(\iota-1))^2 - m^2]\ldots[(\alpha+2)^2 - m^2]}].
\end{aligned}$$

If we choose $\alpha = m$ and define $c_0 = \frac{n^m}{2^m \cdot m!}$, then

$$R_{mn}(r) = \gamma \sum_{\iota=0}^{\infty} \frac{n^{2\iota+m} (\frac{1}{2} r)^{m+2\iota}}{\iota!(\iota+m)!} = \gamma J_m(nr) \tag{2.77}$$

where $J_m$ is the **Bessel function** that converges for all r (see [2]).

By the boundary condition (2.76),

$$\gamma = \frac{1}{J_m(nR_1)} \Longrightarrow R_{mn}(r) = \frac{J_m(nr)}{J_m(nR_1)}.$$

We seek the solution u in the form

$$u(r,\theta,z)=\frac{1}{2}\sum_{n=1}^{\infty}c_{n0}\frac{J_0(nr)}{J_0(nR_1)}\sin nz+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{J_m(nr)}{J_m(nR_1)}\sin nz(c_{nm}\cos m\theta+d_{nm}\sin m\theta) \tag{2.78}$$

where

$$c_{nm}=\frac{2}{\pi^2}\int_0^{\pi}\int_0^{2\pi}g(\theta,z)\sin nz\cos m\theta\,\mathrm{d}\theta\,\mathrm{d}z,$$
$$d_{nm}=\frac{2}{\pi^2}\int_0^{\pi}\int_0^{2\pi}g(\theta,z)\sin nz\sin m\theta\,\mathrm{d}\theta\,\mathrm{d}z.$$

Similarly, we should verify the series (2.78) satisfies the problem (2.70), (2.71), (2.72). We must check if $|g(\theta,z)|$ and its second partial derivatives have finite integrals.

Example:

$$\nabla^2 u=0 \qquad \text{for} \quad 0<r<1, 0<z<\pi, \tag{2.79}$$

$$u(r,\theta,0)=0, \tag{2.80}$$

$$u_z(r,\theta,\pi)=0, \tag{2.81}$$

$$u(1,\theta,z)=z\cos^3\theta. \tag{2.82}$$

Answer :

By separation of variables, we obtain three ODE systems

$$\begin{cases} Z''+C_1 Z=0, \\ Z'(\pi)=Z(0)=0. \end{cases} \tag{2.83}$$

$$\begin{cases} \Theta''+C_2\Theta=0, \\ \Theta(0)=\Theta(2\pi) \quad \text{and} \quad \Theta'(0)=\Theta'(2\pi). \end{cases} \tag{2.84}$$

$$\begin{cases} R''_{mn}+\frac{1}{r}R'_{mn}-((n-\frac{1}{2})^2+(\frac{m}{r})^2)R_{mn}=0, \\ R_{mn} \quad \text{remain finite at r=0 and} \quad R_{mn}(1)=1. \end{cases} \tag{2.85}$$

The systems (2.83) , (2.84) imply

$$Z_n(z)=\sin(n-\frac{1}{2}) \text{ with } C_1=(n-\frac{1}{2})^2, n=1,2\ldots$$

and $C_2 = m^2, m = 0, 1, 2, \ldots$ with the corresponding eigenfunctions

$$\Theta_0 = 1,\ \Theta_m = \sin m\theta, \cos m\theta. \qquad (m = 1, 2, \ldots \text{ are double eigenvalues})$$

By the system (2.85), we let

$$R_{mn}(r) = \ell \sum_{k=0}^{\infty} c_k r^{k+\alpha} \text{ with } c_0 \neq 0,\ \ \ell \text{ and } \alpha \text{ are constants.}$$

We have

$$\{(\alpha^2 - m^2)c_0 + [(\alpha+1)^2 - m^2]c_1 r\}r^{\alpha-1} + \sum_{k=2}^{\infty}([(\alpha+k)^2 - m^2]c_k - (n-\frac{1}{2})^2 c_{k-2})r^{\alpha+k-1} = 0.$$

The power series is identically zero only if all its coefficients vanish, it implies

$$\alpha = \pm m, \quad c_1 = 0 \quad \text{and} \quad c_\iota = \frac{(n-\frac{1}{2})^2 c_{\iota-2}}{(\alpha+\iota)^2 - m^2}.$$

$$\Longrightarrow R_{mn}(r) = \gamma \sum_{\iota=0}^{\infty} \frac{(n-\frac{1}{2})^{2\iota+m}(\frac{1}{2}r)^{m+2\iota}}{\iota!(\iota+m)!} = \gamma I_m((n-\frac{1}{2})r).$$

Since $R_{mn}(1) = 1$, it implies $R_{mn}(r) = \frac{J_m((n-\frac{1}{2})r)}{J_m(n-\frac{1}{2})}$. We get the solution u in the form

$$\begin{aligned} u(r,\theta,z) = &\frac{1}{2}\sum_{n=1}^{\infty} c_{n0} \frac{J_0((n-\frac{1}{2})r)}{J_0(n-\frac{1}{2})} \sin(n-\frac{1}{2})z \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{J_m((n-\frac{1}{2})r)}{J_m(n-\frac{1}{2})} \sin(n-\frac{1}{2})z(c_{nm}\cos m\theta + d_{nm}\sin m\theta) \end{aligned}$$

where

$$d_{nm} = \frac{\int_0^\pi \int_0^{2\pi} z\cos^3\theta \sin(n-\frac{1}{2})z \sin m\theta \, \mathrm{d}\theta \, \mathrm{d}z}{\int_0^\pi \int_0^{2\pi} (\sin(n-\frac{1}{2})z)^2 (\sin m\theta)^2 \, \mathrm{d}\theta \, \mathrm{d}z} = 0,$$
$$(\because \int_0^{2\pi} z\cos^3\theta \sin m\theta \, \mathrm{d}\theta = 0 \quad \forall m)$$

$$\begin{aligned} c_{nm} &= \frac{\int_0^\pi \int_0^{2\pi} z\cos^3\theta \sin(n-\frac{1}{2})z \cos m\theta \, \mathrm{d}\theta \, \mathrm{d}z}{\int_0^\pi \int_0^{2\pi} (\sin(n-\frac{1}{2})z)^2 (\cos m\theta)^2 \, \mathrm{d}\theta \, \mathrm{d}z} \\ &= \begin{cases} \dfrac{(-1)^{n+1}}{(n-\frac{1}{2})^2}\dfrac{3\pi}{4} & \text{for} \quad m = 1, \\ \dfrac{(-1)^{n+1}}{(n-\frac{1}{2})^2}\dfrac{\pi}{4} & \text{for} \quad m = 3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then solution u is

$$u(r,\theta,z) = \frac{3}{2\pi}\sum_{n=1}^{\infty} \sin(n-\frac{1}{2})z\cos\theta \frac{J_1((n-\frac{1}{2})r)}{J_1(n-\frac{1}{2})} + \frac{1}{2\pi}\sum_{n=1}^{\infty} \sin(n-\frac{1}{2})z\cos 3\theta \frac{J_3((n-\frac{1}{2})r)}{J_3(n-\frac{1}{2})}.$$

### 2.1.5 The domain is a sphere (dimension $= 3$) in spherical coordinates

If we introduce spherical coordinates r, $\theta,\ \phi$ such that

$$x = r\sin\theta\ \cos\phi,$$
$$y = r\sin\theta\ \sin\phi,$$
$$z = r\cos\theta,$$

then Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \qquad \text{for } r < R, \tag{2.86}$$

$$u(R,\theta,\phi) = f(\theta,\phi). \tag{2.87}$$

Applying separation of variables, we find that the equation (2.86) has solutions of the form $R(r)\Theta(\theta)\Phi(\phi)$. It yields that

$$\frac{R'' + \frac{2}{r}R'}{R} + \frac{(sin\theta\Theta')'}{\Theta r^2\sin\theta} + \frac{\Phi''}{\Phi r^2\sin^2\theta} = 0.$$

Multiplying by $r^2\sin^2\theta$ and transposing the last term, we see

$$(r^2\sin\theta)\frac{R'' + \frac{2}{r}R'}{R} + \frac{(sin\theta\Theta')'}{\Theta} = -\frac{\Phi''}{\Phi} = c \quad \text{where c is a constant and c} \geq 0.(\text{if } c < 0, \text{ we will get } \Phi \equiv 0)$$

Since $\Phi$ must be periodic of period $2\pi$, we have $\Phi = \cos m\phi$ or $\sin m\phi$ where m=0,1,...

Then

$$\frac{R'' + \frac{2}{r}R'}{R} + \frac{(sin\theta\Theta')'}{\Theta r^2\sin\theta} - \frac{m^2}{r^2\sin^2\theta} = 0.$$

We get

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\theta} - \frac{(sin\theta\Theta')'}{\Theta\sin\theta} = \lambda \qquad \text{where } \lambda \text{ is a constant.}$$

We get two equations

$$r^2(R'' + \frac{2}{r}R') - \lambda R = 0. \tag{2.88}$$

$$(\sin\theta\Theta')' - \frac{m^2}{\sin\theta}\Theta + \lambda\sin\theta\Theta = 0. \tag{2.89}$$

The equation (2.89) for $\Theta(\theta)$ is singular at its two endpoints $\theta = 0$ and $\theta = \pi$. We impose the condition that $\Theta$ and $\Theta'$ remain bounded at both ends. This gives an eigenvalue problem with two singular endpoints. Let $t = \cos\theta$ and $\Theta(\theta) = P(\cos\theta)$. Then equation (2.89) becomes

$$\frac{d}{dt}[(1-t^2)\frac{dP}{dt}] - \frac{m^2}{1-t^2}P + \lambda P = 0 \qquad \text{for } -1 < t < 1. \tag{2.90}$$

If m=0 and $\lambda = 0$ in the equation (2.90) , we find the solution $P \equiv 1$ which is continuously differentiable for $-1 \le t \le 1$. But the eigenfunction P=constant isn't included in our discussion. Since the eigenvalue $\lambda = \frac{\int_{-1}^{1}[(1-t^2)P'^2 + \frac{m^2}{1-t^2}P^2]\,\mathrm{d}t}{\int_{-1}^{1}P^2\,\mathrm{d}t} \ge 0$, we first consider the case m=0 and $\lambda \neq 0$. We have

$$\frac{d}{dt}[(1-t^2)\frac{dP}{dt}] + \lambda P = 0. \tag{2.91}$$

We seek a solution in the neighborhood of t=1 as a power series in (t-1)

$$P(t) = (t-1)^{\alpha}\sum_{k=0}^{\infty} c_k (t-1)^k \quad c_0 \neq 0. \tag{2.92}$$

By the equation (2.91), we have

$$-c_0 2\alpha^2 (t-1)^{\alpha-1} - \sum_{k=0}^{\infty}\{2(k+\alpha+1)^2 c_{k+1} + [-\lambda + (k+\alpha+1)(k+\alpha)]c_k\}(t-1)^{k+\alpha} = 0.$$

The power series is identically zero only if its coefficient vanish, we obtain

$$\alpha = 0 \quad \text{and} \quad c_{k+1} = -\frac{[k(k+1)-\lambda]}{2(k+1)^2}c_k.$$

Thus

$$c_k = \frac{[k(k-1)-\lambda][(k-1)(k-2)-\lambda]\cdots[2-\lambda][-\lambda](-1)^k}{2^k(k!)^2}c_0.$$

By ratio test we find that the series (2.92) converges for $|t-1| < 2$ and in general approach $\pm\infty$ as $t \to -1$ except the series terminates. That means the function is bounded for $-1 \le t \le 1$ if and only if

$$\lambda = n(n+1), \qquad n = 0, 1 \ldots$$

Setting $c_0 = 1$ and we obtain the eigenfunction $P_n(t)$ corresponding to the eigenvalue $\lambda_n = n(n+1)$ is

$$P_n(t) = \sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!(k!)^2 2^k}(t-1)^k.$$

$P_n(t)$ is a polynomial of degree n in t. It is called a **Legendre polynomial**. Since $P_k(t)$ of degree k can be expressed as a linear combination of $P_n(t)$ with $n = 0, 1, \ldots k$ and $P_n$ are orthogonal, it implies

$$\int_{-1}^{1} t^k P_n(t)\,\mathrm{d}t = 0 \qquad \text{for } k = 0, 1, \ldots, n-1.$$

So we can verify the identity

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n,$$

which is called the **Rodrigues formula**.

If $m \neq 0$ and $n \ge m$, the eigenfunction corresponds to the eigenvalue $\lambda = n(n+1)$ is

$$P_n^m(t) \equiv (1-t^2)^{m/2}\frac{d^m}{dt^m}[P_n(t)] = \frac{1}{2^n n!}(1-t^2)^{m/2}\frac{d^{m+n}}{dt^{m+n}}(t^2-1)^n$$

and called the **associated Legendre function**.

The method of separation of variables thus gives the harmonic functions

$$r^n P_n^m(\cos\theta)\cos m\phi, \quad r^n P_n^m(\cos\theta)\sin m\phi \tag{2.93}$$

which are regular in the whole $(r,\theta,\phi)$ space and we call these polynomials **spherical harmonics** (see [2]).

To solve the boundary equation (2.87), we expand $f(\theta,\phi)$ in a double Fourier series

$$f(\theta,\phi) \sim \sum_{n=0}^{\infty}[\frac{1}{2}a_{n0}P_n(\cos\theta) + \sum_{m=1}^{n}(a_{nm}\cos m\phi + b_{nm}\sin m\phi)P_n^m(\cos\theta)]$$

where

$$a_{nm} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_n^m(\cos\theta)\cos m\phi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi,$$
$$b_{nm} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_n^m(\cos\theta)\sin m\phi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The formal solution of the problem (2.86) and (2.87) is then

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}(\frac{r}{R})^n[\frac{1}{2}a_{n0}P_n(\cos\theta) + \sum_{m=1}^{n}(a_{nm}\cos m\phi + b_{nm}\sin m\phi)P_n^m(\cos\theta)]. \tag{2.94}$$

Example:

$$\begin{cases} \nabla u = 0 & \text{for r} < 1, \\ u = x^3 & \text{for r} = 1. \end{cases} \tag{2.95}$$

Answer :

Translate x into spherical coordinates $(x = r\sin\theta\cos\phi)$

$$\begin{aligned} \Longrightarrow a_{nm} &= \tfrac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\cos^3\phi P_n^m(\cos\theta)\cos m\phi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \tfrac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\cos^3\phi\cos m\phi\,\mathrm{d}\phi\ \int_0^{\pi}\sin^3\theta P_n^m(\cos\theta)\sin\theta\,\mathrm{d}\theta. \end{aligned}$$

Since

$$\begin{aligned} &\int_0^{2\pi}\cos^3\phi\cos m\phi\,\mathrm{d}\phi \\ &= \int_0^{2\pi}(\frac{1}{4}\cos 3\phi + \frac{3}{4}\cos\phi)\cos m\phi\,\mathrm{d}\phi \\ &= \frac{1}{8}\{\int_0^{2\pi}[\cos(3+m)\phi + \cos(3-m)\phi]\,\mathrm{d}\phi + 3\int_0^{2\pi}[\cos(1+m)\phi + \cos(1-m)\phi]\,\mathrm{d}\phi\} \\ &= \begin{cases} \frac{\pi}{4} & \text{for m=3,} \\ \frac{3\pi}{4} & \text{for m=1,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For m=1,

$$\int_0^\pi \sin^3\theta P_n^1(\cos\theta)\sin\theta\,\mathrm{d}\theta \qquad\qquad (\cos\theta = t)$$
$$= \int_{-1}^1 (1-t^2)^{\frac{3}{2}} P_n^1(t)\,\mathrm{d}t$$
$$= \int_{-1}^1 (1-t^2)^2 \frac{dP_n(t)}{dt}\,\mathrm{d}t$$
$$= 4[\int_{-1}^1 tP_n(t)\,\mathrm{d}t) - \int_{-1}^1 t^3 P_n(t)\,\mathrm{d}t]$$
$$= \begin{cases} \frac{16}{15} & \text{for n=1} \quad \Rightarrow a_{11} = \frac{3}{5}, \\ -\frac{16}{35} & \text{for n=3} \quad \Rightarrow a_{31} = -\frac{1}{10}, \\ 0 & \text{otherwise.} \end{cases}$$

For m=3,

$$\int_0^\pi \sin^3\theta P_n^3(\cos\theta)\sin\theta\,\mathrm{d}\theta$$
$$= \begin{cases} \frac{96}{7} & \text{for n=3} \quad \Rightarrow a_{33} = \frac{1}{60}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Longrightarrow u(r,\theta,\phi) = \frac{3}{5}\cos\phi P_1^1(\cos\theta) - \frac{1}{10}\cos\phi P_3^1(\cos\theta) + \frac{1}{60}\cos 3\phi P_3^3(\cos\theta).$$

## 2.2 Finite Fourier transform to construct the solution of system of Laplace's equation

Consider the non-homogeneous problem

$$u_{xx} + u_{yy} = F(x, y) \qquad \text{for } 0 < x < \pi,\ 0 < y < 1, \tag{2.96}$$

$$u(x, 1) = u(0, y) = u(\pi, y) = 0, \tag{2.97}$$

$$u(x, 0) = 0. \tag{2.98}$$

To solve the non-homogeneous problem (2.96), (2.97) and (2.98), we expand the solution in a Fourier sine series for each fixed y.

$$u(x, y) \sim \sum_{n=1}^{\infty} b_n(y) \sin nx.$$

The set of sine coefficients

$$b_n(y) = \frac{2}{\pi} \int_0^{\pi} u(x, y) \sin nx \,\mathrm{d}x$$

which is a function of the integer n and of y, determines u(x,y) uniquely. It is called the **fininte sine transform** of u(x,y).

If $\frac{\partial^2 u}{\partial x^2}$ is continuous, its finite sine transform is given by

$$\frac{2}{\pi} \int_0^{\pi} \frac{\partial^2 u}{\partial x^2} \sin nx \,\mathrm{d}x = \frac{2}{\pi} \left\{ \left[ \frac{\partial u}{\partial x} \sin nx \right]_0^{\pi} - n \int_0^{\pi} \frac{\partial u}{\partial x} \cos nx \,\mathrm{d}x \right\} = -n^2 b_n(y).$$

If $\frac{\partial^2 u}{\partial y^2}$ is continuous, we can interchange integration and differentiation to show that

$$\frac{2}{\pi} \int_0^{\pi} \frac{\partial^2 u}{\partial y^2} \sin nx \,\mathrm{d}x = \frac{\partial^2}{\partial y^2} \left[ \frac{2}{\pi} \int_0^{\pi} u \sin nx \,\mathrm{d}x \right] = b_n^{''}(y).$$

Taking the finite sine transform of both sides of the equation (2.96) leads to the equation

$$b_n''(y) - n^2 b_n(y) = B_n(y) \tag{2.99}$$

where

$$B_n(y) = \frac{2}{\pi} \int_0^{\pi} F(x, y) \sin nx \,\mathrm{d}x.$$

The boundary condition (2.98) means that

$$b_n(0) = 0.$$

Taking sine transforms, we have reduced the problem for a partial differential equation to the problem (2.100) for an ordinary differential equation.

$$\begin{cases} b_n''(y) - n^2 b_n(y) = B_n(y), \\ b_n(0) = b_n(1) = 0. \end{cases} \tag{2.100}$$

We can use Green's function to solve the problem (2.100) and the solution has Fourier sine series form. By Schwarz's inequality and Parseval's equation, we know that the series $\sum b_n(y)\sin nx$ converges uniformly for $0 \le x \le \pi,\ 0 \le y \le 1$, then

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin nx.$$

Example:

Solve

$$\begin{cases} u_{xx} + u_{yy} = y(1-y)\sin^3 x \qquad \text{for } 0 < x < \pi,\ 0 < y < 1, \\ u(x,0) = u(x,1) = u(0,y) = u(\pi,y) = 0. \end{cases} \tag{2.101}$$

Answer :

We expand the solution in a Fourier sine series for each fixed y and we find

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin nx \quad \text{where } b_n(y) = \frac{2}{\pi}\int_0^{\pi} u(x,y)\sin nx \,\mathrm{d}x.$$

We have

$$\frac{2}{\pi}\int_0^{\pi} u_{xx}\sin nx\,\mathrm{d}x = -n^2 b_n(y) \quad \text{and}$$
$$\frac{2}{\pi}\int_0^{\pi} u_{yy}\sin nx\,\mathrm{d}x = b_n''(y).$$

Taking the finite sine transform of both sides of (2.101) leads to the second order differential equation

$$\begin{cases} b_n''(y) - n^2 b_n(y) = \frac{2}{\pi}\int_0^{\pi}[y(1-y)\sin^3 x]\sin nx\,\mathrm{d}x = f(y) \quad \text{, for } 0 < y < 1 \\ \qquad = \begin{cases} \frac{3}{4}y(1-y) & \text{for } n = 1, \\ \frac{-1}{4}y(1-y) & \text{for } n = 3, \\ 0 & \text{otherwise,} \end{cases} \\ b_n(0) = b_n(1) = 0. \end{cases} \tag{2.102}$$

Now, we use Green's function to solve the system (2.102) and we have

$$\begin{cases} p(y) = 1 \\ q(y) = -n^2 \end{cases} \text{and} \begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$$

Let $v_1(y) = e^{ny}$ and $v_2(y) = e^{-ny}$ which satisfy the equation $v'' - n^2 v = 0$. We have

$$k = p(y)\,[v_1'(y)v_2(y) - v_2'(y)v_1(y)] = 2n$$

$$\text{and} \qquad D = v_1(\alpha)v_2(\beta) - v_1(\beta)v_2(\alpha) = e^{-n} - e^{n}.$$

When $\xi \leq x$, we have

$$\begin{aligned} G(x,\xi) &= \frac{1}{kD}[v_1(\xi)v_2(\alpha) - v_1(\alpha)v_2(\xi)][v_1(x)v_2(\beta) - v_1(\beta)v_2(x)] \\ &= \frac{1}{2n(e^n - e^{-n})}[e^{n\xi} - e^{-n\xi}][e^{n(1-x)} - e^{-n(1-x)}]. \end{aligned}$$

When $\xi \geq x$, we have

$$\begin{aligned} G(x,\xi) &= \frac{1}{kD}[v_1(x)v_2(\alpha) - v_1(\alpha)v_2(x)][v_1(\xi)v_2(\beta) - v_1(\beta)v_2(\xi)] \\ &= \frac{1}{2n(e^n - e^{-n})}[e^{nx} - e^{-nx}][e^{n(1-\xi)} - e^{-n(1-\xi)}]. \end{aligned}$$

So

$$\begin{aligned} b_n(x) &= \int_0^1 G(x,\xi)f(\xi)\,\mathrm{d}\xi = \int_0^x G(x,\xi)f(\xi)\,\mathrm{d}\xi + \int_x^1 G(x,\xi)f(\xi)\,\mathrm{d}\xi \\ &= \frac{-1}{2n(e^n - e^{-n})}\{[e^{n(1-x)} - e^{-n(1-x)}]\int_0^x (e^{n\xi} - e^{-n\xi})f(\xi)\,\mathrm{d}\xi \\ &\qquad + (e^{nx} - e^{-nx})\int_x^1 [e^{n(1-\xi)} - e^{-n(1-\xi)}]f(\xi)\,\mathrm{d}\xi\} \\ &= \begin{cases} \frac{3}{4}[-x(1-x) + 2(1 - \frac{\cosh(x-\frac{1}{2})}{\cosh\frac{1}{2}})] & \text{for n=1,} \\ -\frac{1}{4}[\frac{-1}{9}x(1-x) + \frac{2}{81}(1 - \frac{\cosh 3(x-\frac{1}{2})}{\cosh\frac{3}{2}})] & \text{for n=3.} \end{cases} \end{aligned}$$

Therefore, the solution is

$$u(x,y) = \frac{3}{4}[-y(1-y) + 2(1 - \frac{\cosh(y-\frac{1}{2})}{\cosh\frac{1}{2}})]\sin x - \frac{1}{4}[\frac{-1}{9}x(1-x) + \frac{2}{81}(1 - \frac{\cosh 3(y-\frac{1}{2})}{\cosh\frac{3}{2}})]\sin 3x.$$

## 2.3 The Fourier transform to construct the solution of Laplace's equation

Fourier transform provides a way of expanding functions on the whole real line **R**=$(-\infty,\infty)$ as (continuous) superpositions of the basic oscillatory functions $e^{iwx}$ $\quad (w \in \mathbf{R})$ in much the same way that Fourier series are used to expand functions on a finite interval.

To understand this relationship, consider a function f on **R**. For any $\ell > 0$ we can expand f on the interval $[-\ell,\ell]$ in a Fourier series, and we wish to see what happens to this expansion as we let $\ell \to \infty$

For x $\in [-\ell,\ell]$

$$f(x) = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} c_{n,\ell}e^{\frac{-i\pi nx}{\ell}} \qquad \text{where } c_{n,\ell} = \int_{-\ell}^{\ell} f(y)e^{\frac{i\pi ny}{\ell}}\,\mathrm{d}y. \tag{2.103}$$

Let $\Delta w = \frac{\pi}{\ell}$ and $w_n = n\Delta w = \frac{n\pi}{\ell}$; then these formulas become

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_{n,\ell}e^{-iw_nx}\Delta w \qquad \text{where } c_{n,\ell} = \int_{-\ell}^{\ell} f(y)e^{iw_ny}\,\mathrm{d}y.$$

Suppose that f(x) vanishes rapidly as $x \to \pm\infty$; then $c_{n,\ell}$ will not change much if we extend the region of integration from $[-\ell,\ell]$ to $(-\infty,\infty)$:

$$c_{n,\ell} \approx \int_{-\infty}^{\infty} f(y)e^{iw_ny}\,\mathrm{d}y. \tag{2.104}$$

The integral function (2.104) only of $w_n$, which we call $\widehat{f}(w)$, and we now have

$$f(x) \approx \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \widehat{f}(w)e^{-iwx}\Delta w \qquad (|x| < \ell). \tag{2.105}$$

This looks very much like a Riemann sum. If we now let $\ell \to \infty$, so that $\Delta w \to 0$. The $\approx$ should become $=$ and the sum should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(w)e^{-iwx}\,\mathrm{d}w \qquad \text{where } \widehat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{iwx}\,\mathrm{d}x. \tag{2.106}$$

The function $\widehat{f}$ is called the **Fourier transform** of f. It is sometimes denoted by $\mathfrak{F}[f]$. And (2.106) is the **Fourier inversion theorem**. The integral certainly converges if $\int_{-\infty}^{\infty}|f(x)|\,\mathrm{d}x$ does (see [3]).

For function of two variables, say u(x,y), we define

$$\mathfrak{F}[u](w,y) \equiv \int_{-\infty}^{\infty} u(x,y)e^{iwx}\,\mathrm{d}x. \tag{2.107}$$

Example:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } -\infty < x < \infty, 0 < y < 1, \\ u(x,0) = e^{-2|x|} = f(x), & \\ u(x,1) = 0, & \\ u(x,y) \to 0 & \text{uniformly in y, as x} \to \pm\infty. \end{cases}$$

Answer :

$\mathfrak{F}[u_{xx} + u_{yy}] = \mathfrak{F}[u_{xx}] + \mathfrak{F}[u_{yy}]$

where

$$\mathfrak{F}[u_{yy}] = \int_{-\infty}^{\infty} e^{iwx} u_{yy}\, \mathrm{d}x = \frac{\partial}{\partial y^2} \int_{-\infty}^{\infty} e^{iwx} u\, \mathrm{d}x = \widehat{u}_{yy},$$

$$(\because u \to 0 \text{ uniformly in y as x} \to \pm\infty \quad \therefore u_x \to 0 \text{ uniformly in y as x} \to \pm\infty)$$

$$\begin{aligned} \mathfrak{F}[u_{xx}] &= \int_{-\infty}^{\infty} e^{iwx} u_{xx}\, \mathrm{d}x = e^{iwx} u_x \mid_{-\infty}^{\infty} - (iw) \int_{-\infty}^{\infty} e^{iwx} u_x\, \mathrm{d}x \\ &= (-iw)[e^{iwx} u \mid_{-\infty}^{\infty} - (iw) \int_{-\infty}^{\infty} e^{iwx} u\, \mathrm{d}x] = -w^2 \widehat{u}, \\ \Rightarrow \mathfrak{F}[u_{xx} + u_{yy}] &= -w^2 \widehat{u} + \widehat{u}_{yy} = 0, \\ \widehat{u}(w,0) &= \int_{-\infty}^{\infty} e^{iwx} u(x,0)\, \mathrm{d}x = \int_{0}^{\infty} e^{iwx} u(x,0)\, \mathrm{d}x = \int_{-\infty}^{0} e^{iwx} u(x,0)\, \mathrm{d}x. \\ &= \lim_{t \to \infty} [\frac{e^{iw-2}}{iw-2} \mid_0^t] + \lim_{m \to -\infty} [\frac{e^{iw-2}}{iw-2} \mid_m^0] \\ &= \frac{-1}{iw-2} + \frac{1}{iw+2} = \frac{4}{w^2+4}. \end{aligned}$$

Thus we find

$$-w^2 \widehat{u} + \widehat{u}_{yy} = 0, \tag{2.108}$$

$$\widehat{u}(w,0) = \frac{4}{w^2+4} = \widehat{f}(w), \tag{2.109}$$

$$\widehat{u}(w,1) = 0. \tag{2.110}$$

By the equation (2.108), if $\widehat{u}(x,y) = A(x)e^{wy} + B(x)e^{-wy}$,

then $\widehat{u}(w,0) = A(w) + B(w) = \frac{4}{w^2+4}$ and $\widehat{u}(w,1) = A(w)e^{w} + B(w)e^{-w} = 0$.

We get $A(w) = \dfrac{-2e^{-w}}{\sinh w(w^2+4)}$ and $B(w) = \dfrac{2e^{w}}{\sinh w(w^2+4)}$.

$\Rightarrow \widehat{u}(x,y) = \dfrac{4\sinh(x-wy)}{\sinh x(x^2+4)}$.

By inverse fourier theorem,

$$u(x,y) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{-iwx}\widehat{u}(w,y)\,\mathrm{d}w = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-iwx}\frac{\sinh w(1-y)}{\sinh w(w^2+4)}\,\mathrm{d}w.$$

There is another solution for the special case where u(x,0)=f(x). If f(x) is either the even function or odd function, we can use **cosine** or **sine transform**. They are defined respectively as

$$\mathfrak{F}_s[f] \equiv \int_0^\infty f(x)\sin wx\,\mathrm{d}x \quad \text{and} \quad \mathfrak{F}_c[f] \equiv \int_0^\infty f(x)\cos wx\,\mathrm{d}x.$$

Since $e^{iwx} = \cos wx + i\sin wx$,

$$\widehat{f}(w) = 2i\mathfrak{F}_s[f] \qquad \text{if f(x) is an odd function at } (-\infty,\infty),$$
$$\widehat{f}(w) = 2\mathfrak{F}_c[f] \qquad \text{if f(x) is an even function at } (-\infty,\infty).$$

$\because f(x) = e^{-2|x|}$ is an even function $\therefore \widehat{f}(w) = 2\mathfrak{F}_c[f]$.

$$\mathfrak{F}_c[u_{xx}] = \int_0^\infty u_{xx}\cos wx\,\mathrm{d}x = -w^2\mathfrak{F}_c[u] = -w^2U(w,y),$$
$$\mathfrak{F}_c[u_{yy}] = U_{yy}(w,y),$$
$$U(w,0) = \int_0^\infty u(x,0)\cos wx\,\mathrm{d}x = \frac{2}{w^2+4}.$$

Thus we find

$$-w^2U(w,y) + U_{yy}(w,y) = 0, \tag{2.111}$$
$$U(w,o) = \frac{2}{w^2+4}, \tag{2.112}$$
$$U(w,1) = 0. \tag{2.113}$$

To compute the system (2.111), (2.112) and (2.113) with similar way, we have $U(w,y) = \frac{2\sinh w(1-y)}{(w^2+4)\sin hw}$.

Thus

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int_0^\infty U(w,y)\cos wx\,\mathrm{d}w \\ &= \frac{2}{\pi}\int_{-\infty}^\infty (\cos wx - i\sin wx)\frac{\sinh w(1-y)}{(w^2+4)\sin hw}\,\mathrm{d}w \\ &= \frac{2}{\pi}\int_{-\infty}^\infty e^{-iwx}\frac{\sinh w(1-y)}{(w^2+4)\sin hw}\,\mathrm{d}w. \end{aligned}$$

# Chapter 3

# Numerical computations

## _ The finite difference method for Laplace's equation

The partial derivative $\frac{\partial u}{\partial x}$ as the limit of a difference quotient

$$\frac{\partial u}{\partial x}(x, y) = \lim_{h \longrightarrow 0} \frac{u(x+h, y) - u(x, y)}{h}. \tag{3.1}$$

By Taylor's theorem with remainder, we know that if $u$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ are continuous on $\Omega$,

$$u(x+h, y) - u(x, y) = h\frac{\partial u(x, y)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x+\theta h, y)}{\partial x^2}.$$

$$\Longrightarrow \frac{\partial u}{\partial x}(x, y) - \frac{u(x+h, y) - u(x, y)}{h} = -\frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x+\theta h, y)$$

where $0 < \theta < 1$. If h is small, the right-hand side is small.

Similarly,

$$u(x+h, y) - u(x, y) = h\frac{\partial u}{\partial x}(x, y) + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3}(x, y) + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4}(x+\theta_1 h, y),$$

$$u(x-h, y) - u(x, y) = (-h)\frac{\partial u}{\partial x}(x, y) + \frac{(-h)^2}{2!}\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{(-h)^3}{3!}\frac{\partial^3 u}{\partial x^3}(x, y) + \frac{(-h)^4}{4!}\frac{\partial^4 u}{\partial x^4}(x+\theta_2 h, y)$$

where $0 < \theta_1, \theta_2 < 1$.

$$\Longrightarrow \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2} = \frac{1}{h^2}[u(x+h, y) - u(x, y) + u(x-h, y) - u(x, y)]$$

$$= \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x+\theta_3 h, y)$$

$$\equiv \frac{\partial^2 u}{\partial x^2}(x, y) \qquad \text{(as h is small enough)}$$

where $-1 < \theta_3 < 1$ and $\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x+\theta_3 h, y)$ is the truncation error; it is $O(h^2)$ for any $u \in C^4(\Omega)$ (see [6]).

We consider the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{in D,} \tag{3.2}$$

$$u = f \qquad \text{on C} \tag{3.3}$$

in a bounded domain D with boundary C. We introduce the square grid $(x_i, y_j) = (ih, jh) \quad i, j = 0, 1 \ldots$ in the x-y plane. We define $U_{i,j} = u(x_i, y_j)$ and replace the Laplace's equation (3.2) by the finite difference equation

$$\nabla_h^2 u(x_i, y_j) = \frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}). \tag{3.4}$$

To estimate the truncation error $\nabla_h^2 u - \nabla^2 u$ , we again assume $u \in C^4(\Omega)$ and expand in Taylor series about $(x_i, y_j)$ through terms in $h^4$. We find that

$$\nabla_h^2 u(x_i, y_j) - \nabla^2 u(x_i, y_j) = \frac{h^2}{12}[\frac{\partial^4}{\partial x^4} u(x_i + \theta_1, y_j) + \frac{\partial^4}{\partial y^4} u(x_i, y_j + \theta_2)]$$

where $|\theta_k| < h$. The right side is the truncation error; it is $O(h^2)$ for any u $\in C^4(\Omega)$. In this notation, the discrete Laplacian of (3.4) appears as

$$\nabla_h^2 u(x_i, y_j) = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u(x_i, y_j).$$

The mesh function $u(ih, jh)$ is defined at all mesh point in D and C (see [3]). We split these mesh points into two class:

We call the points $((i+1)h, jh), ((i-1)h, jh), (ih, (j+1)h), (ih, (j-1)h)$ the **nearest neighbors** of the mesh point $(ih, jh)$ . If $(ih, jh)$ and all its nearest neighbors lie in D+C, we call $(ih, jh)$ an **interior point**. If $(ih, jh)$ is in D+C, but one of its nearest neighbors is not, we call it a **boundary point** (see [2]).

Example:

For two-dimensional problems the situations are alike, but more complicated. Below we only discuss the simple case of a Laplace's equation on a square with nonhomogeneous Dirichlet boundary condition; i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for } 0 < x < \pi, 0 < y < \pi, \tag{3.5}$$

$$u(\pi, y) = u(x, \pi) = u(0, y) = 0, \tag{3.6}$$

$$u(x, 0) = x^2(\pi - x), \tag{3.7}$$

discretized with central differences on the uniform grid

$$(x_i, y_j) := (ih, jh) \qquad i,j = 0,1,\ldots,m+1, \qquad h := \frac{\pi}{m+1}.$$

The central difference scheme for the equation (3.5) reads

$$\frac{1}{h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}) = 0 \qquad i,j = 1,2,\ldots,m;$$

These equations have to be completed with the numerical boundary conditions (3.6)

$$U_{0,j} = U_{m+1,j} = U_{i,m+1} = 0 \qquad \text{and} \qquad U_{i,0} = (ih)^2(\pi - ih).$$

Introducing the vector of unknowns **u**

$$\mathbf{u} := \begin{pmatrix} U_{1,1} \\ U_{1,2} \\ \vdots \\ U_{1,m} \\ U_{2,1} \\ \vdots \\ U_{m,1} \\ \vdots \\ U_{m,m} \end{pmatrix}_{m^2\times 1}, \quad \mathbf{f} := \begin{pmatrix} -U_{1,0} \\ 0 \\ \vdots \\ -U_{2,0} \\ 0 \\ \vdots \\ -U_{m,0} \\ 0 \\ \vdots \end{pmatrix}_{m^2\times 1} \tag{3.8}$$

We can write (3.8) as the linear system

$$\mathbf{Au} = \mathbf{f}$$

where the matrix **A** can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{I} & & & \\ \mathbf{I} & \mathbf{B} & \mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{I} & \mathbf{B} & \mathbf{I} \\ & & & \mathbf{I} & \mathbf{B} \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{pmatrix}.$$

The partitioning of u and A is based on the numbering along horizontal grid lines. We can get $U_{i,j}$ from $\mathbf{u}=\mathbf{A}^{-1}\mathbf{f}$ for i,j=1,2,...m (see [2]).

Assume that m=29, we see the data $U_{i,j}$ in the Appendix B.2 which is computed by the Mathematica program (see Appendix A.1). The Figure 3.1 shows the numerical solution of the system (3.5), (3.6) and (3.7). One can see the comparison between the numerical solution and the exact solution (2.15) truncated up to 1000 in the Figure 3.2 and Figure 3.3. Moreover, the detailed errors between these two solutions are shown in Appendix B.4.

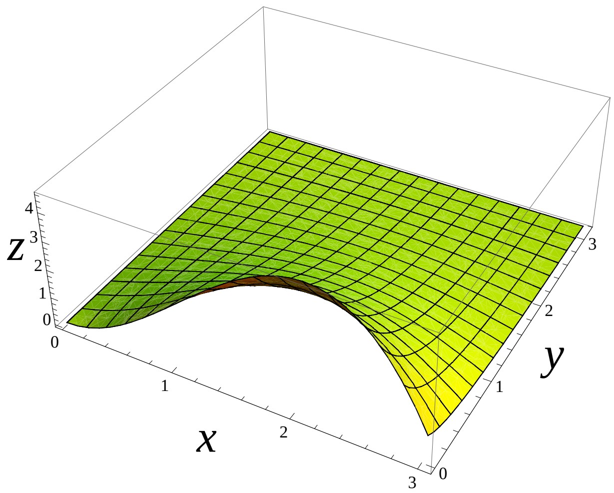


Figure 3.1: The numerical solution of the system (3.5), (3.6) and (3.7) which is computed with the finite difference method.

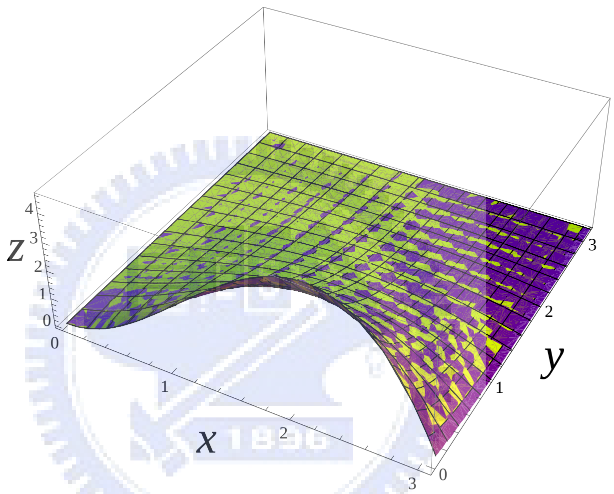


Figure 3.2: The comparison between the numerical solution and the truncation of the exact solution (2.15) up to 1000.

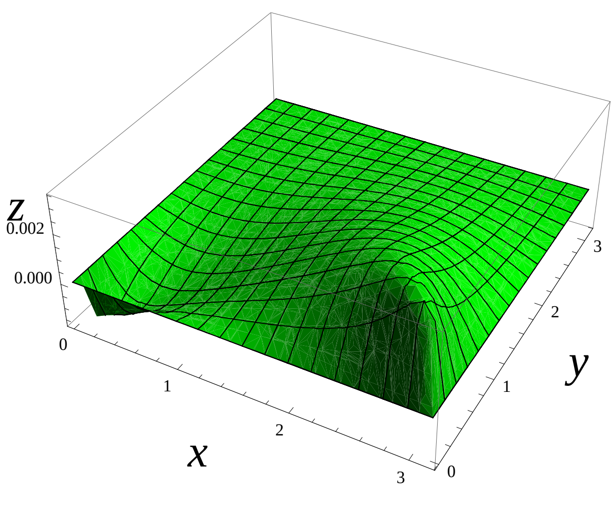


Figure 3.3: The error between two solutions (numerical solution-exact solution).

# Appendix A

# Mathematica codes

## A.1 Constructing Matrix A and f to evaluate $u = A^{-1}f$

Clear[m,A,f];

m=29;

S=Table[If[$k + m \leq m^2 \&\& k - m \leq 0 \&\& \text{Mod}[k, m] == 1$,

$\{\{k, k+m\} \to 1, \{k, k\} \to -4, \{k, k+1\} \to 1\}$,

If[$k + m \leq m^2 \&\& k - m \leq 0 \&\& \text{Mod}[k, m] == 0$,

$\{\{k, k+m\} \to 1, \{k, k\} \to -4, \{k, k-1\} \to 1\}$,

If[$k + m \leq m^2 \&\& k - m \leq 0 \&\& \text{Mod}[k, m] \neq 1 \&\& \text{Mod}[k, m] \neq 0$,

$\{\{k, k+m\} \to 1, \{k, k\} \to -4, \{k, k+1\} \to 1, \{k, k-1\} \to 1\}$,

If[$k + m \leq m^2 \&\& k - m > 0 \&\& \text{Mod}[k, m] == 1$,

$\{\{k, k+m\} \to 1, \{k, k-m\} \to 1, \{k, k\} \to -4, \{k, k+1\} \to 1\}$,

If[$k + m \leq m^2 \&\& k - m > 0 \&\& \text{Mod}[k, m] == 0$,

$\{\{k, k+m\} \to 1, \{k, k-m\} \to 1, \{k, k\} \to -4, \{k, k-1\} \to 1\}$,

If[$k + m \leq m^2 \&\& k - m > 0 \&\& \text{Mod}[k, m] \neq 1 \&\& \text{Mod}[k, m] \neq 0$,

$\{\{k, k+m\} \to 1, \{k, k-m\} \to 1, \{k, k\} \to -4, \{k, k+1\} \to 1, \{k, k-1\} \to 1\}$,

If[$k + m > m^2 \&\& k - m > 0 \&\& \text{Mod}[k, m] == 1$,

$\{\{k, k-m\} \to 1, \{k, k\} \to -4, \{k, k+1\} \to 1\}$,

If[$k + m > m^2 \&\& k - m > 0 \&\& \text{Mod}[k, m] == 0$,

$\{\{k, k-m\} \to 1, \{k, k\} \to -4, \{k, k-1\} \to 1\}$, If$[k+m>m^2$&&$k-m>0$&&Mod$[k,m]\neq 1$&&Mod$[k,m]\neq 0$,

$\{\{k,k-m\}\to 1,\{k,k\}\to -4,\{k,k+1\}\to 1,\{k,k-1\}\to 1\}$,0]]]]]]]], $\{k,1,m^2\}$];

A={};

For$[i=1,i<$Length$[S]+1,i++$,A=Join[A,S[[i]]]];

L=Inverse[SparseArray[A]];

f=Table[If[Mod$[r,m]==1,-((\frac{r-1}{m+1}*\pi)^2)*(m+1-(\frac{r-1}{m+1})*\frac{\pi}{(m+1)^3},0],\{r,m^2\}]$;

u=N[L.f];

Clear[i,j];

$U[x\_,y\_]$ :=If$[x==0||y==m+1||x==m+1,0$,If$[y==0,\frac{(x\pi)^2(m+1-x)\pi}{(m+1)^3}$,

u$[[m*(x-1)+y]]]]]$;

U1[x_, y_]:= U[$\frac{m+1}{\pi}$*x,$\frac{m+1}{\pi}$*y];

U2[w_, $p\_$] := $(-4)*(\sum_{i=1}^{1000}(1+2*(-1)^i)*i^{-3}*\frac{\text{Sinh}[i*(\pi-p)]}{\text{Sinh}[i*\pi]}*\text{Sin}[i*w])$;

## A.2 Using U1[x,y] and U2[w,p] in A.1 to create the figures and tables of the numerical solutions

pic=Table$[\{x,y,U1[x,y]\},\{x,0,\pi,\frac{\pi}{m+1}\},\{y,0,\pi,\frac{\pi}{m+1}\}]$;

picJoin={};

For$[i=1,i<m+2,i++$,picJoin=Join[picJoin,pic[[i]]]];

AA=ListPlot3D[picJoin,PlotStyle→Yellow, AxesLabel→{Style[x,Large],Style[y,Large],

Style[z,Large]},PlotLabel→"Numerical solution(Finite Difference Method)"]

BB=Plot3D[$U2[x,y],\{x,0,\pi\},\{y,0,\pi\}$,PlotStyle→Purple,AxesLabel→{Style[x,Large],

Style[y,Large],Style[z,Large]},PlotLabel→"Truncation of exactly solution up to 1000"]

DD = Show[AA, BB]

pic2=Table$[\{x,y,U1[x,y]-N[U2[x,y],7]\},\{x,0,\pi,\frac{\pi}{m+1}\},\{y,0,\pi,\frac{\pi}{m+1}\}]$;

pic2Join={};

For$[i=1,i<m+3,i++$,pic2Join=Join[pic2Join,pic2[[i]]]];

CC=ListPlot3D[pic2Join,PlotStyle→Green,AxesLabel→{Style[x,Large],Style[y,Large],

Style[z, Large]},PlotLabel→"The error between numerical solution and

Truncation of exactly solution up to 1000(nu.-ex.)"]

Tn=TableForm[Table$[U[x, y], \{x, 0, m + 1\}, \{y, 0, m + 1\}],$

TableHeadings $\rightarrow$ {Table$[u_{i,j}, \{i, 0, m + 1\}]$,Table$[u_{i,j}, \{j, 0, m + 1\}]\}$,

TableSpacing $\rightarrow \{1, 2\}]$

Te=TableForm[Table$[N[U2[x, y], 7], \{x, 0, \pi, \frac{\pi}{m+1}\}, \{y, 0, \pi, \frac{\pi}{m+1}\}],$

TableHeadings $\rightarrow$Table$[u_{i,j}, \{i, 0, m + 1\}]$,Table$[u_{i,j}, \{j, 0, m + 1\}]\}$,

TableSpacing$\rightarrow \{1, 2\}]$

Tdiffer=TableForm[Table$[U[x, y], \{x, 0, m + 1\}, \{y, 0, m + 1\}]-$Table$[N[U2[x, y], 7],$

$\{x, 0, \pi, \frac{\pi}{m+1}\}, \{y, 0, \pi, \frac{\pi}{m+1}\}]$,TableHeadings$\rightarrow$ {Table$[u_{i,j}, \{i, 0, m + 1\}]$,

Table$[u_{i,j}, \{j, 0, m + 1\}]\}$, TableSpacing $\rightarrow \{1, 2\}]$

## Appendix B

# Tables of numerical solutions

| | $y=0$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1. | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0. | 0.056406 | 0.0830309 | 0.0988724 | 0.108243 | 0.113329 | 0.115494 | 0.115674 | 0.114551 | 0.112645 | 0.110365 | 0.10803 | 0.105881 | 0.104091 | 0.102781 | 0.102033 | 0.101899 | 0.10239 | 0.103479 | 0.10509 | 0.10711 |
| $x=1.4$ | 0. | 0.0162212 | 0.029954 | 0.0402176 | 0.0471909 | 0.0514585 | 0.0536485 | 0.0543209 | 0.0539448 | 0.0529034 | 0.051504 | 0.0499902 | 0.0485507 | 0.0473284 | 0.0464262 | 0.045912 | 0.045822 | 0.0461624 | 0.0469096 | 0.0480092 | 0.0493733 |
| | 0. | 0.00562905 | 0.0107249 | 0.0149041 | 0.0179944 | 0.0200055 | 0.0210636 | 0.0213512 | 0.0210662 | 0.0203977 | 0.0195141 | 0.0185583 | 0.0176466 | 0.0168698 | 0.0162948 | 0.0159666 | 0.0159091 | 0.0161265 | 0.0166031 | 0.0173028 | 0.018168 |
| | 0. | 0.0481944 | 0.0700175 | 0.0823566 | 0.0890382 | 0.0920095 | 0.0924905 | 0.0913219 | 0.0891174 | 0.0863469 | 0.0833806 | 0.0805088 | 0.07795 | 0.0758611 | 0.0743521 | 0.0734983 | 0.0733459 | 0.073907 | 0.0751546 | 0.0770213 | 0.0794037 |
| $x=1.42$ | 0. | 0.0130219 | 0.0238615 | 0.0316673 | 0.036611 | 0.0392218 | 0.0400661 | 0.0396491 | 0.0383944 | 0.0366482 | 0.034689 | 0.0327371 | 0.0309638 | 0.0294986 | 0.0284348 | 0.0278342 | 0.0277294 | 0.0281262 | 0.0290032 | 0.0303108 | 0.0319695 |
| | 0. | 0.00408922 | 0.00771768 | 0.0105568 | 0.0124638 | 0.0134561 | 0.0136531 | 0.0132233 | 0.0123477 | 0.0111988 | 0.00992965 | 0.00867018 | 0.00752601 | 0.00657959 | 0.00589164 | 0.00550285 | 0.00543504 | 0.00569193 | 0.00625928 | 0.00710439 | 0.00817505 |
| | 0. | 0.0396113 | 0.0566064 | 0.0654219 | 0.0693958 | 0.0702351 | 0.0690155 | 0.0664824 | 0.0631831 | 0.0595377 | 0.0558771 | 0.0524611 | 0.0494858 | 0.047093 | 0.045382 | 0.0444211 | 0.0442512 | 0.0448827 | 0.0462912 | 0.0484169 | 0.0511668 |
| $x=1.44$ | 0. | 0.00980383 | 0.0177319 | 0.0230622 | 0.0259599 | 0.0268986 | 0.0263836 | 0.024865 | 0.022721 | 0.0202611 | 0.0177344 | 0.0153384 | 0.0132264 | 0.0115147 | 0.010287 | 0.00959851 | 0.0094788 | 0.00993285 | 0.0109415 | 0.0124604 | 0.014418 |
| | 0. | 0.00253623 | 0.00468446 | 0.0061711 | 0.00688319 | 0.00684593 | 0.00617222 | 0.00501647 | 0.00354286 | 0.00190703 | 0.000247007 | −0.00132048 | −0.00270051 | −0.0038191 | −0.00462175 | −0.00507201 | −0.00515032 | −0.00485333 | −0.0041938 | −0.00320106 | −0.00192194 |
| | 0. | 0.0313788 | 0.0435293 | 0.0488011 | 0.0500469 | 0.0487358 | 0.0458 | 0.0418894 | 0.0374837 | 0.0329526 | 0.0285884 | 0.0246208 | 0.0212238 | 0.0185234 | 0.0166078 | 0.0155378 | 0.0153495 | 0.0160525 | 0.0176248 | 0.0200134 | 0.0231357 |
| $x=1.46$ | 0. | 0.00656808 | 0.0115673 | 0.0144056 | 0.0152416 | 0.0144939 | 0.0126065 | 0.00997479 | 0.00693151 | 0.00374933 | 0.000648016 | −0.00219808 | −0.00465328 | −0.00661493 | −0.00800895 | −0.00878644 | −0.00892134 | −0.00840921 | −0.00726703 | −0.00553398 | −0.00327332 |
| | 0. | 0.000970825 | 0.00162668 | 0.00174917 | 0.00125537 | 0.000178469 | −0.00137509 | −0.00326495 | −0.00534372 | −0.00747248 | −0.00952849 | −0.0114081 | −0.0130273 | −0.0143205 | −0.0152395 | −0.0157521 | −0.015841 | −0.0155034 | −0.0147504 | −0.0136079 | −0.0121174 |
| | 0. | 0.0231609 | 0.0304475 | 0.0321568 | 0.0306565 | 0.0271785 | 0.0225117 | 0.0172106 | 0.0116871 | 0.00626036 | 0.00118408 | −0.00334186 | −0.00716574 | −0.0101777 | −0.0123006 | −0.0134813 | −0.0136881 | −0.0129128 | −0.0111743 | −0.0085194 | −0.0050201 |
| $x=1.48$ | 0. | 0.0033157 | 0.00536973 | 0.00570043 | 0.00446001 | 0.00201249 | −0.0012596 | −0.00501532 | −0.00896741 | −0.01288 | −0.0165628 | −0.0198645 | −0.0226673 | −0.0248821 | −0.0264446 | −0.0273124 | −0.0274627 | −0.0268917 | −0.0256142 | −0.0236641 | −0.0210965 |
| | 0. | −0.00060626 | −0.00145423 | −0.00270687 | −0.00441691 | −0.006543 | −0.00898507 | −0.0116167 | −0.0143074 | −0.0169348 | −0.0193916 | −0.0215874 | −0.0234488 | −0.0249188 | −0.0259558 | −0.0265316 | −0.0266313 | −0.0262524 | −0.0254046 | −0.0241105 | −0.0224058 |
| | 0. | 0.0145319 | 0.0169317 | 0.0150606 | 0.0107992 | 0.00514008 | −0.00127189 | −0.00797698 | −0.0146298 | −0.0209611 | −0.0267565 | −0.0318469 | −0.0361032 | −0.0394311 | −0.0417638 | −0.0430561 | −0.0432812 | −0.0424326 | −0.0405267 | −0.0376026 | −0.0337215 |
| $x=1.5$ | 0. | 0.0000477101 | −0.000858706 | −0.00305032 | −0.00638091 | −0.0105411 | −0.0152094 | −0.0200994 | −0.0249693 | −0.02962 | −0.0338907 | −0.0376532 | −0.0408078 | −0.0432789 | −0.045012 | −0.0459711 | −0.046137 | −0.0455064 | −0.0440919 | −0.0419222 | −0.039044 |
| | 0. | −0.00219432 | −0.00455687 | −0.00719498 | −0.010131 | −0.0133153 | −0.016654 | −0.0200347 | −0.0233436 | −0.0264752 | −0.0293374 | −0.0318529 | −0.0339595 | −0.0356087 | −0.036765 | −0.0374048 | −0.0375155 | −0.0370948 | −0.0361511 | −0.0347033 | −0.0327819 |
| | 0. | 0.00617442 | 0.00367329 | −0.00179461 | −0.008834 | −0.0166897 | −0.0248598 | −0.0329798 | −0.0407718 | −0.0480167 | −0.0545392 | −0.0602003 | −0.0648935 | −0.0685404 | −0.0710854 | −0.0724909 | −0.0727348 | −0.0718122 | −0.0697363 | −0.0665396 | −0.0622726 |
| $x=1.52$ | 0. | −0.0032349 | −0.00711608 | −0.0118438 | −0.0172775 | −0.0231623 | −0.0292377 | −0.0352716 | −0.0410677 | −0.0464639 | −0.0513285 | −0.0555569 | −0.0590672 | −0.0617975 | −0.0637032 | −0.0647547 | −0.0649364 | −0.0642455 | −0.0626923 | −0.0603005 | −0.0571081 |
| | 0. | −0.00379268 | −0.0076799 | −0.0117132 | −0.0158843 | −0.0201352 | −0.0243782 | −0.0285149 | −0.0324479 | −0.0360888 | −0.0393607 | −0.0421996 | −0.0445541 | −0.0463845 | −0.0476616 | −0.0483662 | −0.0484879 | −0.048025 | −0.0469842 | −0.045381 | −0.0432403 |
| | 0. | −0.00205387 | −0.00947034 | −0.0185507 | −0.0283833 | −0.0384494 | −0.04839 | −0.0579356 | −0.0668762 | −0.0750431 | −0.0822999 | −0.0885375 | −0.0936718 | −0.0976409 | −0.1004 | −0.101921 | −0.102184 | −0.101186 | −0.0989384 | −0.0954661 | −0.0908092 |
| $x=1.54$ | 0. | −0.00653115 | −0.0134005 | −0.0206771 | −0.0282261 | −0.0358468 | −0.0433394 | −0.0505263 | −0.0572566 | −0.063405 | −0.0688693 | −0.0735682 | −0.0774379 | −0.0804302 | −0.0825103 | −0.0836553 | −0.0838529 | −0.0831011 | −0.0814078 | −0.0787914 | −0.0752815 |
| | 0. | −0.00540065 | −0.010822 | −0.0162595 | −0.0216742 | −0.0269997 | −0.0321543 | −0.0370532 | −0.0416161 | −0.0457711 | −0.0494567 | −0.0526223 | −0.0552274 | −0.0572409 | −0.0586402 | −0.0594103 | −0.0595432 | −0.0590375 | −0.0578985 | −0.0561382 | −0.0537759 |
| | 0. | −0.0107007 | −0.0230522 | −0.0357595 | −0.0483973 | −0.0606854 | −0.072408 | −0.0833906 | −0.0934893 | −0.102586 | −0.110582 | −0.117402 | −0.122982 | −0.127277 | −0.130253 | −0.131889 | −0.132171 | −0.131098 | −0.128677 | −0.124926 | −0.119876 |
| $x=1.56$ | 0. | −0.00984012 | −0.0197101 | −0.0295476 | −0.0392231 | −0.0485901 | −0.0575095 | −0.0658579 | −0.0735299 | −0.0804368 | −0.0865063 | −0.0916801 | −0.0959127 | −0.0991695 | −0.101426 | −0.102665 | −0.102879 | −0.102066 | −0.100231 | −0.0973874 | −0.093557 |
| | 0. | −0.00701759 | −0.0139818 | −0.020832 | −0.0274984 | −0.0339057 | −0.0399786 | −0.0456459 | −0.0508439 | −0.0555175 | −0.0596207 | −0.0631161 | −0.0659742 | −0.0681727 | −0.0696953 | −0.0705316 | −0.0706758 | −0.070127 | −0.0688888 | −0.0669698 | −0.0643837 |
| | 0. | −0.0191722 | −0.0364706 | −0.0528182 | −0.0682747 | −0.0827971 | −0.0963124 | −0.108741 | −0.120006 | −0.130039 | −0.138783 | −0.146188 | −0.152218 | −0.156842 | −0.160037 | −0.161789 | −0.162091 | −0.160941 | −0.158345 | −0.154314 | −0.148866 |
| $x=1.58$ | 0. | −0.0131609 | −0.026043 | −0.0384526 | −0.0502652 | −0.0613882 | −0.0717431 | −0.081261 | −0.0898815 | −0.097553 | −0.104233 | −0.109885 | −0.114484 | −0.118008 | −0.120442 | −0.121777 | −0.122007 | −0.121131 | −0.119154 | −0.116081 | −0.111928 |
| | 0. | −0.00864285 | −0.0171581 | −0.0254288 | −0.0333543 | −0.0408505 | −0.0478478 | −0.054289 | −0.060127 | −0.0653235 | −0.0698479 | −0.073676 | −0.0767894 | −0.0791746 | −0.0808219 | −0.0817249 | −0.0818805 | −0.0812881 | −0.0799498 | −0.0778706 | −0.0750586 |
| | 0. | −0.0274181 | −0.0496734 | −0.0696738 | −0.0879618 | −0.10473 | −0.120048 | −0.133931 | −0.146369 | −0.157346 | −0.166842 | −0.17484 | −0.181323 | −0.186278 | −0.189693 | −0.191563 | −0.191885 | −0.190658 | −0.187885 | −0.183571 | −0.177722 |
| $x=1.6$ | 0. | −0.0164925 | −0.0323975 | −0.0473894 | −0.0613488 | −0.0742368 | −0.0860355 | −0.0967302 | −0.106306 | −0.114747 | −0.122042 | −0.128177 | −0.133145 | −0.136939 | −0.139552 | −0.140983 | −0.141229 | −0.140291 | −0.138169 | −0.134866 | −0.130386 |
| | 0. | −0.0102758 | −0.0203497 | −0.0300482 | −0.0392396 | −0.0478309 | −0.0557586 | −0.0629789 | −0.0694615 | −0.0751848 | −0.0801336 | −0.0842972 | −0.0876682 | −0.0902417 | −0.0920145 | −0.092985 | −0.0931521 | −0.0925157 | −0.0910764 | −0.0888356 | −0.0857958 |

Table B.1: The value $U_{i,j}$ of the solution (2.69) which project at $z=0,\frac{1}{4},\frac{1}{2}$

| | $U_{i,0}$ | $U_{i,1}$ | $U_{i,2}$ | $U_{i,3}$ | $U_{i,4}$ | $U_{i,5}$ | $U_{i,6}$ | $U_{i,7}$ | $U_{i,8}$ | $U_{i,9}$ | $U_{i,10}$ | $U_{i,11}$ | $U_{i,12}$ | $U_{i,13}$ | $U_{i,14}$ | $U_{i,15}$ | $U_{i,16}$ | $U_{i,17}$ | $U_{i,18}$ | $U_{i,19}$ | $U_{i,20}$ | $U_{i,21}$ | $U_{i,22}$ | $U_{i,23}$ | $U_{i,24}$ | $U_{i,25}$ | $U_{i,26}$ | $U_{i,27}$ | $U_{i,28}$ | $U_{i,29}$ | $U_{i,30}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | 0.03330 | 0.08932 | 0.117493 | 0.13116 | 0.136448 | 0.136587 | 0.133471 | 0.12829 | 0.121828 | 0.114621 | 0.107036 | 0.0993303 | 0.091682 | 0.0842121 | 0.0770003 | 0.0700961 | 0.0635269 | 0.0573034 | 0.051424 | 0.0458785 | 0.0406497 | 0.0357162 | 0.031053 | 0.0266329 | 0.0224272 | 0.0184061 | 0.0145393 | 0.0107962 | 0.00714587 | 0.00355746 | 0 |
| $U_{2,j}$ | 0.1286 | 0.206484 | 0.249493 | 0.2707 | 0.278043 | 0.276431 | 0.269008 | 0.25786 | 0.244403 | 0.229618 | 0.214192 | 0.198603 | 0.183186 | 0.168166 | 0.153693 | 0.139857 | 0.126708 | 0.114263 | 0.102514 | 0.0914401 | 0.0810042 | 0.0711621 | 0.0618629 | 0.0530514 | 0.0446697 | 0.0366578 | 0.0289549 | 0.0214995 | 0.0142299 | 0.00708397 | 0 |
| $U_{3,j}$ | 0.2791 | 0.358504 | 0.403293 | 0.424105 | 0.428593 | 0.422084 | 0.408272 | 0.389739 | 0.368305 | 0.345258 | 0.32151 | 0.297705 | 0.274292 | 0.251574 | 0.229748 | 0.208932 | 0.189186 | 0.170524 | 0.152931 | 0.136363 | 0.120765 | 0.106065 | 0.0921851 | 0.0790403 | 0.0665423 | 0.0546006 | 0.0431231 | 0.0320172 | 0.02119 | 0.0105486 | 0 |
| $U_{4,j}$ | 0.4777 | 0.545182 | 0.581072 | 0.593834 | 0.590142 | 0.575039 | 0.552255 | 0.52452 | 0.493821 | 0.461598 | 0.428885 | 0.396416 | 0.364703 | 0.334089 | 0.304792 | 0.276937 | 0.250579 | 0.225719 | 0.20232 | 0.180318 | 0.159627 | 0.140148 | 0.121772 | 0.104382 | 0.0878587 | 0.0720791 | 0.0569196 | 0.0422562 | 0.0279644 | 0.0139203 | 0 |
| $U_{5,j}$ | 0.7177 | 0.763425 | 0.781978 | 0.780017 | 0.763101 | 0.735675 | 0.701189 | 0.662264 | 0.620862 | 0.578428 | 0.536015 | 0.494371 | 0.454015 | 0.415286 | 0.378395 | 0.343447 | 0.310474 | 0.279451 | 0.250313 | 0.222961 | 0.197278 | 0.173129 | 0.150373 | 0.128858 | 0.108431 | 0.0889374 | 0.0702202 | 0.0521234 | 0.0344912 | 0.0171683 | 0 |
| $U_{6,j}$ | 0.9922 | 1.0088 | 1.0034 | 0.981157 | 0.946568 | 0.90337 | 0.854562 | 0.802487 | 0.748933 | 0.695237 | 0.642376 | 0.59104 | 0.541698 | 0.494647 | 0.450053 | 0.40798 | 0.368418 | 0.331301 | 0.296519 | 0.263936 | 0.233393 | 0.204718 | 0.177731 | 0.152245 | 0.128071 | 0.105019 | 0.0829004 | 0.061526 | 0.0407086 | 0.0202618 | 0 |
| $U_{7,j}$ | 1.294 | 1.27618 | 1.24166 | 1.19464 | 1.13865 | 1.07668 | 1.0112 | 0.944188 | 0.877146 | 0.811213 | 0.747212 | 0.685714 | 0.62709 | 0.571552 | 0.51919 | 0.470003 | 0.423919 | 0.380813 | 0.340526 | 0.30287 | 0.26764 | 0.23462 | 0.203589 | 0.17432 | 0.146588 | 0.120168 | 0.0948363 | 0.0703717 | 0.0465554 | 0.0231701 | 0 |
| $U_{8,j}$ | 1.617 | 1.56002 | 1.49243 | 1.41711 | 1.33669 | 1.25349 | 1.16939 | 1.08592 | 1.00425 | 0.925255 | 0.849544 | 0.777516 | 0.709397 | 0.64528 | 0.585153 | 0.528924 | 0.47644 | 0.427508 | 0.381902 | 0.339378 | 0.299676 | 0.262534 | 0.227684 | 0.194859 | 0.163794 | 0.134229 | 0.105905 | 0.0785692 | 0.0519712 | 0.0258633 | 0 |
| $U_{9,j}$ | 1.953 | 1.85454 | 1.75094 | 1.64466 | 1.53754 | 1.43119 | 1.32694 | 1.22584 | 1.12868 | 1.03601 | 0.948195 | 0.865407 | 0.787701 | 0.715018 | 0.647218 | 0.584098 | 0.525411 | 0.470877 | 0.420198 | 0.373062 | 0.329154 | 0.288156 | 0.249753 | 0.213636 | 0.1795 | 0.147047 | 0.115985 | 0.086029 | 0.0568968 | 0.0283119 | 0 |
| $U_{10,j}$ | 2.297 | 2.1538 | 2.01214 | 1.87304 | 1.73761 | 1.60679 | 1.48134 | 1.36183 | 1.24862 | 1.14193 | 1.04181 | 0.948217 | 0.860981 | 0.779873 | 0.704603 | 0.63484 | 0.570227 | 0.510391 | 0.454949 | 0.403519 | 0.355721 | 0.311182 | 0.269538 | 0.230433 | 0.193524 | 0.158475 | 0.124961 | 0.0926644 | 0.0612752 | 0.0304876 | 0 |
| $U_{11,j}$ | 2.640 | 2.45174 | 2.27079 | 2.09776 | 1.93307 | 1.77702 | 1.62981 | 1.4915 | 1.36204 | 1.24126 | 1.12891 | 1.02467 | 0.928134 | 0.83889 | 0.75648 | 0.680432 | 0.610268 | 0.54551 | 0.485689 | 0.430344 | 0.37903 | 0.331314 | 0.286782 | 0.245035 | 0.205687 | 0.168367 | 0.132718 | 0.0983929 | 0.0650518 | 0.0323633 | 0 |
| $U_{12,j}$ | 2.977 | 2.74226 | 2.52151 | 2.31414 | 2.11988 | 1.93842 | 1.76938 | 1.61234 | 1.46678 | 1.33217 | 1.20791 | 1.0934 | 0.987998 | 0.891074 | 0.801996 | 0.72014 | 0.644902 | 0.575694 | 0.511952 | 0.453139 | 0.398739 | 0.348262 | 0.301243 | 0.257237 | 0.215821 | 0.176589 | 0.139153 | 0.103137 | 0.0681759 | 0.0339139 | 0 |
| $U_{13,j}$ | 3.299 | 3.0192 | 2.75887 | 2.5174 | 2.29388 | 2.08738 | 1.89697 | 1.72168 | 1.56058 | 1.41272 | 1.27717 | 1.15302 | 1.03938 | 0.935413 | 0.840288 | 0.753232 | 0.673505 | 0.60041 | 0.533288 | 0.47152 | 0.414525 | 0.361753 | 0.31269 | 0.266849 | 0.22377 | 0.183015 | 0.144168 | 0.106825 | 0.0706011 | 0.0351164 | 0 |
| $U_{14,j}$ | 3.601 | 3.27636 | 2.97735 | 2.70272 | 2.45087 | 2.22027 | 2.00942 | 1.81684 | 1.64113 | 1.48096 | 1.33502 | 1.20212 | 1.08111 | 0.970906 | 0.870513 | 0.778994 | 0.695477 | 0.619152 | 0.549269 | 0.48513 | 0.426086 | 0.371535 | 0.320915 | 0.2737 | 0.229395 | 0.187535 | 0.147676 | 0.109396 | 0.0722866 | 0.0359508 | 0 |
| $U_{15,j}$ | 3.876 | 3.50756 | 3.17148 | 2.86524 | 2.58661 | 2.33342 | 2.10358 | 1.89512 | 1.70616 | 1.53495 | 1.37985 | 1.23934 | 1.11202 | 0.996589 | 0.891864 | 0.796754 | 0.710256 | 0.631454 | 0.559507 | 0.493644 | 0.433155 | 0.377387 | 0.325735 | 0.277639 | 0.232575 | 0.190053 | 0.149607 | 0.110797 | 0.0731981 | 0.0364 | 0 |
| $U_{16,j}$ | 4.116 | 3.70663 | 3.33575 | 3.00015 | 2.69691 | 2.42321 | 2.17638 | 1.95391 | 1.75345 | 1.57284 | 1.41008 | 1.26337 | 1.13103 | 1.01157 | 0.903602 | 0.805901 | 0.717339 | 0.6369 | 0.563661 | 0.496783 | 0.435503 | 0.379122 | 0.327 | 0.278547 | 0.233215 | 0.190493 | 0.149902 | 0.110986 | 0.0733092 | 0.0364512 | 0 |
| $U_{17,j}$ | 4.314 | 3.8674 | 3.46474 | 3.10271 | 2.77767 | 2.48613 | 2.22482 | 1.99067 | 1.78088 | 1.59286 | 1.42428 | 1.27302 | 1.13718 | 1.01504 | 0.905078 | 0.805907 | 0.716299 | 0.635146 | 0.561453 | 0.494325 | 0.432951 | 0.376598 | 0.324597 | 0.276333 | 0.231244 | 0.188803 | 0.148522 | 0.109935 | 0.0726019 | 0.0360955 | 0 |
| $U_{18,j}$ | 4.465 | 3.98377 | 3.5531 | 3.16829 | 2.82493 | 2.51884 | 2.24609 | 2.00309 | 1.78654 | 1.59345 | 1.42116 | 1.26726 | 1.12962 | 1.00635 | 0.895758 | 0.796352 | 0.706804 | 0.625931 | 0.552681 | 0.486112 | 0.42538 | 0.369723 | 0.318455 | 0.270946 | 0.226623 | 0.184955 | 0.145446 | 0.107631 | 0.0710677 | 0.035329 | 0 |
| $U_{19,j}$ | 4.560 | 4.04969 | 3.59558 | 3.19241 | 2.83492 | 2.51819 | 2.23762 | 1.98906 | 1.76872 | 1.57325 | 1.39963 | 1.24523 | 1.1077 | 0.984974 | 0.875253 | 0.77694 | 0.688633 | 0.609094 | 0.537228 | 0.472062 | 0.412732 | 0.358461 | 0.308552 | 0.262373 | 0.219348 | 0.178947 | 0.140677 | 0.104076 | 0.0687085 | 0.0341527 | 0 |
| $U_{20,j}$ | 4.594 | 4.0592 | 3.58713 | 3.17085 | 2.80417 | 2.48136 | 2.19715 | 1.94679 | 1.72605 | 1.53119 | 1.3589 | 1.20633 | 1.07096 | 0.950597 | 0.843339 | 0.747522 | 0.661695 | 0.584584 | 0.515074 | 0.452178 | 0.395024 | 0.342837 | 0.29492 | 0.250647 | 0.20945 | 0.170807 | 0.134237 | 0.0992892 | 0.0655374 | 0.0325732 | 0 |
| $U_{21,j}$ | 4.558 | 4.00644 | 3.5229 | 3.09967 | 2.72956 | 2.40594 | 2.12283 | 1.87491 | 1.65751 | 1.46654 | 1.29847 | 1.15024 | 1.01921 | 0.903116 | 0.799983 | 0.708115 | 0.626039 | 0.552475 | 0.486305 | 0.426551 | 0.372351 | 0.322941 | 0.277643 | 0.235845 | 0.196997 | 0.160595 | 0.126175 | 0.093306 | 0.0615786 | 0.0306028 | 0 |
| $U_{22,j}$ | 4.447 | 3.88576 | 3.39836 | 2.97537 | 2.60845 | 2.29 | 2.01331 | 1.77251 | 1.56252 | 1.37899 | 1.21819 | 1.07694 | 0.952538 | 0.842672 | 0.745363 | 0.658916 | 0.581872 | 0.512971 | 0.451121 | 0.395371 | 0.344887 | 0.298935 | 0.256865 | 0.218094 | 0.182098 | 0.1484 | 0.116564 | 0.0861808 | 0.0568681 | 0.0282594 | 0 |
| $U_{23,j}$ | 4.252 | 3.69169 | 3.2094 | 2.79502 | 2.43886 | 2.1323 | 1.86791 | 1.63931 | 1.44109 | 1.26871 | 1.11835 | 0.986793 | 0.871328 | 0.76967 | 0.679881 | 0.600314 | 0.529561 | 0.466415 | 0.409837 | 0.358924 | 0.312891 | 0.271048 | 0.232788 | 0.197567 | 0.1649 | 0.134345 | 0.105498 | 0.0779854 | 0.0514535 | 0.0255668 | 0 |
| $U_{24,j}$ | 3.969 | 3.41914 | 2.95255 | 2.55646 | 2.21965 | 1.93245 | 1.68673 | 1.47571 | 1.29381 | 1.13641 | 0.999713 | 0.880553 | 0.776312 | 0.6848 | 0.604178 | 0.532897 | 0.469642 | 0.413293 | 0.362887 | 0.317597 | 0.276704 | 0.239579 | 0.20567 | 0.174487 | 0.145589 | 0.118581 | 0.0930997 | 0.0688089 | 0.0453937 | 0.0225542 | 0 |
| $U_{25,j}$ | 3.589 | 3.06352 | 2.62519 | 2.25861 | 1.95083 | 1.69112 | 1.47083 | 1.283 | 1.12203 | 0.983423 | 0.863534 | 0.759395 | 0.668567 | 0.589038 | 0.519134 | 0.457455 | 0.402819 | 0.354226 | 0.310822 | 0.271874 | 0.236749 | 0.204895 | 0.175828 | 0.14912 | 0.124389 | 0.101291 | 0.0795102 | 0.0587568 | 0.0387584 | 0.0192562 | 0 |
| $U_{26,j}$ | 3.105 | 2.62105 | 2.22609 | 1.90198 | 1.63392 | 1.41038 | 1.22249 | 1.06343 | 0.927887 | 0.811713 | 0.711607 | 0.624924 | 0.549522 | 0.483651 | 0.425867 | 0.374969 | 0.329953 | 0.28997 | 0.2543 | 0.222328 | 0.193523 | 0.167423 | 0.143626 | 0.121777 | 0.101557 | 0.0826828 | 0.0648934 | 0.0479496 | 0.0316269 | 0.0157123 | 0 |
| $U_{27,j}$ | 2.512 | 2.08938 | 1.75613 | 1.4893 | 1.27249 | 1.09398 | 0.945317 | 0.820327 | 0.714378 | 0.623936 | 0.546257 | 0.479172 | 0.420947 | 0.370178 | 0.325713 | 0.286602 | 0.252053 | 0.2214 | 0.194082 | 0.169616 | 0.147591 | 0.127649 | 0.109478 | 0.092803 | 0.0773801 | 0.0629895 | 0.0494312 | 0.0365212 | 0.0240873 | 0.0119662 | 0 |
| $U_{28,j}$ | 1.801 | 1.46883 | 1.21976 | 1.0266 | 0.872766 | 0.747721 | 0.644477 | 0.558188 | 0.485361 | 0.423396 | 0.370311 | 0.32456 | 0.284918 | 0.2504 | 0.220204 | 0.193672 | 0.170256 | 0.149497 | 0.13101 | 0.114463 | 0.0995756 | 0.0861031 | 0.0738328 | 0.0625774 | 0.0521709 | 0.0424638 | 0.0333207 | 0.0246166 | 0.016235 | 0.00806503 | 0 |
| $U_{29,j}$ | 0.9658 | 0.76552 | 0.627464 | 0.524579 | 0.444252 | 0.379664 | 0.326682 | 0.282588 | 0.245482 | 0.213978 | 0.187032 | 0.163839 | 0.143764 | 0.126299 | 0.111032 | 0.0976265 | 0.0858015 | 0.0753237 | 0.0659961 | 0.0576512 | 0.0501455 | 0.0433554 | 0.0371728 | 0.0315031 | 0.026262 | 0.0213743 | 0.0167712 | 0.0123897 | 0.0081709 | 0.00405898 | 0 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table B.2: The value $U_{i,j}$ are computed by the Finite Difference method.

| | $U_{i,0}$ | $U_{i,1}$ | $U_{i,2}$ | $U_{i,3}$ | $U_{i,4}$ | $U_{i,5}$ | $U_{i,6}$ | $U_{i,7}$ | $U_{i,8}$ | $U_{i,9}$ | $U_{i,10}$ | $U_{i,11}$ | $U_{i,12}$ | $U_{i,13}$ | $U_{i,14}$ | $U_{i,15}$ | $U_{i,16}$ | $U_{i,17}$ | $U_{i,18}$ | $U_{i,19}$ | $U_{i,20}$ | $U_{i,21}$ | $U_{i,22}$ | $U_{i,23}$ | $U_{i,24}$ | $U_{i,25}$ | $U_{i,26}$ | $U_{i,27}$ | $U_{i,28}$ | $U_{i,29}$ | $U_{i,30}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | 0.03330 | 0.09090 | 0.1189 | 0.1322 | 0.1372 | 0.1372 | 0.1339 | 0.1287 | 0.1221 | 0.1149 | 0.1072 | 0.09947 | 0.09179 | 0.08429 | 0.07706 | 0.07014 | 0.06355 | 0.05732 | 0.05143 | 0.04588 | 0.04064 | 0.03571 | 0.03104 | 0.02662 | 0.02241 | 0.01839 | 0.01453 | 0.01079 | 0.007140 | 0.003555 | 0 |
| $U_{2,j}$ | 0.1286 | 0.2078 | 0.2512 | 0.2722 | 0.2793 | 0.2775 | 0.2699 | 0.2585 | 0.2450 | 0.2301 | 0.2145 | 0.1989 | 0.1834 | 0.1683 | 0.1538 | 0.1399 | 0.1268 | 0.1143 | 0.1025 | 0.09143 | 0.08099 | 0.07114 | 0.06184 | 0.05302 | 0.04464 | 0.03663 | 0.02893 | 0.02148 | 0.01422 | 0.007078 | 0 |
| $U_{3,j}$ | 0.2791 | 0.3595 | 0.4048 | 0.4257 | 0.4301 | 0.4234 | 0.4094 | 0.3906 | 0.3690 | 0.3459 | 0.3220 | 0.2981 | 0.2746 | 0.2518 | 0.2299 | 0.2090 | 0.1892 | 0.1706 | 0.1529 | 0.1363 | 0.1207 | 0.1060 | 0.09214 | 0.07900 | 0.06650 | 0.05456 | 0.04309 | 0.03199 | 0.02117 | 0.01054 | 0 |
| $U_{4,j}$ | 0.4777 | 0.5460 | 0.5823 | 0.5953 | 0.5916 | 0.5764 | 0.5534 | 0.5255 | 0.4947 | 0.4623 | 0.4294 | 0.3968 | 0.3650 | 0.3343 | 0.3050 | 0.2770 | 0.2506 | 0.2257 | 0.2023 | 0.1803 | 0.1596 | 0.1401 | 0.1217 | 0.1043 | 0.08780 | 0.07203 | 0.05687 | 0.04222 | 0.02794 | 0.01391 | 0 |
| $U_{5,j}$ | 0.7177 | 0.7640 | 0.7830 | 0.7813 | 0.7644 | 0.7369 | 0.7023 | 0.6633 | 0.6217 | 0.5791 | 0.5366 | 0.4948 | 0.4543 | 0.4155 | 0.3786 | 0.3435 | 0.3105 | 0.2794 | 0.2503 | 0.2229 | 0.1972 | 0.1730 | 0.1503 | 0.1288 | 0.1084 | 0.08887 | 0.07016 | 0.05208 | 0.03446 | 0.01715 | 0 |
| $U_{6,j}$ | 0.9922 | 1.009 | 1.004 | 0.9822 | 0.9477 | 0.9045 | 0.8556 | 0.8034 | 0.7497 | 0.6959 | 0.6429 | 0.5915 | 0.5420 | 0.4949 | 0.4502 | 0.4080 | 0.3684 | 0.3313 | 0.2964 | 0.2638 | 0.2333 | 0.2046 | 0.1776 | 0.1521 | 0.1280 | 0.1049 | 0.08282 | 0.06147 | 0.04067 | 0.02024 | 0 |
| $U_{7,j}$ | 1.294 | 1.277 | 1.242 | 1.195 | 1.140 | 1.078 | 1.012 | 0.9450 | 0.8778 | 0.8118 | 0.7477 | 0.6861 | 0.6273 | 0.5717 | 0.5193 | 0.4700 | 0.4239 | 0.3807 | 0.3404 | 0.3027 | 0.2675 | 0.2345 | 0.2034 | 0.1742 | 0.1465 | 0.1201 | 0.09474 | 0.07030 | 0.04651 | 0.02314 | 0 |
| $U_{8,j}$ | 1.617 | 1.560 | 1.493 | 1.418 | 1.337 | 1.254 | 1.170 | 1.087 | 1.005 | 0.9257 | 0.8499 | 0.7777 | 0.7095 | 0.6453 | 0.5851 | 0.5288 | 0.4763 | 0.4273 | 0.3817 | 0.3392 | 0.2995 | 0.2623 | 0.2275 | 0.1947 | 0.1636 | 0.1341 | 0.1058 | 0.07848 | 0.05191 | 0.02583 | 0 |
| $U_{9,j}$ | 1.953 | 1.855 | 1.751 | 1.645 | 1.538 | 1.432 | 1.327 | 1.226 | 1.129 | 1.036 | 0.9484 | 0.8655 | 0.7877 | 0.7150 | 0.6471 | 0.5839 | 0.5252 | 0.4706 | 0.4199 | 0.3728 | 0.3289 | 0.2879 | 0.2495 | 0.2134 | 0.1793 | 0.1469 | 0.1158 | 0.08592 | 0.05682 | 0.02828 | 0 |
| $U_{10,j}$ | 2.297 | 2.154 | 2.012 | 1.873 | 1.738 | 1.607 | 1.482 | 1.362 | 1.249 | 1.142 | 1.042 | 0.9481 | 0.8608 | 0.7797 | 0.7043 | 0.6345 | 0.5699 | 0.5101 | 0.4546 | 0.4032 | 0.3554 | 0.3109 | 0.2692 | 0.2302 | 0.1933 | 0.1583 | 0.1248 | 0.09254 | 0.06119 | 0.03045 | 0 |
| $U_{11,j}$ | 2.640 | 2.452 | 2.271 | 2.098 | 1.933 | 1.777 | 1.630 | 1.492 | 1.362 | 1.241 | 1.129 | 1.024 | 0.9278 | 0.8385 | 0.7561 | 0.6800 | 0.6098 | 0.5451 | 0.4853 | 0.4299 | 0.3786 | 0.3309 | 0.2864 | 0.2447 | 0.2054 | 0.1681 | 0.1325 | 0.09825 | 0.06496 | 0.03232 | 0 |
| $U_{12,j}$ | 2.977 | 2.742 | 2.521 | 2.314 | 2.120 | 1.938 | 1.769 | 1.612 | 1.466 | 1.332 | 1.207 | 1.093 | 0.9875 | 0.8905 | 0.8014 | 0.7196 | 0.6443 | 0.5751 | 0.5114 | 0.4526 | 0.3983 | 0.3478 | 0.3008 | 0.2569 | 0.2155 | 0.1763 | 0.1389 | 0.1030 | 0.06807 | 0.03386 | 0 |
| $U_{13,j}$ | 3.299 | 3.019 | 2.759 | 2.517 | 2.294 | 2.087 | 1.897 | 1.721 | 1.560 | 1.412 | 1.277 | 1.152 | 1.039 | 0.9347 | 0.8396 | 0.7525 | 0.6728 | 0.5997 | 0.5327 | 0.4709 | 0.4140 | 0.3612 | 0.3122 | 0.2664 | 0.2234 | 0.1827 | 0.1439 | 0.1066 | 0.07048 | 0.03506 | 0 |
| $U_{14,j}$ | 3.601 | 3.276 | 2.977 | 2.702 | 2.450 | 2.220 | 2.009 | 1.816 | 1.640 | 1.480 | 1.334 | 1.201 | 1.080 | 0.9700 | 0.8696 | 0.7781 | 0.6947 | 0.6184 | 0.5485 | 0.4844 | 0.4255 | 0.3710 | 0.3204 | 0.2732 | 0.2290 | 0.1872 | 0.1474 | 0.1092 | 0.07216 | 0.03589 | 0 |
| $U_{15,j}$ | 3.876 | 3.507 | 3.171 | 2.865 | 2.586 | 2.333 | 2.103 | 1.894 | 1.705 | 1.534 | 1.379 | 1.238 | 1.111 | 0.9955 | 0.8908 | 0.7958 | 0.7093 | 0.6306 | 0.5587 | 0.4929 | 0.4325 | 0.3768 | 0.3252 | 0.2771 | 0.2322 | 0.1897 | 0.1493 | 0.1106 | 0.07306 | 0.03633 | 0 |
| $U_{16,j}$ | 4.116 | 3.706 | 3.335 | 2.999 | 2.696 | 2.422 | 2.175 | 1.953 | 1.752 | 1.572 | 1.409 | 1.262 | 1.130 | 1.010 | 0.9024 | 0.8048 | 0.7163 | 0.6359 | 0.5627 | 0.4959 | 0.4347 | 0.3784 | 0.3264 | 0.2780 | 0.2328 | 0.1901 | 0.1496 | 0.1108 | 0.07316 | 0.03638 | 0 |
| $U_{17,j}$ | 4.314 | 3.867 | 3.464 | 3.102 | 2.777 | 2.485 | 2.223 | 1.989 | 1.779 | 1.591 | 1.423 | 1.271 | 1.136 | 1.014 | 0.9037 | 0.8046 | 0.7151 | 0.6341 | 0.5605 | 0.4934 | 0.4321 | 0.3759 | 0.3239 | 0.2758 | 0.2308 | 0.1884 | 0.1482 | 0.1097 | 0.07244 | 0.03602 | 0 |
| $U_{18,j}$ | 4.465 | 3.983 | 3.552 | 3.167 | 2.824 | 2.517 | 2.244 | 2.001 | 1.785 | 1.592 | 1.419 | 1.266 | 1.128 | 1.005 | 0.8943 | 0.7950 | 0.7055 | 0.6248 | 0.5516 | 0.4851 | 0.4245 | 0.3690 | 0.3178 | 0.2704 | 0.2261 | 0.1845 | 0.1451 | 0.1074 | 0.07091 | 0.03525 | 0 |
| $U_{19,j}$ | 4.560 | 4.049 | 3.595 | 3.191 | 2.833 | 2.516 | 2.236 | 1.987 | 1.767 | 1.571 | 1.398 | 1.243 | 1.106 | 0.9833 | 0.8737 | 0.7755 | 0.6873 | 0.6079 | 0.5361 | 0.4711 | 0.4118 | 0.3577 | 0.3079 | 0.2618 | 0.2188 | 0.1785 | 0.1403 | 0.1038 | 0.06854 | 0.03407 | 0 |
| $U_{20,j}$ | 4.594 | 4.059 | 3.586 | 3.169 | 2.802 | 2.479 | 2.195 | 1.945 | 1.724 | 1.529 | 1.357 | 1.204 | 1.069 | 0.9488 | 0.8417 | 0.7460 | 0.6603 | 0.5833 | 0.5139 | 0.4512 | 0.3941 | 0.3420 | 0.2942 | 0.2500 | 0.2089 | 0.1704 | 0.1339 | 0.09904 | 0.06537 | 0.03249 | 0 |
| $U_{21,j}$ | 4.558 | 4.006 | 3.522 | 3.098 | 2.727 | 2.404 | 2.120 | 1.872 | 1.655 | 1.464 | 1.296 | 1.148 | 1.017 | 0.9013 | 0.7983 | 0.7066 | 0.6246 | 0.5512 | 0.4852 | 0.4255 | 0.3714 | 0.3221 | 0.2770 | 0.2353 | 0.1965 | 0.1602 | 0.1259 | 0.09306 | 0.06142 | 0.03052 | 0 |
| $U_{22,j}$ | 4.447 | 3.885 | 3.397 | 2.973 | 2.606 | 2.287 | 2.011 | 1.770 | 1.560 | 1.376 | 1.216 | 1.075 | 0.9505 | 0.8408 | 0.7437 | 0.6574 | 0.5805 | 0.5117 | 0.4500 | 0.3944 | 0.3440 | 0.2982 | 0.2562 | 0.2175 | 0.1816 | 0.1480 | 0.1163 | 0.08595 | 0.05672 | 0.02818 | 0 |
| $U_{23,j}$ | 4.252 | 3.691 | 3.208 | 2.793 | 2.436 | 2.129 | 1.865 | 1.636 | 1.438 | 1.266 | 1.116 | 0.9846 | 0.8693 | 0.7679 | 0.6782 | 0.5988 | 0.5282 | 0.4652 | 0.4088 | 0.3580 | 0.3121 | 0.2703 | 0.2322 | 0.1970 | 0.1644 | 0.1340 | 0.1052 | 0.07777 | 0.05131 | 0.02550 | 0 |
| $U_{24,j}$ | 3.969 | 3.418 | 2.951 | 2.554 | 2.217 | 1.929 | 1.684 | 1.473 | 1.291 | 1.134 | 0.9974 | 0.8784 | 0.7744 | 0.6831 | 0.6026 | 0.5315 | 0.4684 | 0.4122 | 0.3619 | 0.3167 | 0.2759 | 0.2389 | 0.2051 | 0.1740 | 0.1452 | 0.1182 | 0.09284 | 0.06861 | 0.04526 | 0.02249 | 0 |
| $U_{25,j}$ | 3.589 | 3.062 | 2.623 | 2.256 | 1.948 | 1.688 | 1.468 | 1.280 | 1.119 | 0.9810 | 0.8614 | 0.7574 | 0.6668 | 0.5875 | 0.5177 | 0.4562 | 0.4017 | 0.3532 | 0.3100 | 0.2711 | 0.2361 | 0.2043 | 0.1753 | 0.1487 | 0.1240 | 0.1010 | 0.07928 | 0.05859 | 0.03865 | 0.01920 | 0 |
| $U_{26,j}$ | 3.105 | 2.619 | 2.223 | 1.899 | 1.631 | 1.407 | 1.220 | 1.061 | 0.9255 | 0.8096 | 0.7097 | 0.6232 | 0.5480 | 0.4823 | 0.4247 | 0.3739 | 0.3290 | 0.2891 | 0.2536 | 0.2217 | 0.1930 | 0.1669 | 0.1432 | 0.1214 | 0.1013 | 0.08244 | 0.06470 | 0.04781 | 0.03153 | 0.01567 | 0 |
| $U_{27,j}$ | 2.512 | 2.087 | 1.753 | 1.486 | 1.269 | 1.091 | 0.9427 | 0.8180 | 0.7124 | 0.6222 | 0.5447 | 0.4778 | 0.4197 | 0.3691 | 0.3248 | 0.2858 | 0.2513 | 0.2208 | 0.1935 | 0.1691 | 0.1472 | 0.1273 | 0.1092 | 0.09253 | 0.07715 | 0.06280 | 0.04928 | 0.03641 | 0.02401 | 0.01193 | 0 |
| $U_{28,j}$ | 1.801 | 1.466 | 1.216 | 1.023 | 0.8700 | 0.7454 | 0.6425 | 0.5565 | 0.4839 | 0.4221 | 0.3692 | 0.3236 | 0.2841 | 0.2497 | 0.2195 | 0.1931 | 0.1697 | 0.1491 | 0.1306 | 0.1141 | 0.09928 | 0.08584 | 0.07361 | 0.06239 | 0.05201 | 0.04234 | 0.03322 | 0.02454 | 0.01619 | 0.008041 | 0 |
| $U_{29,j}$ | 0.9658 | 0.7623 | 0.6247 | 0.5225 | 0.4426 | 0.3783 | 0.3256 | 0.2817 | 0.2447 | 0.2133 | 0.1865 | 0.1633 | 0.1433 | 0.1259 | 0.1107 | 0.09733 | 0.08554 | 0.07510 | 0.06580 | 0.05748 | 0.04999 | 0.04322 | 0.03706 | 0.03141 | 0.02618 | 0.02131 | 0.01672 | 0.01235 | 0.008146 | 0.004047 | 0 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table B.3: The value $U_{i,j}$ of truncation of the exact solution up to 1000.

| | $U_{i,0}$ | $U_{i,1}$ | $U_{i,2}$ | $U_{i,3}$ | $U_{i,4}$ | $U_{i,5}$ | $U_{i,6}$ | $U_{i,7}$ |
|---|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | 0 | −0.00157805 | −0.00135948 | −0.00102277 | −0.000774399 | −0.000598755 | −0.000470348 | −0.000372774 |
| $U_{2,j}$ | 0 | −0.00135902 | −0.00166639 | −0.00151872 | −0.00127444 | −0.00104312 | −0.000846956 | −0.000684834 |
| $U_{3,j}$ | 0 | −0.00102092 | −0.00151641 | −0.00159358 | −0.00147076 | −0.00128054 | −0.00108224 | −0.000898252 |
| $U_{4,j}$ | 0 | −0.000769798 | −0.0012671 | −0.00146439 | −0.00145639 | −0.00133947 | −0.0011769 | −0.00100345 |
| $U_{5,j}$ | 0 | −0.000589634 | −0.00102723 | −0.00126251 | −0.00132607 | −0.0012744 | −0.00115765 | −0.00101098 |
| $U_{6,j}$ | 0 | −0.000454557 | −0.000818219 | −0.00104616 | −0.00114165 | −0.00113375 | −0.00105683 | −0.000940263 |
| $U_{7,j}$ | 0 | −0.00034781 | −0.000638279 | −0.000836728 | −0.000936598 | −0.000951142 | −0.000902084 | −0.000811513 |
| $U_{8,j}$ | 0 | −0.000259335 | −0.000481181 | −0.000640023 | −0.000726987 | −0.000747128 | −0.00071352 | −0.00064183 |
| $U_{9,j}$ | 0 | −0.000182929 | −0.000340964 | −0.000455899 | −0.000519557 | −0.000533179 | −0.000504457 | −0.000444143 |
| $U_{10,j}$ | 0 | −0.000114571 | −0.00021279 | −0.000282171 | −0.000316342 | −0.000315131 | −0.000283108 | −0.000227548 |
| $U_{11,j}$ | 0 | −0.0000515008 | −0.0000928439 | −0.000116123 | −0.000117109 | −0.0000954962 | −0.0000541906 | 0.0000018837 |
| $U_{12,j}$ | 0 | 0.000008304 | 0.0000219344 | 0.0000449753 | 0.0000794482 | 0.000125116 | 0.000179841 | 0.000240263 |
| $U_{13,j}$ | 0 | 0.0000664283 | 0.000134104 | 0.000203732 | 0.000275131 | 0.000347165 | 0.000417944 | 0.000485177 |
| $U_{14,j}$ | 0 | 0.000124215 | 0.000245925 | 0.000362647 | 0.00047198 | 0.000571703 | 0.000659909 | 0.000735123 |
| $U_{15,j}$ | 0 | 0.000182902 | 0.000359533 | 0.000524186 | 0.000672168 | 0.00080009 | 0.000905953 | 0.000989062 |
| $U_{16,j}$ | 0 | 0.00024373 | 0.000477088 | 0.000690873 | 0.000877968 | 0.00103381 | 0.00115641 | 0.00124603 |
| $U_{17,j}$ | 0 | 0.000308052 | 0.000600921 | 0.000865403 | 0.00109174 | 0.00127431 | 0.00141144 | 0.00150476 |
| $U_{18,j}$ | 0 | 0.000377449 | 0.000733709 | 0.00105075 | 0.00131592 | 0.00152279 | 0.00167066 | 0.00176324 |
| $U_{19,j}$ | 0 | 0.00045389 | 0.000878674 | 0.00125027 | 0.00155289 | 0.00177993 | 0.0019327 | 0.00201815 |
| $U_{20,j}$ | 0 | 0.000539957 | 0.00103986 | 0.00146779 | 0.00180477 | 0.00204528 | 0.00219446 | 0.00226412 |
| $U_{21,j}$ | 0 | 0.000639207 | 0.00122248 | 0.00170758 | 0.00207288 | 0.00231639 | 0.00244999 | 0.00249268 |
| $U_{22,j}$ | 0 | 0.000756751 | 0.00143341 | 0.00197405 | 0.00235651 | 0.00258708 | 0.00268885 | 0.00269092 |
| $U_{23,j}$ | 0 | 0.000900279 | 0.00168164 | 0.00227068 | 0.00265056 | 0.0028446 | 0.00289362 | 0.00283982 |
| $U_{24,j}$ | 0 | 0.00108189 | 0.00197852 | 0.00259694 | 0.00294062 | 0.00306502 | 0.00303671 | 0.00291271 |
| $U_{25,j}$ | 0 | 0.00132142 | 0.00233599 | 0.00294048 | 0.00319367 | 0.00320632 | 0.00307697 | 0.00287452 |
| $U_{26,j}$ | 0 | 0.00165214 | 0.00275715 | 0.0032579 | 0.00334197 | 0.00320037 | 0.00295825 | 0.00268399 |
| $U_{27,j}$ | 0 | 0.00212557 | 0.00319873 | 0.00343215 | 0.00326109 | 0.0029495 | 0.00261498 | 0.00230144 |
| $U_{28,j}$ | 0 | 0.00277354 | 0.00344277 | 0.00319988 | 0.00276082 | 0.00234394 | 0.00199289 | 0.00170492 |
| $U_{29,j}$ | 0 | 0.00318376 | 0.00277377 | 0.0021265 | 0.00165444 | 0.00132598 | 0.00108978 | 0.000912761 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $U_{i,8}$ | $U_{i,9}$ | $U_{i,10}$ | $U_{i,11}$ | $U_{i,12}$ | $U_{i,13}$ | $U_{i,14}$ | $U_{i,15}$ |
|---|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | −0.00029629 | −0.000234987 | −0.000185119 | −0.000144196 | −0.000110481 | −0.0000827054 | −0.0000599069 | −0.0000413262 |
| $U_{2,j}$ | −0.000551144 | −0.000440487 | −0.000348546 | −0.000272007 | −0.000208324 | −0.000155504 | −0.000111952 | −0.0000763557 |
| $U_{3,j}$ | −0.000735261 | −0.000593912 | −0.000472728 | −0.000369649 | −0.000282596 | −0.000209647 | −0.000149082 | −0.0000993726 |
| $U_{4,j}$ | −0.000836282 | −0.000683093 | −0.000546733 | −0.000427684 | −0.000325299 | −0.00023842 | −0.000165695 | −0.00010572 |
| $U_{5,j}$ | −0.000856199 | −0.000705829 | −0.00056654 | −0.000441506 | −0.000331866 | −0.000237593 | −0.000158014 | −0.000092105 |
| $U_{6,j}$ | −0.000805535 | −0.000667024 | −0.000533703 | −0.000410784 | −0.000300985 | −0.000205414 | −0.000124165 | −0.0000567101 |
| $U_{7,j}$ | −0.000697961 | −0.000575355 | −0.000453382 | −0.000338331 | −0.000233971 | −0.000142283 | −0.0000640234 | 0.0000008652 |
| $U_{8,j}$ | −0.000546803 | −0.000440622 | −0.000332492 | −0.000228876 | −0.000133978 | −0.0000502614 | 0.0000210873 | 0.0000797797 |
| $U_{9,j}$ | −0.000363384 | −0.000272117 | −0.000178351 | −0.0000880363 | −0.0000052571 | 0.0000674488 | 0.000128745 | 0.000178181 |
| $U_{10,j}$ | −0.000156577 | −0.000077862 | 0.0000021293 | 0.0000784036 | 0.000147427 | 0.000206947 | 0.000255757 | 0.000293444 |
| $U_{11,j}$ | 0.0000669988 | 0.000135578 | 0.000202801 | 0.000264893 | 0.000319172 | 0.000363962 | 0.000398425 | 0.000422376 |
| $U_{12,j}$ | 0.000302539 | 0.000362954 | 0.000418338 | 0.000466274 | 0.000505139 | 0.000534048 | 0.000552726 | 0.000561374 |
| $U_{13,j}$ | 0.000546551 | 0.000600042 | 0.000644107 | 0.00067777 | 0.000700609 | 0.00071268 | 0.000714419 | 0.00070653 |
| $U_{14,j}$ | 0.00079639 | 0.000843296 | 0.00087594 | 0.000894859 | 0.000900927 | 0.000895253 | 0.000879078 | 0.000853691 |
| $U_{15,j}$ | 0.00104982 | 0.00108949 | 0.00110986 | 0.00111307 | 0.00110139 | 0.00107702 | 0.00104208 | 0.000998477 |
| $U_{16,j}$ | 0.00130464 | 0.00133532 | 0.00134175 | 0.00132776 | 0.00129706 | 0.00125301 | 0.00119858 | 0.00113628 |
| $U_{17,j}$ | 0.00155821 | 0.00157706 | 0.00156704 | 0.00153384 | 0.00148262 | 0.0014179 | 0.00134343 | 0.00126227 |
| $U_{18,j}$ | 0.00180708 | 0.00181007 | 0.00178035 | 0.0017255 | 0.00165217 | 0.0015659 | 0.00147116 | 0.00137139 |
| $U_{19,j}$ | 0.00204634 | 0.0020284 | 0.00197508 | 0.00189596 | 0.00179905 | 0.00169074 | 0.00157598 | 0.00145842 |
| $U_{20,j}$ | 0.00226902 | 0.00222415 | 0.0021431 | 0.00203728 | 0.00191581 | 0.00178562 | 0.00165184 | 0.00151806 |
| $U_{21,j}$ | 0.00246525 | 0.00238705 | 0.0022744 | 0.0021402 | 0.00199418 | 0.00184335 | 0.00169259 | 0.00154514 |
| $U_{22,j}$ | 0.00262134 | 0.00250387 | 0.00235698 | 0.00219427 | 0.00202534 | 0.00185662 | 0.00169229 | 0.00153485 |
| $U_{23,j}$ | 0.00271898 | 0.00255828 | 0.00237703 | 0.00218826 | 0.00200036 | 0.00181849 | 0.0016456 | 0.00148316 |
| $U_{24,j}$ | 0.00273486 | 0.00253128 | 0.00231976 | 0.002111 | 0.00191107 | 0.00172312 | 0.00154845 | 0.00138728 |
| $U_{25,j}$ | 0.00264161 | 0.00240272 | 0.00217099 | 0.00195293 | 0.00175128 | 0.00156669 | 0.00139867 | 0.00124619 |
| $U_{26,j}$ | 0.00241116 | 0.00215464 | 0.00191997 | 0.00170818 | 0.00151829 | 0.00134852 | 0.00119681 | 0.00106111 |
| $U_{27,j}$ | 0.00202167 | 0.00177659 | 0.00156306 | 0.00137703 | 0.00121453 | 0.00107209 | 0.000946738 | 0.000835965 |
| $U_{28,j}$ | 0.00146839 | 0.00127225 | 0.00110774 | 0.000968256 | 0.000848838 | 0.000745725 | 0.000656026 | 0.000577478 |
| $U_{29,j}$ | 0.000775229 | 0.000665236 | 0.000575231 | 0.000500237 | 0.000436842 | 0.000382619 | 0.000335791 | 0.000295016 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $U_{i,16}$ | $U_{i,17}$ | $U_{i,18}$ | $U_{i,19}$ | $U_{i,20}$ | $U_{i,21}$ | $U_{i,22}$ | $U_{i,23}$ |
|---|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | −0.0000263458 | −0.0000144512 | −0.0000052052 | 0.0000017691 | 0.0000067999 | 0.0000101749 | 0.0000121475 | 0.0000129421 |
| $U_{2,j}$ | −0.000047616 | −0.0000247942 | −0.000007077 | 0.0000062425 | 0.0000157915 | 0.0000221205 | 0.0000257183 | 0.0000270194 |
| $U_{3,j}$ | −0.000059162 | −0.0000272409 | −0.0000025302 | 0.0000159348 | 0.0000290175 | 0.0000374898 | 0.0000420424 | 0.000043293 |
| $U_{4,j}$ | −0.0000571238 | −0.0000186018 | 0.0000110646 | 0.0000330009 | 0.0000482373 | 0.0000577111 | 0.0000622708 | 0.0000626818 |
| $U_{5,j}$ | −0.0000386784 | 0.0000035128 | 0.0000357086 | 0.0000591021 | 0.0000748175 | 0.0000838984 | 0.0000873026 | 0.0000859033 |
| $U_{6,j}$ | −0.0000021572 | 0.000040581 | 0.0000726771 | 0.0000953171 | 0.000109655 | 0.000116786 | 0.00011773 | 0.000113428 |
| $U_{7,j}$ | 0.0000529587 | 0.0000931451 | 0.000122482 | 0.0001421 | 0.000153135 | 0.000156689 | 0.000153804 | 0.00014545 |
| $U_{8,j}$ | 0.000126134 | 0.000160859 | 0.00018489 | 0.00019928 | 0.000205121 | 0.00020349 | 0.000195418 | 0.00018187 |
| $U_{9,j}$ | 0.000215931 | 0.000242582 | 0.000258976 | 0.000266095 | 0.000264974 | 0.000256647 | 0.000242109 | 0.000222295 |
| $U_{10,j}$ | 0.000320173 | 0.000336495 | 0.00034321 | 0.000341247 | 0.000331589 | 0.000315219 | 0.000293078 | 0.000266052 |
| $U_{11,j}$ | 0.000436105 | 0.000440224 | 0.000435541 | 0.000422967 | 0.000403445 | 0.000377902 | 0.000347218 | 0.000312207 |
| $U_{12,j}$ | 0.000560527 | 0.000550941 | 0.000533489 | 0.000509092 | 0.000478666 | 0.000443081 | 0.000403148 | 0.000359599 |
| $U_{13,j}$ | 0.00068989 | 0.00066546 | 0.000634222 | 0.000597128 | 0.000555075 | 0.000508878 | 0.000459265 | 0.000406875 |
| $U_{14,j}$ | 0.000820366 | 0.000780308 | 0.000734628 | 0.000684322 | 0.000630264 | 0.000573206 | 0.000513786 | 0.000452532 |
| $U_{15,j}$ | 0.000947888 | 0.000891777 | 0.00083138 | 0.000767724 | 0.000701651 | 0.000633833 | 0.000564802 | 0.000494964 |
| $U_{16,j}$ | 0.00106819 | 0.00099598 | 0.000920992 | 0.000844254 | 0.000766545 | 0.000688439 | 0.000610339 | 0.000532513 |
| $U_{17,j}$ | 0.00117681 | 0.0010889 | 0.000999888 | 0.000910769 | 0.000822221 | 0.000734686 | 0.000648417 | 0.000563525 |
| $U_{18,j}$ | 0.00126919 | 0.00116645 | 0.00106448 | 0.00096415 | 0.000865997 | 0.000770292 | 0.000677115 | 0.000586408 |
| $U_{19,j}$ | 0.00134069 | 0.00122458 | 0.00111125 | 0.00100139 | 0.000895322 | 0.000793114 | 0.000694652 | 0.000599694 |
| $U_{20,j}$ | 0.00138673 | 0.00125938 | 0.00113691 | 0.00101972 | 0.000907885 | 0.000801239 | 0.000699457 | 0.000602109 |
| $U_{21,j}$ | 0.00140299 | 0.00126728 | 0.0011385 | 0.00101673 | 0.000901728 | 0.000793085 | 0.000690258 | 0.000592639 |
| $U_{22,j}$ | 0.00138564 | 0.00124523 | 0.00111363 | 0.000990527 | 0.000875372 | 0.000767496 | 0.00066616 | 0.000570595 |
| $U_{23,j}$ | 0.00133166 | 0.00119099 | 0.00106064 | 0.000939908 | 0.000827943 | 0.000723851 | 0.000626726 | 0.000535675 |
| $U_{24,j}$ | 0.00123919 | 0.00110337 | 0.000978832 | 0.000864501 | 0.000759294 | 0.000662147 | 0.000572041 | 0.000488012 |
| $U_{25,j}$ | 0.00110795 | 0.000982562 | 0.000868658 | 0.000764916 | 0.000670102 | 0.000583069 | 0.000502761 | 0.000428208 |
| $U_{26,j}$ | 0.000939526 | 0.000830305 | 0.000731876 | 0.000642835 | 0.000561925 | 0.000488026 | 0.000420133 | 0.000357344 |
| $U_{27,j}$ | 0.000737665 | 0.000650054 | 0.000571614 | 0.000501042 | 0.000437212 | 0.000379146 | 0.000325983 | 0.000276964 |
| $U_{28,j}$ | 0.000508277 | 0.000446961 | 0.000392328 | 0.000343372 | 0.000299244 | 0.000259218 | 0.000222664 | 0.000189032 |
| $U_{29,j}$ | 0.000259253 | 0.00022768 | 0.000199631 | 0.000174559 | 0.000152007 | 0.000131587 | 0.000112967 | 0.0000958586 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $U_{i,24}$ | $U_{i,25}$ | $U_{i,26}$ | $U_{i,27}$ | $U_{i,28}$ | $U_{i,29}$ | $U_{i,30}$ |
|---|---|---|---|---|---|---|---|
| $U_{0,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_{1,j}$ | 0.0000127582 | 0.0000117737 | 0.000010149 | 0.0000080295 | 0.000005549 | 0.0000028327 | 0 |
| $U_{2,j}$ | 0.000026411 | 0.0000242399 | 0.0000208182 | 0.00001643 | 0.0000113364 | 0.000005782 | 0 |
| $U_{3,j}$ | 0.0000417951 | 0.0000380457 | 0.0000324937 | 0.0000255478 | 0.0000175845 | 0.0000089564 | 0 |
| $U_{4,j}$ | 0.0000596343 | 0.0000537508 | 0.0000455952 | 0.0000356818 | 0.0000244851 | 0.0000124497 | 0 |
| $U_{5,j}$ | 0.0000804931 | 0.0000717904 | 0.0000604481 | 0.0000470628 | 0.0000321859 | 0.0000163338 | 0 |
| $U_{6,j}$ | 0.000104739 | 0.0000924455 | 0.0000772605 | 0.0000598373 | 0.0000407798 | 0.0000206538 | 0 |
| $U_{7,j}$ | 0.000132518 | 0.000115822 | 0.0000961068 | 0.000074055 | 0.000050297 | 0.0000254239 | 0 |
| $U_{8,j}$ | 0.000163738 | 0.000141838 | 0.000116918 | 0.0000896615 | 0.0000607001 | 0.000030625 | 0 |
| $U_{9,j}$ | 0.000198069 | 0.000170223 | 0.000139479 | 0.000106496 | 0.0000718824 | 0.0000362039 | 0 |
| $U_{10,j}$ | 0.000234953 | 0.000200522 | 0.000163433 | 0.000124296 | 0.0000836696 | 0.0000420738 | 0 |
| $U_{11,j}$ | 0.000273614 | 0.000232107 | 0.000188289 | 0.000142698 | 0.0000958239 | 0.0000481169 | 0 |
| $U_{12,j}$ | 0.00031309 | 0.000264199 | 0.000213438 | 0.000161255 | 0.000108051 | 0.0000541872 | 0 |
| $U_{13,j}$ | 0.00035226 | 0.000295892 | 0.000238174 | 0.000179449 | 0.00012001 | 0.0000601156 | 0 |
| $U_{14,j}$ | 0.00038988 | 0.00032618 | 0.000261714 | 0.000196702 | 0.000131322 | 0.0000657152 | 0 |
| $U_{15,j}$ | 0.000424622 | 0.000353993 | 0.000283224 | 0.000212407 | 0.000141589 | 0.0000707878 | 0 |
| $U_{16,j}$ | 0.000455119 | 0.00037823 | 0.000301851 | 0.000225936 | 0.0001504 | 0.000075131 | 0 |
| $U_{17,j}$ | 0.000480014 | 0.000397804 | 0.000316753 | 0.000236675 | 0.000157353 | 0.0000785463 | 0 |
| $U_{18,j}$ | 0.000498006 | 0.000411676 | 0.000327129 | 0.000244043 | 0.00016207 | 0.0000808463 | 0 |
| $U_{19,j}$ | 0.000507908 | 0.000418906 | 0.000332259 | 0.000247516 | 0.000164209 | 0.0000818639 | 0 |
| $U_{20,j}$ | 0.000508699 | 0.000418693 | 0.000331534 | 0.000246656 | 0.000163488 | 0.0000814595 | 0 |
| $U_{21,j}$ | 0.00049958 | 0.000410419 | 0.00032449 | 0.000241133 | 0.000159694 | 0.0000795294 | 0 |
| $U_{22,j}$ | 0.000480026 | 0.00039369 | 0.000310838 | 0.000230745 | 0.0001527 | 0.0000760124 | 0 |
| $U_{23,j}$ | 0.000449833 | 0.000368371 | 0.000290494 | 0.00021544 | 0.000142477 | 0.0000708951 | 0 |
| $U_{24,j}$ | 0.000409153 | 0.000334613 | 0.00026359 | 0.000195326 | 0.0001291 | 0.0000642164 | 0 |
| $U_{25,j}$ | 0.000358518 | 0.000292867 | 0.000230493 | 0.000170679 | 0.000112753 | 0.0000560687 | 0 |
| $U_{26,j}$ | 0.000298842 | 0.000243887 | 0.000191797 | 0.000141943 | 0.0000937304 | 0.0000465977 | 0 |
| $U_{27,j}$ | 0.00023141 | 0.000188714 | 0.000148319 | 0.000109715 | 0.0000724256 | 0.0000359991 | 0 |
| $U_{28,j}$ | 0.000157837 | 0.000128645 | 0.000101064 | 0.0000747347 | 0.0000493227 | 0.0000245124 | 0 |
| $U_{29,j}$ | 0.0000800076 | 0.0000651888 | 0.0000511991 | 0.000037853 | 0.0000249783 | 0.0000124127 | 0 |
| $U_{30,j}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table B.4: The value $U_{i,j}$ are the error between numerical solution and the truncation of the exact solution up to 1000.

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