國 立 交 通 大 學

應用數學系

碩 士 論 文

N相黎曼空間的理論與應用

Theory and Applications of Riemann Surfaces of genus N

研 究 生：凃偉隆

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中 華 民 國 九 十 九 年 九 月

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N相黎曼空間的理論與應用

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摘要

我們利用代數與幾何分析的方法建構多值函數(開方函數)的黎曼空間使得一個定義在複數平面上是多值的函數在黎曼空間上是唯一值且可解析的函數。在黎曼空間上對封閉曲線 a，b cycles 的積分可以解決許多微分方程上的問題，而且可以找到 a，b cycles 之等價路徑，再經由 Cauchy Integral Theorem 可得知 a，b， cycles 之積分值與它們的等價路徑積分值會相等。藉由這樣的方法，當我們執行黎曼空間的積分時，無論是數值上或是理論上，我們都可以解決問題進而求得解答。

Theory and Applications of Riemann Surfaces of genus N

Student：Wei-Long Tu **Advisor**：Jong-Eao Lee

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Abstract

We use algebraic and geometric analysis to develop two-sheet Riemann surface R of genus N such that muti-valued function on the complex plane C become single-valued and analytic on R. The integrals over a, b cycles on R can solve many problems in Differential Equations. By Cauchy Integral Theorem, we can find equivalent paths of a, b cycles such that their integrals are equal. When we do the integral on the Riemann surface ,no matter what on theoretically or in value , by the principle ,we could solves the problem and get the solution.

Contents

I.1 Construct the corresponding Riemann Surface

Where ω , $z \in C$ *and* $\omega^k = z$

$$\omega^k = z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)} \quad ,\theta \in [-\pi,\pi) \quad ,n \in \mathrm{Z}$$

$$\Rightarrow \quad \omega = z^{\frac{1}{k}} = |z|e^{\frac{i(\theta+2n\pi)}{k}} \quad ,\theta \in [-\pi,\pi) \quad ,n \in \mathrm{Z}$$

First, we take a example $f(z)=\sqrt{z}$, $f : C \to C$

$$f(z) = z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{\frac{(\theta+2n\pi)i}{2}} = \begin{cases} |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} & if \quad n \in \quad even \\ -|z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} & if \quad n \in \quad odd \end{cases}$$

Where $z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}$ $n \in Z$

Because f f is a two-valued function.

We will take $f(z)$ becomes a single valued function, so that we modify its domain C to develop the corresponding Riemann Surface such that f becomes a single-valued on Riemann Surface .

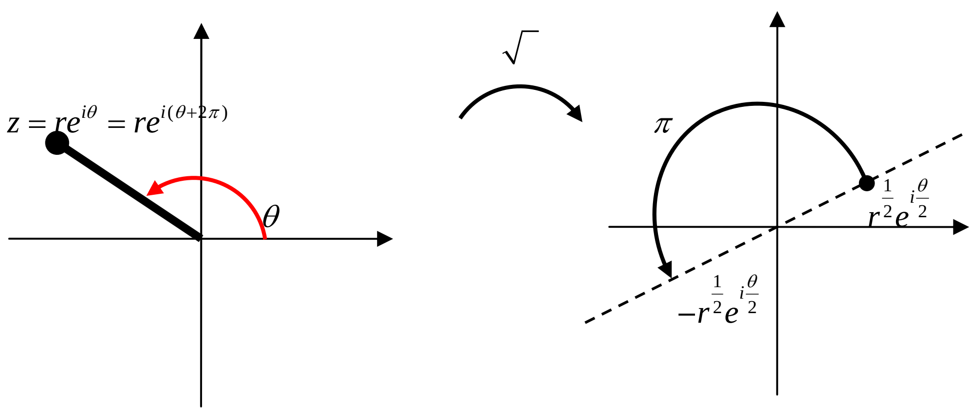


Figure 1: The idea of two sheets

Now define that

$$
f(z) = \begin{cases} |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} & -\pi \le \theta < \pi \qquad (*) \\ |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} & \pi \le \theta < 3\pi \qquad (**) \end{cases}
$$

Where (*) called sheet-I and (**) called sheet-II

The each sheet has two edges, the starting edge with (−) and terminal edge with (+). Next we will use the stereographic sphere help us to consider the two sheets work process.

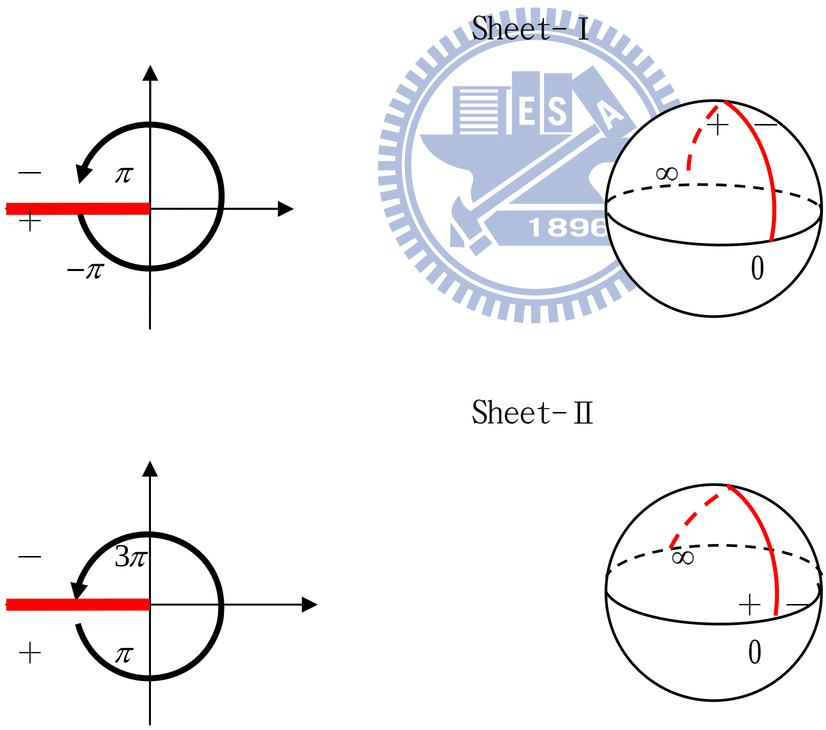


Figure 2: complex plane and extended complex plane

Example 1: Construct the Riemann Surface of $f(z) = \sqrt{\prod_{k=1}^{7} (z - z_k)} = \prod_{k=1}^{7} \sqrt{(z - z_k)}$

where k=1.2…...7

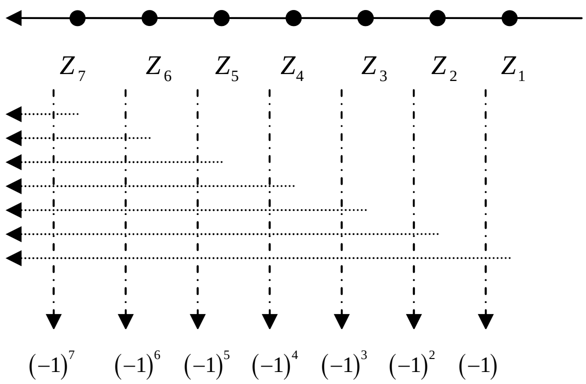


Figure 5: The cut from $z_k \to \infty$

We have crossing one cut need to change the sign by " -1".
So that when crosse even cuts we will not change the sing and crosse odd cuts will change sign.

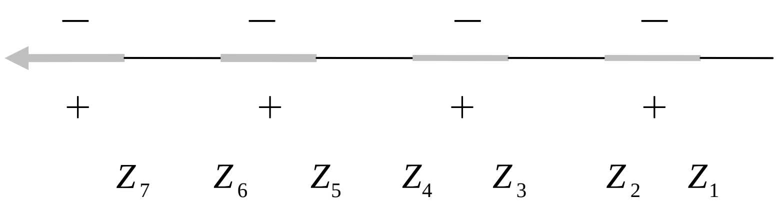


Figure 6：The cut plane

There are branch cuts in $[z_1, z_2], [z_3, z_4], [z_5, z_6], [z_7, -\infty)$ and then using same idea to construct the corresponding Riemann Surface.

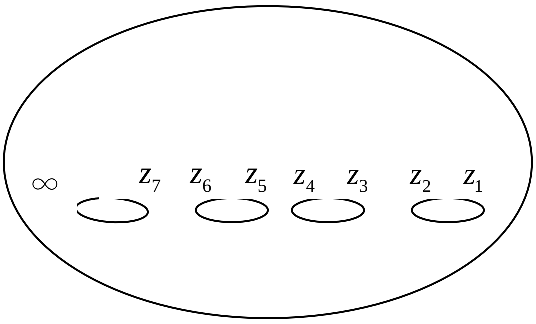


Figure 7: Placing the cuts open

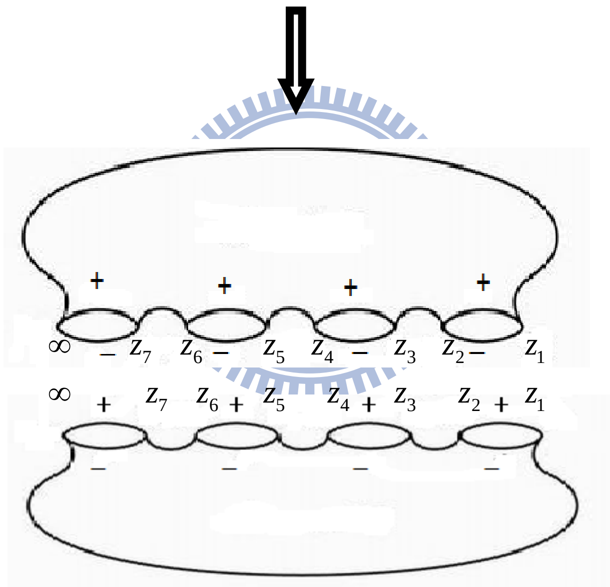


Figure 8：Take the two sheet together with (+) edge to (－) edge

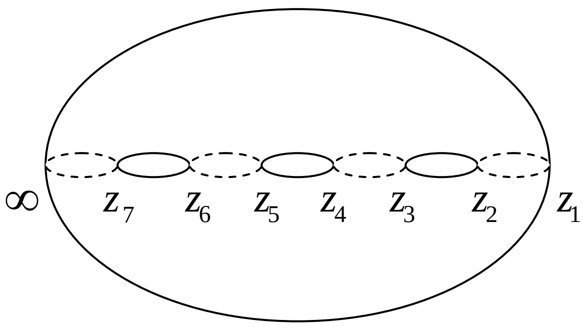


Figure 9: Riemann Surface of genus 3

Example 2：Construct the Riemann Surface of $f(z)=\sqrt{\prod_{k=1}^{8}(z-z_k)}=\prod_{k=1}^{8}\sqrt{(z-z_k)}$

where k=1.2…...8

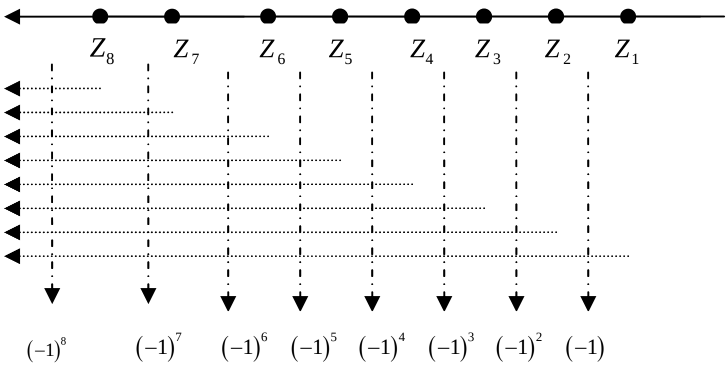


Figure 10 : The cut from $z_k \to \infty$

There are branch cuts in $[z_1, z_2],[z_3, z_4],[z_5, z_6],[z_7, z_8]$.

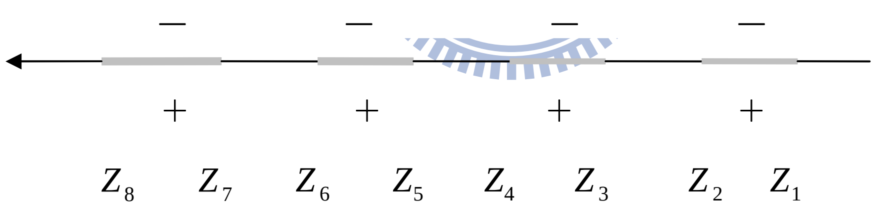


Figure 11 : The cut plane

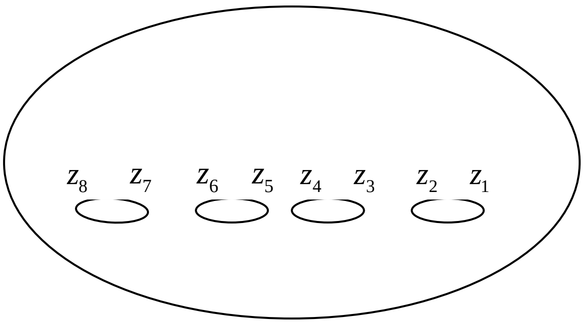


Figure 11 : Placing the cuts open

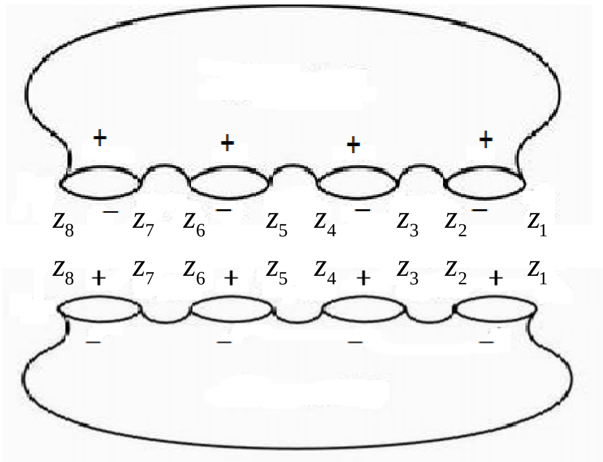


Figure 12：Take the two sheet together with (+) edge to (－) edge

⇓

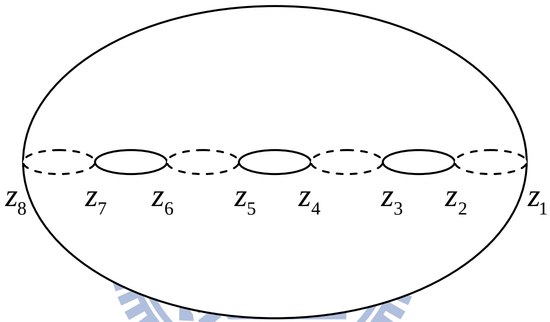


Figure 13: Riemann Surface of genus 3

In general

$$f(z)=\sqrt{\prod_{k=1}^{n}(z-z_k)}=\prod_{k=1}^{n}\sqrt{(z-z_k)} \quad \text{, where k=1.2......n}$$

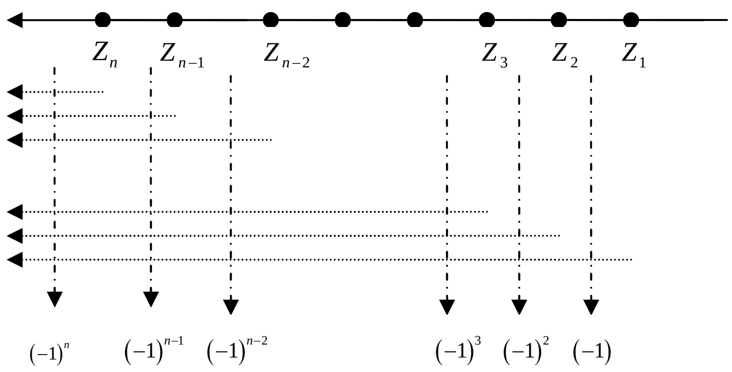


Figure 14: cut plane $z_k \to \infty$

Case 1. if the $n \in$ odd

There are branch cuts in $[z_1,z_2],[z_3,z_4],\cdots\cdots,[z_{2N-3},z_{2N-2}],[z_{2N-1},-\infty)$

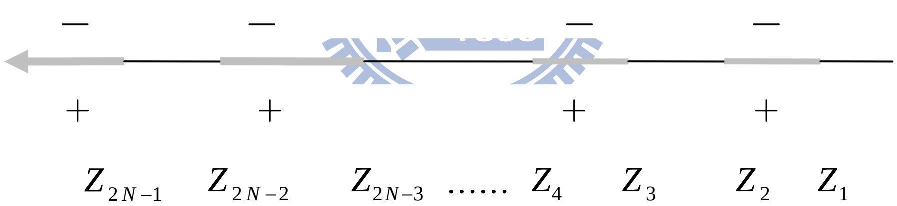


Figure 15: The cut plane of n=2N-1

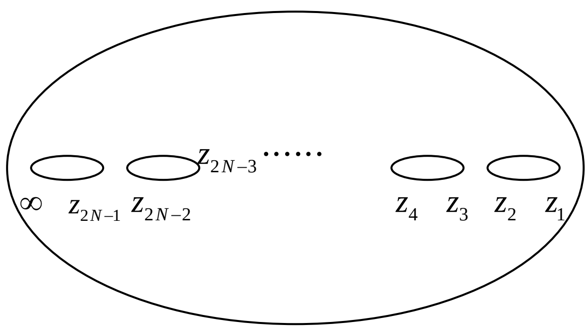


Figure 16: Placing cuts open in both sheets

⇓

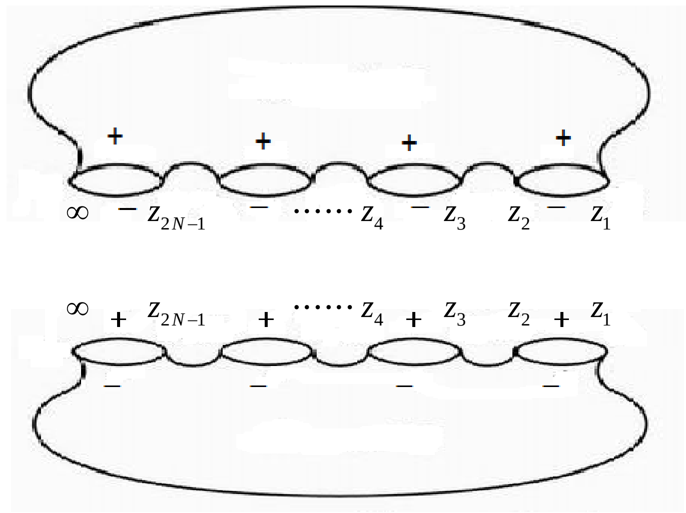


Figure 17: Together two sheets

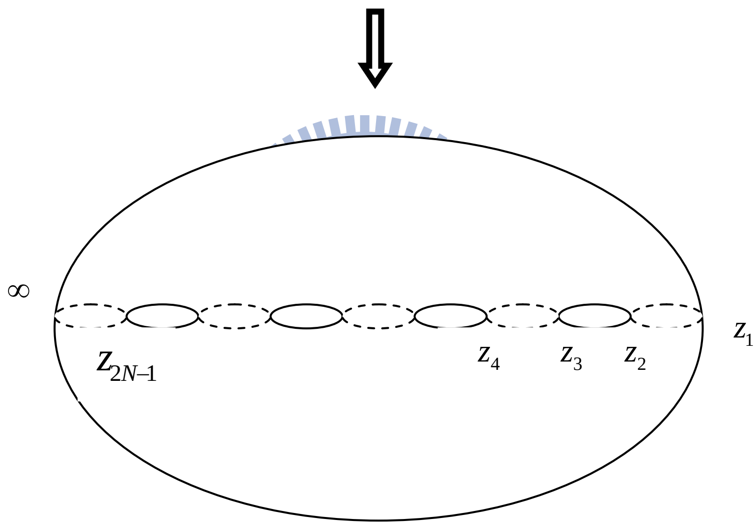


Figure 18: n=2N-1 and N-1 holes

Case 2. if the n∈ even

There are branch cuts in $[z_1, z_2],[z_3, z_4],\cdots\cdots,[z_{2N-3}, z_{2N-2}],[z_{2N-1}, z_{2N}]$

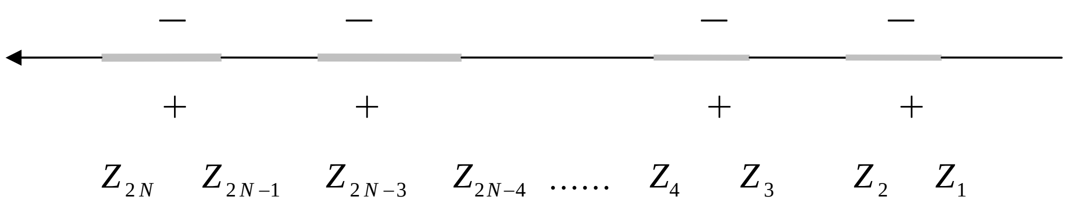


Figure 19: The cut of n=2N

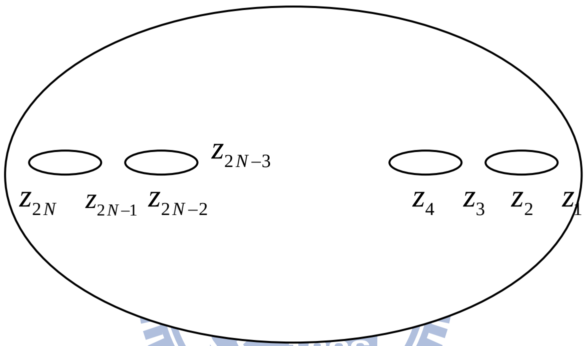


Figure 20: Placing cuts open in both sheets

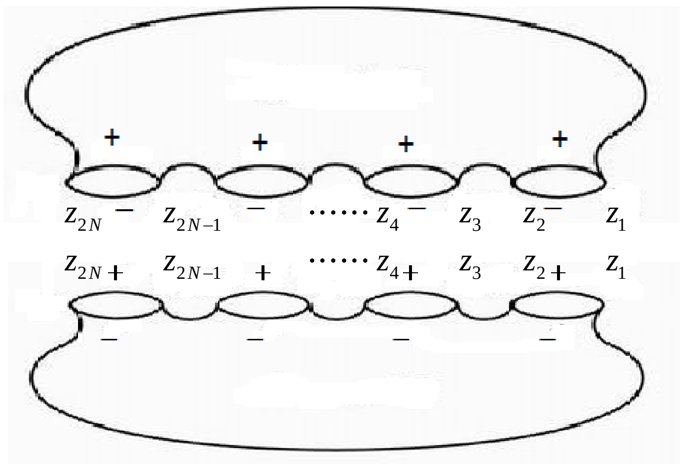


Figure 21: Together two sheets

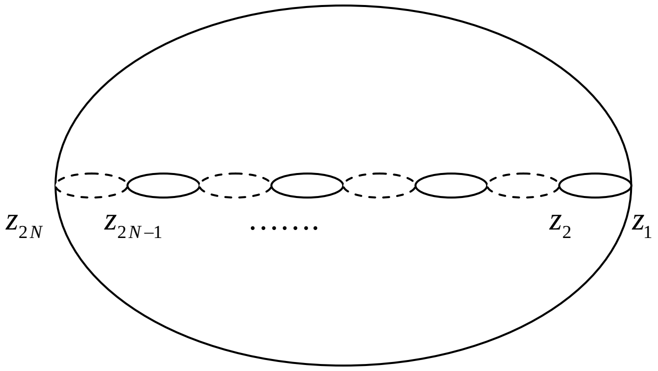


Figure 22: n=2N and n-1 holes

So the conclusion , no matter the n=2N or n=2N-1 , there are N cuts and N-1 holes called **Riemann surface** of genus N-1.

1.2 The a, b cycles

Example $f(z)=\sqrt{z(z-1)(z-2)(z-3)}$

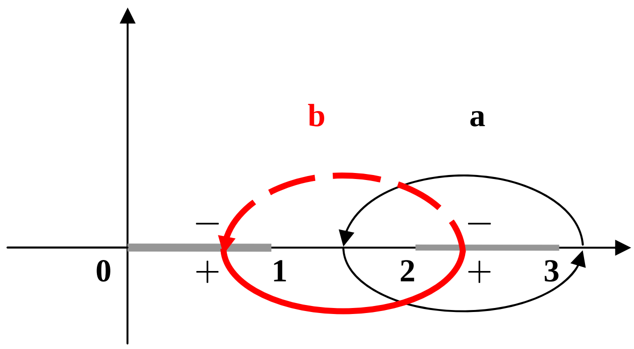


Figure 23: a, b cycle of $f(z)=\sqrt{z(z-1)(z-2)(z-3)}$

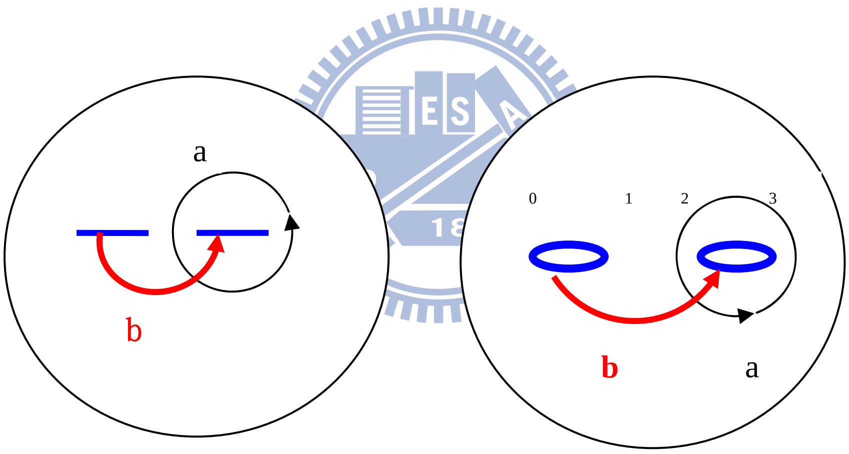


Figure 24 : The cut construct of sheet Ⅰ

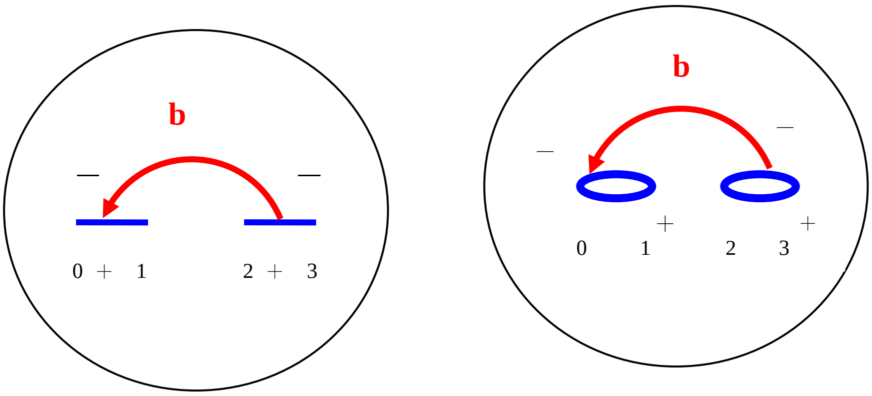


Figure 25 : The cut construct of sheet Ⅱ

⇓

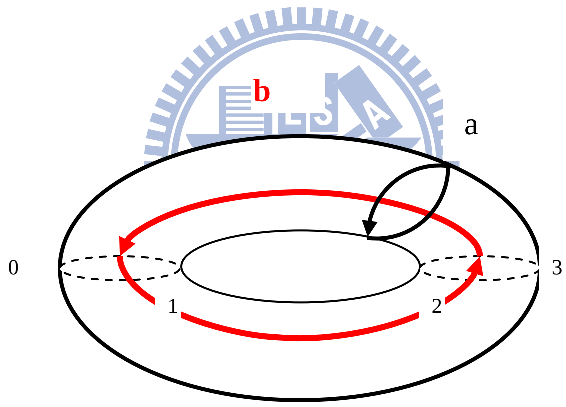
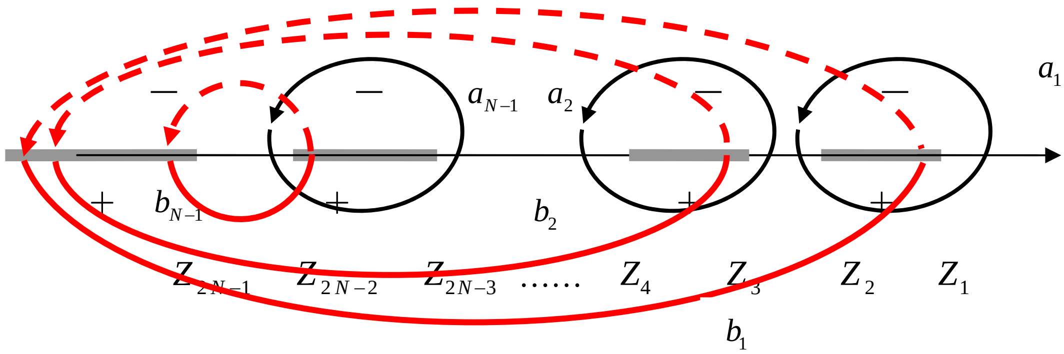


Figure 26 : The geometric structure

In general

If n=2N-1



If n=2N

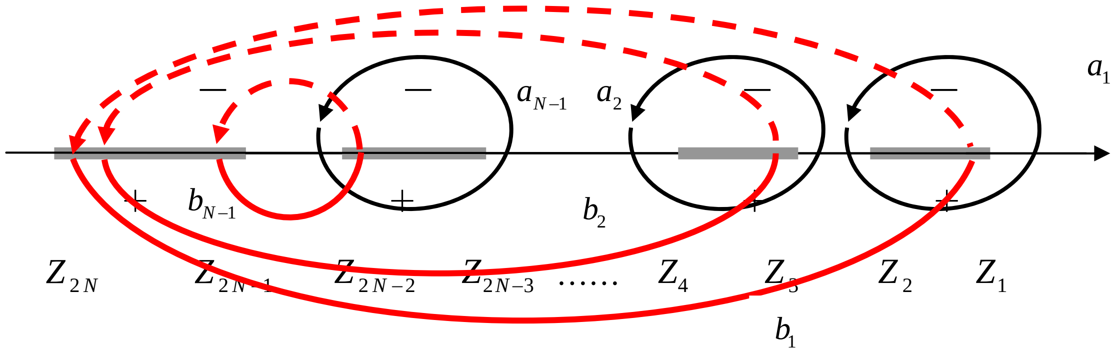


Figure 27 :a,b cycles on complex plane

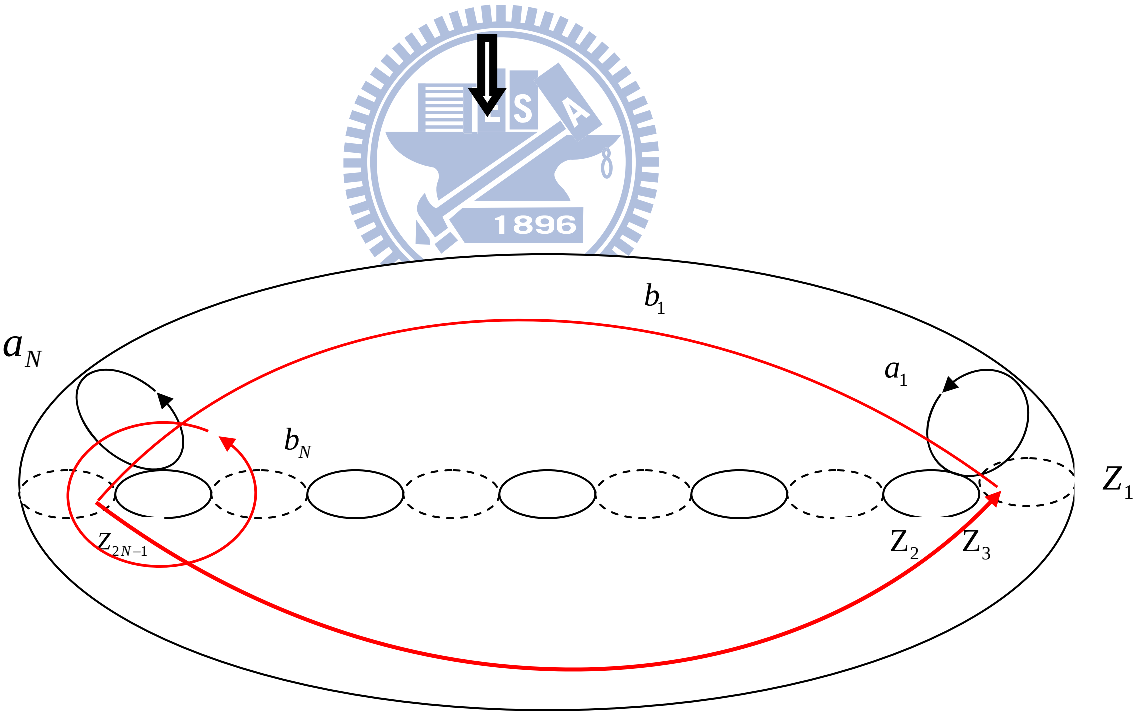


Figure 28: The a,b cycles on Riemann Surface

1.3 homotopic

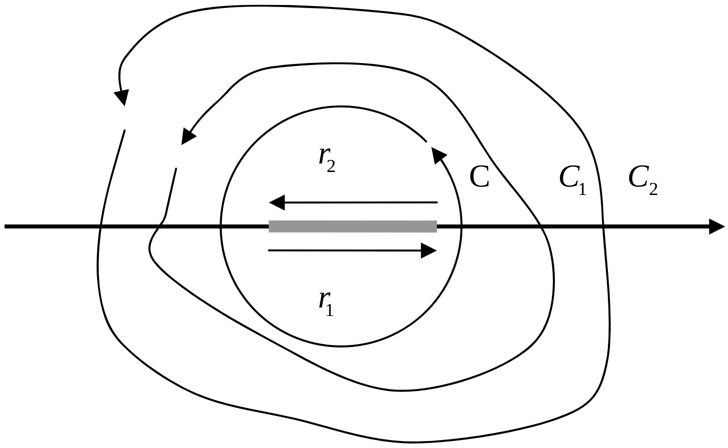


Figure 29 : Homotopic path

In the homotopic case ,C is homotopic to C_1 and C_2.

So that $\int_C \frac{1}{f(z)}dz = \int_{C_1} \frac{1}{f(z)}dz = \int_{C_2} \frac{1}{f(z)}dz$, and we compress the curve C ,finally the equivalent paths C homotopic to $r_1 \cup r_2$

$$\Rightarrow \int_C \frac{1}{f(z)}dz = \int_{r_1} \frac{1}{f(z)}dz + \int_{r_2} \frac{1}{f(z)}dz .$$

1.4 The conclusion of Riemann Surface

For any branch points 2N-1 or 2N we have

I.   There are N branch cuts in complex plane.

II. There are N-1 holes and that called Riemann Surface of genus N-1.

III. There are N-1 a-cycles and N-1 b-cycles

2. The integrals of $\frac{1}{f(z)}$ over a, b cycles for horizontal cut

First we discuss the value different in sheet-I of theory and on Mathematica .

$\sqrt{-1}$ in sheet-I, $\sqrt{-1}=-i$ but $\sqrt{-1}$ in Mathematica, $\sqrt{-1}=i$.

We will find the mistake in this.

We know that $\theta\in(-\pi,\pi]$ in Mathematica

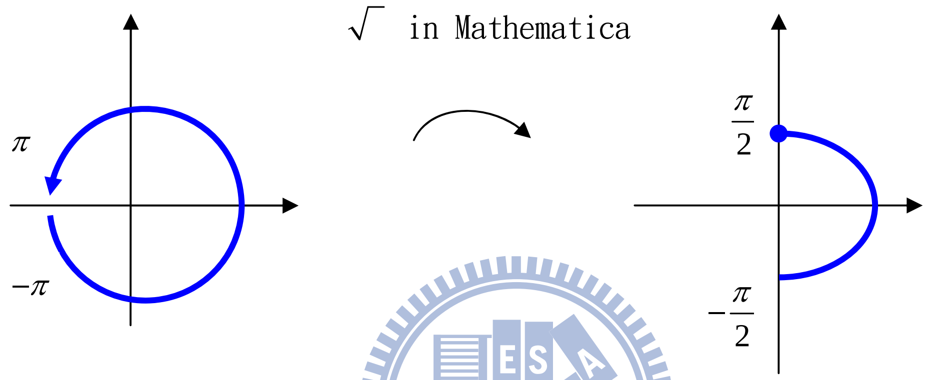


Figure 30: Domain and range in Mathematica

Lemma I: $\theta\in[-\pi,\pi)$ in sheet-I for horizontal cut

$$f(z)|_{sheet-I}=\begin{cases} f(z)|_{MATH} & \theta\in(-\pi,\pi) \\ -f(z)|_{MATH} & \theta=-\pi \end{cases}$$

Proof:

Theory : $-1=e^{-\pi i}\quad\Rightarrow\sqrt{-1}=e^{-\frac{\pi}{2}i}=-i$

Mathematica : $-1=e^{\pi i}\quad\Rightarrow\sqrt{-1}=e^{\frac{\pi}{2}i}=i$

Thus $f(z)\overset{math}{=}-f(z)$ where $\theta=-\pi$ in Mathematica.

Now we take a example to test and verify the Lemma Ⅰ.

To evaluate $\int \frac{1}{f(z)} dz$, *where* $f(z) = \sqrt{z(z-1)(z-2)}$

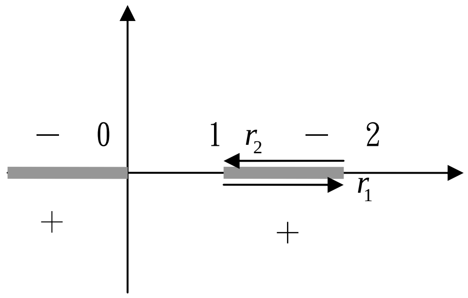


Figure 31: cut plane of $f(z) = \sqrt{z(z-1)(z-2)}$

Proof :

1. $z \in r_1$

(1) theory: $\begin{cases} z \geq 0 \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} \\ z-1 \geq 0 \Rightarrow \sqrt{z-1} = |z-1|^{\frac{1}{2}} \\ z-2 < 0 \Rightarrow z-2 = |z-2| e^{-\pi i} \Rightarrow \sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{-\frac{\pi}{2} i} = -i|z-2|^{\frac{1}{2}} \end{cases}$

$$\Rightarrow \int_{r_1} \frac{1}{f(z)} = i\int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

(2)mathematica: $\begin{cases} z \geq 0 \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} \\ z-1 \geq 0 \Rightarrow \sqrt{z-1} = |z-1|^{\frac{1}{2}} \\ z-2 < 0 \Rightarrow z-2 = |z-2| e^{\pi i} \Rightarrow \sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2} i} = i|z-2|^{\frac{1}{2}} \end{cases}$

$$\Rightarrow \int_{r_1} \frac{1}{f(z)} = -i\int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

The(1)and(2) difference a ” －” sign.

2. $z \in r_2$

(1) theory: $\begin{cases} z \ge 0 \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} \\ z-1 \ge 0 \Rightarrow \sqrt{z-1} = |z-1|^{\frac{1}{2}} \\ z-2 < 0 \Rightarrow z-2 = |z-2|e^{\pi i} \Rightarrow \sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2}i} = i|z-2|^{\frac{1}{2}} \end{cases}$

$$\Rightarrow \int_{r_2} \frac{1}{f(z)} = -i\int_2^1 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

(2) mathematica: $\begin{cases} z \ge 0 \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} \\ z-1 \ge 0 \Rightarrow \sqrt{z-1} = |z-1|^{\frac{1}{2}} \\ z-2 < 0 \Rightarrow z-2 = |z-2|e^{\pi i} \Rightarrow \sqrt{z-2} = |z-2|^{\frac{1}{2}} e^{\frac{\pi}{2}i} = i|z-2|^{\frac{1}{2}} \end{cases}$

$$\Rightarrow \int_{r_2} \frac{1}{f(z)} = -i\int_2^1 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

The value of (1) and (2) are the same.

So that $\int \frac{1}{f(z)}dz = \begin{cases} 2i\int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} & in \quad theory \\ 0 & in \quad mathematica \end{cases}$

$$= \begin{cases} 0 + 5.24412i & in \quad theory \\ 0 & in \quad mathematica \end{cases}$$

When $\theta = -\pi$ we see that have a problem between theory and mathematica, they difference a '—' sign.

Now we get the result Lemma I, we can use the result to do some example.

Example 1.

1. $z \in r_1$

$$\begin{cases} z \ge 0 \ \Rightarrow \arg(z) = 0 \Rightarrow \sqrt{z} \overset{Math.}{=} \sqrt{z} \\ z-1 \ge 0 \ \Rightarrow \arg(z-1) = 0 \Rightarrow \sqrt{z-1} \overset{Math.}{=} \sqrt{z-1} \\ z-2 < 0 \ \Rightarrow \arg(z-2) = -\pi \Rightarrow \sqrt{z-2} \overset{Math.}{=} -\sqrt{z-2} \end{cases}$$

$$\int_{r_1} \frac{1}{f(z)} \overset{math.}{=} -\int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

2. $z \in r_2$

$$\begin{cases} z \ge 0 \ \Rightarrow \arg(z) = 0 \Rightarrow \sqrt{z} \overset{Math.}{=} \sqrt{z} \\ z-1 \ge 0 \ \Rightarrow \arg(z-1) = 0 \Rightarrow \sqrt{z-1} \overset{Math.}{=} \sqrt{z-1} \\ z-2 < 0 \ \Rightarrow \arg(z-2) = \pi \Rightarrow \sqrt{z-2} \overset{Math.}{=} \sqrt{z-2} \end{cases}$$

$$\int_{r_2} \frac{1}{f(z)} \overset{math.}{=} \int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}}$$

So that we have $\int_r \frac{1}{f(z)} \overset{math.}{=} -2\int_1^2 |z|^{-\frac{1}{2}} |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} = 0 + 5.24412i$

Example 2. Evaluate $\int \frac{1}{f(z)} dz$ and over a, b

cycle. where $f(z) = \sqrt{(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)(z-9)}$

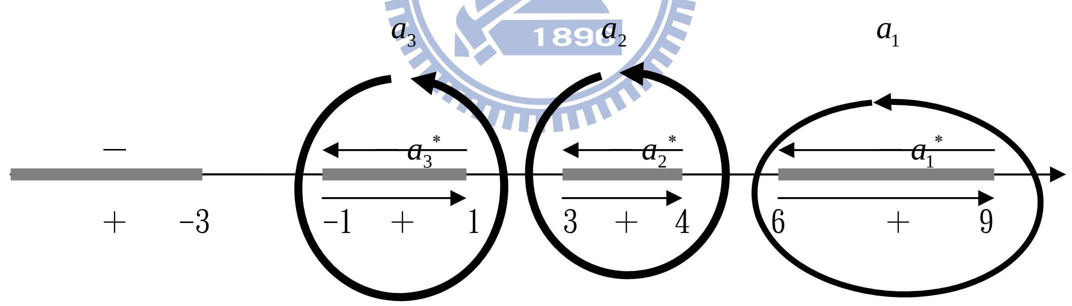


Figure 32: a-cycle and homotopic path a^*

Proof:

Take $z_1 = 9, z_2 = 6, z_3 = 4, z_4 = 3, z_5 = 1, z_6 = -1, z_7 = 3$

1. $z \in [6,9]$

Let a_1 be a cycle center at $\frac{15}{2}$ with r=2 and $z = \frac{15}{2} + 2e^{i\theta}$.

(1) $$\int_{a_1} \frac{1}{f(z)} dz = \int_{-\pi}^{\pi} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{(\frac{15}{2} + e^{i\theta}) - z_k}} d\theta \quad \text{(by Cauchy Thm)}$$

$$= 1.0842 \times 10^{-19} + 0.0776642i$$

(2)theory: $z \in a_1^*$

(Ⅰ) $z \in 6 \rightarrow 9$

$$\begin{cases} z-9<0 \Rightarrow z-9=|z-9|e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{i\frac{\pi}{2}}=i|z-9|^{-\frac{1}{2}} \\ z-z_k=|z-z_k| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{6\to 9}\frac{1}{f(z)}dz=\int_6^9 i\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

(Ⅱ) $z \in 6 \leftarrow 9$

$$\begin{cases} z-9<0 \Rightarrow z-9=|z-9|e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{-i\frac{\pi}{2}}=-i|z-9|^{-\frac{1}{2}} \\ z-z_k=|z-z_k| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{9\to 6}\frac{1}{f(z)}dz=\int_9^6 (-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

by(Ⅰ)(Ⅱ) we know that in theory $\Rightarrow \int_{a_1^*}\frac{1}{f(z)}dz=\int_9^6(-2i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$

= 0.0776642i

(3) mathematic : $z \in a_1^*$

(Ⅰ) $z \in 6 \rightarrow 9$

$$\begin{cases} z-9<0 \Rightarrow z-9=|z-9|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{i\frac{-\pi}{2}}=-i|z-9|^{-\frac{1}{2}} \\ z-z_k=|z-z_k| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{6\to 9}\frac{1}{f(z)}dz=\int_6^9(-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

(Ⅱ) $z \in 6 \leftarrow 9$

$$\begin{cases} z-9<0 \Rightarrow z-9=|z-9|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-9}}=|z-9|^{-\frac{1}{2}}e^{i\frac{-\pi}{2}}=-i|z-9|^{-\frac{1}{2}} \\ z-z_k=|z-z_k| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}},\ k=2,3,4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{6\leftarrow 9}\frac{1}{f(z)}dz=\int_9^6(-i)\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}dz$$

In mathematic the value of $\Rightarrow \int_{a_1^*}\frac{1}{f(z)}dz=0$

(4)by the Lemma (Ⅰ) $z \in a_1^*$

(Ⅰ) $z \in 6 \rightarrow 9$

$$\begin{cases} \arg(z-z_1)=-\pi \Rightarrow \sqrt{z-z_1} \overset{math.}{=} -\sqrt{z-z_1} \\ \arg(z-z_k)=0 \quad \Rightarrow \sqrt{z-z_k} \overset{math.}{=} \sqrt{z-z_k}\ \ k=2.....7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} -f(z)$$

(Ⅱ) $z \in 6 \leftarrow 9$

$$\begin{cases} \arg(z-z_1)=\pi \Rightarrow \sqrt{z-z_1} \overset{math.}{=} \sqrt{z-z_1} \\ \arg(z-z_k)=0 \quad \Rightarrow \sqrt{z-z_k} \overset{math.}{=} \sqrt{z-z_k}\ \ k=2.....7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} f(z)$$

So that $\int_{a_1^*}\frac{1}{f(z)}dz=-2\int_6^9\frac{1}{f(z)}dz=0.0776642i$

2. $z \in [3,4]$

(1)Let a_2 be a cycle center at $\frac{7}{2}$ with r=1 and $z = \frac{7}{2} + e^{i\theta}$

$$\int_{a_2} \frac{1}{f(z)} dz = \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{(\frac{7}{2} + e^{i\theta}) - z_k}} d\theta \quad \text{(by Cauchy Thm)}$$

$$= 0 - 0.200969i$$

(2)theory $z \in a_2^*$

(I) $z \in 3 \to 4$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{-i\pi} \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z - z_k|^{-\frac{1}{2}} \ k = 1,2,3 \\ z - z_k = |z - z_k| \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{3\to 4} \frac{1}{f(z)} dz = \int_3^4 i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(II) $z \in 3 \leftarrow 4$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{i\pi} \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{-\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}} \ k = 1,2,3 \\ z - z_k = |z - z_k| \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{3\leftarrow 4} \frac{1}{f(z)} dz = \int_4^3 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

by(I) (II) we know that in theory $\Rightarrow \int_{a_2^*} \frac{1}{f(z)} dz = -2\int_4^3 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$

= 0-0.200969i

(3)mathematic $z \in a_2^*$

(I) $z \in 3 \to 4$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{-\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}} \ k = 1,2,3 \\ z - z_k = |z - z_k| \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{3\to 4} \frac{1}{f(z)} dz = \int_3^4 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(II) $z \in 3 \leftarrow 4$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{-\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}} \ k = 1,2,3 \\ z - z_k = |z - z_k| \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow \int_{3\leftarrow 4} \frac{1}{f(z)} dz = \int_4^3 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

By (I)(II) $\int_{a_2^*} \frac{1}{f(z)} dz = 0$ in mathematic.

(4) by Lemma (I)

(I) $z \in 3 \to 4$

$$\begin{cases} \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math.}{=} -\sqrt{z - z_k} \, k = 1,2,3 \\ \arg(z - z_k) = 0 \ \Rightarrow \sqrt{z - z_k} \overset{math.}{=} \sqrt{z - z_k} \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} -f(z)$$

(II) $z \in 3 \leftarrow 4$

$$\begin{cases} \arg(z - z_k) = \pi \Rightarrow \sqrt{z - z_k} \overset{math.}{=} \sqrt{z - z_k} \, k = 1,2,3 \\ \arg(z - z_k) = 0 \ \Rightarrow \sqrt{z - z_k} \overset{math.}{=} \sqrt{z - z_k} \ k = 4,5,6,7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} f(z)$$

So that $\int_{a_2^*} \frac{1}{f(z)} dz = -2\int_3^4 \frac{1}{f(z)} dz$ = 0−0.200969i

3. $z \in [-1,1]$

Let a_3 is a cycle center at 0 with r = 2 $\Rightarrow z = 2e^{i\theta}$

(1)

$$\int_{a_3} \frac{1}{f(z)} dz = \int_{-\pi}^{\pi} \frac{2ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{(2e^{i\theta}) - z_k}} d\theta \qquad \text{(by Cauchy thm)}$$

$$= 3.46945 \times 10^{18} + 0.151409i$$

(2)theory : $z \in a_3^*$

(I) $z \in -1 \rightarrow 1$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{-i\pi} \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z - z_k|^{-\frac{1}{2}} \quad k = 1,2,3,4,5 \\ z - z_k = |z - z_k| \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \; k = 6,7 \end{cases}$$

$$\Rightarrow \int_{-1 \to 1} \frac{1}{f(z)} dz = \int_{-1}^{1} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(II) $z \in -1 \leftarrow 1$

$$\begin{cases} z - z_k < 0 \Rightarrow z - z_k = |z - z_k| e^{i\pi} \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i\frac{-\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}} \quad k = 1,2,3,4,5 \\ z - z_k = |z - z_k| \Rightarrow \frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, \; k = 6,7 \end{cases}$$

$$\Rightarrow \int_{-1 \leftarrow 1} \frac{1}{f(z)} dz = \int_{1}^{-1} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

$$\int_{a_3^*}\frac{1}{f(z)}dz = 2\int_{-1}^{1} i^5\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz = 0.0151409i$$

(3)mathematic: $z \in a_3^*$

(Ⅰ) $z \in -1 \to 1$

$$\begin{cases} z-z_k<0 \Rightarrow z-z_k=\left|z-z_k\right|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{i\frac{-\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}} \quad k=1,2,3,4,5 \\ z-z_k=\left|z-z_k\right| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}},\ k=6,7 \end{cases}$$

$$\Rightarrow \int_{-1\to 1}\frac{1}{f(z)}dz = \int_{-1}^{1}(-i)^5\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅱ) $z \in -1 \leftarrow 1$

$$\begin{cases} z-z_k<0 \Rightarrow z-z_k=\left|z-z_k\right|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{i\frac{-\pi}{2}}=-i\left|z-z_k\right|^{-\frac{1}{2}} \quad k=1,2,3,4,5 \\ z-z_k=\left|z-z_k\right| \Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}},\ k=6,7 \end{cases}$$

$$\Rightarrow \int_{-1\leftarrow 1}\frac{1}{f(z)}dz = \int_{1}^{-1}(-i)^5\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

By (Ⅰ)(Ⅱ) $\int_{a_3^*}\frac{1}{f(z)}dz = 0$ in mathematic.

(4)by the Lemma（Ⅰ）

（Ⅰ）$z \in -1 \to 1$

$$\begin{cases} \arg(z-z_k) = -\pi \Rightarrow \sqrt{z-z_k} \overset{math.}{=} -\sqrt{z-z_k} k=1,2,3,4,5 \\ \arg(z-z_k) = 0 \quad \Rightarrow \sqrt{z-z_k} \overset{math.}{=} \sqrt{z-z_k} \ k=6,7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} -f(z)$$

（Ⅱ） $z \in -1 \leftarrow 1$

$$\begin{cases} \arg(z-z_k) = \pi \Rightarrow \sqrt{z-z_k} \overset{math.}{=} \sqrt{z-z_k} k=1,2,3,4,5 \\ \arg(z-z_k) = 0 \quad \Rightarrow \sqrt{z-z_k} \overset{math.}{=} \sqrt{z-z_k} \ k=6,7 \end{cases}$$

$$\Rightarrow f(z) \overset{math.}{=} f(z)$$

So that $\int_{a_3^*} \frac{1}{f(z)} dz = -2\int_{-1}^{1} \frac{1}{f(z)} dz = 0.151409i$

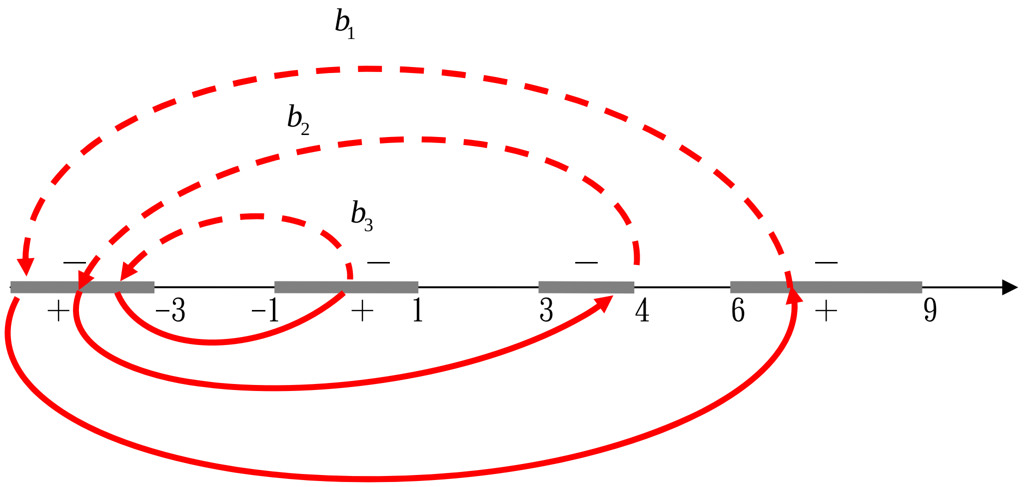


Figure 33:b-cycles

4.

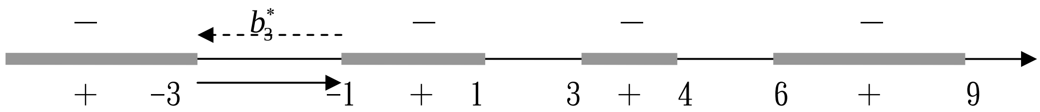


Figure 34: b_3's equivalent path b_3^*

(1)theory $z \in b_3^*$

(Ⅰ) $z \in -3 \to -1$

$$\begin{cases} z - z_7 \geq 0 \Rightarrow \dfrac{1}{\sqrt{z - z_7}} = \left|z - z_7\right|^{-\frac{1}{2}} \\ z - z_k < 0 \Rightarrow = z - z_k = \left|z - z_k\right| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_k\right|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i\left|z - z_k\right|^{-\frac{1}{2}}, k = 1,2,3,4,5,6 \end{cases}$$

$$\int_{-3\to -1}\frac{1}{f(z)}dz=-\int_{-3}^{-1}i^{6}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅱ) $z\in -3\leftarrow -1$

$$\begin{cases} z-z_7\geq 0\Rightarrow \dfrac{1}{\sqrt{z-z_7}}=\left|z-z_7\right|^{-\frac{1}{2}} \\ z-z_k<0\Rightarrow= z-z_k=\left|z-z_k\right|e^{-i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{i\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4,5,6 \end{cases}$$

$$\int_{-3\leftarrow -1}\frac{1}{f(z)}dz=-\int_{-3\leftarrow -1}\frac{1}{f(z)}dz=\int_{-1}^{-3}i^{6}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

In the theory we have

$$\int_{b_3^*}\frac{1}{f(z)}dz=2\int_{-1}^{-3}i^{6}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz=-2\int_{-1}^{-3}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz=-0.0765026$$

(2) mathematic $z\in b_3^*$

(Ⅰ) $z\in -3\to -1$

$$\begin{cases} z-z_7\geq 0\Rightarrow \dfrac{1}{\sqrt{z-z_7}}=\left|z-z_7\right|^{-\frac{1}{2}} \\ z-z_k<0\Rightarrow= z-z_k=\left|z-z_k\right|e^{i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{-i\pi}{2}}=-i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4,5,6 \end{cases}$$

$$\int_{-3\to -1}\frac{1}{f(z)}dz=\int_{-3}^{-1}(-i)^{6}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz=-\int_{-3}^{-1}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅱ) $z\in -3\leftarrow -1$

$$\begin{cases} z-z_7\geq 0\Rightarrow \dfrac{1}{\sqrt{z-z_7}}=\left|z-z_7\right|^{-\frac{1}{2}} \\ z-z_k<0\Rightarrow= z-z_k=\left|z-z_k\right|e^{i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{-i\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4,5,6 \end{cases}$$

$$\int_{-3\leftarrow -1}\frac{1}{f(z)}dz=-\int_{-1}^{-3}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

In mathematic

$$\int_{b_3^*} \frac{1}{f(z)} dz = 0$$

(3)by the Lemma（Ⅰ）

（Ⅰ） $z \in -3 \rightarrow -1$

$$\begin{cases} \arg(z - z_7) = 0 \Rightarrow \sqrt{z - z_7} \overset{math}{=} \sqrt{z - z_7} \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

（Ⅱ）$z \in -3 \leftarrow -1$

Because $f(z)|_{\text{II}} = -f(z)|_{\text{I}}$

Consider the $z \in -3 \rightarrow -1$ in sheet-Ⅰ

$$\begin{cases} \arg(z - z_7) = 0 \Rightarrow \sqrt{z - z_7} \overset{math}{=} \sqrt{z - z_7} \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1 \ldots 6 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow f(z)|_{-1 \dashrightarrow 3} = -f(z)|_{-1 \rightarrow 3} \overset{math}{=} -f(z)$$

$$\int_{b_3^*} \frac{1}{f(z)} dz = 2\int_{-3}^{-1} \frac{1}{f(z)} dz = -0.0765026$$

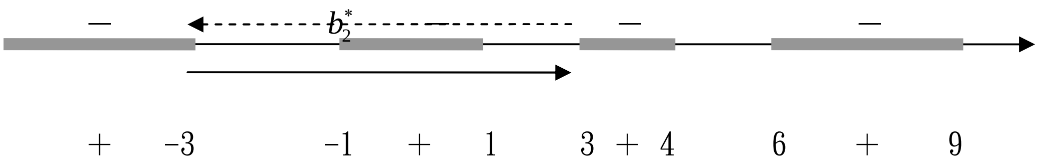


Figure 35: b_2's equivalent path b_2^*

(1)theory $z \in b_2^*$

(Ⅰ) $z \in -1 \to 1$

$$\left\{\left\{\begin{array}{l} z - z_k \geq 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_7|^{-\frac{1}{2}}, k = 6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = |z - z_k| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i|z - z_k|^{-\frac{1}{2}}, k = 1,2,3,4,5 \end{array}\right.\right.$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{f(z)} dz = \int_{-1}^{1} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(Ⅱ) $z \in$-1 <---1

Because $\int_{-1<--1} \dfrac{1}{f(z)} = \int_{-1\leftarrow 1} \dfrac{1}{f(z)}$

Consider $z \in -1 \leftarrow 1$

$$\left\{\left\{\begin{array}{l} z - z_k \geq 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_7|^{-\frac{1}{2}}, k = 6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = |z - z_k| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i|z - z_k|^{-\frac{1}{2}}, k = 1,2,3,4,5 \end{array}\right.\right.$$

$$\int_{-1<--1}\frac{1}{f(z)}=\int_{-1\leftarrow 1}\frac{1}{f(z)}=\int_{1}^{-1}i^{5}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

（Ⅲ） $z\in 1\to 3$

$$\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=6,7,5 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{-i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{i\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4 \end{cases}$$

$$\Rightarrow \int_{1\to 3}\frac{1}{f(z)}=\int_{1}^{3}i^{4}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

（Ⅳ） $z\in 1$ <---- 3

$$\int_{1<--3}\frac{1}{f(z)}=-\int_{1\leftarrow 3}\frac{1}{f(z)}$$

$$\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=6,7,5 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{-i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{i\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4 \end{cases}$$

$$\int_{1<--3}\frac{1}{f(z)}=-\int_{1\leftarrow 3}\frac{1}{f(z)}=-\int_{3}^{1}i^{4}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

By（Ⅰ)(Ⅱ)(Ⅲ)(Ⅳ)

$$\Rightarrow \int_{b_2^*}\frac{1}{f(z)}=2\int_{-1}^{1}i^{5}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz+2\int_{1}^{3}i^{4}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz = 0.157328$$

(2)mathematic $z\in b_2^*$

（Ⅰ） $z\in -1\to 1$

$$\left\{\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=6,7 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{-i\pi}{2}}=-i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3,4,5 \end{cases}\right.$$

$$\Rightarrow \int_{-1}^{1}\frac{1}{f(z)}dz=\int_{-1}^{1}(-i)^{5}\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅱ) $z \in -1 \leftarrow\!\!---1$

$$\left\{\left\{\begin{array}{l} z-z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_7|^{-\frac{1}{2}}, k=6,7 \\ z-z_k < 0 \Rightarrow = z-z_k = |z-z_k|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}} e^{\frac{-i\pi}{2}} = -i|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4,5 \end{array}\right.\right.$$

$$\int_{-1<--1} \frac{1}{f(z)} = \int_1^{-1} (-i)^5 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

(Ⅲ) $z \in 1 \rightarrow 3$

$$\left\{\begin{array}{l} z-z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_7|^{-\frac{1}{2}}, k=6,7,5 \\ z-z_k < 0 \Rightarrow = z-z_k = |z-z_k|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}} e^{\frac{-i\pi}{2}} = -i|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4 \end{array}\right.$$

$$\Rightarrow \int_{1\to 3} \frac{1}{f(z)} = \int_1^{3} (-i)^4 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

(Ⅳ) $z \in 1 \leftarrow\!\!--- 3$

$$\left\{\begin{array}{l} z-z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_7|^{-\frac{1}{2}}, k=6,7,5 \\ z-z_k < 0 \Rightarrow = z-z_k = |z-z_k|e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}} e^{\frac{-i\pi}{2}} = -i|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4 \end{array}\right.$$

$$\int_{1<--3} \frac{1}{f(z)} = \int_3^{1} (-i)^4 \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

By (Ⅰ)(Ⅱ)(Ⅲ)(Ⅳ)

$$\Rightarrow \int_{b_2^*} \frac{1}{f(z)} = 0$$

(3)by Lemma （Ⅰ）

（Ⅰ） $z \in -1 \to 1$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2,3,4,5 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -f(z)$$

（Ⅱ） $z \in -1 \; \text{<---} 1$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2,3,4,5 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -f(z)$$

（Ⅲ） $z \in 1 \to 3$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=5,6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2,3,4 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

（Ⅳ） $z \in 1 \; \text{<----} \; 3$

$$\int_{1<--3} \frac{1}{f(z)} = -\int_{1\leftarrow 3} \frac{1}{f(z)}$$

Consider $z \in 1 \leftarrow 3$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=5,6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2,3,4 \end{cases}$$

$$f(z)\big|_{1\leftarrow 3} \overset{math}{=} f(z)$$

$$\Rightarrow f(z)\big|_{1<--3} = -f(z)\big|_{1\leftarrow 3} \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_2^*} \frac{1}{f(z)} = -2\int_{-1}^{1} \prod_{k=1}^{7} \left|z - z_k\right|^{-\frac{1}{2}} dz + 2\int_{1}^{3} \prod_{k=1}^{7} \left|z - z_k\right|^{-\frac{1}{2}} dz$$

=0.157328

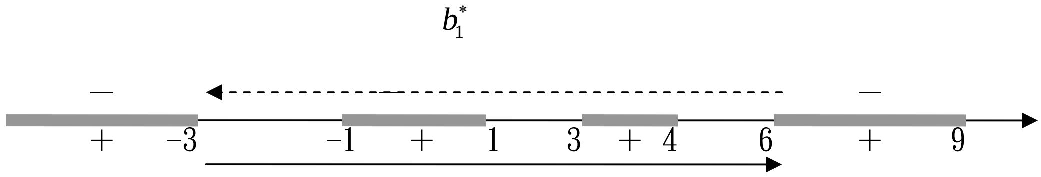


Figure 36: $b_1's$ equivalent path b_1^*

(1) theory $z \in b_1^*$

(Ⅰ) $z \in 3 \to 4$

$$\begin{cases} z - z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_7\right|^{-\frac{1}{2}}, k = 4,5,6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = \left|z - z_k\right| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_k\right|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i\left|z - z_k\right|^{-\frac{1}{2}}, k = 1,2,3, \end{cases}$$

$$\Rightarrow \int_{3 \to 4} \frac{1}{f(z)} = \int_{3}^{4} i^3 \prod_{k=1}^{7} \left|z - z_k\right|^{-\frac{1}{2}} dz$$

(Ⅱ) $z \in$ 3 <---- 4 $\equiv 3 \leftarrow 4$

$$\begin{cases} z - z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_7\right|^{-\frac{1}{2}}, k = 4,5,6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = \left|z - z_k\right| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_k\right|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i\left|z - z_k\right|^{-\frac{1}{2}}, k = 1,2,3, \end{cases}$$

$$\Rightarrow \int_{3 <\text{--}\ 4} \frac{1}{f(z)} = \int_{4}^{3} i^3 \prod_{k=1}^{7} \left|z - z_k\right|^{-\frac{1}{2}} dz$$

(Ⅲ) $z \in 4 \to 6$

$$\begin{cases} z - z_k \ge 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_7\right|^{-\frac{1}{2}}, k = 3,4,5,6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = \left|z - z_k\right| e^{-i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = \left|z - z_k\right|^{-\frac{1}{2}} e^{\frac{i\pi}{2}} = i\left|z - z_k\right|^{-\frac{1}{2}}, k = 1,2 \end{cases}$$

$$\Rightarrow \int_{4\to 6}\frac{1}{f(z)}=\int_4^6 i^2\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

（Ⅳ） $z\in 4<\text{--}\ 6$

$$\int_{4<\text{--}\,6}\frac{1}{f(z)}=-\int_{4\leftarrow\,6}\frac{1}{f(z)}$$

$$\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=3,4,5,6,7 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{-i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{i\pi}{2}}=i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2 \end{cases}$$

$$\int_{4<\text{--}\,6}\frac{1}{f(z)}=-\int_{4\leftarrow\,6}\frac{1}{f(z)}=-\int_6^4 i^2\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

$$\int_{b_1^*}\frac{1}{f(z)}=2\int_{-1}^{-3} i^6\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz+2\int_1^3 i^4\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz+2\int_4^6 i^2\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(2)mathematic $z\in b_1^*$

(Ⅰ) $z\in 3\to 4$

$$\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=4,5,6,7 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{-i\pi}{2}}=-i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3, \end{cases}$$

$$\Rightarrow \int_{3\to 4}\frac{1}{f(z)}=\int_3^4 (\text{-}i)^3\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅱ) $z\in\ 3 <\text{----}\ 4\ \equiv 3\leftarrow 4$

$$\begin{cases} z-z_k\ge 0\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_7\right|^{-\frac{1}{2}},k=4,5,6,7 \\ z-z_k<0\Rightarrow = z-z_k=\left|z-z_k\right|e^{i\pi}\Rightarrow \dfrac{1}{\sqrt{z-z_k}}=\left|z-z_k\right|^{-\frac{1}{2}}e^{\frac{-i\pi}{2}}=-i\left|z-z_k\right|^{-\frac{1}{2}},k=1,2,3, \end{cases}$$

$$\Rightarrow \int_{3<\text{--}\,4}\frac{1}{f(z)}=\int_4^3 (-i)^3\prod_{k=1}^{7}\left|z-z_k\right|^{-\frac{1}{2}}dz$$

(Ⅲ) $z \in 4 \to 6$

$$\begin{cases} z - z_k \geq 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_7|^{-\frac{1}{2}}, k = 3,4,5,6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = |z - z_k| e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{-i\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}}, k = 1,2 \end{cases}$$

$$\Rightarrow \int_{4\to 6} \frac{1}{f(z)} = \int_4^6 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(Ⅳ) $z \in 4 <\text{--}\ 6$

$$\begin{cases} z - z_k \geq 0 \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_7|^{-\frac{1}{2}}, k = 3,4,5,6,7 \\ z - z_k < 0 \Rightarrow = z - z_k = |z - z_k| e^{i\pi} \Rightarrow \dfrac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{\frac{-i\pi}{2}} = -i|z - z_k|^{-\frac{1}{2}}, k = 1,2 \end{cases}$$

$$\int_{4<\text{--}\,6} \frac{1}{f(z)} = \int_6^4 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(3)by the Lemma (Ⅰ)

(Ⅰ) $z \in 3 \to 4$

$$\begin{cases} \arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_7} \overset{math}{=} \sqrt{z - z_7}, k = 4,5,6,7 \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k}, k = 1,2,3, \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -f(z)$$

(Ⅱ) $z \in\ 3 \Leftarrow\text{----}\ 4\ \equiv 3 \leftarrow 4$

$$\begin{cases} \arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_7} \overset{math}{=} \sqrt{z - z_7}, k = 4,5,6,7 \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k}, k = 1,2,3, \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -f(z)$$

(Ⅲ) $z \in 4 \to 6$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=3,4,5,6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2, \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(Ⅳ) $z \in 4 <\text{--}\ 6$

$$f(z)\big|_{\text{II}} = -f(z)\big|_{\text{I}}$$

Consider $z \in 4 \leftarrow 6$

$$\begin{cases} \arg(z-z_k)=0 \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7}, k=3,4,5,6,7 \\ \arg(z-z_k)=-\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2, \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

So $f(z)\big|_{4<\text{--}6} = -f(z)\big|_{4\leftarrow 6} \overset{math}{=} -f(z)$

By (1) (2) (3)

$$\int_{b_1^*} \frac{1}{f(z)}dz = -2\int_{-1}^{-3} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz + 2\int_{1}^{3} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz - 2\int_{4}^{6} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}} dz$$

Now we discuss in general situation:

Compute $\int \frac{1}{f(z)}dz$ over a,b cycles for horizontal cut where $f(z) = \sqrt{\prod_{k=1}^{m}(z-z_k)}$,

$\forall k \in 1.....m$, *where* $z_1 > z_2 > ..... > z_k$

1.a-cycle

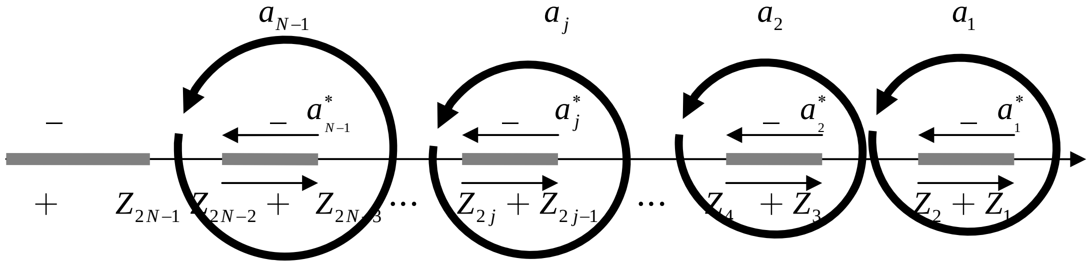


Figure 37: a-cycles for 2N-1 points

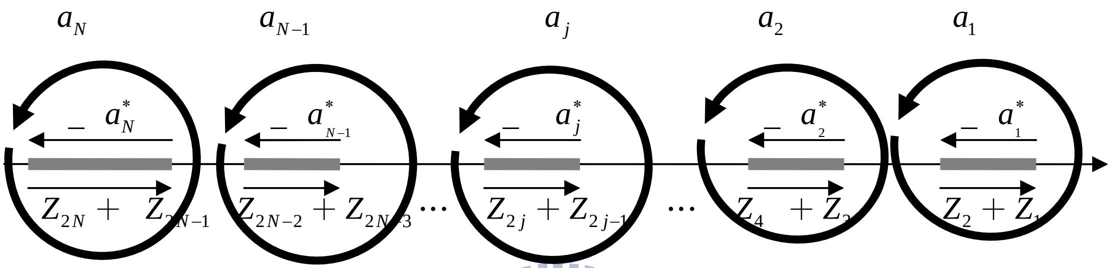


Figure 38:a-cycles for 2N points

By Cauchy theorem, we can get that

$$\int_{a_j} \frac{1}{f(z)} dz = \int_{a_j^*} \frac{1}{f(z)} dz$$

Now we consider the path by the Lemma I

$$\begin{cases} z_{2j} \xrightarrow{+} z_{2j-1}\text{:}\textit{the path } z_{2j} \to z_{2j-1} \textit{ on } (+) \textit{ edge} \\ z_{2j} \xleftarrow{+} z_{2j-1}\text{:}\textit{the path } z_{2j} \leftarrow z_{2j-1} \textit{ on } (-) \textit{ edge} \end{cases}$$

(1) consider $z_{2j} \to z_{2j-1}$ *on* $(+)$ *edge*

$$\begin{cases} \arg(z-z_k)=0 & \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k} \ , where\ k=2j,2j+1,\ldots\ldots,m \\ \arg(z-z_k)=-\pi & \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k} \ , where\ k=1,2,\ldots\ldots,2j-1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$$

So that $\int_{2j \,\underline{+}\, 2j-1} \frac{1}{f(z)} \overset{math}{=} -\int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)}$

(2) consider $z_{2j} \leftarrow z_{2j-1}$ *on* $(-)$ *edge*

$$\begin{cases} \arg(z-z_k)=0 & \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k} \ , where\ k=2j,2j+1,\ldots\ldots,m \\ \arg(z-z_k)=\pi & \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k} \ , where\ k=1,2,\ldots\ldots,2j-1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

So that $\int_{2j \,\underline{-}\, 2j-1} \frac{1}{f(z)} \overset{math}{=} \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)}$

So we have the conclusion

$$\int_{a_j^*} \frac{1}{f(z)} \overset{math}{=} -2\int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)}$$

2. b-cycle

(1) Give b_j is a circle centered at x with radius r and enclosed

the $\left[Z_{2N-1}, Z_{2j}\right]$ and intersect at the points on $\left[Z_{2j-1}, Z_{2j}\right]$ and $\left[Z_{2N-1}, \infty\right)$.

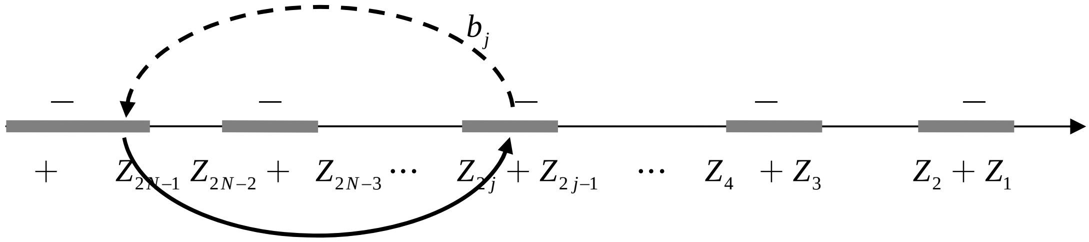


Figure 39: b_j-cycle for 2N-1 points

(2) Give b_j is a circle centered at x with radius r and enclosed the $\left[Z_{2N-1}, Z_{2j}\right]$ and intersect at the points on $\left[Z_{2j-1}, Z_{2j}\right]$ and $[Z_{2N-1}, Z_{2N}]$.

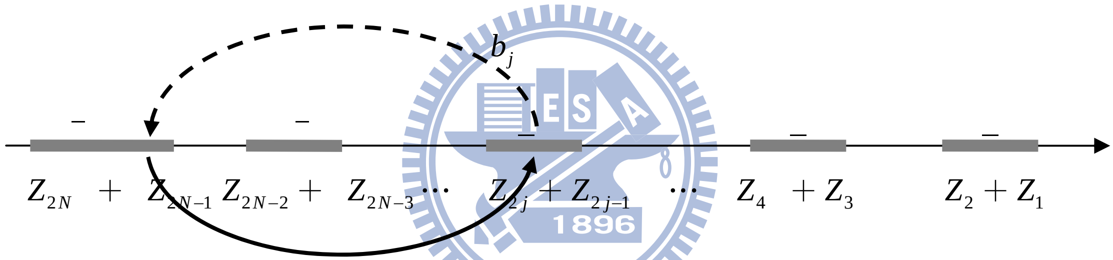


Figure 40: b_j-cycle for 2N points

If $z \in b_j$ *and* $z = x + re^{i\theta}$ *where* $\theta \in [-\pi, 0) \cup [2\pi, 3\pi)$.

Note that $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

$$\int_{b_j} \frac{1}{f(z)} dz = \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta + \int_{2\pi}^{3\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta$$

$$= \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta - \int_{0}^{\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta$$

3. Now we consider the equivalent path b_j^*

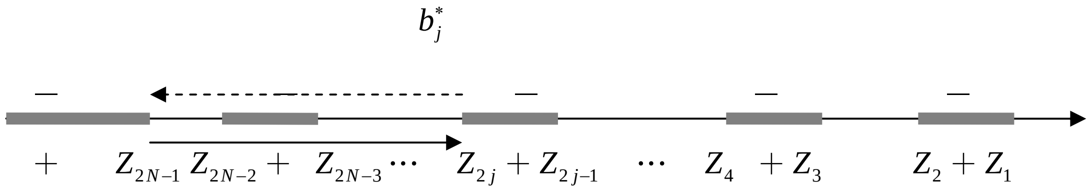


Figure 41: b_j^*-cycle for 2N-1 points

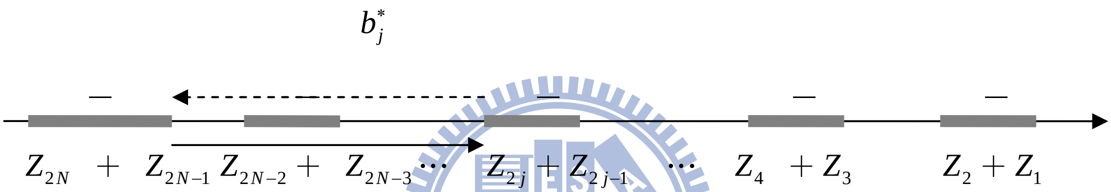


Figure 42: b_j^*-cycle for 2N points

By Cauchy Theorem, we know that

$$\int_{b_j} \frac{1}{f(z)} = \int_{b_j^*} \frac{1}{f(z)}$$

The path we must have two part to discuss.

(1) path on the cut is

$$\begin{cases} z_{2s+2} \to z_{2s+1} \text{ , } where\ s = j, j+1, \ldots\ldots, N-2\ on\ the\ (+)\ edge\ in\ sheet - \mathrm{I} \\ z_{2s+2} \leftarrow z_{2s+1} \text{ , } where\ s = j, j+1, \ldots\ldots, N-2\ on\ the\ (-)\ edge\ in\ sheet - \mathrm{II} \end{cases}$$

Now we use Lemma I to computation the question.

(i) $z \in z_{2s+2} \xrightarrow{+} z_{2s+1}$

$$\begin{cases} \arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \;\; k = 2s + 2, \ldots\ldots, m \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \;\; k = 1, 2, \ldots\ldots 2s + 1 \end{cases}$$

We have that

$$f(z) \overset{math}{=} (-1)^{2s+1} f(z) = -f(z)$$

(ii) $z \in z_{2s+2} \xleftarrow{-} z_{2s+1}$ *in the sheet* - II $\equiv z \in z_{2s+2} \xleftarrow{+} z_{2s+1}$ *in the sheet* – I

$$\begin{cases} \arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \;\; k = 2s + 2, \ldots\ldots, m \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \;\; k = 1, 2, \ldots\ldots 2s + 1 \end{cases}$$

We have that

$$f(z) \overset{math}{=} (-1)^{2s+1} f(z) = -f(z)$$

(2) no cut on the path is

$$\begin{cases} z_{2s+1} \rightarrow z_{2s} \;, \; where \; s = j, j+1, \ldots\ldots, N-2 \;\; in \; sheet - \text{I} \\ z_{2s+1} \leftarrow z_{2s} \;, \; where \; s = j, j+1, \ldots\ldots, N-2 \;\; in \; sheet - \text{II} \end{cases}$$

Now we use Lemma I to computation the question.

(i) $z \in z_{2s+1} \rightarrow z_{2s}$

$$\begin{cases} \arg(z - z_k) = 0 \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \;\; k = 2s + 1, \ldots\ldots, m \\ \arg(z - z_k) = -\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \;\; k = 1, 2, \ldots\ldots 2s \end{cases}$$

We have that

$$f(z) \overset{math}{=} (-1)^{2s} f(z) = f(z)$$

(ii) $z \in z_{2s+1} \leftarrow - - z_{2s}$ *because* $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

So we consider $z_{2s+1} \leftarrow z_{2s}$

$$\begin{cases} \arg(z-z_k) = 0 \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k} \quad k = 2s+1, \ldots\ldots, m \\ \arg(z-z_k) = -\pi \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k} \quad k = 1, 2, \ldots\ldots 2s \end{cases}$$

We have that

$$f(z) \overset{math}{=} -(-1)^{2s} f(z) = -f(z)$$

$$\int_{b_j} \frac{1}{f(z)} = \sum_{s=j}^{N-1} \left[(-1)^s 2 \int_{z_{2s+1}}^{z_{2s}} \prod_{k=1}^{m} \left| z - z_k \right|^{-\frac{1}{2}} dz \right]$$

$$\overset{math}{=} \sum_{s=j}^{N-1} \left[2 \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz \right]$$

3. The integrals of $\frac{1}{f(z)}$ over a, b cycles for vertical cut

First define that

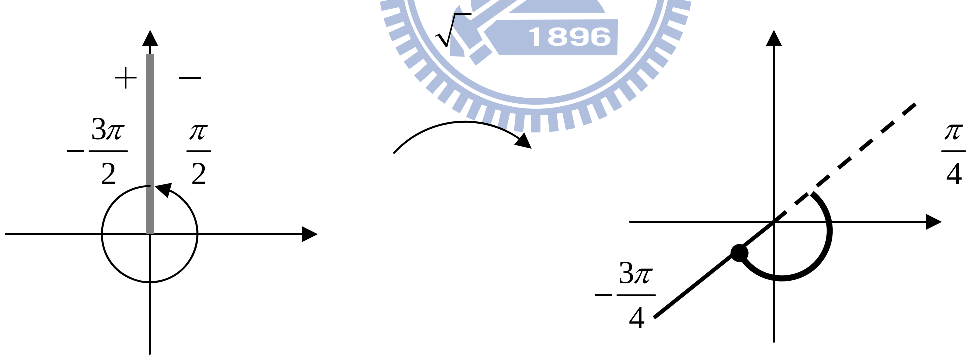
$$z - z_k = \begin{cases} re^{i\theta} & \theta \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right) \text{ where } z \in \text{sheet-I} \\ re^{i\theta} & \theta \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right) \text{ where } z \in \text{sheet-II} \end{cases}$$

Each sheet has “+” and” −” edge, z_k is the endpoint of the cut.

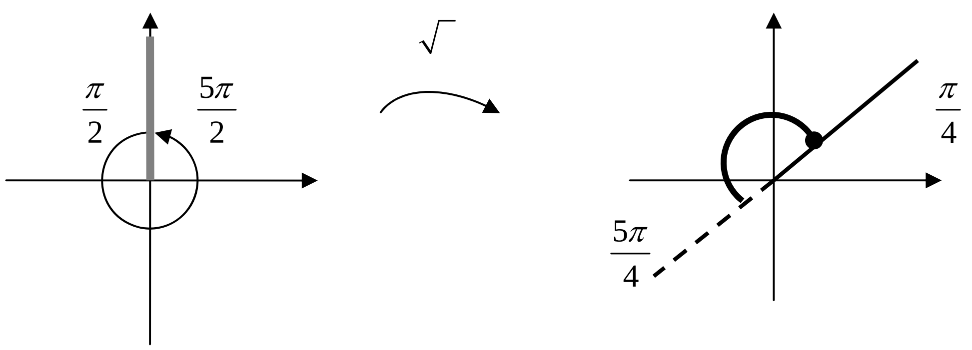
Example:

Take $f(z) = \sqrt{z}$

$$\begin{cases} z = |z|e^{i\theta} \quad \theta \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} \quad \frac{\theta}{2} \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right) \text{ where } z \in \text{I} \\ z = |z|e^{i\theta} \quad \theta \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right) \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}} \quad \frac{\theta}{2} \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right) \text{ where } z \in \text{II} \end{cases}$$



$z \in \text{I}$



$z \in \text{II}$

Figure 43:Example of $f(z) = \sqrt{z}$

Because $\theta_2 = \theta_1 + 2\pi$ *where* $\theta_1 \in \mathrm{I}$, $\theta_2 \in \mathrm{II}$

$$\Rightarrow f(z)\big|_{\mathrm{II}} = \sqrt{r}e^{\frac{i\theta_2}{2}} = \sqrt{r}e^{\frac{i(\theta_1+2\pi)}{2}} = \sqrt{r}e^{\frac{i\theta_1}{2}}e^{i\pi} = -f(z)\big|_{\mathrm{I}}$$

Now we discuss the value different between theory and mathematic.

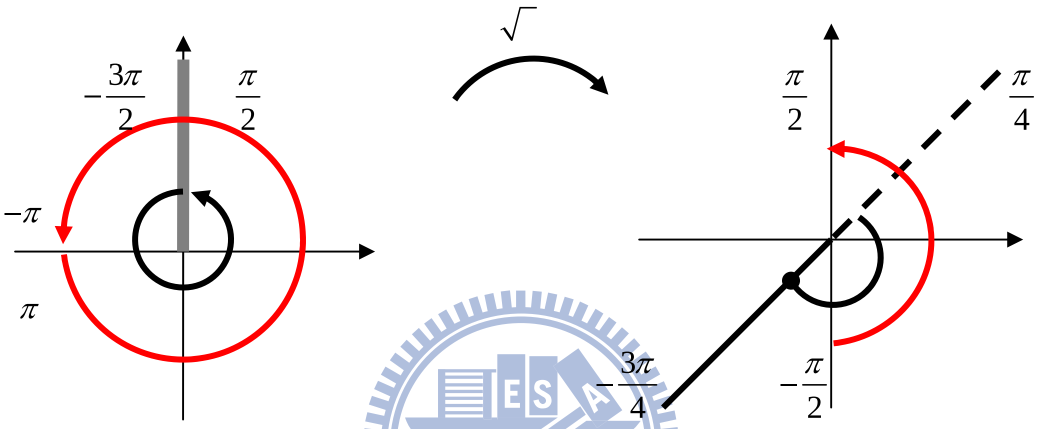


Figure 44：the value of $\sqrt{z}$ in theory and mathematic

We see that have the problem at $\theta \in \left[\frac{-3\pi}{2}, -\pi\right]$, next we prove that how to solve the problem .

Lemma 2 Take $z \in \mathrm{I}$ for the vertical cut .

$$\sqrt{z - z_k} \overset{math}{=} \begin{cases} \sqrt{z - z_k} & \arg(z - z_k) \in \left(-\pi, \frac{\pi}{2}\right) \\ -\sqrt{z - z_k} & \arg(z - z_k) \in \left[-\frac{3\pi}{2}, -\pi\right] \end{cases}$$

Proof:

Take $z - z_k = re^{i\theta}$

(1) $\theta \in \left(-\pi, \frac{\pi}{2}\right)$ is OK.

(2)when $\theta \in \left[-\frac{3\pi}{2}, -\pi\right]$

In mathematic , it will regard as θ *to* $\theta + 2\pi \Rightarrow \left[-\frac{3\pi}{2}, -\pi\right] \to \left[\frac{\pi}{2}, \pi\right]$, and $re^{i\theta} = re^{i(\theta+2\pi)}$.

So that $\sqrt{z - z_k} = \begin{cases} \sqrt{r}e^{i\frac{\theta}{2}} & \textit{theory} \\ \sqrt{r}e^{i\frac{\theta}{2}}e^{i\pi} = -\sqrt{r}e^{i\frac{\theta}{2}} & \textit{mathematic} \end{cases}$

$$\Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \qquad \textit{where} \quad \theta \in \left[-\frac{3\pi}{2}, -\pi\right].$$

Now we use the Lemma 2 to do some example .

Example 1: Evaluate the integrals of $\frac{1}{f(z)}$ over a, b cycles for vertical cut . where $f(z) = \sqrt{(z-i)(z-2i)(z-3i)(z-4i)(z-5i)(z-6i)}$

Let $z_k = ki \quad k = 1,2,3,4,5,6$

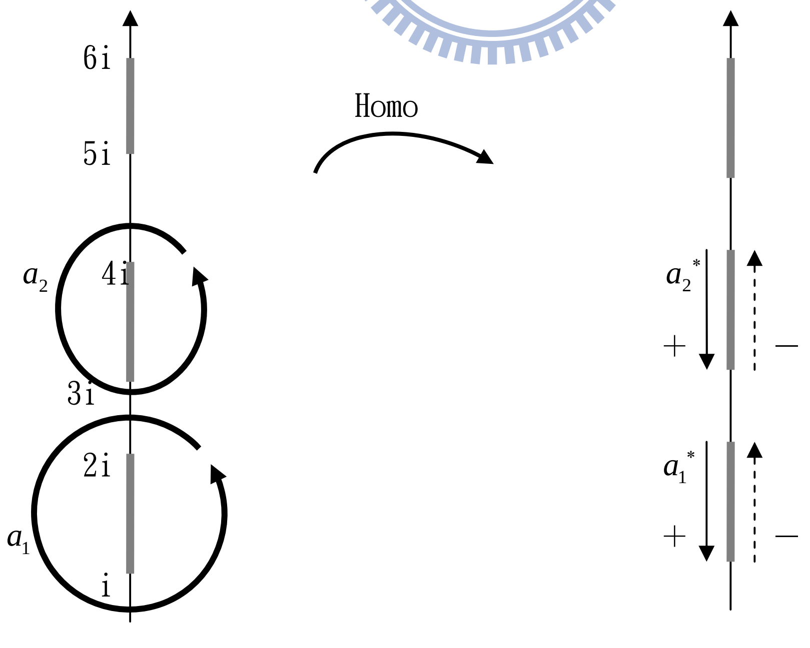


Figure 45: a cycle and a_1 homotopic path

(1) $z \in a_1^*$ $2i \to i$ Let $z = ri$, $r = 2 \to 1$, $dz = idr$

(Ⅰ) theory $z - ki = |z - ki| e^{i\arg(z-ki)}$

$$\begin{cases} \arg(z-i) = -\dfrac{3\pi}{2} \Rightarrow \arg\sqrt{z-i} = -\dfrac{3\pi}{4} \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 2 \ldots 6 \end{cases}$$

$$\frac{1}{f(z)} = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}} (e^{-\frac{3\pi}{4}i})(e^{-\frac{\pi}{4}i})^5 = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}}$$

(Ⅱ) mathematic $z - ki = |z - ki| e^{i\arg(z-ki)}$

$$\begin{cases} \arg(z-i) = \dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-i} = \dfrac{\pi}{4} \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 2 \ldots 6 \end{cases}$$

$$\frac{1}{f(z)} = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}} (e^{\frac{\pi}{4}i})(e^{-\frac{\pi}{4}i})^5 = -\prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}}$$

By (Ⅰ)(Ⅱ) we see that theory and math difference a "—"

$\Rightarrow f(z)|_{\mathrm{I}} = -f(z)|_{math}$.

(Ⅲ) by the Lemma 2

$$\begin{cases} \arg(z-i) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z-i} \overset{math}{=} -\sqrt{z-i} \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} = \sqrt{z-ki} \ , \ k = 2 \ldots 6 \end{cases}$$

$$f(z) \overset{math}{=} -f(z)$$

(2) i ----► 2i $Let\ z = ri$, $r = 1 \to 2$, $dz = idr$

(I) theory $z-ki=|z-ki|e^{i\arg(z-ki)}$

$$\begin{cases}\arg(z-i)=\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-i}=\dfrac{\pi}{4}\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{\pi}{4},k=2\ldots6\end{cases}$$

$$\frac{1}{f(z)}=\prod_{k=1}^{6}|z-z_k|^{-\frac{1}{2}}(e^{\frac{\pi}{4}i})(e^{-\frac{\pi}{4}i})^5=-\prod_{k=1}^{6}|z-z_k|^{-\frac{1}{2}}$$

(II) mathematic $z-ki=|z-ki|e^{i\arg(z-ki)}$

$$\begin{cases}\arg(z-i)=\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-i}=\dfrac{\pi}{4}\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{\pi}{4},k=2\ldots6\end{cases}$$

$$\frac{1}{f(z)}=\prod_{k=1}^{6}|z-z_k|^{-\frac{1}{2}}(e^{\frac{\pi}{4}i})(e^{-\frac{\pi}{4}i})^5=-\prod_{k=1}^{6}|z-z_k|^{-\frac{1}{2}}$$

$$\Rightarrow f(z)|_{\mathrm{I}}=f(z)|_{math}.$$

(III) by the Lemma 2

$$\begin{cases}\arg(z-i)=\dfrac{\pi}{2}\Rightarrow \sqrt{z-i}\overset{math}{=}\sqrt{z-i}\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \sqrt{z-ki}\overset{math}{=}\sqrt{z-ki}\ ,\ k=2\ldots6\end{cases}$$

$$f(z)\overset{math}{=}f(z)$$

$$\int_{a_1^*}\frac{1}{f(z)}=2\int_1^2 i|ri-ki|^{-\frac{1}{2}}dr\overset{math}{=}2\int_1^2\frac{i}{f(ri)}dr$$

=0+0.871563i

(1) $z \in a_2^* \quad 4i \to 3i \quad Let\ z = ri\ ,\ r = 4 \to 3\ ,\ dz = idr$

(I)theory

$$z - ki = |z - ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki) = -\dfrac{3\pi}{2} \Rightarrow \arg\sqrt{z-i} = -\dfrac{3\pi}{4}, k = 1,2,3 \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 4,5,6 \end{cases}$$

$$\frac{1}{f(z)} = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}} (e^{-\frac{3\pi}{4}i})^3 (e^{-\frac{\pi}{4}i})^3 = -\prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}}$$

(II) mathematic $z - ki = |z - ki| e^{i\arg(z-ki)}$

$$\begin{cases} \arg(z-i) = \dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-i} = \dfrac{\pi}{4}, k = 1,2,3 \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 4,5,6 \end{cases}$$

$$\frac{1}{f(z)} = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}} (e^{\frac{\pi}{4}i})^3 (e^{-\frac{\pi}{4}i})^3 = \prod_{k=1}^{6} |z - z_k|^{-\frac{1}{2}}$$

$$\Rightarrow f(z)\big|_{\mathrm{I}} = f(z)\big|_{math}$$

(III) by the Lemma 2

$$\begin{cases} \arg(z-ki) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} -\sqrt{z-ki}, k = 1,2,3 \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}\ ,\ k = 4,5,6 \end{cases}$$

$$f(z) \overset{math}{=} -f(z)$$

(2) 4i ◄---- 3i $Let\ z = ri\ ,\ r = 4 \leftarrow 3\ ,\ dz = idr$

(I)theory

$$z-ki=\left|z-ki\right|e^{i\arg(z-ki)}$$

$$\begin{cases}\arg\left(z-ki\right)=\frac{\pi}{2}\Rightarrow\arg\sqrt{z-i}=\frac{\pi}{4},k=1,2,3\\ \arg(z-ki)=-\frac{\pi}{2}\Rightarrow\arg\sqrt{z-ki}=-\frac{\pi}{4},k=4,5,6\end{cases}$$

$$f(z)=\prod_{k=1}^{6}\left|z-z_k\right|^{\frac{1}{2}}(e^{\frac{\pi}{4}i})^3(e^{-\frac{\pi}{4}i})^3=\prod_{k=1}^{6}\left|z-z_k\right|^{\frac{1}{2}}$$

(II) mathematic $z-ki=\left|z-ki\right|e^{i\arg(z-ki)}$

$$\begin{cases}\arg\left(z-ki\right)=\frac{\pi}{2}\Rightarrow\arg\sqrt{z-i}=\frac{\pi}{4},k=1,2,3\\ \arg(z-ki)=-\frac{\pi}{2}\Rightarrow\arg\sqrt{z-ki}=-\frac{\pi}{4},k=4,5,6\end{cases}$$

$$f(z)=\prod_{k=1}^{6}\left|z-z_k\right|^{\frac{1}{2}}(e^{\frac{\pi}{4}i})^3(e^{-\frac{\pi}{4}i})^3=\prod_{k=1}^{6}\left|z-z_k\right|^{\frac{1}{2}}$$

$$\Rightarrow f(z)\big|_{\mathrm{I}}=f(z)\big|_{math}$$

(III) by the Lemma 2

$$\begin{cases}\arg\left(z-ki\right)=\frac{\pi}{2}\Rightarrow\sqrt{z-ki}\overset{math}{=}\sqrt{z-ki},k=1,2,3\\ \arg(z-ki)=-\frac{\pi}{2}\Rightarrow\sqrt{z-ki}\overset{math}{=}\sqrt{z-ki}\ ,\ k=4,5,6\end{cases}$$

$$f(z)\overset{math}{=}f(z)$$

$$\int_{a_2^*}\frac{1}{f(z)}=2\int_3^4 i\left|ri-ki\right|^{-\frac{1}{2}}dr\overset{math}{=}2\int_3^4\frac{i}{f(ri)}dr$$

=0+1.74313i

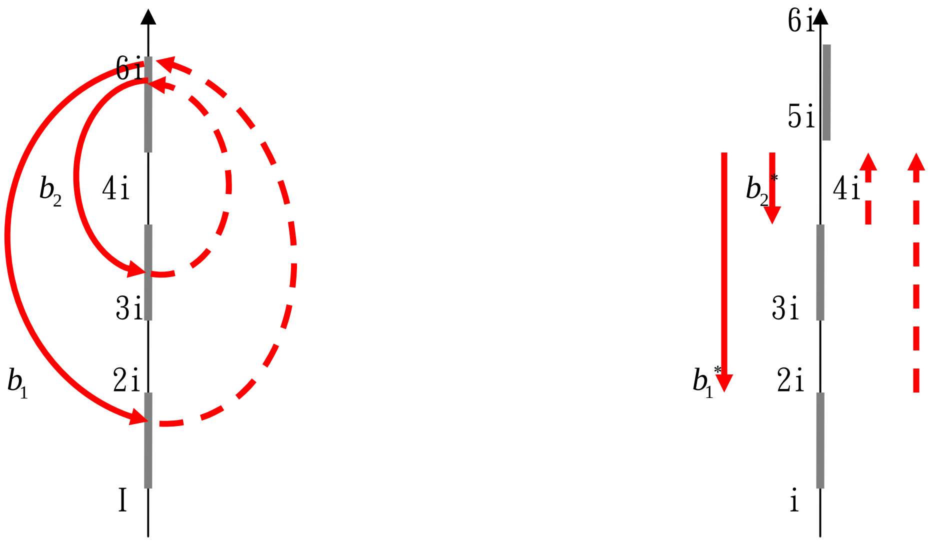


Figure 46: b cycle and homotopic path b^*

1.(Ⅰ)theory $z \in b_2^*$ $Let\ z = ri\ ,\ r = 5 \to 4\ ,\ dz = idr$

$$z - ki = |z - ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki) = -\dfrac{3\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{3\pi}{4}, k = 1,2,3,4 \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 5,6 \end{cases}$$

$$f(z) = \prod_{k=1}^{6} |z - z_k|^{\frac{1}{2}} (e^{-\frac{3\pi}{4}i})^4 (e^{-\frac{\pi}{4}i})^2 = i\prod_{k=1}^{6} |z - z_k|^{\frac{1}{2}}$$

(Ⅱ)mathematic $z \in b_2^*$

$$z - ki = |z - ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki) = \dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = \dfrac{\pi}{4}, k = 1,2,3,4 \\ \arg(z-ki) = -\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k = 5,6 \end{cases}$$

$$f(z) = \prod_{k=1}^{6} |z - z_k|^{\frac{1}{2}} (e^{-\frac{3\pi}{4}i})^4 (e^{-\frac{\pi}{4}i})^2 = i\prod_{k=1}^{6} |z - z_k|^{\frac{1}{2}}$$

$$\Rightarrow f(z)\big|_{\text{I}} = f(z)\big|_{math}$$

(Ⅲ) by the Lemma 2

$$\begin{cases} \arg(z-ki)=\frac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}, k=1,2,3,4 \\ \arg(z-ki)=-\frac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki} \ , \ k=5,6 \end{cases}$$

$$f(z) \overset{math}{=} f(z)$$

(2)we know that $\Rightarrow f(z)|_{\text{II}} = -f(z)|_{\text{I}}$

So we consider 4i→5i

Let z = ri , r = 5 ← 4 , dz = idr

(I)theory $z \in b_2^*$ *Let z = ri , r = 5 ← 4 , dz = idr*

$$z-ki = |z-ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki)=-\frac{3\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\frac{3\pi}{4}, k=1,2,3,4 \\ \arg(z-ki)=-\frac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\frac{\pi}{4}, k=5,6 \end{cases}$$

$$f(z)=\prod_{k=1}^{6} |z-z_k|^{\frac{1}{2}} (e^{-\frac{3\pi}{4}i})^4 (e^{-\frac{\pi}{4}i})^2 = i\prod_{k=1}^{6} |z-z_k|^{\frac{1}{2}}$$

$$\Rightarrow f(z)|_{4-->5} = -f(z)|_{4\to5} = -i\prod_{k=1}^{6} |z-z_k|^{\frac{1}{2}}$$

(Ⅱ)mathematic $z \in b_2^*$ *Let z = ri , r = 5 ← 4 , dz = idr*

$$z-ki = |z-ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki)=\frac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = \frac{\pi}{4}, k=1,2,3,4 \\ \arg(z-ki)=-\frac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\frac{\pi}{4}, k=5,6 \end{cases}$$

$$f(z)=\prod_{k=1}^{6} |z-z_k|^{\frac{1}{2}} (e^{-\frac{3\pi}{4}i})^4 (e^{-\frac{\pi}{4}i})^2 = i\prod_{k=1}^{6} |z-z_k|^{\frac{1}{2}}$$

$$\Rightarrow f(z)|_{theory} = -f(z)|_{math}$$

(Ⅲ) by the Lemma 2

$$\begin{cases} \arg(z-ki)=\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}, k=1,2,3,4 \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}\ ,\ k=5,6 \end{cases}$$

$$f(z) \overset{math}{=} f(z)$$

$$\int_{b_2^*} \frac{1}{f(z)} = -2\int_4^5 i\prod_{k=1}^{6} |ri-ki|^{-\frac{1}{2}} dr \overset{math}{=} -2\int_4^5 \frac{i}{f(ri)} dr = -1.48065$$

2. evaluate $\int_{b_1^*} \frac{1}{f(z)} dz$, take $b_1^* = b_2^* \cup b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{14}^*$

Where $\begin{cases} b_{11}^* = \text{the path along vertical cut from } 4i \text{ to } 3i \text{ on } (+)\text{edge of sheet-I} \\ b_{12}^* = \text{the path along vertical cut from } 3i \text{ to } 4i \text{ on } (-)\text{edge of sheet-II} \\ b_{13}^* = \text{the path along vertical cut from } 3i \text{ to } 2i \text{ on sheet-I} \\ b_{14}^* = \text{the path along vertical cut from } 2i \text{ to } 3i \text{ on sheet-II} \end{cases}$

(1) $b_{11}^* \equiv a_{2(4i\to 3i)}^*$

(2) $b_{12}^* \equiv \text{the path along vertical cut from } 3i \text{ to } 4i \text{ on } (+)\text{edge of sheet-I}$

(Ⅰ) theory $Let\ z = ri\ ,\ r = 3 \to 4\ ,\ dz = idr$

$$z-ki = |z-ki|e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki)=-\dfrac{3\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{3\pi}{4}, k=1,2,3 \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k=4,5,6 \end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{-\frac{3\pi}{4}i})^3(e^{-\frac{\pi}{4}i})^3 = -\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅱ)mathematic $Let\ z = ri\ ,\ r = 3 \to 4\ ,\ dz = idr$

$$z-ki=|z-ki|e^{i\arg(z-ki)}$$

$$\begin{cases}\arg(z-ki)=\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=\dfrac{\pi}{4},k=1,2,3\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{\pi}{4},k=4,5,6\end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{\frac{\pi}{4}i})^3(e^{-\frac{\pi}{4}i})^3=-\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅲ)by Lemma 2

$$\begin{cases}\arg(z-ki)=\dfrac{\pi}{2}\Rightarrow \sqrt{z-ki}\overset{math}{=}\sqrt{z-ki},k=1,2,3,\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \sqrt{z-ki}\overset{math}{=}\sqrt{z-ki}\ ,\ k=4,5,6\end{cases}$$

(3) $b_{13}^{*}\equiv$ *the path along vertical cut from* $3i$ *to* $2i$ *on* $(+)$*edge of sheet*-Ⅰ

(Ⅰ) theory $Let\ z = ri\ ,\ r = 3 \to 2\ ,\ dz = idr$

$$z-ki=|z-ki|e^{i\arg(z-ki)}$$

$$\begin{cases}\arg(z-ki)=-\dfrac{3\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{3\pi}{4},k=1,2\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{\pi}{4},k=3,4,5,6\end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{-\frac{3\pi}{4}i})^2(e^{-\frac{\pi}{4}i})^4=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}e^{-\frac{5\pi}{2}i}=-i\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅱ)mathematic $Let\ z = ri\ ,\ r = 3 \to 2\ ,\ dz = idr$

$$z-ki=|z-ki|e^{i\arg(z-ki)}$$

$$\begin{cases}\arg(z-ki)=\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=\dfrac{\pi}{4},k=1,2\\ \arg(z-ki)=-\dfrac{\pi}{2}\Rightarrow \arg\sqrt{z-ki}=-\dfrac{\pi}{4},k=3,4,5,6\end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{\frac{\pi}{4}i})^2(e^{-\frac{\pi}{4}i})^4=-i\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅲ)by Lemma 2

$$\begin{cases} \arg(z-ki)=\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}, k=1,2, \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}\ ,\ k=3,4,5,6 \end{cases}$$

$$f(z) \overset{math}{=} f(z)$$

(4) b_{14}^{*}: we know that $f(z)|_{\text{II}} = -f(z)|_{\text{I}}$, so we consider that the path $2i \to 3i$ on sheet-Ⅰ .

(Ⅰ) theory $Let\ z = ri\ ,\ r = 3 \leftarrow 2\ ,\ dz = idr$

$$z-ki = |z-ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki)=-\dfrac{3\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{3\pi}{4}, k=1,2 \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k=3,4,5,6 \end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{-\frac{3\pi}{4}i})^2(e^{-\frac{\pi}{4}i})^4=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}e^{-\frac{5\pi}{2}i}=-i\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

$$f(z)|_{b_{14}^{*}} = -f(z)|_{2i\to 3i} = i\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅱ)mathematic $Let\ z = ri\ ,\ r = 3 \leftarrow 2\ ,\ dz = idr$

$$z-ki = |z-ki| e^{i\arg(z-ki)}$$

$$\begin{cases} \arg(z-ki)=\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = \dfrac{\pi}{4}, k=1,2 \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \arg\sqrt{z-ki} = -\dfrac{\pi}{4}, k=3,4,5,6 \end{cases}$$

$$f(z)=\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}(e^{\frac{\pi}{4}i})^2(e^{-\frac{\pi}{4}i})^4=-i\prod_{k=1}^{6}|z-z_k|^{\frac{1}{2}}$$

(Ⅲ)by Lemma 2

$$\begin{cases} \arg(z-ki)=\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki}, k=1,2, \\ \arg(z-ki)=-\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-ki} \ , \ k=3,4,5,6 \end{cases}$$

$$f(z)\big|_{2i\to 3i} \overset{math}{=} f(z) \Rightarrow f(z)\big|_{b_{14}^*} = -f(z)\big|_{2i\to 3i} \overset{math}{=} -f(z)$$

By (1) (2) (3) (4)

$$\int_{b_1^*} \frac{1}{f(z)} = -2i\int_2^3 \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} dr - 2i\int_4^5 \prod_{k=1}^{6} |z-ki|^{-\frac{1}{2}} dr$$

$$= -2i\int_2^3 \frac{i}{f(ri)} dr - 2i\int_4^5 \frac{i}{f(ri)} dr$$

$$=-2.9613$$

Now we give a more easier method to reduce the work process.

Take a example $f(z)=\sqrt{(z-i)(z-2i)}$

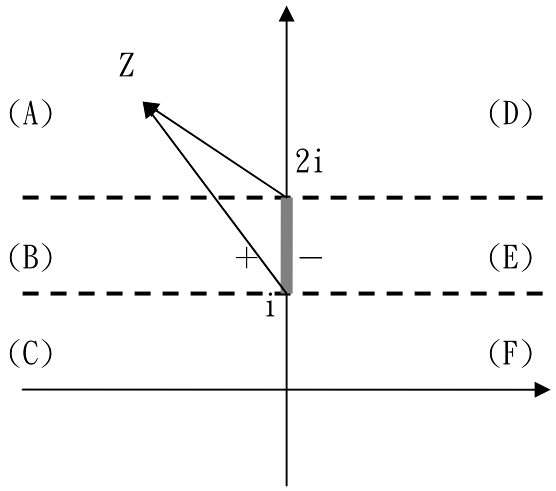


Figure 47: the 6 area

1. $z\in A$

$$\begin{cases}\sqrt{z-i}\overset{math}{=}-\sqrt{z-i},\arg(z-i)\in\left(-\dfrac{3\pi}{2},-\pi\right)\\ \sqrt{z-2i}\overset{math}{=}-\sqrt{z-2i},\arg(z-2i)\in\left(-\dfrac{3\pi}{2},-\pi\right)\end{cases}$$

$$\Rightarrow f(z)\overset{math}{=}f(z)$$

2. $z\in B$

$$\begin{cases}\sqrt{z-i}\overset{math}{=}-\sqrt{z-i},\arg(z-i)\in\left(-\dfrac{3\pi}{2},-\pi\right)\\ \sqrt{z-2i}\overset{math}{=}\sqrt{z-2i},\arg(z-2i)\in\left(-\pi,-\dfrac{\pi}{2}\right)\end{cases}$$

$$\Rightarrow f(z)\overset{math}{=}-f(z)$$

3. $z \in C$

$$\begin{cases} \sqrt{z-i} \overset{math}{=} \sqrt{z-i}, \arg(z-i) \in \left(-\pi, -\dfrac{\pi}{2}\right) \\ \sqrt{z-2i} \overset{math}{=} \sqrt{z-2i}, \arg(z-2i) \in \left(-\pi, -\dfrac{\pi}{2}\right) \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

4. $z \in D, E, F$

$$\begin{cases} \arg(z-i) \in \left(-\pi, \dfrac{\pi}{2}\right) \\ \arg(z-2i) \in \left(-\pi, \dfrac{\pi}{2}\right) \end{cases} \Rightarrow \begin{cases} \sqrt{z-i} \overset{math}{=} \sqrt{z-i} \\ \sqrt{z-2i} \overset{math}{=} \sqrt{z-2i} \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

5. $z \in +$ *edge of* $2i \to i$

$$\begin{cases} \arg(z-i) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z-i} \overset{math}{=} -\sqrt{z-i} \\ \arg(z-2i) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z-ki} \overset{math}{=} \sqrt{z-2i} \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -f(z)$$

6. $z \in -$ *edge of* $i \to 2i$

$$\begin{cases} \arg(z-i) = \dfrac{\pi}{2} \Rightarrow \sqrt{z-i} \overset{math}{=} \sqrt{z-i} \\ \arg(z-2i) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z-2i} \overset{math}{=} \sqrt{z-2i} \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

So that $f(z) \overset{math}{=} \begin{cases} -f(z) & \textit{where } z \in B \textit{ or } + \textit{edge of } (i, 2i) \\ f(z) & \textit{otherwise} \end{cases}$

Next ,we give a example to test the method .

Example Compute $\int \frac{1}{f(z)}$ *and* a_1, a_2, b_1, b_2 *cycle*

Let

$z_1 = 1-3i,\ z_2 = 1+3i,\ z_3 = -i,\ z_4 = i,\ z_5 = -1-3i,\ z_6 = -1+3i$ *and* $f(z) = \sqrt{\prod_{k=1}^{6}(z-z_k)}$

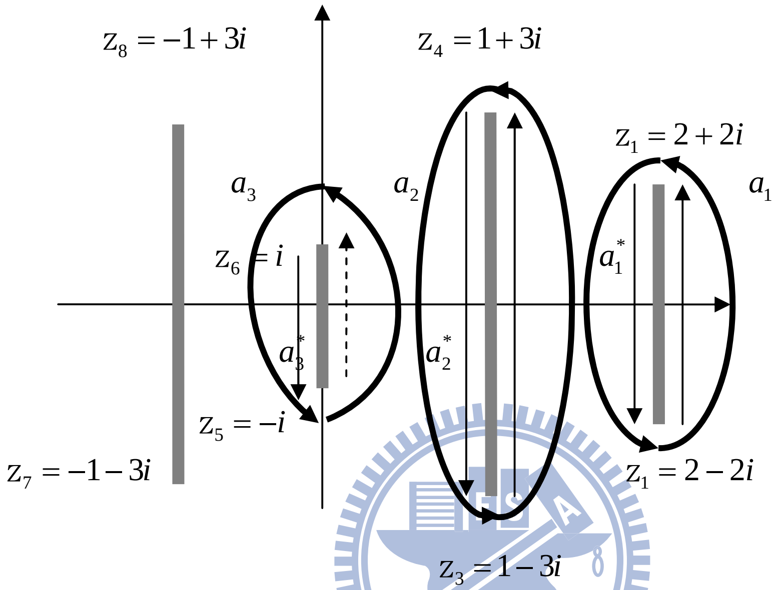


Figure 48: a-cycle and the equivalent path

1. $\int_{a_1^*} \frac{1}{f(z)}$ and $a_1^* = a_{11}^* \cup a_{12}^*$

$$\begin{cases} a_{11}^* = \text{ the path vertical cut from } 2+2i \rightarrow 2-2i \text{ on } (+) \text{ edge in sheet} - \mathrm{I} \\ a_{12}^* = \text{ the path vertical cut from } 2-2i \rightarrow 2+2i \text{ in } (-) \text{ edge of sheet} - \mathrm{I} \end{cases}$$

(1) $z \in a_{11}^*$

$Let\ z = 2 + ri\ ,\ r = 2 \rightarrow -2\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$\int_{a_{11}^*} \frac{1}{f(z)} dz \overset{math}{=} -\int_2^{-2} \frac{1}{f(2+ri)} dr$$

(2) $z \in a_{12}^*$

$Let\ z = 2 + ri\ ,\ r = -2 \rightarrow 2\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\int_{a_{12}^*} \frac{1}{f(z)} dz \overset{math}{=} \int_{-2}^{2} \frac{1}{f(2+ri)} dr$$

$$\Rightarrow \int_{a_1^*} \frac{1}{f(z)} \overset{math}{=} 2\int_{-2}^{2} \frac{1}{f(2+ri)} dr = 1.04083 \times 10^{-17} + 0.119738i$$

2. $\int_{a_2^*} \frac{1}{f(z)}$ and $a_2^* = a_{21}^* \cup a_{22}^* \cup a_{23}^* \cup a_{24}^* \cup a_{25}^* \cup a_{26}^*$

$$\begin{cases} a_{21}^* = \textit{the path vertical cut from } 1+3i \rightarrow 1+2i \textit{ on } (+) \textit{ edge in sheet} - \mathrm{I} \\ a_{22}^* = \textit{the path vertical cut from } 1+2i \rightarrow 1+3i \textit{ in } (-) \textit{ edge of sheet} - \mathrm{I} \\ a_{23}^* = \textit{the path vertical cut from } 1+2i \rightarrow 1-2i \textit{ in } (+) \textit{ edge of sheet} - \mathrm{I} \\ a_{24}^* = \textit{the path vertical cut from } 1-2i \rightarrow 1+2i \textit{ in } (-) \textit{ edge of sheet} - \mathrm{I} \\ a_{25}^* = \textit{the path vertical cut from } 1-2i \rightarrow 1-3i \textit{ in } (+) \textit{ edge of sheet} - \mathrm{I} \\ a_{26}^* = \textit{the path vertical cut from } 1-3i \rightarrow 1-2i \textit{ in } (-) \textit{ edge of sheet} - \mathrm{I} \end{cases}$$

(1) $z \in a_{21}^*$

Let $z = 1 + ri$, $r = 3 \rightarrow 2$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(2) $z \in a_{22}^*$

Let $z = 1 + ri$, $r = 3 \leftarrow 2$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(3) $z \in a_{23}^{*}$

$Let\ z = 1 + ri\ ,\ r = 2 \to -2\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(4) $z \in a_{24}^{*}$

$Let\ z = 1 + ri\ ,\ r = 2 \leftarrow -2\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(5) $z \in a_{25}^{*}$

$Let\ z = 1 + ri\ ,\ r = -2 \to -3\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(6) $z \in a_{26}^{*}$

$Let\ z = 1 + ri\ ,\ r = -2 \leftarrow -3\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

By (1)(2)(3)(4)(5)(6)

$$\int_{a_2^*} \frac{1}{f(z)} \overset{math}{=} 2\int_2^3 \frac{i}{f(1-ri)}dr - 2\int_{-2}^{2} \frac{i}{f(1-ri)}dr + 2\int_{-3}^{-2} \frac{i}{f(1-ri)}dr = 0 - 0.103156i$$

3. $\int_{a_3^*} \frac{1}{f(z)}$ and $a_3^* = a_{31}^* \cup a_{32}^*$

$$\begin{cases} a_{31}^* = \text{ the path vertical cut from } i \rightarrow -i \text{ on } (+) \text{ edge in sheet} - \mathrm{I} \\ a_{32}^* = \text{ the path vertical cut from } -i \rightarrow i \text{ on } (-) \text{ edge in sheet} - \mathrm{I} \end{cases}$$

(1) $z \in a_{31}^{*}$

$Let\ z = ri\ ,\ r = 1 \rightarrow -1\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} -\sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(2) $z \in a_{32}^{*}$

$Let\ z = ri\ ,\ r = 1 \leftarrow -1\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_3^*} \frac{1}{f(z)} \overset{math}{=} 2\int_{-1}^{1} \frac{1}{f(ri)} dr = -1.73472 \times 10^{-18} + 0.227188i$$

Next we evaluate the b-cycle .

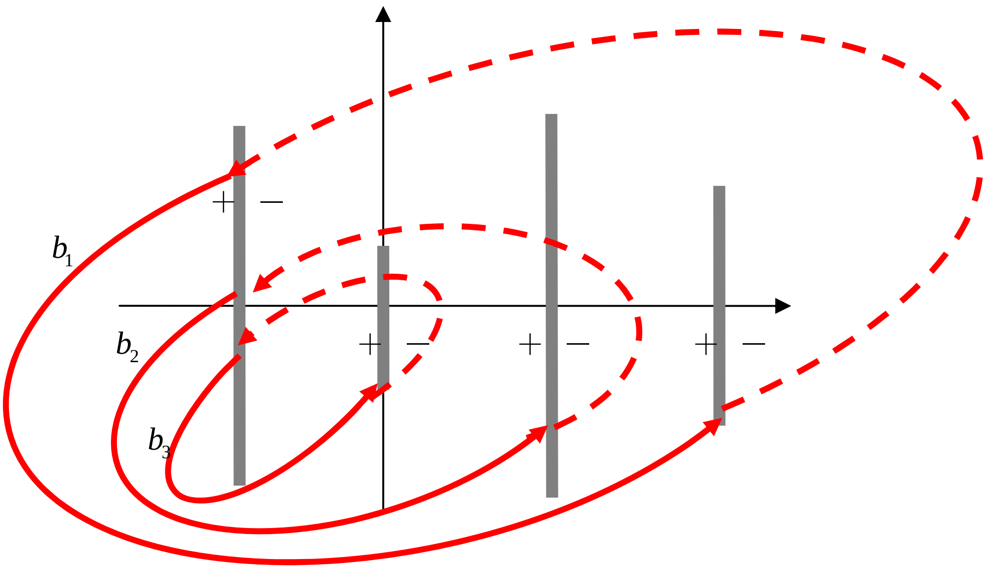


Figure 49: the b-cycles

Evaluate $\int_{b_3^*} \frac{1}{f(z)}$

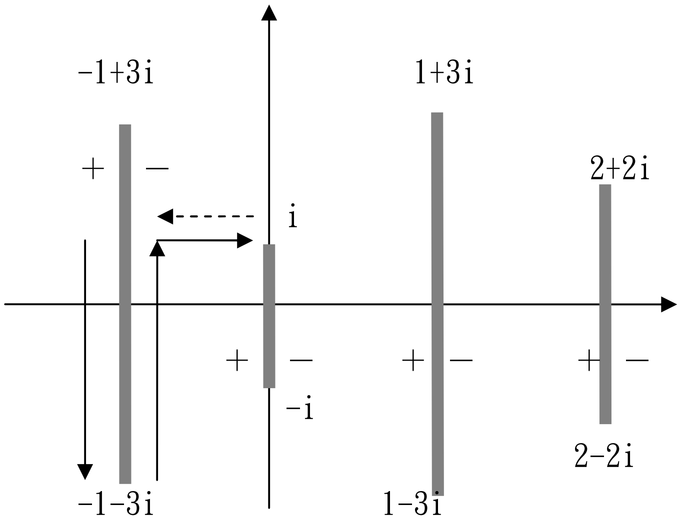


Figure 50: the path of b_3^*

$$\begin{cases} b_{31}^* = \text{the path horizontal line from } i \to -1+i \text{ on sheet} - \text{II} \\ b_{32}^* = \text{the path horizontal line from } -1+i \to i \text{ on sheet} - \text{I} \\ b_{33}^* = \text{the path vertical cut from } -1+i \to -1-i \text{ in } (+) \text{ edge of sheet} - \text{I} \\ b_{34}^* = \text{the path vertical cut from } -1-i \to -1+i \text{ in } (-) \text{ edge of sheet} - \text{I} \\ b_{35}^* = \text{the path vertical cut from } -1-i \to -1-2i \text{ in } (+) \text{ edge of sheet} - \text{I} \\ b_{36}^* = \text{the path vertical cut from } -1-2i \to -1-i \text{ in } (-) \text{ edge of sheet} - \text{I} \\ b_{37}^* = \text{the path vertical cut from } -1-2i \to -1-3i \text{ in } (+) \text{ edge of sheet} - \text{I} \\ b_{38}^* = \text{the path vertical cut from } -1-3i \to -1-2i \text{ in } (-) \text{ edge of sheet} - \text{I} \end{cases}$$

(1) $z \in b_{31}^*$ $\quad Let\ z = r + i\ ,\ r = -1 \to 0\ \ dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(2) $z \in b_{32}^{*}(-1+\mathrm{i} \leftarrow\!-- \ \mathrm{i}\ )$ $Let\ z = r + i\ ,\ r = -1 \leftarrow 0\ \ dz = dr$

Since $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$, we consider $-1+i \leftarrow i$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{-1+i<--i} \frac{1}{f(z)} = -\int_{-1+i\leftarrow i} \frac{1}{f(z)} = -\int_{0}^{-1} \frac{1}{f(r+1)} dr$$

(3) $z \in b_{33}^{*}$ $Let\ z = -1 + ri\ ,\ r = 1 \rightarrow -1\ \ dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} -\sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} -\sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(4) $z \in b_{34}^{*}$ $Let\ z = -1 + ri\ ,\ r = 1 \leftarrow -1\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} -\sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

（5） $z \in b_{35}^{*}$ $\quad Let\ z=-1+ri$ ， $r=-1 \to -2 \quad dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} -\sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

（6） $z \in b_{36}^{*}$ $\quad Let\ z=-1+ri$ ， $r=-1 \leftarrow -2 \quad dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

（7） $z \in b_{37}^{*}$ $\quad Let\ z=-1+ri$ ， $r=-2 \to -3 \quad dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} -\sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

（8） $z \in b_{38}^{*}$ $\quad Let\ z=-1+ri$ ， $r=-2 \leftarrow -3 \quad dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

By (1)(2)(3)(4)(5)(6)(7)(8),

$$\int_{b_3^*} \frac{1}{f(z)} dz \overset{math}{=} 2\int_0^2 \frac{i}{f(r+i)} dr - 2\int_{-1}^{1} \frac{i}{f(-1+ri)} dr + 2\int_{-2}^{-1} \frac{i}{f(-1+ri)} dr - 2\int_{-3}^{-2} \frac{i}{f(-1+ri)} dr$$

$$= 0.0944734 + 0.0000942992i$$

Evaluate $\int_{b_2^*} \frac{1}{f(z)}$

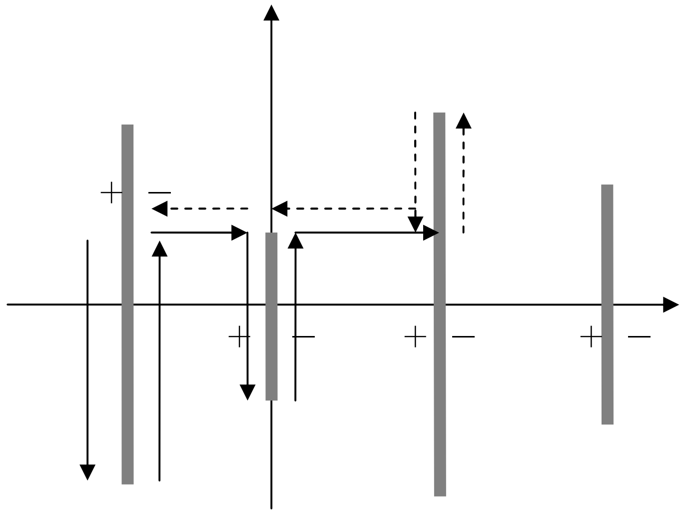


Figure 51: the path of b_2^*

$$\begin{cases} b_{21}^* = \text{the path horizontal line from } i \rightarrow 1+i \text{ on sheet} - \mathrm{I} \\ b_{22}^* = \text{the path horizontal line from } 1+i \rightarrow i \text{ on sheet} - \mathrm{II} \\ b_{23}^* = \text{the path vertical cut from } 1+3i \rightarrow 1+2i \text{ in } (+) \text{ edge of sheet} - \mathrm{II} \\ b_{24}^* = \text{the path vertical cut from } 1+2i \rightarrow 1+i \text{ in } (+) \text{ edge of sheet} - \mathrm{II} \\ b_{25}^* = \text{the path vertical cut from } 1+2i \rightarrow 1+3i \text{ in } (-) \text{ edge of sheet} - \mathrm{II} \\ b_{26}^* = \text{the path vertical cut from } 1+i \rightarrow 1+2i \text{ in } (-) \text{ edge of sheet} - \mathrm{II} \end{cases}$$

(1) $z \in b_{21}^*$ Let $z = r + i$, $r = 0 \rightarrow 1$ $dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(2) $z \in b_{22}^{*}$ (i ←--1+i)

Since $f(z)|_{\mathrm{II}} = -f(z)|_{\mathrm{I}}$, we consider $i \leftarrow 1+i$

$Let\ z = r+i\ ,\ r = 1 \to 0, dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{i\ <--\ 1+i} \frac{1}{f(z)} = -\int_{i \leftarrow 1+i} \frac{1}{f(z)} = -\int_{1}^{0} \frac{1}{f(r+i)} dr$$

(3)

Because $\begin{cases} (+) edge\ of\ \ sheet\text{-}\mathrm{II} \equiv (-) edge\ of\ \ sheet\text{-}\mathrm{I} \\ (-) edge\ of\ \ sheet\text{-}\mathrm{II} \equiv (+) edge\ of\ \ sheet\text{-}\mathrm{I} \end{cases}$

$$\Rightarrow \begin{cases} b_{23}^{*} \equiv vertical\ cut\ from\ 1+3i \to 1+2i\ \ in\ (-)\ edge\ of\ \ sheet\text{-}\mathrm{I} \\ b_{24}^{*} \equiv vertical\ cut\ from\ 1+2i \to 1+i\ \ in\ (-)\ edge\ of\ \ sheet\text{-}\mathrm{I} \\ b_{25}^{*} \equiv vertical\ cut\ from\ 1+2i \to 1+3i\ in\ (+)\ edge\ of\ \ sheet\text{-}\mathrm{I} \\ b_{26}^{*} \equiv vertical\ cut\ from\ 1+i \to 1+2i\ in\ (+)\ edge\ of\ \ sheet\text{-}\mathrm{I} \end{cases}$$

Next ,we consider all in sheet- I .

$z \in b_{23}^{*}\quad Let\ z = 1+ri\ ,\ r = 3 \to 2\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(4) $z \in b_{24}^{*}$ $\quad Let\ z = 1 + ri$, $r = 2 \to 1$ $\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(5) $z \in b_{25}^{*}$ $\quad Let\ z = 1 + ri$, $r = 2 \to 3$ $\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(6) $z \in b_{26}^{*}$ $\quad Let\ z = 1 + ri$, $r = 1 \to 2$ $\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

By (1)⋯(6)

$$\int_{b_2^*} \frac{1}{f(z)} dz \overset{math}{=} 2\int_0^2 \frac{i}{f(r+i)} dr - 2\int_{-1}^{1} \frac{i}{f(-1+ri)} dr + 2\int_{-2}^{-1} \frac{i}{f(-1+ri)} dr - 2\int_{-3}^{-2} \frac{i}{f(-1+ri)} dr$$
$$+2\int_{-1}^{1} \frac{i}{f(ri)} dr + 2\int_0^1 \frac{i}{f(r+i)} dr - 2\int_2^3 \frac{i}{f(r+i)} dr + 2\int_1^2 \frac{i}{f(r+i)} dr$$

3. next evaluate $\int_{b_1^*} \frac{1}{f(z)}$ *and* $b_{1k}^* - cycle$

$$b_1^* = b_2^* \cup b_{11}^* \cup \ldots \cup b_{18}^*$$

$$\begin{cases} b_{11}^* &= \text{the path on a vertical cut from } 1+i \to 1-2i \text{ with } (+)\text{edge of sheet - I} \\ b_{12}^* &= \text{the path on a vertical cut from } 1-2i \to 1+i \text{ with } (-)\text{edge of sheet - I} \\ b_{13}^* &= \text{the path on a vertical cut from } 1-2i \to 1-3i \text{ with } (+)\text{edge of sheet - I} \\ b_{14}^* &= \text{the path on a vertical cut from } 1-3i \to 1-2i \text{ with } (-)\text{edge of sheet - I} \\ b_{15}^* &= \text{the path on a horizontal line from } 1+i \to 2+i \text{ on sheet - I} \\ b_{16}^* &= \text{the path on a horizontal line from } 2+i \to 1+i \text{ on sheet - II} \\ b_{17}^* &= \text{the path on a vertical cut from } 2+2i \to 2+i \text{ with } (+)\text{edge of sheet - II} \\ b_{18}^* &= \text{the path on a vertical cut from } 2+i \to 2+2i \text{ with } (-)\text{edge of sheet - II} \end{cases}$$

(1) $z \in b_{11}^*$ *Let* $z = 1 + ri$, $r = 1 \to -2$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(2) $z \in b_{12}^*$ *Let* $z = 1 + ri$, $r = 1 \leftarrow -2$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(3) $z \in b_{13}^{*}$ $Let\ z = 1 + ri$, $r = -2 \to -3$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(4) $z \in b_{14}^{*}$ $Let\ z = 1 + ri$, $r = -3 \to -2$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(5) $z \in b_{15}^{*}$ $Let\ z = r + i$, $r = 1 \to 2$ $dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(6) $z \in b_{16}^{*}$ since $f(z)|_{\mathrm{II}} = -f(z)|_{\mathrm{I}}$

So we consider $2+i \to 1+i \in \mathrm{I}$

$Let\ z = r + i\ ,\ r = 2 \to 1\ \ dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z)|_{\mathrm{I}} \overset{math}{=} -f(z)$$

So that $f(z)|_{b_{16}^{*}} = -f(z)|_{\mathrm{I}} \overset{math}{=} f(z)$

(7)

Because $\begin{cases} (+)edge\ of\ sheet\text{-}\mathrm{II} \equiv (-)edge\ of\ sheet\text{-}\mathrm{I} \\ (-)edge\ of\ sheet\text{-}\mathrm{II} \equiv (+)edge\ of\ sheet\text{-}\mathrm{I} \end{cases}$

$$\Rightarrow \begin{cases} b_{17}^{*} \equiv vertical\ cut\ from\ 2+2i \to 2+i\ \ in\ (-)\ edge\ of\ sheet\text{-}\mathrm{I} \\ b_{18}^{*} \equiv vertical\ cut\ from\ 2+i \to 2+2i\ \ in\ (+)\ edge\ of\ sheet\text{-}\mathrm{I} \end{cases}$$

Next ,we consider all in sheet-Ⅰ.

$z \in b_{17}^{*}\ \ Let\ z = 2 + ri\ ,\ r = 2 \to 1\ \ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

(8) $z \in b_{18}^{*}$ $Let\ z = 2 + ri$, $r = 2 \leftarrow 1$ $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

Now we discuss the general situation.

Compute $\int \frac{1}{f(z)}$ *over* a^*, b^* *for vertical cut where* $f(z) = \sqrt{\prod_{k=1}^{m}(z - z_k)}$ and $z_k = a_k i \ \ a_k \in R$.

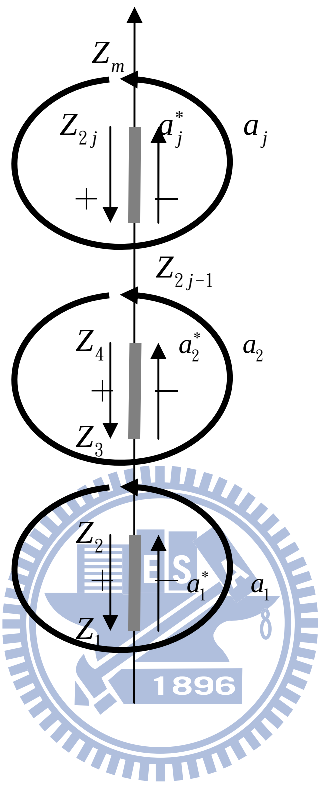


Figure 52: a-cycle and their equivalent path a^*

1. a-cycle: The equivalent path

a_j =the path on a vertical cut $z_{2j-1} \rightarrow z_{2j}$ with (+)edge of sheet-I

and $z_{2j-1} \rightarrow z_{2j}$ with (−)edge of sheet-I.

Now we use the Lemma to computation .

(i) $z \in Z_{2j} \xrightarrow{+} Z_{2j-1}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \ \ where\ \ k = 1, 2, \ldots 2j\text{-}1 \\ \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \ \ where\ \ k = 2j, \ldots, m \end{cases}$$

$$\Rightarrow\ f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$$

(ii) $z \in Z_{2j} \underleftarrow{\;-\;} Z_{2j-1}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad where\ k = 1, 2, \ldots .2j\text{-}1 \\ \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad where\ k = 2j, \ldots ., m \end{cases}$$

$$\Rightarrow\ f(z) \overset{math}{=} f(z)$$

By (i)(ii) and Cauchy

$$\int_{a_j} \frac{1}{f(z)} = \int_{a_j^*} \frac{1}{f(z)} \overset{math}{=} 2\int_{\mathrm{Im}(z_{2j-1})}^{\mathrm{Im}(z_{2j})} \prod_{k=1}^{m} \frac{i}{\sqrt{z - z_k}}\, dr$$

2. b-cycle

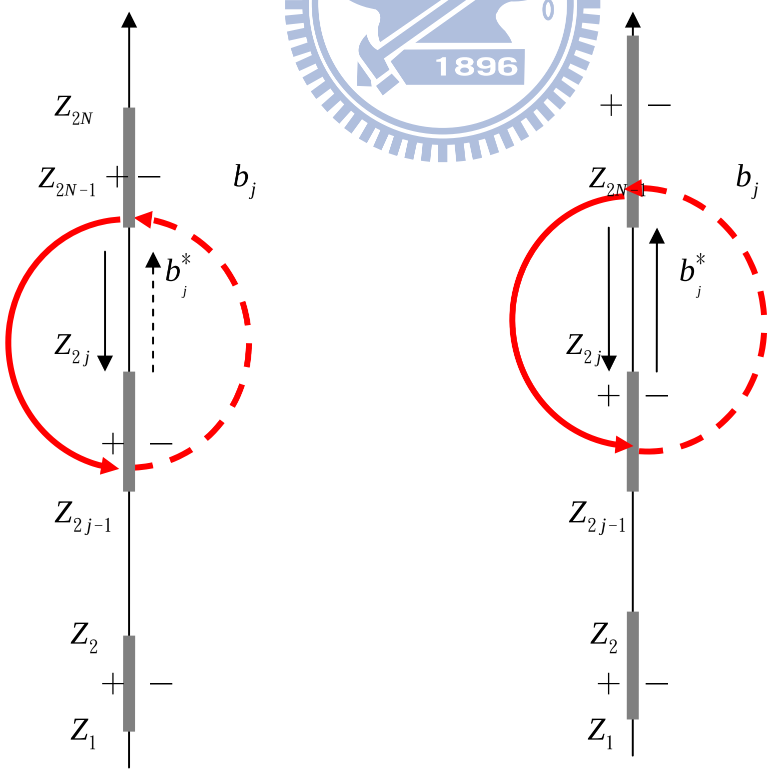


Figure 53: b_j and b_j^* of 2N and 2N points

Now we discuss the two part .

(1)The path on the cut.

$$\begin{cases} z_{2s+2} \rightarrow z_{2s+1} \ , \ s = j, j+1, \ldots, N-2 \ \ on \ the \ (+) \ edge \ in \ sheet - \text{I} \\ z_{2s+2} \leftarrow z_{2s+1} \ , \ s = j, j+1, \ldots, N-2 \ \ on \ the \ (-) \ edge \ in \ sheet - \text{II} \end{cases}$$

(i) $z \in z_{2s+2} \xrightarrow{+} z_{2s+1}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \ \ where \ k = 2s+2, \ldots, m \\ \arg(z - z_k) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \ \ where \ k = 1, 2, \ldots, 2s+1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2s+1} f(z) = -f(z)$$

(ii) $z \in z_{2s+2} \xleftarrow{-} z_{2s+1}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \ \ where \ k = 2s+2, \ldots, m \\ \arg(z - z_k) = \dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \ \ where \ k = 1, 2, \ldots, 2s+1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(2)The path on no cuts.

$$\begin{cases} z_{2s+1} \rightarrow z_{2s} \ , \ s = j, j+1, \ldots, N-2 \ \ in \ sheet - \text{I} \\ z_{2s+1} \leftarrow z_{2s} \ , \ s = j, j+1, \ldots, N-2 \ \ in \ sheet - \text{II} \end{cases}$$

(i) $z \in z_{2s+1} \rightarrow z_{2s}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \ \ where \ k = 2s+1, \ldots, m \\ \arg(z - z_k) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \ \ where \ k = 1, 2, \ldots, 2s \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2s} f(z) = f(z)$$

(ii) $z \in z_{2s+1} \leftarrow\!\!-\!\!-\!\!- z_{2s}$ since $f(z)|_{II} = -f(z)|_{I}$,

so we consider $z \in z_{2s+1} \leftarrow z_{2s}$

$$\begin{cases} \arg(z - z_k) = -\dfrac{\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \;\; where \;\; k = 2s+1, \ldots, m \\ \arg(z - z_k) = -\dfrac{3\pi}{2} \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \;\; where \;\; k = 1, 2, \ldots, 2s \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} -(-1)^{2s} f(z) = -f(z)$$

By (i)(ii) we have

$$\int_{b_j} \frac{1}{f(z)} = \int_{b_j^*} \frac{1}{f(z)} \overset{math}{=} \sum_{s=j}^{N-1} (2\int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz)$$

4 The integrals of $\frac{1}{f(z)}$ over a, b cycle for slant cut

Defnition : The cut with α means the slope of the straight line $m=\tan\alpha$, where $0<\alpha\le\pi$.

1. For the first example we take $\alpha=\frac{\pi}{4}$.

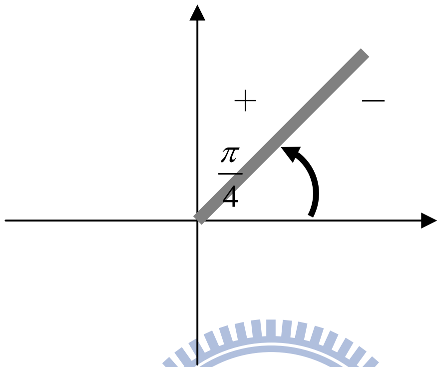


Figure 54: $\alpha=\frac{\pi}{4}$

The same to the horizontal and vertical cut, we define the sheet-Ⅰ and sheet-Ⅱ first .

$$z=re^{i\theta}\begin{cases}\theta\in\left[-\frac{7\pi}{4},\frac{\pi}{4}\right) & z\in\mathrm{I}\\ \theta\in\left[\frac{\pi}{4},\frac{9\pi}{4}\right) & z\in\mathrm{II}\end{cases}$$

$$\Rightarrow z=|z|e^{i\theta}\begin{cases}\theta\in\left[-\frac{7\pi}{4},\frac{\pi}{4}\right)\\ \theta\in\left[\frac{\pi}{4},\frac{9\pi}{4}\right)\end{cases}\Rightarrow\sqrt{z}=|z|^{\frac{1}{2}}e^{i\frac{\theta}{2}}\begin{cases}\frac{\theta}{2}\in\left[-\frac{7\pi}{8},\frac{\pi}{8}\right)\\ \frac{\theta}{2}\in\left[\frac{\pi}{8},\frac{9\pi}{8}\right)\end{cases}$$

Now we to do the same work , discuss the dfference between the value in theory and in Mathematica.

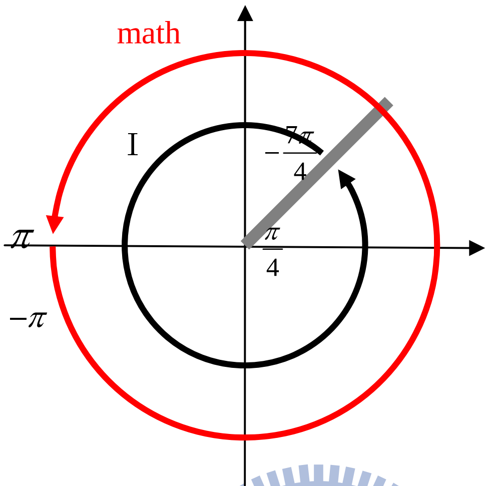


Figure 55:The value in sheet-I

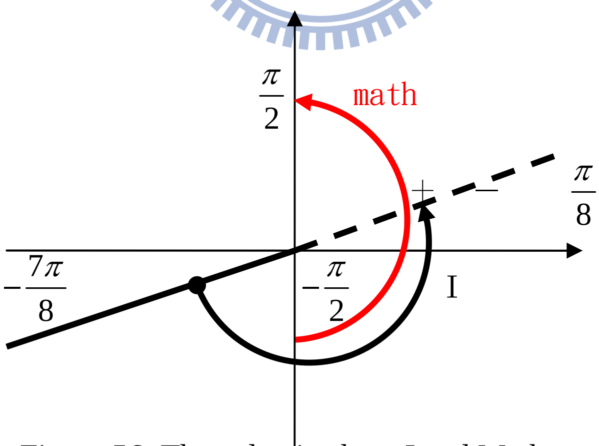


Figure 56: The value in sheet-I and Mathematica

Since there have some problem in $\theta \in \left[-\frac{7\pi}{4}, -\pi\right]$

$$\Rightarrow z = \begin{cases} |z|e^{i\theta} & \theta \in \left[-\frac{7\pi}{4}, \frac{\pi}{4}\right) & \theta \in \mathrm{I} \\ |z|e^{i(\theta+2\pi)} & \theta \in \left[\frac{\pi}{4}, \frac{9\pi}{4}\right) & \theta \in mathematic \end{cases}$$

$$\Rightarrow \sqrt{z} = \begin{cases} |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \frac{\theta}{2} \in \left[-\frac{7\pi}{8}, \frac{\pi}{8}\right) \quad \theta \in \mathrm{I} \\ |z|^{\frac{1}{2}} e^{i\frac{(\theta+2\pi)}{2}} = -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \theta \in mathematic \end{cases}$$

So that $\sqrt{z - z_k} \overset{math}{=} \begin{cases} \sqrt{z - z_k} & \arg(z - z_k) \in \left(-\pi, \frac{\pi}{4}\right) \\ -\sqrt{z - z_k} & \arg(z - z_k) \in \left[-\frac{7\pi}{4}, -\pi\right] \end{cases}$

2. second we take the $\alpha = \frac{2\pi}{3}$.

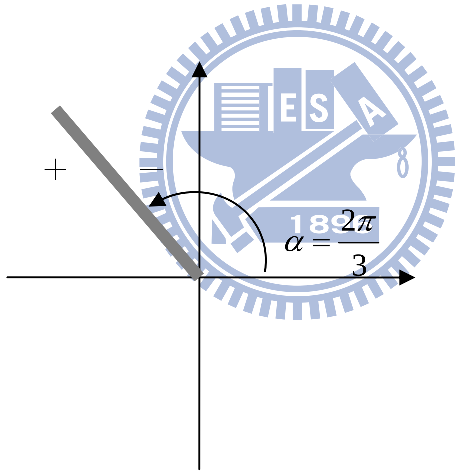


Figure 57: the cut of $\alpha = \frac{2\pi}{3}$

Define that

$$z = re^{i\theta} \begin{cases} \theta \in \left[-\frac{4\pi}{3}, \frac{2\pi}{3}\right) & z \in \mathrm{I} \\ \theta \in \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right) & z \in \mathrm{II} \end{cases}$$

$$\Rightarrow z = |z| e^{i\theta} \begin{cases} \theta \in \left[-\dfrac{4\pi}{3}, \dfrac{2\pi}{3} \right) \\ \theta \in \left[\dfrac{2\pi}{3}, \dfrac{8\pi}{3} \right) \end{cases} \quad \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} \begin{cases} \dfrac{\theta}{2} \in \left[-\dfrac{2\pi}{3}, \dfrac{\pi}{3} \right) \\ \dfrac{\theta}{2} \in \left[\dfrac{\pi}{3}, \dfrac{4\pi}{3} \right) \end{cases}$$

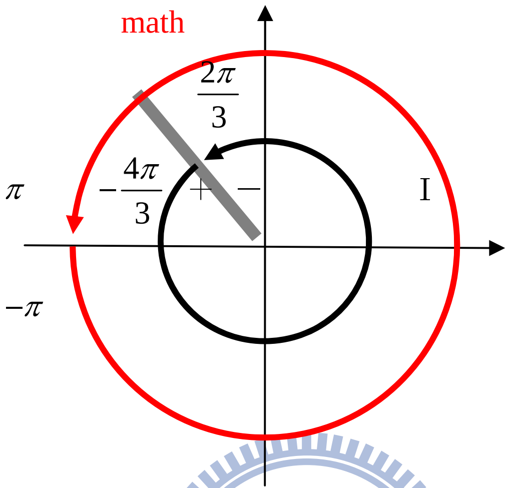


Figure 58:The value in sheet-I and mathematica

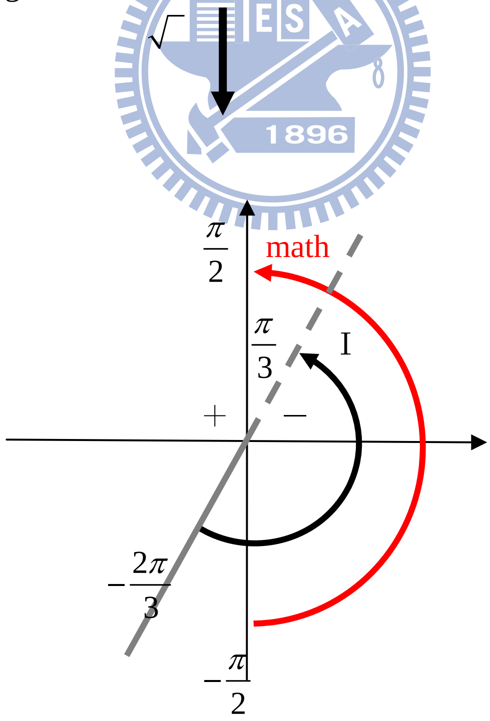


Figure 59:The value in sheet-I and mathematica

Since there have some problem in $\theta \in \left[-\frac{4\pi}{3}, -\pi\right]$

$$\Rightarrow z = \begin{cases} |z| e^{i\theta} & \theta \in \left[-\frac{4\pi}{3}, \frac{2\pi}{3}\right) \quad \theta \in \mathrm{I} \\ |z| e^{i(\theta+2\pi)} & \theta \in \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right) \quad \theta \in mathematic \end{cases}$$

$$\Rightarrow \sqrt{z} = \begin{cases} |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \frac{\theta}{2} \in \left[-\frac{2\pi}{3}, \frac{\pi}{3}\right) \quad \theta \in \mathrm{I} \\ |z|^{\frac{1}{2}} e^{i\frac{(\theta+2\pi)}{2}} = -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \theta \in \left[-\frac{4\pi}{3}, \frac{2\pi}{3}\right) \quad \theta \in mathematic \end{cases}$$

So that $\sqrt{z - z_k} \overset{math}{=} \begin{cases} \sqrt{z - z_k} & \arg(z - z_k) \in \left(-\pi, \frac{2\pi}{3}\right) \\ -\sqrt{z - z_k} & \arg(z - z_k) \in \left[-\frac{4\pi}{3}, -\pi\right] \end{cases}$

So that we give a conclusion .

Definition : The cut slope line $m = \tan\alpha$ *where* $0 < \alpha \le \pi$

$$\text{Define that } z = re^{i\theta} \begin{cases} \theta \in [\alpha - 2\pi, \alpha) & \theta \in \text{I} \\ \theta \in [\alpha, \alpha + 2\pi) & \theta \in \text{II} \end{cases}$$

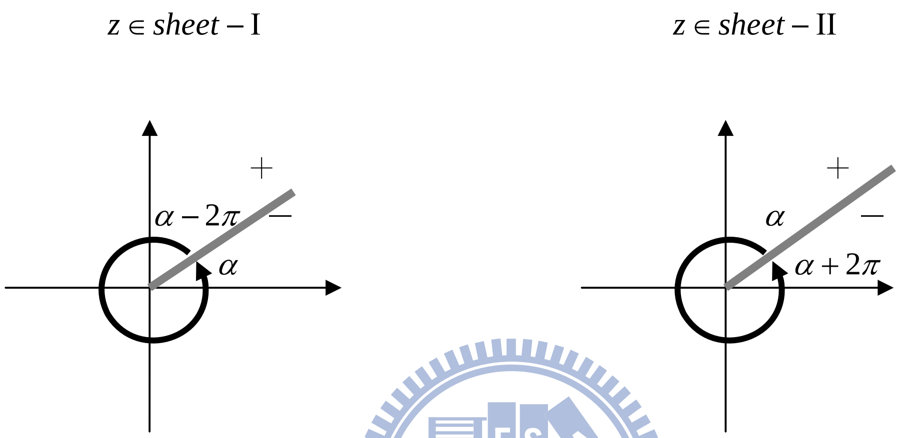


Figure 60: The I and II sheet

$$\Rightarrow z = |z| e^{i\theta} \begin{cases} \theta \in [\alpha - 2\pi,\ \alpha) \\ \theta \in [\alpha\ , \alpha + 2\pi) \end{cases} \Rightarrow \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} \begin{cases} \dfrac{\theta}{2} \in \left[\dfrac{\alpha - 2\pi}{2},\ \dfrac{\alpha}{2}\right) \\ \dfrac{\theta}{2} \in \left[\dfrac{\alpha}{2}\ , \dfrac{\alpha + 2\pi}{2}\right) \end{cases}$$

$$\text{So that } \sqrt{z - z_k} \overset{math}{=} \begin{cases} \sqrt{z - z_k} & \arg(z - z_k) \in (-\pi, \alpha) \\ -\sqrt{z - z_k} & \arg(z - z_k) \in [\alpha - 2\pi, -\pi] \end{cases}$$

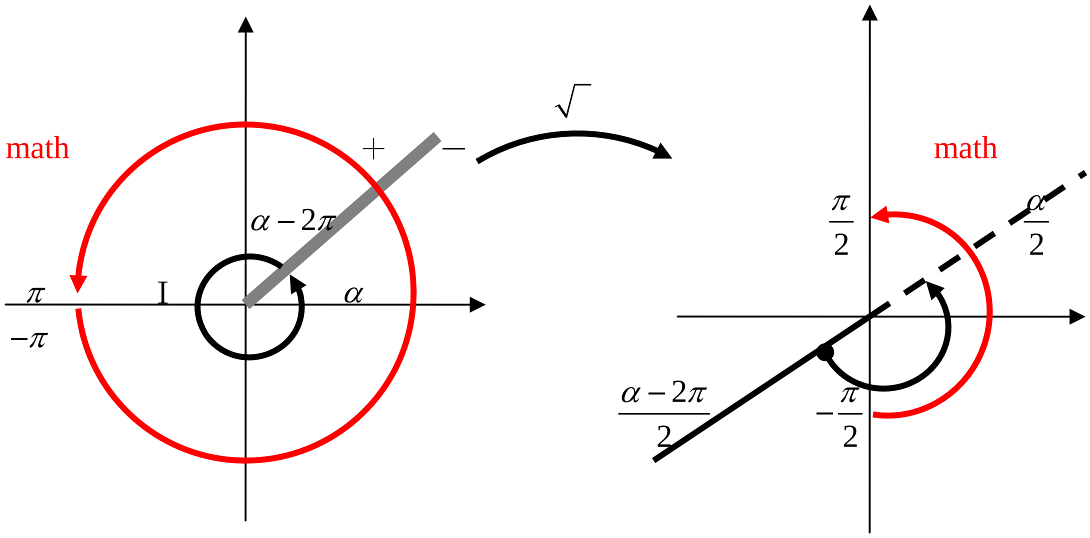


Figure 61:Example of $f(z) = \sqrt{z}$

Give an example

$\int \frac{1}{f(z)}$ *and* a, b *cycle*

Let

$z_1 = 3 - i,\ z_2 = 4 + (\sqrt{3} - 1)i,\ z_3 = 2 + i, z_4 = (2 + \sqrt{3} + 2i),\ z_5 = 0,\ z_6 = 2 + 2i, z_7 = -i$

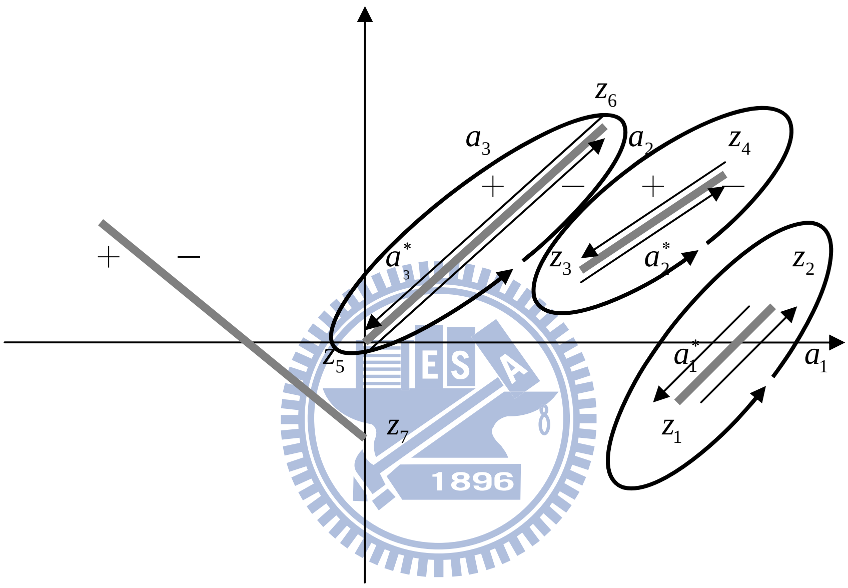


Figure 62: a-cycle and the equivalent path

1. $a_1^* = a_{11}^* \cup a_{12}^*$

$$\begin{cases} a_{11}^* = 4 + (\sqrt{3} - 1)i \rightarrow 3 - i \text{ on } (+) \text{edge of sheet} - \text{I} \\ a_{12}^* = 4 + (\sqrt{3} - 1)i \leftarrow 3 - i \text{ on } (-) \text{edge of sheet} - \text{I} \end{cases}$$

(1) $z \in a_{11}^*$: Let $z = 3 - i + r(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$, $r = 2 \rightarrow 0$, $dz = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)dr$

$$\begin{cases} \arg(z-z_1) = -\dfrac{5\pi}{3} \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_2) = -\dfrac{2\pi}{3} \Rightarrow \sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_2} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=3,4,5,6,7 \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$\int_{a_{11}^*} \frac{1}{f(z)} \overset{math}{=} -\int_2^0 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr$$

(2) $z \in a_{12}^*$: $Let\ z = 3-i+r(\frac{1}{2}+\frac{\sqrt{3}}{2}i)\ , r=2 \leftarrow 0\ , dz = (\frac{1}{2}+\frac{\sqrt{3}}{2}i)dr$

$$\begin{cases} \arg(z-z_1) = \dfrac{\pi}{3} \Rightarrow \sqrt{z-z_1} \overset{math}{=} \sqrt{z-z_1} \\ \arg(z-z_2) = -\dfrac{2\pi}{3} \Rightarrow \sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_2} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=3,4,5,6,7 \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\int_{a_{12}^*} \frac{1}{f(z)} \overset{math}{=} \int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr$$

By (1)(2)

$$\Rightarrow \int_{a_1^*} \frac{1}{f(z)} \overset{math}{=} 2\int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr = -0.226932 + 0.0125601i$$

2. $a_2^* = a_{21}^* \bigcup a_{22}^*$

$$\begin{cases} a_{21}^* &= (2+\sqrt{3})+2i \;\rightarrow\; 2+i \; on \; (+)edge \; of \;\; sheet-\mathrm{I} \\ a_{22}^* &= (2+\sqrt{3})+2i \;\leftarrow\; 2+i \; on \; (-)edge \; of \;\; sheet-\mathrm{I} \end{cases}$$

(1) $z \in a_{21}^*$: $Let\ z = (2+i)+(\frac{\sqrt{3}}{2}+\frac{1}{2})r \ ,r = 2 \rightarrow 0 \ ,dz = (\frac{\sqrt{3}}{2}+\frac{1}{2}i)dr$

$$\left.\begin{cases} \arg(z-z_k) \in \left[-\frac{5\pi}{3},-\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1}, k=1,2 \\ \arg(z-z_3) = -\frac{11\pi}{6} \Rightarrow \sqrt{z-z_3} \overset{math}{=} -\sqrt{z-z_3} \\ \arg(z-z_4) = -\frac{5\pi}{6} \Rightarrow \sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_4} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=5,6,7 \end{cases}\right. \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$\int_{a_{21}^*} \frac{1}{f(z)} \overset{math}{=} -\int_2^0 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+(\frac{\sqrt{3}}{2}+\frac{1}{2}i)r)} dr$$

(2) $z \in a_{22}^*$: $Let\ z = (2+i)+(\frac{\sqrt{3}}{2}+\frac{1}{2})r \ ,r = 2 \leftarrow 0 \ ,dz = (\frac{\sqrt{3}}{2}+\frac{1}{2}i)dr$

$$\left.\begin{cases} \arg(z-z_k) \in \left[-\frac{5\pi}{3},-\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1}, k=1,2 \\ \arg(z-z_3) = \frac{\pi}{6} \Rightarrow \sqrt{z-z_3} \overset{math}{=} \sqrt{z-z_3} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=4,5,6,7 \end{cases}\right. \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\int_{a_{22}^*} \frac{1}{f(z)} \overset{math}{=} \int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+(\frac{\sqrt{3}}{2}+\frac{1}{2}i)r)} dr$$

$$\int_{a_2^*} \frac{1}{f(z)} \overset{math}{=} 2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+(\frac{\sqrt{3}}{2}+\frac{1}{2}i)r)} dr = -0.0584232+0.842766i$$

3. $a_3^* = a_{31}^* \cup a_{32}^* \cup .... \cup a_{36}^*$

$$\begin{cases} a_{31}^* = 2+2i \to 1+i \text{ on } (+)\text{edge of sheet - I} \\ a_{32}^* = 1+i \to (\sqrt{3}-1)+(\sqrt{3}-1)i \text{ on } (+)\text{edge of sheet - I} \\ a_{33}^* = (\sqrt{3}-1)+(\sqrt{3}-1)i \to 0 \text{ on } (+)\text{edge of sheet - I} \\ a_{34}^* = (\sqrt{3}-1)+(\sqrt{3}-1)i \leftarrow 0 \text{ on } (-)\text{edge of sheet - I} \\ a_{35}^* = 1+i \leftarrow (\sqrt{3}-1)+(\sqrt{3}-1)i \text{ on } (-)\text{edge of sheet - I} \\ a_{36}^* = 2+2i \leftarrow 1+i \text{ on } (-)\text{edge of sheet - I} \end{cases}$$

(1) $z \in a_{31}^*$ Let $z = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)r,\ r = 2\sqrt{2} \to 2 \quad dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)dr$

$$\begin{cases} \arg(z-z_k) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2 \\ \arg(z-z_4) \in \left[-\frac{11\pi}{6}, -\pi\right] \Rightarrow \sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_4} \\ \arg(z-z_5) = -\frac{7\pi}{4} \Rightarrow \sqrt{z-z_5} \overset{math}{=} -\sqrt{z-z_5} \\ \arg(z-z_6) = -\frac{3\pi}{4} \Rightarrow \sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_6} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=3,7 \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_{31}^*} \frac{1}{f(z)} \overset{math}{=} \int_{2\sqrt{2}}^{\sqrt{2}} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)r)} dr$$

$$(2)\quad z\in a_{32}^{*}\ \ Let\ z=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r,\ r=\sqrt{2}\rightarrow\sqrt{2}(\sqrt{3}-1)\quad dz=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$$

$$\left\{\begin{array}{l}\arg(z-z_k)\in\left[-\frac{5\pi}{3},-\pi\right]\Rightarrow\sqrt{z-z_k}\overset{math}{=}-\sqrt{z-z_k},k=1,2\\ \arg(z-z_5)=-\frac{7\pi}{4}\Rightarrow\sqrt{z-z_5}\overset{math}{=}-\sqrt{z-z_5}\\ \arg(z-z_6)=-\frac{3\pi}{4}\Rightarrow\sqrt{z-z_6}\overset{math}{=}\sqrt{z-z_6}\\ \arg(z-z_k)\in(-\pi,\pi)\Rightarrow\sqrt{z-z_k}\overset{math}{=}\sqrt{z-z_k},k=3,4,7\end{array}\right.\qquad\Rightarrow f(z)\overset{math}{=}-f(z)$$

$$\Rightarrow\int_{a_{32}^{*}}\frac{1}{f(z)}\overset{math}{=}-\int_{\sqrt{2}}^{\sqrt{2}(\sqrt{3}-1)}\frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

$$(3)\quad z\in a_{33}^{*}\ \ Let\ z=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r,\ r=\sqrt{2}(\sqrt{3}-1)\rightarrow 0\quad dz=(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$$

$$\left\{\begin{array}{l}\arg(z-z_1)\in\left[-\frac{5\pi}{3},-\pi\right]\Rightarrow\sqrt{z-z_1}\overset{math}{=}-\sqrt{z-z_1},\\ \arg(z-z_5)=-\frac{7\pi}{4}\Rightarrow\sqrt{z-z_5}\overset{math}{=}-\sqrt{z-z_5}\\ \arg(z-z_6)=-\frac{3\pi}{4}\Rightarrow\sqrt{z-z_6}\overset{math}{=}\sqrt{z-z_6}\\ \arg(z-z_k)\in(-\pi,\pi)\Rightarrow\sqrt{z-z_k}\overset{math}{=}\sqrt{z-z_k},k=2,3,4,7\end{array}\right.\qquad\Rightarrow f(z)\overset{math}{=}f(z)$$

$$\Rightarrow\int_{a_{33}^{*}}\frac{1}{f(z)}\overset{math}{=}\int_{\sqrt{2}(\sqrt{3}-1)}^{0}\frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(4) $z \in a_{34}^{*}$ $Let\ z = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) r,\ r = \sqrt{2}(\sqrt{3}-1) \leftarrow 0 \quad dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1}, \\ \arg(z-z_5) = \frac{\pi}{4} \Rightarrow \sqrt{z-z_5} \overset{math}{=} \sqrt{z-z_5} \\ \arg(z-z_6) = -\frac{3\pi}{4} \Rightarrow \sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_6} \\ \arg(z-z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 2,3,4,7 \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{a_{34}^{*}} \frac{1}{f(z)} \overset{math}{=} -\int_{0}^{\sqrt{2}(\sqrt{3}-1)} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) r)} dr$$

(5) $z \in a_{35}^{*}$ $Let\ z = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) r,\ r = \sqrt{2}(\sqrt{3}-1) \rightarrow \sqrt{2} \quad dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_2) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_2} \\ \arg(z-z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 3,4,5,6,7 \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_{35}^{*}} \frac{1}{f(z)} \overset{math}{=} \int_{\sqrt{2}(\sqrt{3}-1)}^{\sqrt{2}} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) r)} dr$$

(6) $z \in a_{35}^{*}$ $Let\ z = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) r,\ r = \sqrt{2} \rightarrow 2\sqrt{2} \quad dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i) dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_2) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_2} \\ \arg(z-z_3) \in \left[-\frac{11\pi}{6}, -\pi\right] \Rightarrow \sqrt{z-z_3} \overset{math}{=} -\sqrt{z-z_3} \\ \arg(z-z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 4,5,6,7 \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{a_{36}^*} \frac{1}{f(z)} \overset{math}{=} -\int_{\sqrt{2}}^{2\sqrt{2}} \frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)} dr$$

So we have

$$\int_{a_3^*} \frac{1}{f(z)} \overset{math}{=} -2\int_0^{\sqrt{2}(\sqrt{3}-1)} \frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)} dr + 2\int_{\sqrt{2}(\sqrt{3}-1)}^{\sqrt{2}} \frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)} dr - 2\int_{\sqrt{2}}^{2\sqrt{2}} \frac{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)} dr$$

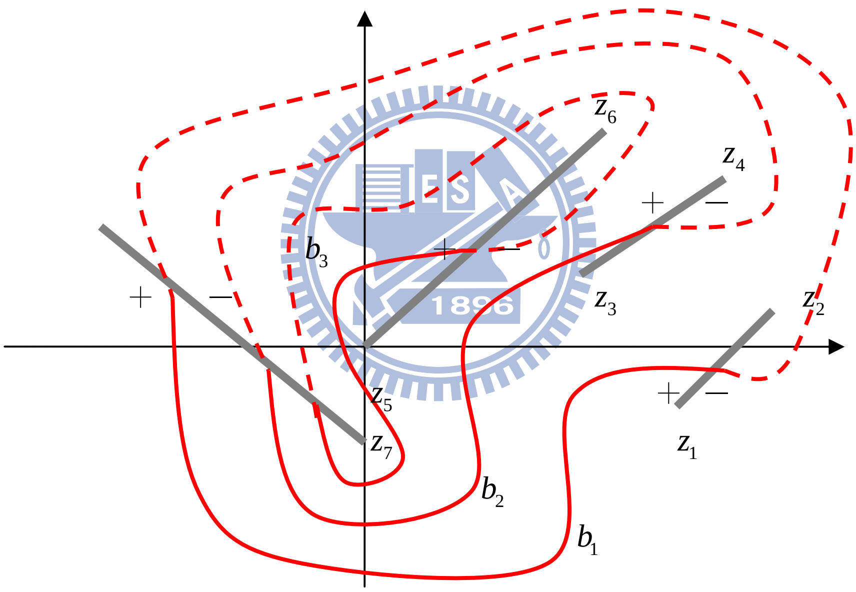


Figure 63: The path b-cycle

1. $b_3^* = b_{31}^* \cup b_{32}^* \cup .... \cup b_{36}^*$

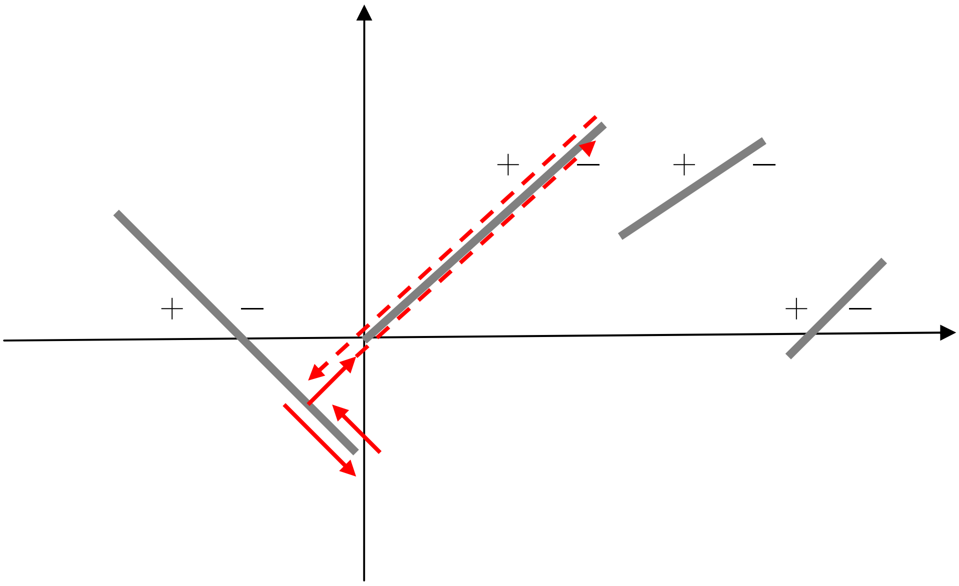


Figure 64 : the path of b_3^*

$$\begin{cases} b_{31}^* = \textit{slant cut from } (-\frac{1}{2}-\frac{1}{2}i) \to -i \textit{ on } (+)\textit{edge of sheet} - \text{I} \\ b_{32}^* = \textit{slant cut from } (-\frac{1}{2}-\frac{1}{2}i) \leftarrow -i \textit{ on } (-)\textit{edge of sheet} - \text{I} \\ b_{33}^* = \textit{slant line from } (-\frac{1}{2}-\frac{1}{2}i) \to 0 \textit{ on sheet} - \text{I} \\ b_{34}^* = \textit{slant cut from } 0 \to (2+2i) \textit{ on } (-)\textit{edge of sheet} - \text{II} \\ b_{35}^* = \textit{slant cut from } 0 \leftarrow (2+2i) \textit{ on } (+)\textit{edge of sheet} - \text{II} \\ b_{36}^* = \textit{slant line from } (-\frac{1}{2}-\frac{1}{2}i) \leftarrow 0 \textit{ on sheet} - \text{II} \end{cases}$$

(1) $z \in b_{31}^*$ Let $z = -i + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r$, $r = \frac{1}{\sqrt{2}} \to 0$, $dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_7) = -\frac{5\pi}{4} \Rightarrow \sqrt{z-z_7} \overset{math}{=} -\sqrt{z-z_7} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=2,3,4,5,6 \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{b_{31}^*} \frac{1}{f(z)} \overset{math}{=} \int_{\frac{1}{\sqrt{2}}}^{0} \frac{(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(2) $z \in b_{32}^*$ *Let* $z = -i + (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r$, $r = \frac{1}{\sqrt{2}} \leftarrow 0$, $dz = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_7) = \frac{3\pi}{4} \Rightarrow \sqrt{z-z_7} \overset{math}{=} \sqrt{z-z_7} \qquad\qquad \Rightarrow f(z) \overset{math}{=} -f(z) \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 2,3,4,5,6 \end{cases}$$

$$\Rightarrow \int_{b_{32}^*} \frac{1}{f(z)} \overset{math}{=} -\int_{0}^{\frac{1}{\sqrt{2}}} \frac{(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f(-i+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

(3) $z \in b_{33}^*$ *Let* $z = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r$, $r = -\frac{1}{\sqrt{2}} \to 0$, $dz = (\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)dr$

$$\begin{cases} \arg(z-z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_k) = -\frac{3\pi}{4} \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 5,6 \qquad \Rightarrow f(z) \overset{math}{=} -f(z) \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k = 2,3,4,7 \end{cases}$$

$$\Rightarrow \int_{b_{33}^*} \frac{1}{f(z)} \overset{math}{=} -\int_{-\frac{1}{\sqrt{2}}}^{0} \frac{(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{f((-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)r)}dr$$

$$(4)\begin{cases} b_{34}^* \equiv slant\ cut\ 0 \to 2+2i\ on\ (+)\ edge\ of\ sheet\text{-I} \\ b_{35}^* \equiv slant\ cut\ 2+2i \to 0\ on\ (-)\ edge\ of\ sheet\text{-I} \end{cases}$$

$$\int_{b_{34}^* \cup b_{35}^*} \frac{1}{f(z)} = -\int_{a_3^*} \frac{1}{f(z)}$$

(5) $z \in b_{36}^{*}$

Because $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

So we consider the path $0 \to (-\frac{1}{2} - \frac{1}{2}i)$ $on\ sheet - \mathrm{I}$

$$Let\ z = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r,\ r = 0 \to -\frac{1}{\sqrt{2}},\ dz = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)dr$$

$$\begin{cases} \arg(z - z_1) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z - z_1} \overset{math}{=} -\sqrt{z - z_1} \\ \arg(z - z_k) = -\frac{3\pi}{4} \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}, k = 5,6 \qquad \Rightarrow f(z) \overset{math}{=} -f(z) \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}, k = 2,3,4,7 \end{cases}$$

$$\Rightarrow \int_{b_{36}^{*}} \frac{1}{f(z)} \overset{math}{=} -\int_{-\frac{1}{\sqrt{2}}}^{0} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r)} dr = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r)} dr$$

By (1)(2)(3)(4)(5)

$$\int_{b_3^{*}} \frac{1}{f(z)} \overset{math}{=} 2\int_{\frac{1}{\sqrt{2}}}^{0} \frac{(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)}{f(-i + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r)} dr + \int_{0}^{-\frac{1}{\sqrt{2}}} \frac{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)}{f((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)r)} dr - \int_{a_3^{*}} \frac{1}{f(z)}$$

$2.\ b_2^* = a_{32}^* \cup b_{21}^* \cup b_{22}^* \cup b_{23}^* \cup b_{24}^* \cup b_{31}^* \cup b_{32}^* \cup b_{34}^* \cup b_{35}^* \cup b_{36}^*$

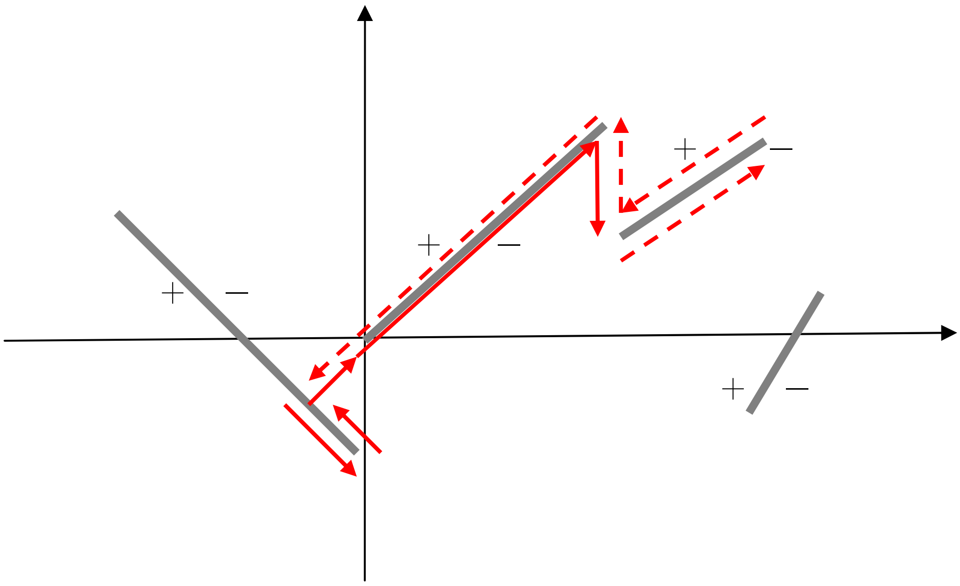


Figure 65 : The path of b_2^*

$$\begin{cases} b_{21}^* = vertical\ line\ \ 2+2i \to 2+i\ on\ sheet\text{-I} \\ b_{22}^* = slant\ cut\ \ 2+i \to (2+\sqrt{3})+2i\ on\ (-)edge\ of\ \ sheet\text{-II} \\ b_{23}^* = slant\ cut\ \ (2+\sqrt{3})+2i \to 2+i\ on\ (+)edge\ of\ \ sheet\text{-II} \\ b_{24}^* = vertical\ line\ 2+i \to 2+2i\ on\ sheet\text{-II} \end{cases}$$

(1) $z \in b_{21}^*$ Let $z = 2 + ri$, $r = 2 \to 1$ $dz = idr$

$$\begin{cases} \arg(z-z_k) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2 \\ \arg(z-z_k) \in \left[-\frac{11\pi}{6}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=3,4 \Rightarrow f(z) \overset{math}{=} f(z) \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=5,6,7 \end{cases}$$

$$\int_{b_{21}^*} \frac{1}{f(z)} = -\int_2^1 \frac{i}{f(2+ri)} dr$$

$$(2)\quad \begin{cases} b_{22}^{*} \equiv slant\ cut\ from\ 2+i \to (2\ +\sqrt{3})+2i\ on\ (+)edge\ of\ \ sheet-\mathrm{I} \\ b_{23}^{*} \equiv slant\ cut\ from(2\ +\sqrt{3})+2i \to 2+i\ on\ (-)edge\ of\ \ sheet-\mathrm{I} \end{cases}$$

$$\Rightarrow \int_{b_{22}^{*}\cup b_{23}^{*}} \frac{1}{f(z)} = -\int_{a_2^{*}} \frac{1}{f(z)}$$

(3) $z \in b_{24}^{*}$ since we know $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

We consider the path $2+i \to 2+2i\ on\ sheet-\mathrm{I}$

$Let\ z\ =\ 2\ +\ ri\ ,\ r=1\to 2\quad dz\ =\ idr$

$$\begin{cases} \arg(z-z_k) \in \left[-\frac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2 \\ \arg(z-z_k) \in \left[-\frac{11\pi}{6}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=3,4 \Rightarrow f(z) \overset{math}{=} f(z) \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}, k=5,6,7 \end{cases}$$

$$\int_{b_{24}^{*}} \frac{1}{f(z)} = \int_1^2 \frac{i}{f(2+ri)} dr$$

By (1)(2)(3) we have

$$\int_{b_2^{*}} \frac{1}{f(z)} = \int_{b_3^{*}} \frac{1}{f(z)} + 2\int_1^2 \frac{i}{f(2+ri)} dr + 2\int_0^2 \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{f(2+i+(\frac{\sqrt{3}}{2}+\frac{1}{2}i)r)} dr$$

3. $b_1^* = b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{14}^* \cup b_{15}^* \cup b_{31}^* \cup b_{35}^* \cup b_{36}^*$

$$\begin{cases} b_{11}^* = horizontal\ line\ from\ -i \to 3-i\ on\ sheet-\text{I} \\ b_{12}^* = slant\ cut\ from\ 3-i \to 4+(\sqrt{3}-1)i\ on\ (+)edge\ of\ sheet-\text{I} \\ b_{13}^* = slant\ line\ from\ 4+(\sqrt{3}-1)i \to (3+\sqrt{3})+2i\ on\ sheet-\text{II} \\ b_{14}^* = horizontal\ line\ from\ (3+\sqrt{3})+2i \to 2\sqrt{3}+2i\ on\ sheet-\text{II} \\ b_{15}^* = horizontal\ line\ from\ 2\sqrt{3}+2i \to 2+2i\ on\ sheet-\text{II} \end{cases}$$

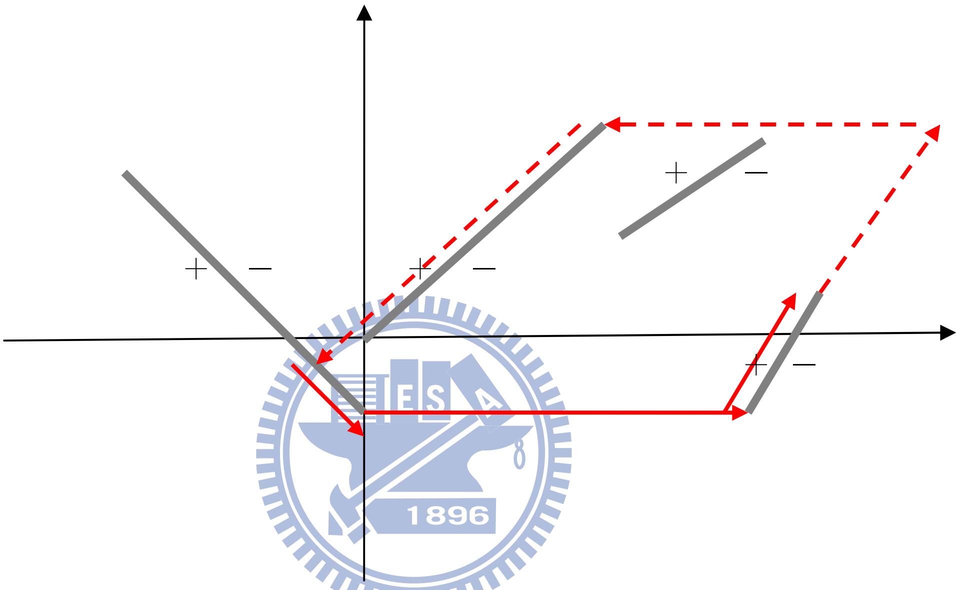


Figure 66: the path of b_1^*

(1) $z \in b_{11}^*$ Let $z = r - i$, $r = 0 \to 3$, $dz = dr$

$$\begin{cases} \arg(z-z_1) = -\pi \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}\ , k=2,3,4,5,6,7 \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_{11}^*} \frac{1}{f(z)} = -\int_0^3 \frac{1}{f(r-i)} dr$$

(2) $b_{12}^{*} = -a_{11}^{*} \Rightarrow \int_{b_{12}^{*}} \frac{1}{f(z)} = -\int_{a_{11}^{*}} \frac{1}{f(z)}$

(3) $z \in b_{13}^{*}$ since we know $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

So we consider $-i \to 3-i$ *on sheet* $-\mathrm{I}$

$$Let\ z = (3-i) + (\frac{1}{2} + \frac{\sqrt{3}}{2}i)r,\ r = 2 \to 2\sqrt{3}\ and\ \ dz = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)dr$$

$$\begin{cases} \arg(z-z_k) = -\dfrac{5\pi}{3} \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k = 1,2 \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}\ , k = 3,4,5,6,7 \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\int_{b_{13}^{*}} \frac{1}{f(z)} = -\int_{2}^{2\sqrt{3}} \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr$$

(4) $z \in b_{14}^{*}$ since we know $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

So we consider $(3+\sqrt{3})+2i \to (2+\sqrt{3})+2i$ *on sheet* $-\mathrm{I}$

$$let\ z = r + 2i,\ r = 3+\sqrt{3} \to 2+\sqrt{3}\ ,\ dz = dr$$

$$\begin{cases} \arg(z-z_k) = \left[-\dfrac{5\pi}{3}, -\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k = 1,2 \\ \arg(z-z_k) \in (-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}\ , k = 3,4,5,6,7 \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\int_{b_{14}^{*}} \frac{1}{f(z)} = -\int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)} dr$$

(5) $z \in b_{15}^{*}$ since we know $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

So we consider the path horizontal line $2+\sqrt{3}+2i \to 2+2i$ on sheet- I

$$let\ z = r + 2i\ ,\ r = 2+\sqrt{3} \to 2,\ dz = dr$$

$$\begin{cases} \arg(z-z_k)=\left[-\dfrac{5\pi}{3},-\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=1,2 \\ \arg(z-z_k)=\left[-\dfrac{11\pi}{6},-\pi\right] \Rightarrow \sqrt{z-z_k} \overset{math}{=} -\sqrt{z-z_k}, k=3,4 \quad f(z) \overset{math}{=} f(z) \\ \arg(z-z_k)\in(-\pi,\pi) \Rightarrow \sqrt{z-z_k} \overset{math}{=} \sqrt{z-z_k}\ , k=5,6,7 \end{cases}$$

$$\int_{b_{15}^*} \frac{1}{f(z)} = -\int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)} dr$$

So by (1)…(5)

$$\int_{b_1^*} \frac{1}{f(z)} = \frac{1}{2}\int_{b_3^*} \frac{1}{f(z)} - \int_0^3 \frac{1}{f(r-i)} dr - \int_0^2 \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr - \int_2^{2\sqrt{3}} \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{f(3-i+(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r)} dr$$

$$-\int_{3+\sqrt{3}}^{2+\sqrt{3}} \frac{1}{f(r+2i)} dr - \int_{2+\sqrt{3}}^{2} \frac{1}{f(r+2i)} dr$$

Similarly now we give a more easier method to reduce the work process. We divided C by many blocks to discuss the slant cuts.

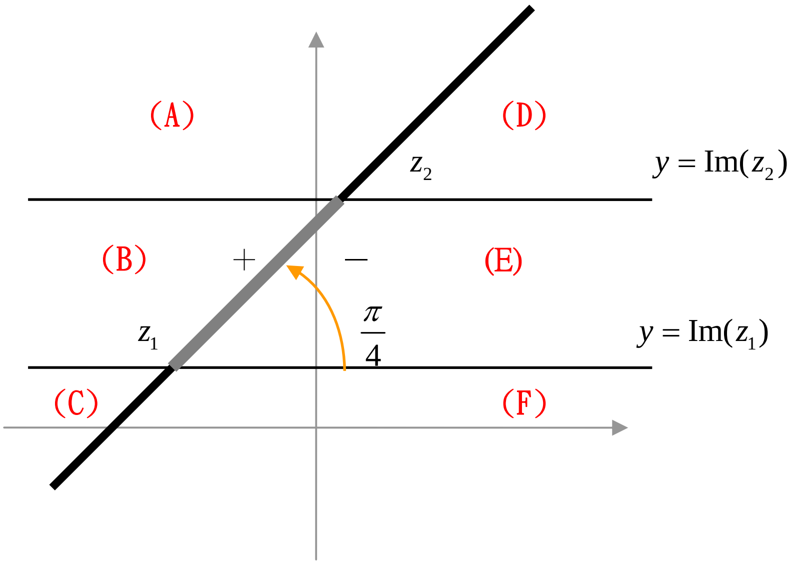


Figure 67: The 6 blocks of $\alpha = \frac{\pi}{4}$

1. For the first example we take $\alpha = \frac{\pi}{4}$, and $f(z)\sqrt{(z-z_1)(z-z_2)}$

$$(1)\ z \in (A): \begin{cases} \arg(z-z_1) \in \left[-\frac{7\pi}{4}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_2) \in \left[-\frac{7\pi}{4}, -\pi\right] \Rightarrow \sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$$(2)\ z \in (B): \begin{cases} \arg(z-z_1) \in \left[-\frac{7\pi}{4}, -\pi\right] \Rightarrow \sqrt{z-z_1} \overset{math}{=} -\sqrt{z-z_1} \\ \arg(z-z_2) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$$(3)\ z \in (C)(D)(E)(F): \begin{cases} \arg(z-z_1) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_1} \overset{math}{=} \sqrt{z-z_1} \\ \arg(z-z_2) \in (-\pi, \pi) \Rightarrow \sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

$(4)\ z \in (+)\ edge\ of\ \ sheet\text{-}I$

$$\begin{cases} \arg(z - z_1) = -\dfrac{7\pi}{4} \Rightarrow \sqrt{z - z_1} \overset{math}{=} -\sqrt{z - z_1} \\ \arg(z - z_2) = -\dfrac{3\pi}{4} \Rightarrow \sqrt{z - z_2} \overset{math}{=} \sqrt{z - z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

$(5)\ \ z \in (-)\ edge\ of\ \ sheet\text{-}I$

$$\begin{cases} \arg(z - z_1) = \dfrac{\pi}{4} \Rightarrow \sqrt{z - z_1} \overset{math}{=} \sqrt{z - z_1} \\ \arg(z - z_2) = -\dfrac{3\pi}{4} \Rightarrow \sqrt{z - z_2} \overset{math}{=} \sqrt{z - z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

So that the conclusion

$$f(z) \overset{math}{=} \begin{cases} -f(z) & where\ z \in (B) \bigcup z \in (+)\ of\ \ sheet - \mathrm{I} \\ f(z) & otherwise \end{cases}$$

2. For the second example we take $\alpha=\dfrac{2\pi}{3}$. and $f(z)\sqrt{(z-z_1)(z-z_2)}$

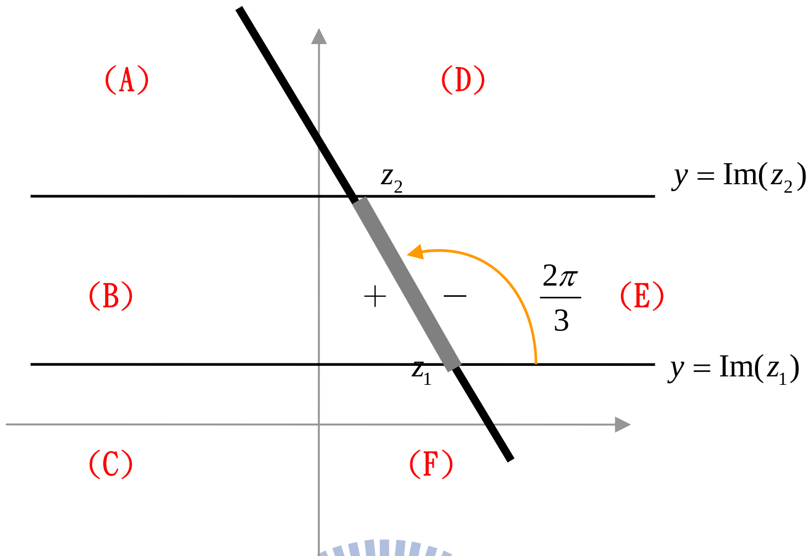


Figure 68 : The 6 blocks of $\alpha=\dfrac{2\pi}{3}$

(1) $z\in(A):\begin{cases}\arg(z-z_1)\in\left[-\dfrac{4\pi}{3},-\pi\right]\Rightarrow\sqrt{z-z_1}\overset{math}{=}-\sqrt{z-z_1}\\ \arg(z-z_2)\in\left[-\dfrac{4\pi}{3},-\pi\right]\Rightarrow\sqrt{z-z_2}\overset{math}{=}-\sqrt{z-z_2}\end{cases}\Rightarrow f(z)\overset{math}{=}f(z)$

(2) $z\in(B):\begin{cases}\arg(z-z_1)\in\left[-\dfrac{4\pi}{3},-\pi\right]\Rightarrow\sqrt{z-z_1}\overset{math}{=}-\sqrt{z-z_1}\\ \arg(z-z_2)\in(-\pi,\pi)\Rightarrow\sqrt{z-z_2}\overset{math}{=}\sqrt{z-z_2}\end{cases}\Rightarrow f(z)\overset{math}{=}-f(z)$

(3) $z\in(C)(D)(E)(F):\begin{cases}\arg(z-z_1)\in(-\pi,\pi)\Rightarrow\sqrt{z-z_1}\overset{math}{=}\sqrt{z-z_1}\\ \arg(z-z_2)\in(-\pi,\pi)\Rightarrow\sqrt{z-z_2}\overset{math}{=}\sqrt{z-z_2}\end{cases}\Rightarrow f(z)\overset{math}{=}f(z)$

(4) $z \in (+)\ edge\ of\ \ sheet\text{-}I$

$$\begin{cases} \arg(z - z_1) = -\dfrac{4\pi}{3} \Rightarrow \sqrt{z - z_1} \overset{math}{=} -\sqrt{z - z_1} \\ \arg(z - z_2) = -\dfrac{\pi}{3} \Rightarrow \sqrt{z - z_2} \overset{math}{=} \sqrt{z - z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} -f(z)$$

(5) $z \in (-)\ edge\ of\ \ sheet\text{-}I$

$$\begin{cases} \arg(z - z_1) = \dfrac{2\pi}{3} \Rightarrow \sqrt{z - z_1} \overset{math}{=} \sqrt{z - z_1} \\ \arg(z - z_2) = -\dfrac{\pi}{3} \Rightarrow \sqrt{z - z_2} \overset{math}{=} \sqrt{z - z_2} \end{cases} \Rightarrow f(z) \overset{math}{=} f(z)$$

So that we have the conclusion

$$f(z) \overset{math}{=} \begin{cases} -f(z) & where\ z \in (B) \cup z \in (+)\ of\ \ sheet - \mathrm{I} \\ f(z) & otherwise \end{cases}$$

Now we discuss the general of this special case: for any slant cut which has slope $m = \tan\alpha$, $0 < \alpha \le \pi$.

For $f(z) = \sqrt{\prod_{k=1}^{m}(z - z_k)}$, where $z_k = x_k + y_k i$.

For case Ⅰ: m=2N-1

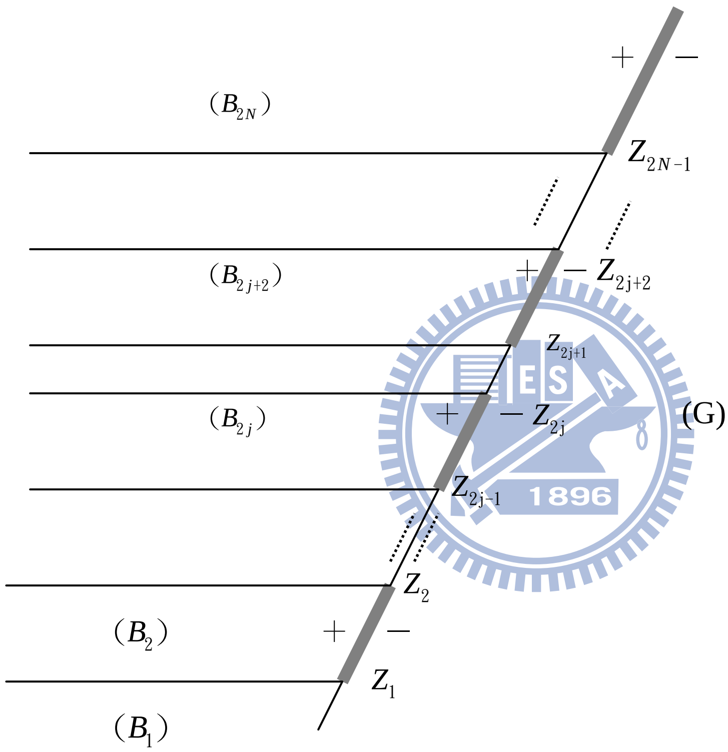


Figure 69: m=2N-1 for slant cuts

$$\begin{cases} (B_1) = \{(x, y) : (x, y) \in C \text{ and } y < y_1\} \\ (B_2) = \{(x, y) : (x, y) \in C \text{ and } y_1 \le y < y_2\} \\ (B_{2j}) = \{(x, y) : (x, y) \in C \text{ and } y_{2j-1} \le y < y_{2j}\} \\ (B_{2j+1}) = \{(x, y) : (x, y) \in C \text{ and } y_{2j} \le y < y_{2j+1}\} \\ (B_{2N-1}) = \{(x, y) : (x, y) \in C \text{ and } y_{2N-2} \le y < y_{2N-1}\} \\ (B_{2N}) = \{(x, y) : (x, y) \in C \text{ and } y_{2N-1} \le y\} \\ (G) = \{(x, y) : (x, y) \in R \setminus C\} \\ L : \{(x, y) : y - y_1 = \tan\alpha(x - x_1)\} \end{cases}$$

(1) $z \in (B_1)$: $\arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}$

$k = 1, 2, 3, \ldots, 2N - 1$

$\Rightarrow f(z) \overset{math}{=} f(z)$

(2)

$$z \in (B_2) : \begin{cases} \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_1} \overset{math}{=} -\sqrt{z - z_1} \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 2, \ldots 2N - 1 \end{cases}$$

$\Rightarrow f(z) \overset{math}{=} -f(z)$

(3)

$$z \in (B_{2j}) : \begin{cases} \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1, 2, \ldots 2j - 1 \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 2j, \ldots 2N - 1 \end{cases}$$

$\Rightarrow f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$

(4)

$$z \in (B_{2j+1}) : \begin{cases} \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1, 2, \ldots 2j \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 2j+1, \ldots 2N - 1 \end{cases}$$

$\Rightarrow f(z) \overset{math}{=} (-1)^{2j} f(z) = f(z)$

(5) $z \in (G)$: $\arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}$

$k = 1, 2, 3, \ldots, 2N - 1$

$\Rightarrow f(z) \overset{math}{=} f(z)$

(6) If $z \in L : y - y_1 = \tan\alpha(x - x_1)$

(i) The path on no cuts: $z \in L \cap \{(x,y) : y < y_1\}$:

$$\arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(ii) The path on no cuts: $z \in L \cap \{(x,y) : y_{2j} \le y < y_{2j+1}\}$ $j = 1, 2 \ldots, N - 1$

$$\begin{cases} \arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j + 1, \ldots, 2N - 1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j} f(z) = f(z)$$

(iii) The path on the cut:

$z \in L \cap \{(x,y) : y_{2j-1} \le y < y_{2j}\}$ *on* $(+)$ *edge where* $j = 1, 2 \ldots, N - 1$

$$\begin{cases} \arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j - 1 \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j, \ldots, 2N - 1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$$

(iiii) The path on the cut: $z \in L \cap \{(x,y) : y_{2N-1} \le y\}$ *on* $(+)$ *edge*

$$\arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1, 2, \ldots 2N - 1$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2N-1} f(z) = -f(z)$$

(v)The path on the cut:

$$z \in L \cap \{(x,y) : y_{2j-1} \le y < y_{2j}\} \quad on \ (-) \ edge \ where \ j = 1,2...,N-1$$

$$\begin{cases} \arg(z - z_k) = \alpha \quad \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 1,2,...2j-1 \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 2j,...,2N-1 \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(vi) The path on the cut: $z \in L \cap \{(x,y) : y_{2N-1} \le y\} \quad on \ (-) \ edge$

$$\arg(z - z_k) = \alpha \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} \quad k = 1,2,...2N-1$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

When m=2N-1 the conclusion

$$f(z) \overset{math}{=} \begin{cases} -f(z) \ if \ z \in \bigcup_{j=1}^{N-1} B_{2j} \cup \bigcup_{j=1}^{N-1} \{z \in L \cup \{(x,y) : y_{2j-1} \le y < y_{2j}\} \quad on \ (+) \ edge\} \\ \qquad \cup \{z \in L \cup \{(x,y) : y_{2N-1} \le y\} \quad on \ (+) \ edge\} \\ f(z) \quad otherwise \end{cases}$$

For case Ⅱ: m=2N

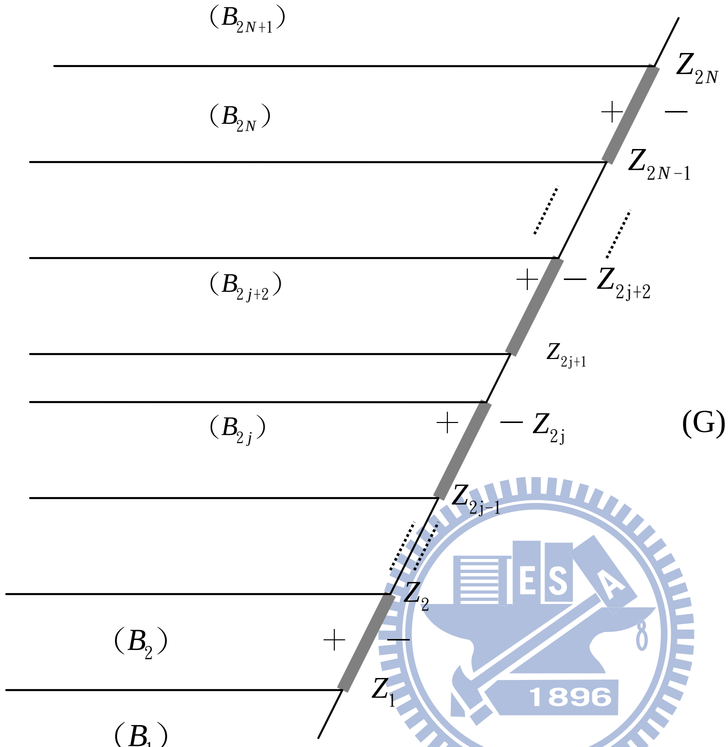


Figure 70: m=2N for slant cuts

Note that

$$\begin{cases} (B_1) = \{(x, y) : (x, y) \in C \ \ and \ \ y < y_1\} \\ (B_2) = \{(x, y) : (x, y) \in C \ \ and \ \ y_1 \le y < y_2\} \\ (B_{2j}) = \{(x, y) : (x, y) \in C \ \ and \ \ y_{2j-1} \le y < y_{2j}\} \\ (B_{2j+1}) = \{(x, y) : (x, y) \in C \ \ and \ \ y_{2j} \le y < y_{2j+1}\} \\ (B_{2N-1}) = \{(x, y) : (x, y) \in C \ \ and \ \ y_{2N-2} \le y < y_{2N-1}\} \\ (B_{2N}) = \{(x, y) : (x, y) \in C \ \ and \ \ y_{2N-1} \le y < y_{2N}\} \\ (B_{2N+1}) = \{(x, y) : (x, y) \in C \ \ and \ \ y_{2N} \le y\} \\ (G) = \{(x, y) : (x, y) \in R \setminus C\} \\ L : \{(x, y) : y - y_1 = \tan\alpha(x - x_1)\} \end{cases}$$

(1) $z \in (B_1)(B_2)$ *the same to the* $m = 2N - 1$

(2)

$$z \in (B_{2j}) : \begin{cases} \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j - 1 \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j, \ldots 2N \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$$

(3)

$$z \in (B_{2j+1}) : \begin{cases} \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j \\ \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j+1, \ldots 2N \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j} f(z) = f(z)$$

(4)

$$z \in (B_{2N+1}) : \arg(z - z_k) \in (\alpha - 2\pi, -\pi] \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1, 2, \ldots 2N$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2N} f(z) = f(z)$$

(5) $z \in (G) : \arg(z - z_k) \in (-\pi, \pi) \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}$

$k = 1, 2, 3, \ldots, 2N$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(6) If $z \in L : y - y_1 = \tan\alpha(x - x_1)$

(i)The path on no cuts: $z \in L \cap \{(x, y) : y < y_1\}$:

$$\arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

(ii) The path on no cuts: $z \in L \cap \{(x, y) : y_{2j} \le y < y_{2j+1}\}$

$$\begin{cases} \arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j + 1, \ldots, 2N \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j} f(z) = f(z)$$

(iii) The path on the cut: $z \in L \cap \{(x, y) : y_{2N} \le y\}$

$$\arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} \quad k = 1, 2, \ldots 2N$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2N} f(z) = f(z)$$

(iiii)The path on the cut:

$z \in L \cap \{(x, y) : y_{2j-1} \le y < y_{2j}\}$ *on* (+) *edge where* $j = 1, 2 \ldots, N$

$$\begin{cases} \arg(z - z_k) = \alpha - 2\pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} -\sqrt{z - z_k} & k = 1, 2, \ldots 2j - 1 \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j, \ldots, 2N \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} (-1)^{2j-1} f(z) = -f(z)$$

(v) The path on the cut:

$z \in L \cap \{(x, y) : y_{2j-1} \le y < y_{2j}\}$ *on* (−) *edge where* $j = 1, 2 \ldots, N$

$$\begin{cases} \arg(z - z_k) = \alpha \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 1, 2, \ldots 2j - 1 \\ \arg(z - z_k) = \alpha - \pi \Rightarrow \sqrt{z - z_k} \overset{math}{=} \sqrt{z - z_k} & k = 2j, \ldots, 2N \end{cases}$$

$$\Rightarrow f(z) \overset{math}{=} f(z)$$

When m=2N the conclusion

$$f(z) \overset{math}{=} \begin{cases} -f(z) \ \ if \ \ z \in \bigcup_{j=1}^{N} B_{2j} \cup \bigcup_{j=1}^{N-1} \left\{ z \in L \cup \left\{ (x, y) : y_{2j-1} \le y < y_{2j} \right\} on \ (+) \ edge \right\} \\ f(z) \quad otherwise \end{cases}$$

Example:

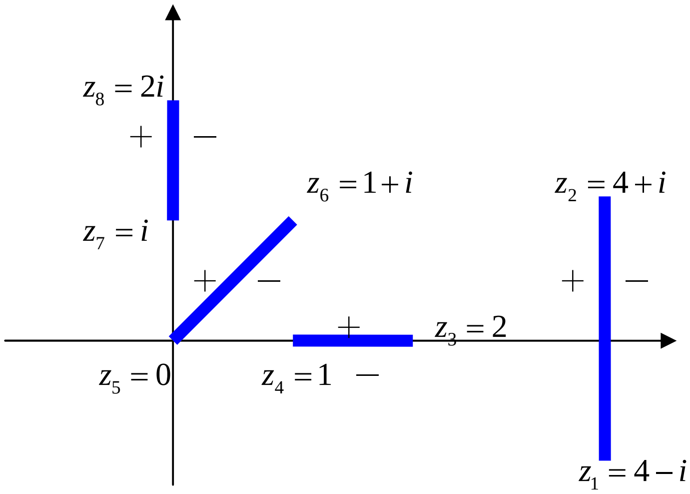


Figure 71: The cut of $f(z)$

Evaluate $\int \frac{1}{f(z)}$ *and* $a, b-cycle$

$$let\ f(z) = \sqrt{(z-4+i)(z-4-i)(z-2)(z-1)(z-0)(z-1-i)(z-i)(z-2i)}$$

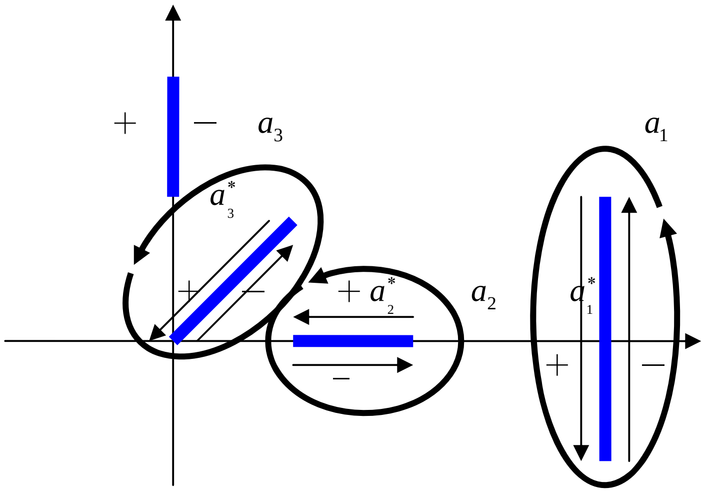


Figure72 : *The cut of* $f(z)$ *in complex plane*

1. $z \in a_1 \equiv a_1^* = a_{11}^* \cup a_{12}^*$

$$\begin{cases} a_{11}^* = vertical\ cut\ from\ 4+i \to 4-i\ on\ (+)edge\ of\ sheet-\mathrm{I} \\ a_{12}^* = vertical\ cut\ from\ 4-i \to 4+i\ on\ (-)edge\ of\ sheet-\mathrm{I} \end{cases}$$

(1) $z \in a_{11}^*$ $Let\ z = 4+ri\ , r = 1 \to -1,\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{a_{11}^*} \frac{1}{f(z)} \overset{math}{=} -\int_{1}^{-1} \frac{1}{f(4+ri)} dr$$

(2) $z \in a_{12}^*$ $Let\ z = 4+ri\ , r = 1 \leftarrow -1,\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_{12}^*} \frac{1}{f(z)} \overset{math}{=} \int_{-1}^{1} \frac{1}{f(4+ri)} dr$$

So that $\int_{a_1^*} \frac{1}{f(z)} \overset{math}{=} 2\int_{-1}^{1} \frac{1}{f(4+ri)} dr = 0.0553983-0.116615i$

2. $z \in a_2 \equiv a_2^* = a_{21}^* \cup a_{22}^*$

$$\begin{cases} a_{21}^* = horizon\ cut\ from\ 1 \to 2\ on\ (+)edge\ of\ sheet - \mathrm{I} \\ a_{22}^* = horizon\ cut\ from\ 2 \to 1\ on\ (-)edge\ of\ sheet - \mathrm{I} \end{cases}$$

(1) $z \in a_{21}^*$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} -\sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_{21}^*} \frac{1}{f(z)} \overset{math}{=} \int_1^2 \frac{1}{f(z)} dz$$

(2) $z \in a_{22}^*$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{a_{22}^*} \frac{1}{f(z)} \overset{math}{=} -\int_2^1 \frac{1}{f(z)} dz$$

So that $\int_{a_2^*} \frac{1}{f(z)} \overset{math}{=} 2\int_1^2 \frac{1}{f(z)} dz$ =0.828531 -0.150956

3. $z \in a_3 \equiv a_3^* = a_{31}^* \cup a_{32}^*$

$$\begin{cases} a_{31}^* = slant\ cut\ from\ 1+i \to 0\ on\ (+)edge\ of\ \ sheet-\mathrm{I} \\ a_{32}^* = slant\ cut\ from\ 0 \to 1+i\ on\ (-)edge\ of\ \ sheet-\mathrm{I} \end{cases}$$

(1) $z \in a_{31}^*$ $Let\ z = r(1\ +\ i),\ r = 1 \to 0,\ dz = (1\ +\ i)dr$

$$\left\{\begin{array}{l} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} -\sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{array}\right. f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{a_{31}^*} \frac{1}{f(z)} \overset{math}{=} \int_1^0 \frac{1+i}{f((1+i)r)} dr$$

(2) $z \in a_{32}^*$ $Let\ z = r(1\ +\ i),\ r = 0 \to 1,\ dz = (1\ +\ i)dr$

$$\left\{\begin{array}{l} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{array}\right. f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{a_{32}^*} \frac{1}{f(z)} \overset{math}{=} -\int_0^1 \frac{1+i}{f((1+i)r)} dr$$

So that we have $\int_{a_3^*} \frac{1}{f(z)} \overset{math}{=} -2\int_0^1 \frac{1+i}{f((1+i)r)} dr = -5.47602+6.09113$

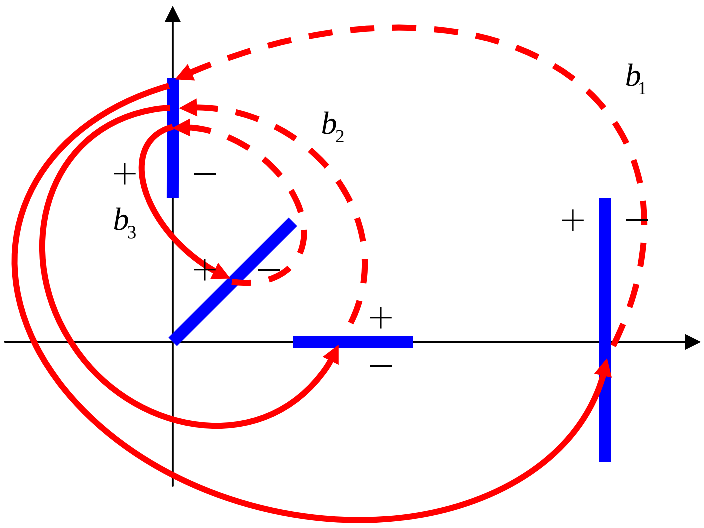


Figure 73: the path of b-cycle

1. $z \in b_3^*$

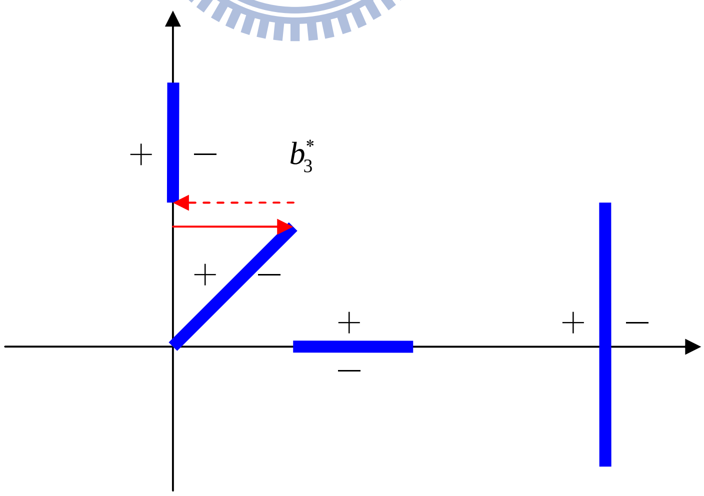


Figure 74:The path of b_3^*

$z \in b_3 \equiv b_3^* = b_{31}^* \bigcup b_{32}^*$

$$\begin{cases} b_{31}^* = horizontal\ line\ i \to 1+i\ on\ sheet\text{-I} \\ b_{32}^* = horizontal\ line\ 1+i \to i\ on\ sheet\text{-II} \end{cases}$$

(1) $z \in b_{31}^*$ $Let\ z = r+i\ ,\ r = 0 \to 1\ ,\ dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{b_{31}^*} \frac{1}{f(z)} \overset{math}{=} \int_1^0 \frac{1}{f(r+i)} dr$$

(2) since $f(z)\big|_{\text{II}} = -f(z)\big|_{\text{I}}$

We consider the path $1+i \to i\ on\ sheet-\text{I}$

$Let\ z = r+i\ ,\ r = 0 \leftarrow 1\ ,\ dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{b_{32}^*} \frac{1}{f(z)} \overset{math}{=} -\int_1^0 \frac{1}{f(r+i)} dr$$

$$\Rightarrow \int_{b_3^*} \frac{1}{f(z)} \overset{math}{=} 2\int_0^1 \frac{1}{f(r+i)} dr = 17.2369\ +11.8361$$

2. $z \in b_2^*$

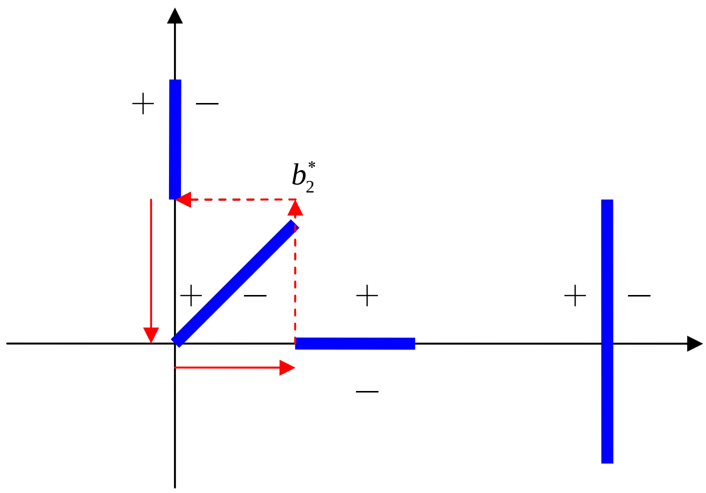


Figure 75: The path of b_2^*

$$z \in b_2 \equiv b_2^* = b_{21}^* \cup b_{22}^* \cup b_{23}^* \cup b_{32}^*$$

$$\begin{cases} b_{21}^* = vertical\ \ line\ \ i \to 0\ \ on\ \ sheet - \mathrm{I} \\ b_{22}^* = horizontal\ \ line\ \ 0 \to 1\ \ on\ \ sheet - \mathrm{I} \\ b_{23}^* = vertical\ \ line\ \ 1 \to 1+i\ \ on\ \ sheet - \mathrm{II} \end{cases}$$

(1) $z \in b_{21}^*$ $Let\ z\ =\ ri,\ r = 1 \to 0\ ,\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} -\sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{b_{21}^*} \frac{1}{f(z)} \overset{math}{=} \int_1^0 \frac{1}{f(ri)} dr$$

(2) $z \in b_{22}^{*}$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_{22}^{*}} \frac{1}{f(z)} \overset{math}{=} -\int_0^1 \frac{1}{f(z)} dr$$

(3) since $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

We consider the path $1 \to 1+i$ *on sheet* $-\mathrm{I}$

$Let\ z = 1+ri\ ,\ r = 0 \to 1\ ,\ dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_{23}^{*}} \frac{1}{f(z)} \overset{math}{=} \int_0^1 \frac{i}{f(1+ri)} dr$$

By (1)(2)(3) we have

$$\int_{b_2^{*}} \frac{1}{f(z)} \overset{math}{=} \int_1^0 \frac{1}{f(ri)} dr - \int_0^1 \frac{1}{f(r)} dr + \int_0^1 \frac{i}{f(1+ri)} dr - \int_1^0 \frac{1}{f(r+i)} dr$$

$= 0.317812+0.747643i$

3. $z \in b_1$

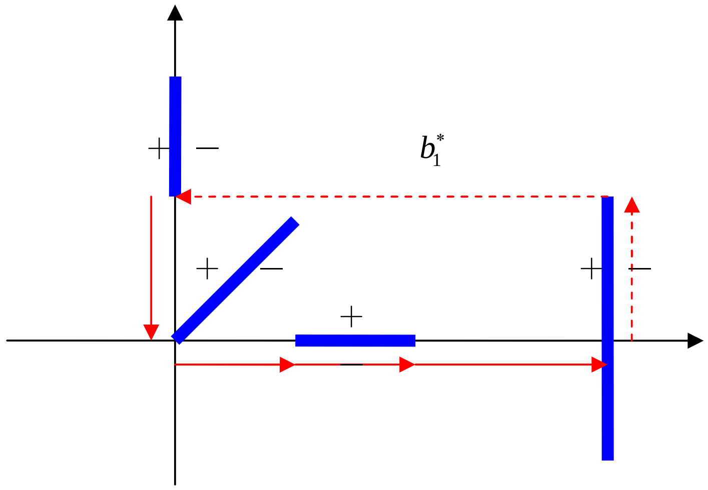


Figure 76:The path of b_1^*

$$z \in b_1 \equiv b_1^* = b_{21}^* \cup b_{22}^* \cup a_{21}^* \cup b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{24}^*$$

$$\begin{cases} b_{11}^* = horizontal\ line\ 2 \to 4\ on\ sheet - \mathrm{I} \\ b_{12}^* = vertical\ cut\ 4 \to 4+i\ on\ (-)edge\ of\ sheet - \mathrm{II} \\ b_{13}^* = horizontal\ line\ 4+i \to 1+i\ on\ sheet - \mathrm{II} \end{cases}$$

(1) $z \in b_{11}^*$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_{11}^*} \frac{1}{f(z)} \overset{math}{=} -\int_2^3 \frac{1}{f(z)} dz$$

(2) $b_{12}^{*} \equiv$ *vertical cut from* 4 *to* $4+i$ *on* $(+)$*edge of sheet - I*

Let $z = 4 + ri$, $r = 0 \to 1$, $dz = idr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} -f(z)$$

$$\Rightarrow \int_{b_{12}^{*}} \frac{1}{f(z)} \overset{math}{=} -\int_0^1 \frac{i}{f(4+ri)} dr$$

(3) $z \in b_{13}^{*}$ since $f(z)\big|_{\mathrm{II}} = -f(z)\big|_{\mathrm{I}}$

We consider the path $4+i \to 1+i$ *on sheet* $-$ I

Let $z = r + i$, $r = 4 \to 1$, $dz = dr$

$$\begin{cases} \sqrt{z-z_1}\sqrt{z-z_2} \overset{math}{=} \sqrt{z-z_1}\sqrt{z-z_2} \\ \sqrt{z-z_3}\sqrt{z-z_4} \overset{math}{=} \sqrt{z-z_3}\sqrt{z-z_4} \\ \sqrt{z-z_5}\sqrt{z-z_6} \overset{math}{=} \sqrt{z-z_5}\sqrt{z-z_6} \\ \sqrt{z-z_7}\sqrt{z-z_8} \overset{math}{=} \sqrt{z-z_7}\sqrt{z-z_8} \end{cases} \quad f(z) \overset{math}{=} f(z)$$

$$\Rightarrow \int_{b_{13}^{*}} \frac{1}{f(z)} \overset{math}{=} -\int_4^1 \frac{1}{f(r+i)} dr$$

So that

$$\int_{b_1^{*}} \frac{1}{f(z)} = \int_1^0 \frac{1}{f(ri)} dr - \int_0^1 \frac{1}{f(z)} dz - \int_1^0 \frac{1}{f(r+i)} dr + \int_1^0 \frac{1}{f(ri)} dr - \int_2^3 \frac{1}{f(z)} dz$$
$$-\int_0^1 \frac{i}{f(4+ri)} dr - \int_4^1 \frac{1}{f(r+i)} dr$$

=0.103448+0.944865

5. Applications of Differential Equations

Differential Equations Example :

$u'' + \cos u = 0$ is second order differential equations.

Know that $\cos u \sim 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4$

We want to slove the equation , $u'' + 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 = 0$

So that $E = \frac{1}{2}(u')^2 + u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5$ where E is constant.

Take E=5 we have

$$(u')^2 = 10 - 2u + \frac{2}{3!}u^3 - \frac{2}{5!}u^5$$

$$= R(u-4.54)(u-1.59-2.24i)(u-1.59+2.24i)(u+3.86-1.63i)(u+3.86+1.63i)$$

where R is constant.

Let

$$f(u) = \sqrt{(u-4.54)(u-1.59-2.24i)(u-1.59+2.24i)(u+3.86-1.63i)(u+3.86+1.63i)}$$

compute $\int \frac{1}{f(u)}$ *and* a,b*-cycles* .

take $u_1 = 4.54,\ u_2 = 1.59 - 2.24i,\ u_3 = 1.59 + 2.24i, u_4 = -3.86 - 1.63i,\ u_5 = -3.86 + 1.63i$

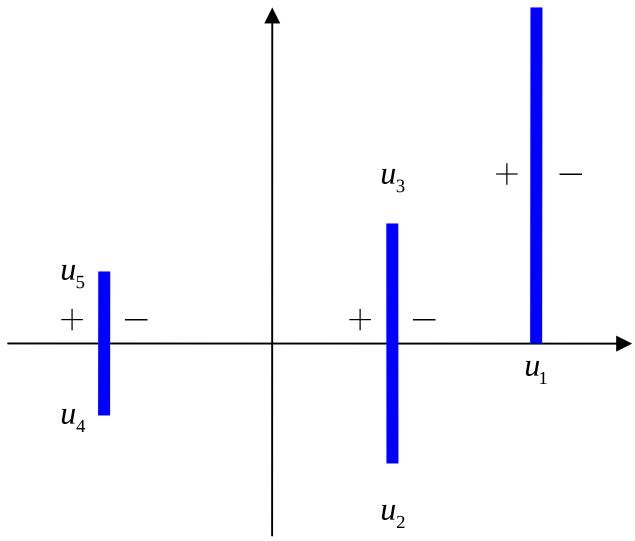


Figure 77: the cut plan of $f(u)$

Solution:

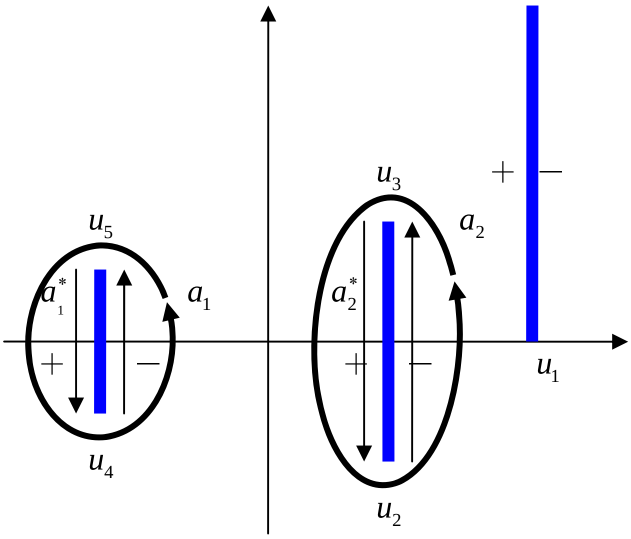


Figure 78: The path of a-cycle and a^*

1. $z \in a_1 = a_1^* = a_{11}^* \cup a_{12}^* \cup a_{13}^* \cup a_{14}^*$

$$\begin{cases} a_{11}^* = vertical\ cut\ from\ -3.86+1.63i \to 0\ on\ (+)edge\ of\ sheet-I \\ a_{12}^* = vertical\ cut\ from\ 0 \to -3.86-1.63i\ on\ (+)edge\ of\ sheet-I \\ a_{13}^* = vertical\ cut\ from\ -3.86-1.63i \to 0\ on\ (-)edge\ of\ sheet-I \\ a_{14}^* = vertical\ cut\ from\ 0 \to -3.86+1.63i\ on\ (-)edge\ of\ sheet-I \end{cases}$$

The region needs change sign of

$$\begin{cases} A = \sqrt{u-4.45}\ is\ \{u = x+iy : x < 4.54\ , 0 \le y\} \cup \{(+)edge\ \} \\ B = \sqrt{u-1.59-2.24i}\sqrt{u-1.59+2.24i}\ is\ \{u = x+iy : x < 1.59, -2.24 \le y \le 2.24\} \cup \{(+)edge\} \\ C = \sqrt{u+3.86+1.63i}\sqrt{u+3.86-1.63i}\ is\ \{u = x+iy : x < -3.86, -1.63 \le y \le 1.63\} \cup \{(+)edge\} \end{cases}$$

(1) $a_{11}^* : Let\ u = -3.86 + ri,\ du = idr,\ r: 1.63 \to 0$

$$u \in A, B, C \Rightarrow f(u) \overset{math}{=} (-1)^3 f(u) = -f(u)$$

$$\int_{a_{11}^*} \frac{1}{f(u)} du \overset{math}{=} -\int_{1.63}^{0} \frac{i}{f(-3.86+ri)} dr$$

(2) $a_{12}^* : Let\ u = -3.86 + ri,\ du = idr,\ r = 0 \to -1.63$

$$u \in B, C \Rightarrow f(u) \overset{math}{=} (-1)^2 f(u) = f(u)$$

$$\int_{a_{12}^*} \frac{1}{f(u)}du \overset{math}{=} \int_0^{-1.63} \frac{i}{f(-3.86+ri)}dr$$

(3) $a_{13}^* : Let\ u=-3.86+ri,\ du=idr,\ r=-1.63 \to 0$

$$u \in B \Rightarrow f(u) \overset{math}{=} -f(u)$$

$$\int_{a_{13}^*} \frac{1}{f(u)}du \overset{math}{=} -\int_{-1.63}^{0} \frac{i}{f(-3.86+ri)}dr$$

(4) $a_{14}^* : Let\ u=-3.86+ri,\ du=idr,\ r=0 \to 1.63$

$$u \in B,C \Rightarrow f(u) \overset{math}{=} (-1)^2 f(u) = f(u)$$

$$\int_{a_{14}^*} \frac{1}{f(u)}du \overset{math}{=} \int_0^{1.63} \frac{i}{f(-3.86+ri)}dr$$

By (1)⋯(4)

$$\int_{a_1^*} \frac{1}{f(u)}du \overset{math}{=} 2\int_0^{1.63} \frac{i}{f(-3.86+ri)}dr - 2\int_{-1.63}^{0} \frac{i}{f(-3.86+ri)}dr$$

$$= 0.35043 - 5.55112 \times 10^{-17} i$$

2. $z \in a_2 = a_2^* = a_{21}^* \cup a_{22}^* \cup a_{23}^* \cup a_{24}^*$

$$\begin{cases} a_{21}^* = vertical\ cut\ from\ 1.59+2.24i \to 0\ on\ (+)edge\ of\ sheet-I \\ a_{22}^* = vertical\ cut\ from\ 0 \to 1.59-2.24i\ on\ (+)edge\ of\ sheet-I \\ a_{23}^* = vertical\ cut\ from\ 1.59-2.24i \to 0\ on\ (-)edge\ of\ sheet-I \\ a_{24}^* = vertical\ cut\ from\ 0 \to 1.59+2.24i\ on\ (-)edge\ of\ sheet-I \end{cases}$$

(1) $a_{21}^* : Let\ u=1.59+ri,\ du=idr,\ r=2.24 \to 0$

$$u \in B,C \Rightarrow f(u) \overset{math}{=} (-1)^2 f(u) = f(u)$$

$$\int_{a_{21}^*} \frac{1}{f(u)}du \overset{math}{=} \int_{2.24}^{0} \frac{i}{f(1.59+ri)}dr$$

(2) $a_{22}^{*}: Let\ u = 1.59 + ri,\ du = idr,\ r = 0 \to -2.24$

$$u \in B \Rightarrow f(u) \overset{math}{=} -f(u)$$

$$\int_{a_{22}^{*}} \frac{1}{f(u)} du \overset{math}{=} -\int_{0}^{-2.24} \frac{i}{f(1.59 + ri)} dr$$

(3) $a_{23}^{*}: Let\ u = 1.59 + ri,\ du = idr,\ r = -2.24 \to 0$

$$\Rightarrow f(u) \overset{math}{=} f(u)$$

$$\int_{a_{23}^{*}} \frac{1}{f(u)} du \overset{math}{=} \int_{-2.24}^{0} \frac{i}{f(1.59 + ri)} dr$$

(4) $a_{24}^{*}: Let\ u = 1.59 + ri,\ du = idr,\ r = 0 \to 2.24$

$$u \in C \Rightarrow f(u) \overset{math}{=} -f(u)$$

$$\int_{a_{24}^{*}} \frac{1}{f(u)} du \overset{math}{=} -\int_{0}^{2.24} \frac{i}{f(1.59 + ri)} dr$$

By (1)…(4)

$$\int_{a_{2}^{*}} \frac{1}{f(u)} du \overset{math}{=} -2\int_{0}^{2.24} \frac{i}{f(1.59 + ri)} dr + 2\int_{2.24}^{0} \frac{i}{f(1.59 + ri)} dr$$

$$= -0.587776 + 2.77556 \times 10^{-17} i$$

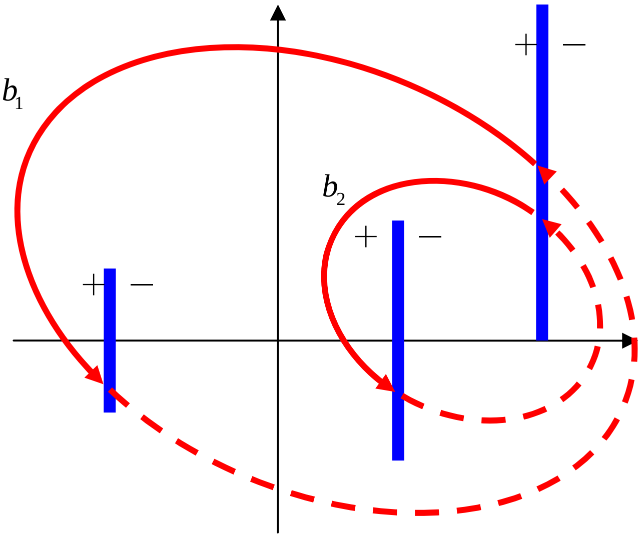


Figure 79: The path of b-cycle

1. $u \in b_2$

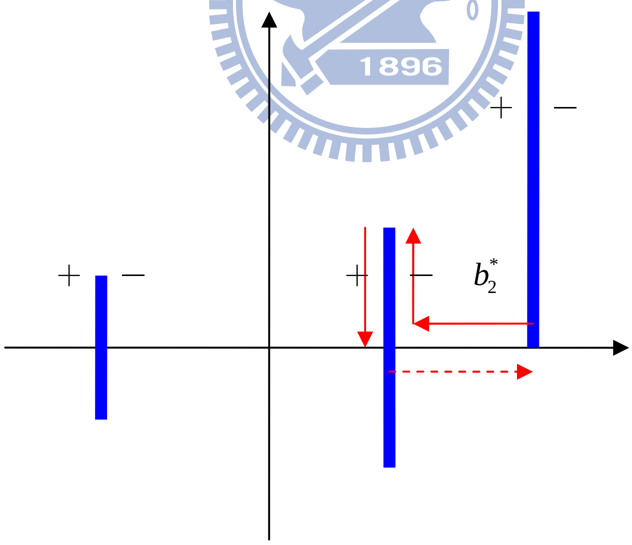


Figure 80:The path of the b_2 equivalent path b_2^*

$u \in b_2 = b_2^* = a_{21}^* \cup b_{22}^* \cup b_{21}^* \cup a_{24}^*$

$$\begin{cases} b_{21}^* = \textit{horizon line from } 4.54 \rightarrow 1.59 \textit{ on sheet} - \mathrm{I} \\ b_{22}^* = \textit{horizon line from } 1.59 \rightarrow 4.54 \textit{ on sheet} - \mathrm{II} \end{cases}$$

(1) $u \in b_{21}^*$

$$u \in C \Rightarrow f(u) \overset{math}{=} -f(u)$$

$$\int_{b_{21}^*} \frac{1}{f(u)} \overset{math}{=} -\int_{4.54}^{1.59} \frac{1}{f(u)} du$$

(2) $u \in b_{22}^*$ Since we know $f(u)\big|_{\mathrm{II}} = -f(u)\big|_{\mathrm{I}}$

So we consider that *horizon line from* $1.59 \rightarrow 4.54$ *on sheet* $-\mathrm{I}$

$$u \in C \Rightarrow f(u) \overset{math}{=} -f(u)$$

$$\int_{b_{22}^*} \frac{1}{f(u)} \overset{math}{=} \int_{1.59}^{4.54} \frac{1}{f(u)} du$$

By (1)(2)

$$\int_{b_2^*} \frac{1}{f(u)} \overset{math}{=} -\int_0^{2.24} \frac{i}{f(1.59+ri)} dr - 2\int_{4.54}^{1.59} \frac{1}{f(u)} du = -0.293888 - 0.170468i$$

2. $u \in b_1$

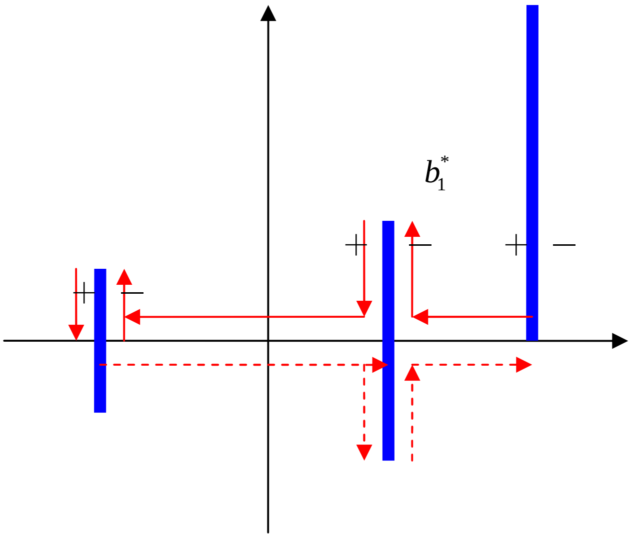


Figure 81:The path of the b_1 equivalent path b_1^*

$$u \in b_1 = b_1^* = a_{11}^* \cup a_{14}^* \cup a_{21}^* \cup a_{24}^* \cup b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{14}^* \cup b_{21}^* \cup b_{22}^*$$

$$\begin{cases} b_{11}^* = horizontal\ line\ from\ 1.59 \to -3.86\ on\ sheet - \text{I} \\ b_{12}^* = horizontal\ line\ from - 3.86 \to 1.59\ on\ sheet - \text{II} \\ b_{13}^* = vertical\ cut\ from\ 1.59 \to 1.59 - 2.24i\ on\ (+)edge\ of\ sheet - \text{II} \\ b_{14}^* = vertical\ cut\ from\ 1.59 - 2.24i \to 1.59\ on\ (-)edge\ of\ sheet - \text{II} \end{cases}$$

(1) $u \in b_{11}^* \Rightarrow u \in B, C$

$$f(u) \overset{math}{=} (-1)^2 f(u) = f(u)$$

$$\int_{b_{11}^*} \frac{1}{f(u)} \overset{math}{=} \int_{1.59}^{-3.86} \frac{1}{f(u)} du$$

(2) $u \in b_{12}^{*}$ Since we know $f(u)\big|_{\text{II}} = -f(u)\big|_{\text{I}}$

So we consider that *horizon line from* $-3.86 \to 1.59$ *on sheet* $-\text{I}$

$$u \in B, C \Rightarrow f(u) \overset{math}{=} (-1)^2 f(u) = f(u)$$

$$\int_{b_{12}^{*}} \frac{1}{f(u)} \overset{math}{=} \int_{1.59}^{-3.86} \frac{1}{f(u)} du$$

(3)

$$\begin{cases} b_{13}^{*} \equiv \textit{the path along vertical cut from } 1.59 \leftarrow 1.59 - 2.24i \textit{ on } (-)\textit{edge of sheet} - \text{I} \\ b_{14}^{*} \equiv \textit{the path along vertical cut from } 1.59 - 2.24i \rightarrow 1.59 \textit{ on } (+)\textit{edge of sheet} - \text{I} \end{cases}$$

So that $\int_{b_{13}^{*} \cup b_{14}^{*}} \frac{1}{f(u)} du = -\int_{a_{22}^{*} \cup a_{23}^{*}} \frac{1}{f(u)} du$

By (1)(2)(3)

So we have

$$\int_{b_1^{*}} \frac{1}{f(u)} \overset{math}{=} -2\int_{-3.86}^{1.59} \frac{1}{f(u)} du + 2\int_{-2.24}^{0} \frac{1}{f(u)} du + 2\int_{0}^{1.63} \frac{i}{f(-3.86 + ri)} dr$$
$$-2\int_{0}^{2.24} \frac{i}{f(1.59 + ri)} dr - 2\int_{4.54}^{1.59} \frac{1}{f(u)} du$$

=0.175215-0.12607i

References

1. Wu, J. R., NCTU, *The Path-Integral Computations on Two-sheeted Riemann Surface of Genus N*, Master thesis, NCTU, Taiwan, 2000.

2. Wolfram Stephen, *Mathematica : a system for doing mathematics by computer* , 2nd ed, Redwood city, 1991

3. George Springer, *Introduction to Rieamnn Surface* , 2nd ed. , Chelsea, 1981.

4. James Ward Brown and Ruel V.Churchill, *Complex Variables and Applications*, 7^{th} ed, Mc Graw Hill, 2003