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Stein-Lovász 定 理 的 推 廣 及 其 應 用

An Extension of Stein-Lovász Theorem and Some of its Applications

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Stein-Lovász 定 理 的 推 廣 及 其 應 用

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摘　要

Stein-Lovász定理提供一個演算法的方式來找出好的覆蓋(covering)，並且可以用來處理一些組合問題，找出它們的上界。為了可以用來處理更多的組合問題，在這篇論文裡，我們將原先的Stein-Lovász定理作推廣。此外，我們將利用推廣後的Stein-Lovász定理來處理一些模型，這些模型包括：分離矩陣(disjunct matrices)、選擇器(selectors)以及系統集(set systems)，在固定行(column)數的前提之下，分別去找出這些矩陣的最小列(row)數的上界。其中分離矩陣和選擇器可以應用在匯集設計(pooling design)上。

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Abstract

The Stein-Lovász theorem provides an algorithmic way to deal with the existence of good coverings and then to derive some upper bounds related to some combinatorial structures. In order to deal with more combinatorial problems, an extension of the classical Stein-Lovász theorem, called the extended Stein-Lovász theorem, will be given in this thesis. Moreover, we will also discuss applications of the extended Stein-Lovász theorem to various models stated as follows:

1. Several disjunct matrices (for group testing purpose)
   - d-disjunct matrices, $(d; z]$-disjunct matrices;
   - $(d, r]$-disjunct matrices, $(d, r; z]$-disjunct matrices;
   - (d, r)-disjunct matrices, $(d, r; z)$-disjunct matrices;
   - $(d, s$ out of $r]$-disjunct matrices, $(d, s$ out of $r; z]$-disjunct matrices.
2. Several selectors (for group testing purpose)
   - (k, m, n)-selectors, $(k, m, n; z)$-selectors;
   - (k, m, c, n)-selectors, $(k, m, c, n; z)$-selectors.
3. Some set systems (for others)
   - uniform (m, t)-splitting systems, uniform $(m, t; z)$-splitting systems;
   - uniform (m, t_1, t_2)-separating systems, uniform $(m, t_1, t_2; z)$-separating systems;
   - (v, k, t)-covering designs, $(v, k, t; z)$-covering designs;
   - (v, k, t, p)-lotto designs, $(v, k, t, p; z)$-lotto designs.

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Chapter 1

Introduction

Let X be a finite set and let Γ be a family of subsets of X. We denote by $H = (X, \Gamma)$ the hypergraph having X as the set of vertices and Γ as the set of hyperedges. The *degree* of $x \in X$ is the number of hyperedges containing x. Denoted by $d(H)$ the maximum degree in the hypergraph H. For $d(H)$ and other functions to be defined we remove the argument H if no confusion can arise.

A binary matrix $M = (m_{ij})$ of order $|\Gamma| \times |X|$ can be interpreted as a *block-point incidence matrix* of the hypergraph H, i.e., the rows of M correspond to the hyperedge set $\{E_1, E_2, ..., E_{|\Gamma|}\}$, and the columns correspond to the vertex set $\{x_1, x_2, ..., x_{|X|}\}$, where

$$m_{ij} = \begin{cases} 1 & \text{if the hyperedge } E_i \text{ contains the vertex } x_j \\ 0 & \text{otherwise.} \end{cases}$$

The *weight* of a binary matrix M is the number of entries with a "1".

A subset $M \subseteq \Gamma$ (the same hyperedge may occur more than once) such that each vertex belongs to at most k of its members is called a *k-matching* of the hypergraph H. The maximum size over all k-matchings of the hypergraph H is denoted by $\nu_k(H)$. A k-matching is *simple* if no hyperedge occurs in it more than once. Denoted by $\widetilde{\nu_k}$ the maximum number of hyperedges in simple k-matchings, then $\widetilde{\nu_k} \leq \nu_k$.

A subset $T \subseteq X$ (in this thesis, the same vertex does not occur more than once) such that $|T \bigcap E| \geq k$ for any hyperedge E is called a *k-cover* of the hypergarph H. The minimum size over all k-covers of the hypergraph H is denoted by $\tau_k(H)$. Thus $\tau(H) = \tau_1(H)$ is the minimum size of a vertex cover of the hypergraph H.

A vector $(w_{E_1}, w_{E_2}, ..., w_{E_{|\Gamma|}})$ with $w_{E_i} \geq 0$ for each $E_i \in \Gamma$ is called a *fractional matching* of the hypergraph H if each entry of the vector $(w_{E_1}, w_{E_2}, ..., w_{E_{|\Gamma|}})M$ is at most 1. A vector $(w_{x_1}, w_{x_2}, ..., w_{x_{|X|}})$ with $w_{x_i} \geq 0$ for each $x_i \in X$ is called a *fractional cover* of the hypergraph H if each entry of the vector $M(w_{x_1}, w_{x_2}, ..., w_{x_{|X|}})^t$ is at least 1. Define

$$\nu^*(H) = \max \sum_{E_i \in \Gamma} w_{E_i} \quad \text{and} \quad \tau^*(H) = \min \sum_{x_i \in X} w_{x_i},$$

where the extrema are taken over all fractional matchings $(w_{E_1}, w_{E_2}, ..., w_{E_{|\Gamma|}})$ and all fractional covers $(w_{x_1}, w_{x_2}, ..., w_{x_{|X|}})$, respectively. By the duality theorem of linear programming, we have $\nu^* = \tau^*$. Then it is easy to see that $\nu \leq \nu_k/k \leq \nu^* = \tau^* \leq \tau_k/k$.

One of the most natural methods to produce a small vertex cover of a given hypergraph H is the so-called "*Greedy Cover Algorithm*", which we describe as follows:

1. Let x_1 be a vertex with maximum degree.

2. Suppose that $x_1, x_2, ..., x_i$ have been already selected,

   if $x_1, x_2, ..., x_i$ cover all hyperedges, then we stop; otherwise,

   let x_{i+1} be a vertex which covers the most number of uncovered hyperedges.

Generally, the greedy cover algorithm is not the best, but we can expect that it gives a rather good estimate. By the greedy cover algorithm, an upper bound for $\tau(H)$ was given by Lovász [10].

Theorem 1. [10] If H is a hypergraph and any greedy cover algorithm produces t covering vertices, then

$$t \leq \frac{\widetilde{\nu_1}}{1 \times 2} + \frac{\widetilde{\nu_2}}{2 \times 3} + \cdots + \frac{\widetilde{\nu}_{d-1}}{(d-1) \times d} + \frac{\widetilde{\nu_d}}{d}.$$

Use the facts that $\tau \leq t$ and $\widetilde{\nu_i} \leq \nu_i \leq i\nu^* = i\tau^*$, we have the following corollary.

Corollary 1. [10] For a hypergraph H,

$$\tau(H) \leq (1 + \frac{1}{2} + \cdots + \frac{1}{d})\tau^*(H) < (1 + \ln d)\tau^*(H).$$

Hence we have the following theorem (for completeness, we also give a proof).

Theorem 2. For a hypergraph $H = (X, \Gamma)$,

$$\tau(H) < \frac{|X|}{\min_{E \in \Gamma} |E|}(1 + \ln \triangle),$$

where $\triangle = \max_{x \in X} |\{E : E \in \Gamma \text{ with } x \in E\}|$.

Proof. Let M be the block-point incidence matrix of H. Define

$$w_{x_i} = \frac{1}{\min_{E \in \Gamma} |E|}$$

for each $x_i \in X$. Then each E_i-entry of the vector $M(w_{x_1}, w_{x_2}, ..., w_{x_{|X|}})^t$ is

$$\frac{|E_i|}{\min_{E \in \Gamma} |E|} \geq 1,$$

i.e., $(w_{x_1}, w_{x_2}, ..., w_{x_{|X|}})$ is a fractional cover of H. Hence

$$\tau^*(H) \leq \sum_{x_i \in X} w_{x_i} = \frac{|X|}{\min_{E \in \Gamma} |E|}.$$

By Corollary 1,

$$\tau(H) < (1 + \ln d)\tau^*(H) \leq \frac{|X|}{\min_{E \in \Gamma} |E|}(1 + \ln \triangle),$$

as required.

□

Similarly, by the greedy cover algorithm, an equivalent statement in terms of the point-block incidence matrices of the corresponding hypergraphs was given by Stein [11] independently.

Theorem 3. [11] Let X be a finite set of cardinality n, and let $\Gamma = \{A_1, A_2, ..., A_t\}$ be a family of subsets of X, where $|A_i| \leq a$ for all $1 \leq i \leq t$. Assume that each element of X is in at least q members of the set Γ. Then there is a subfamily of Γ that covers X and has at most

$$\frac{n}{a} + \frac{t}{q}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a})$$

members.

Note that Theorem 3 is closely related to work of Fulkerson and Ryser [9] in the 1-width of a $(0,1)$-matrix. They define the *1-width* of such a matrix, A, as the minimum number of columns that can be selected from A in such a way that each row of the resulting submatrix has at least one 1. In this terminology, Theorem 3 can be restated as follows:

Theorem 4. [11] Let A be a $(0,1)$-matrix with n rows and t columns. Assume that each row contains at least q 1's and each column at most a 1's. Then the 1-width of A is at most

$$\frac{n}{a} + \frac{t}{q}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a}).$$

Theorem 4 was called the Stein-Lovász Theorem in [6] while dealing with the covering problems in coding theory. The Stein-Lovász theorem was first used in dealing with the upper bounds for the sizes of (k,m,n)-selectors [2]. Inspired by this work, it was also used in dealing with the upper bounds for the sizes of $(d,r;z]$-disjunct matrices [5]. Some more applications can also be found in [8]. The notion of (k,m,n)-selectors was first introduced by De Bonis, Gasieniec and Vaccaro in [2], followed by a generalization to the notion of (k,m,c,n)-selectors [1]. A further generalization of (k,m,c,n)-selectors will be given in Chapter 2.

In this thesis, definitions of several properties over binary matrices are considered in Chapter 2 including several disjunct matrices, several selectors and some set systems stated as follows:

1. Several disjunct matrices

   - d-disjunct matrices, $(d; z]$-disjunct matrices;
   - $(d, r]$-disjunct matrices, $(d, r; z]$-disjunct matrices;
   - (d, r)-disjunct matrices, $(d, r; z)$-disjunct matrices;
   - $(d, s$ out of $r]$-disjunct matrices, $(d, s$ out of $r; z]$-disjunct matrices.

2. Several selectors

   - (k, m, n)-selectors, $(k, m, n; z)$-selectors;
   - (k, m, c, n)-selectors, $(k, m, c, n; z)$-selectors.

3. Some set systems

   - uniform (m, t)-splitting systems, uniform $(m, t; z)$-splitting systems;
   - uniform (m, t_1, t_2)-separating systems, uniform $(m, t_1, t_2; z)$-separating systems;
   - (v, k, t)-covering designs, $(v, k, t; z)$-covering designs;
   - (v, k, t, p)-lotto designs, $(v, k, t, p; z)$-lotto designs.

Note that the upper bounds of the sizes of several disjunct matrices and selectors are obtained for group testing purpose, and the upper bounds of the sizes of some set systems are obtained for others. Some formulas are given in Section 2.4 for later simplification purpose used in Chapter 4.

In order to deal with the upper bounds for these binary matrices defined in Chapter 2, an extended Stein-Lovász theorem is derived in Chapter 3. Some applications of the determination of some upper bounds of the sizes of various models are considered in Chapter 4. In Section 4.1 and Section 4.2, the extended Stein-Lovász theorem will be used in dealing the upper bounds for the sizes of several disjunct matrices and selectors, respectively. Those upper bounds for the sizes of uniform splitting systems, uniform separating systems, covering designs and lotto designs are given in Section 4.3 respectively.

Chapter 2

Preliminaries

2.1 Several disjunct matrices

A few types of binary matrices, called disjunct matrices, will be introduced in this section, followed by corresponding associated parameters. These families of disjunct matrices will be used as models for pooling designs.

Definition 2.1.1. A binary matrix M of order $t \times n$ is called d-*disjunct* if the union of any d columns does not contain any other column of M, i.e., for any $d+1$ columns $C_1, C_2, \cdots, C_{d+1}$, $|C_{d+1} \setminus \bigcup_{i=1}^{d} C_i| \geq 1$. The integer t is called the *size* of the d-disjunct matrix. The minimum size over all d-disjunct matrices with n columns is denoted by $t(n, d)$.

Definition 2.1.2. A binary matrix M of order $t \times n$ is called $(d; z]$-*disjunct* if for any $d+1$ columns $C_1, C_2, \cdots, C_{d+1}$, $|C_{d+1} \setminus \bigcup_{i=1}^{d} C_i| \geq z$. The integer t is called the *size* of the $(d; z]$-disjunct matrix. The minimum size over all $(d; z]$-disjunct matrices with n columns is denoted by $t(n, d; z]$.

Definition 2.1.3. A binary matrix M of order $t \times n$ is called $(d, r]$-*disjunct* if the union of any d columns does not contain the intersection of any other r columns of M, i.e., for any $d+r$ columns $C_1, C_2, \cdots, C_{d+r}$, $|\bigcap_{i=1}^{r} C_i \setminus \bigcup_{i=r+1}^{d+r} C_i| \geq 1$. The integer t is called the *size* of the $(d, r]$-disjunct matrix. The minimum size over all $(d, r]$-disjunct matrices with n columns is denoted by $t(n, d, r]$.

Definition 2.1.4. A binary matrix M of order $t \times n$ is called $(d, r; z]$-*disjunct* if for any $d + r$ columns $C_1, C_2, \cdots, C_{d+r}$, $|\bigcap_{i=1}^{r} C_i \setminus \bigcup_{i=r+1}^{d+r} C_i| \geq z$. The integer t is called the *size* of the $(d, r; z]$-disjunct matrix. The minimum size over all $(d, r; z]$-disjunct matrices with n columns is denoted by $t(n, d, r; z]$.

Definition 2.1.5. A binary matrix M of order $t \times n$ is called (d, r)-*disjunct* if the union of any d columns does not contain the union of any other r columns of M, i.e., for any $d + r$ columns $C_1, C_2, \cdots, C_{d+r}$, $|\bigcup_{i=1}^{r} C_i \setminus \bigcup_{i=r+1}^{d+r} C_i| \geq 1$. The integer t is called the *size* of the (d, r)-disjunct matrix. The minimum size over all (d, r)-disjunct matrices with n columns is denoted by $t(n, d, r)$.

Definition 2.1.6. A binary matrix M of order $t \times n$ is called $(d, r; z)$-*disjunct* if for any $d + r$ columns $C_1, C_2, \cdots, C_{d+r}$, $|\bigcup_{i=1}^{r} C_i \setminus \bigcup_{i=r+1}^{d+r} C_i| \geq z$. The integer t is called the *size* of the $(d, r; z)$-disjunct matrix. The minimum size over all $(d, r; z)$-disjunct matrices with n columns is denoted by $t(n, d, r; z)$.

Definition 2.1.7. A binary matrix M of order $t \times n$ is called $(d, s$ out of $r]$-*disjunct*, $1 \leq s \leq r$, if for any d columns and any other r columns of M, there exists a row index in which none of the d columns appear and at least s of the r columns do. The integer t is called the *size* of the $(d, s$ out of $r]$-disjunct matrix. The minimum size over all $(d, s$ out of $r]$-disjunct matrices with n columns is denoted by $t(n, d, r, s]$.

Definition 2.1.8. A binary matrix M of order $t \times n$ is called $(d, s$ out of $r; z]$-*disjunct*, $1 \leq s \leq r$, if for any d columns and any other r columns of M, there exist z row indices in which none of the d columns appear and at least s of the r columns do. The integer t is called the *size* of the $(d, s$ out of $r; z]$-disjunct matrix. The minimum size over all $(d, s$ out of $r; z]$-disjunct matrices with n columns is denoted by $t(n, d, r, s; z]$.

Some subclasses of $(d, s$ out of $r; z]$-disjunct matrices are listed in the following table.

parameters	types	bounds	references
$s = r = 1,\ z = 1$	d-disjunct	$t(d, n)$	[13]
$s = r = 1$	$(d; z]$-disjunct		
$s = r,\ z = 1$	$(d, r]$-disjunct	$t(n, d, r]$	[14]
$s = r$	$(d, r; z]$-disjunct	$t(n, d, r; z]$	[5]
$s = 1,\ z = 1$	(d, r)-disjunct	$t(n, d, r)$	[14]
$s = 1$	$(d, r; z)$-disjunct	$t(n, d, r; z)$	
$z = 1$	$(d, s$ out of $r]$-disjunct	$t(n, d, r, s]$	[14]
	$(d, s$ out of $r; z]$-disjunct	$t(n, d, r, s; z]$	

2.2 Several selectors

A few types of binary matrices, called selectors, will be introduced in this section, followed by corresponding associated parameters. These families of selectors will be used as models for pooling designs.

Definition 2.2.1. For integers k, m and n with $1 \leq m \leq k \leq n$, a binary matrix M of order $t \times n$ is called a (k, m, n)-*selector* if any $t \times k$ submatrix of M contains a submatrix with each row weight exactly one, with at least m distinct rows. The integer t is called the *size* of the (k, m, n)-selector. The minimum size over all (k, m, n)-selectors is denoted by $t_s(k, m, n)$.

Definition 2.2.2. For integers k, m and n with $1 \leq m \leq k \leq n$, a binary matrix M of order $t \times n$ is called a $(k, m, n; z)$-*selector* if any $t \times k$ submatrix of M contains z disjoint submatrices with each row weight exactly one, with at least m distinct rows each. The integer t is called the *size* of the $(k, m, n; z)$-selector. The minimum size over all $(k, m, n; z)$-selectors is denoted by $t_s(k, m, n; z)$.

Definition 2.2.3. For integers k, m, c and n with $1 \leq c \leq k \leq n$ and $1 \leq m \leq \binom{k}{c}$, a $t \times n$ binary matrix M is called a (k, m, c, n)-*selector* if any $t \times k$ submatrix of M contains a submatrix with each row weight exactly c, with at least m distinct rows. The integer t is called the *size* of the (k,m,c,n)-selector. The minimum size over all (k,m,c,n)-selectors is denoted by $t_s(k,m,c,n)$.

Definition 2.2.4. For integers k, m, c and n with $1 \leq c \leq k \leq n$ and $1 \leq m \leq \binom{k}{c}$, a $t \times n$ binary matrix M is called a $(k,m,c,n;z)$-*selector* if any $t \times k$ submatrix of M contains z disjoint submatrices with each row weight exactly c, with at least m distinct rows each. The integer t is called the *size* of the $(k,m,c,n;z)$-selector. The minimum size over all $(k,m,c,n;z)$-selectors is denoted by $t_s(k,m,c,n;z)$.

It is interesting to remark that the notion of (k,m,n)-selectors was first introduced by De Bonis, Gasieniec and Vaccaro [2], and it was then generalized to the notion of (k,m,c,n)-selectors [1], which are equivalent to $(k,m,1,n;1)$-selectors and $(k,m,c,n;1)$-selectors rsepectively. The upper bounds for the sizes of (k,m,n)-selectors and (k,m,c,n)-selectors were studied in [2] and in [1] respectively by the Stein-Lovász theorem. The bounds for the sizes of $(k,m,c,n;z)$-selectors will be derived by the extended Stein-Lovász theorem (Theorem 3.2.1) in Chapter 4 (Theorem 4.1.12).

Some subclasses of $(k,m,c,n;z)$-selectors are listed in the following table.

parameters	types	bounds	references
$c=1,\ z=1$	(k,m,n)-selectors	$t_s(k,m,n)$	[2, 14]
$c=1$	$(k,m,n;z)$-selectors	$t_s(k,m,n;z)$	
$z=1$	(k,m,c,n)-selectors	$t_s(k,m,c,n)$	[1]
	$(k,m,c,n;z)$-selectors	$t_s(k,m,c,n;z)$	

The relationship between various models (disjunct matrices, selectors) and nonadaptive group testing are listed below.

1. A $(d,r]$-disjunct matrix can be used to identify the up-to-d positives on the complex model [4].

2. The property of (h,d)-disjunctness is a necessary condition for identifying the positive set on the (d,h)-inhibitor model [3].

3. There exists a two-state group testing algorithm for finding up-to-d positives out of n items and that uses a number of tests equal to $t+k-1$, where t is the size of a $(k, d+1, n)$-selector [2].

2.3 Some set systems

Most of the combinatorial structures can be viewed as set systems. We present some relevant definitions. A *set system* is a pair (X, Γ), where X is a set of points and Γ is a set of subsets of X, called blocks. A set sysyem (X, Γ) is called k-*uniform* if $|B| = k$ for each $B \in \Gamma$.

Definition 2.3.1. Let m and t be even integers with $2 \leq t \leq m$. An *uniform* (m, t)-*splitting system* is a pair (X, Γ) where $|X| = m$, Γ is a family of $\frac{m}{2}$–subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exists a block $B \in \Gamma$ such that $|T \bigcap B| = \frac{t}{2}$, i.e., B splits T. The system (X, Γ) is also called a t-*splitting system*. The minimum number of blocks over all t-splitting systems is denoted by $SP(m, t)$.

Definition 2.3.2. Let m and t be even integers with $2 \leq t \leq m$, and let z be a positive integer. An *uniform* $(m, t; z)$-*splitting system* is a pair (X, Γ) where $|X| = m$, Γ is a family of $\frac{m}{2}$–subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist z blocks $B \in \Gamma$ such that $|T \bigcap B| = \frac{t}{2}$, i.e., B splits T. The system (X, Γ) is also called a $(t; z)$-*splitting system*. The minimum number of blocks over all $(t; z)$-splitting systems is denoted by $SP(m, t; z)$.

Definition 2.3.3. Let m be an even integer, and let t_1, t_2 be positive integers with $t_1 + t_2 \leq m$. An *uniform* (m, t_1, t_2)-*separating system* is a pair (X, Γ) where $|X| = m$, Γ is a family of $\frac{m}{2}$–subsets of X, called blocks such that for every $T_1, T_2 \subseteq X$, where $|T_i| = t_i$ for $i = 1, 2$ and $|T_1 \bigcap T_2| = \emptyset$, there exists a block $B \in \Gamma$ for which either $T_1 \subseteq B$, $T_2 \bigcap B = \emptyset$ or $T_2 \subseteq B$, $T_1 \bigcap B = \emptyset$, i.e., T_1, T_2 are separated by B. The system (X, Γ) is also called a

(t_1, t_2)-*separating system*. The minimum number of blocks over all (t_1, t_2)-separating systems is denoted by $SE(m, t_1, t_2)$.

Definition 2.3.4. Let m be an even integer, and let t_1, t_2, z be positive integers with $t_1+t_2 \leq m$. An *uniform* $(m, t_1, t_2; z)$-*separating system* is a pair (X, Γ) where $|X| = m$, Γ is a family of $\frac{m}{2}$–subsets of X, called blocks such that for every $T_1, T_2 \subseteq X$, where $|T_i| = t_i$ for $i = 1, 2$ and $|T_1 \bigcap T_2| = \emptyset$, there exist z blocks $B \in \Gamma$ for which either $T_1 \subseteq B$, $T_2 \bigcap B = \emptyset$ or $T_2 \subseteq B$, $T_1 \bigcap B = \emptyset$, i.e., T_1, T_2 are separated by B. The system (X, Γ) is also called a $(t_1, t_2; z)$-*separating system*. The minimum number of blocks over all $(t_1, t_2; z)$-separating systems is denoted by $SE(m, t_1, t_2; z)$.

Definition 2.3.5. Let v, k, and t be positive integers with $t \leq k \leq v$. A (v, k, t)-*covering design* is a pair (X, Γ) where $|X| = v$, Γ is a family of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exists a block $B \in \Gamma$ containing T. The minimum number of blocks over all (v, k, t)-covering designs is denoted by $C(v, k, t)$.

Definition 2.3.6. Let v, k, t and z be positive integers with $t \leq k \leq v$. A $(v, k, t; z)$-covering design is a pair (X, Γ) where $|X| = v$, Γ is a family of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist z blocks $B \in \Gamma$ containing T. The minimum number of blocks over all $(v, k, t; z)$-covering designs is denoted by $C(v, k, t; z)$.

Definition 2.3.7. Let v, k, t, and p be positive integers with $p \leq t, k \leq v$. A (v, k, t, p)-lotto design is a pair (X, Γ) where $|X| = v$, Γ is a family of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exists a block $B \in \Gamma$ such that $|T \bigcap B| \geq p$. The minimum number of blocks over all (v, k, t, p)-lotto designs is denoted by $L(v, k, t, p)$.

Definition 2.3.8. Let v, k, t, p and z be positive integers with $p \leq t, k \leq v$. A $(v, k, t, p; z)$-lotto design is a pair (X, Γ) where $|X| = v$, Γ is a family of k-subsets of X, called blocks such that for every $T \subseteq X$ with $|T| = t$, there exist z blocks $B \in \Gamma$ such that $|T \bigcap B| \geq p$. The minimum number of blocks over all $(v, k, t, p; z)$-lotto designs is denoted by $L(v, k, t, p; z)$.

Note that when $p = t$, a $(v, k, t, t; z)$-lotto design will be reduced to a $(v, k, t; z)$-covering design. The related bounds are summarized in the following table.

types	bounds	references
uniform (m, t)-splitting systems	$SP(m, t)$	[8]
uniform $(m, t; z)$-splitting systems	$SP(m, t; z)$	
uniform (m, t_1, t_2)-separating systems	$SE(m, t_1, t_2)$	
uniform $(m, t_1, t_2; z)$-separating systems	$SE(m, t_1, t_2; z)$	
(v, k, t)-covering designs	$C(v, k, t)$	[8]
$(v, k, t; z)$-covering designs	$C(v, k, t; z)$	
(v, k, t, p)-lotto designs	$L(v, k, t, p)$	[8]
$(v, k, t, p; z)$-lotto designs	$L(v, k, t, p; z)$	

2.4 Some basic counting results

Stein-Lovász theorem and its extension will be used to estimate the upper bounds of the sizes for pooling designs of various models. In order to give upper bounds for the above mentioned parameters, the following results involving binomial coefficients will be involved. Lemma 2.4.3 will be used in showing appropriate values of w for pooling designs of various models. We need information regarding the maximum of the function

$$f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$$

with various r and s when dealing with possible upper bounds for t of various models. Lemmas 2.4.4 $\sim$ 2.4.6 will be used in the simplifications of the bounds $\frac{M}{v}$, and $\ln a$ respectively in the expression $\frac{M}{v}(1 + \ln a)$ found in the Stein-Lovász theorem (Theorem 3.1.1).

Lemma 2.4.1. The function

$$f(x) = \frac{1 + \ln x}{x}$$

is strictly decreasing on $(1, \infty)$.

Proof.

$$f'(x) = \frac{\frac{1}{x} \cdot x - (1 + \ln x) \cdot 1}{x^2} = \frac{-\ln x}{x^2} < 0$$

for all $x \in (1, \infty)$, as required.

□

Lemma 2.4.2.

$$\binom{a}{b} \leq \frac{a^b}{b!} \leq (\frac{ea}{b})^b$$

Proof. Since $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$, we have $e^x \geq \frac{x^b}{b!}$ for each x, thus $e^b \geq \frac{b^b}{b!}$ and hence $1 \leq \frac{b!e^b}{b^b}$. Therefore,

$$\binom{a}{b} = \frac{a!}{(a-b)!b!} = \frac{a(a-1)\cdots(a-b+1)}{b!} \leq \frac{a^b}{b!} \leq \frac{a^b}{b!} \cdot \frac{b!e^b}{b^b} = (\frac{ea}{b})^b,$$

as required. □

Lemma 2.4.3. For any positive integers n, d, r, s with $k = d + r \leq n$ and $1 \leq s \leq r$, the function

$$f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$$

gets its maximum at

$$w = \frac{ns - (k-s)}{k}.$$

Proof. First we note that

$$\begin{aligned}
f(w) &= \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s} \\
&= \frac{(n-w)!}{(n-w-d)!d!} \cdot \frac{(n-w-d)!}{(n-w-d-r+s)!(r-s)!} \cdot \binom{w}{s} \cdot \frac{(d+r-s)!}{(d+r-s)!} \\
&= \binom{n-w}{d+r-s}\binom{w}{s}\binom{d+r-s}{d} = \binom{n-w}{k-s}\binom{w}{s}\binom{k-s}{d}.
\end{aligned}$$

Since

$$f(w+1) = \binom{n-(w+1)}{k-s}\binom{w+1}{s}\binom{k-s}{d}$$
$$= \left(\frac{(n-w)-(k-s)}{n-w}\binom{n-w}{k-s}\right)\cdot\left(\frac{w+1}{w+1-s}\binom{w}{s}\right)\cdot\binom{k-s}{d}$$
$$= \left(\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s}\right)f(w),$$

and

$$\frac{(n-w)-(k-s)}{n-w}\cdot\frac{w+1}{w+1-s} = \frac{(w+1)(n-w)-(w+1)(k-s)}{(w+1)(n-w)-s(n-w)} = 1$$

if and only if $s(n-w) = (w+1)(k-s)$, i.e., $w = \dfrac{ns-(k-s)}{k}$; hence

$\dfrac{(n-w)-(k-s)}{n-w}\cdot\dfrac{w+1}{w+1-s} \geq 1$ for $w \leq \dfrac{ns-(k-s)}{k}$ and

$\dfrac{(n-w)-(k-s)}{n-w}\cdot\dfrac{w+1}{w+1-s} \leq 1$ for $w \geq \dfrac{ns-(k-s)}{k}$.

As a consequence, we then have

$f(w)$ is increasing for $w \leq \dfrac{ns-(k-s)}{k}$, and

$f(w)$ is decreasing for $w \geq \dfrac{ns-(k-s)}{k}$

as required. □

By taking $s = r = c$ in $f(w) = \binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}$, we get the quadratic function $g(w) = \binom{w}{c}\binom{n-w}{k-c}$ (for selectors). Hence we have the following corollary.

Corollary 2.4.1. The function

$$g(w) = \binom{w}{c}\binom{n-w}{k-c}$$

gets its maximum at

$$w = \frac{nc-(k-c)}{k}.$$

Lemma 2.4.4. For any positive integers n, d, r, s with $k = d + r \leq n$ and $1 \leq s \leq r$,

$$\frac{n(n-1)\cdots(n-s+1)(n-s)(n-s-1)\cdots(n-r-d+1)}{n(n-\frac{k}{s})\cdots(n-k+\frac{k}{s})\cdot n\cdot(n-\frac{k}{k-s})\cdots(n-k+\frac{k}{k-s})} \leq 1.$$

Proof. Without loss of generality, let $s \leq k-s$ and thus $1 \leq \frac{k}{k-s} \leq 2 \leq \frac{k}{s}$. Moreover, we note that the left hand side is

$$\begin{aligned}&\frac{n(n-1)\cdots(n-s+1)(n-s)(n-s-1)\cdots(n-r-d+1)}{n(n-\frac{k}{s})\cdots(n-k+\frac{k}{s})\cdot n\cdot(n-\frac{k}{k-s})\cdots(n-k+\frac{k}{k-s})}\\ =&\frac{\prod\limits_{0\leq i\leq r+d-1}(n-i)}{\prod\limits_{0\leq i\leq s-1}(n-i\cdot\dfrac{k}{s})\prod\limits_{0\leq j\leq k-s-1}(n-j\cdot\dfrac{k}{k-s})}.\end{aligned}$$

To prove this inequality, we will rearrange the terms in the denominator so that $\dfrac{n-t}{n-f(t)} \leq 1$ for each t with $0 \leq t \leq r+d-1$, i.e., we will give a bijection

$$f : \{0, 1, ..., r+d-1\} \to \{i\cdot\frac{k}{s} \mid 0 \leq i \leq s-1\}\bigcup\{j\cdot\frac{k}{k-s} \mid 0 \leq j \leq k-s-1\}$$

with the property that $f(t) \leq t$ for each t. Note that the element 0 will be counted twice as $0\cdot\frac{k}{s}$ and $0\cdot\frac{k}{k-s}$ respectively in the range of the function f. Note also that if $i\cdot\frac{k}{s} = i + i\cdot\frac{k-s}{s} > t$, then $\frac{s}{k-s} < \frac{i}{t-i}$ and hence $j\cdot\frac{k}{k-s} = j(1+\frac{s}{k-s}) < j(1+\frac{i}{t-i}) = j(\frac{t}{t-i}) = j(\frac{t}{j}) = t$, where $i+j = t$.

A such function f is defined recursively as follows. For $t = 0, 1, 2$, let $f(0) = 0_{\frac{k}{s}}$, $f(1) = 0_{\frac{k}{k-s}}$, $f(2) = \frac{k}{k-s}$. For $3 \leq t \leq r+d-1$, let i (resp. j) be the smallest positive integers such that $i\cdot\frac{k}{s}$ (resp. $j\cdot\frac{k}{k-s}$)$\notin \{f(0), f(1), ..., f(t-1)\}$ if they exist, it follows that $t = i+j$.

1. Let $f(t) = i\cdot\frac{k}{s}$ if $i\cdot\frac{k}{s} \leq t$.

2. Otherwise, we define $f(t) = j\cdot\frac{k}{k-s}$.

3. Finally, suppose t is large and there is no such i, note that

   $$\frac{n-s}{n} \leq \frac{n-s-1}{n-\frac{k}{k-s}} \leq \cdots \leq \frac{n-r-d+1}{n-k+\frac{k}{k-s}} \text{ and } \frac{n-r-d+1}{n-k+\frac{k}{k-s}} < 1,$$

   we have $\dfrac{n-t}{n-(t-s)\frac{k}{k-s}} \leq 1$ for all $s \leq t \leq r+d-1$, we define $f(t) = (t-s)\frac{k}{k-s}$.

Cearly, the function f defined above is 1-1, onto, and $f(t) \leq t$ for all $0 \leq t \leq r+d-1$ as required. □

Lemma 2.4.5. For any positive integers n, d, r, s with $k = d + r \leq n$ and $1 \leq s \leq r$, let $n' \geq n$ be the smallest positive integer such that $w = \frac{n's}{k}$ is an integer, then

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \leq \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}.$$

Proof.

$$\begin{aligned}
&\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \\
=&\frac{\frac{(n')!}{(n'-d)!d!}\cdot\frac{(n'-d)!}{(n'-d-r)!r!}}{\frac{(n'-w)!}{(n'-w-d)!d!}\cdot\frac{(n'-w-d)!}{(n'-w-d-r+s)!(r-s)!}\cdot\frac{w!}{(w-s)!s!}} \\
=&\frac{\frac{1}{r!}n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n'-d-r+1)}{\frac{1}{(r-s)!s!}w(w-1)\cdots(w-s+1)(n'-w)(n'-w-1)\cdots(n'-w-d-r+s+1)} \\
=&\frac{1}{\binom{r}{s}}\cdot\frac{n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n'-d-r+1)}{\frac{n's}{k}(\frac{n's}{k}-1)\cdots(\frac{n's}{k}-s+1)(n'-\frac{n's}{k})(n'-\frac{n's}{k}-1)\cdots(n'-\frac{n's}{k}-d-r+s+1)} \\
=&\frac{1}{\binom{r}{s}}\cdot\frac{1}{(\frac{s}{k})^s(\frac{k-s}{k})^{k-s}}\cdot\frac{n'(n'-1)\cdots(n'-s+1)(n'-s)(n'-s-1)\cdots(n'-d-r+1)}{n'(n'-\frac{k}{s})\cdots(n'-k+\frac{k}{s})\cdot n'\cdot(n'-\frac{k}{k-s})\cdots(n'-k+\frac{k}{k-s})} \\
\leq&\frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}} \text{ (by Lemma 2.4.4).}
\end{aligned}$$

□

Lemma 2.4.6. For any positive integers n, d, r, s with $k = d + r \leq n$ and $1 \leq s \leq r$, let $n' \geq n$ be the smallest positive integer such that $w = \frac{n's}{k}$ is an integer, then

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}) < k[1 + \ln(\frac{n}{k} + 1)] + \ln\binom{k-s}{d}.$$

Proof. First we note that $n' < n + k$ for such n'. Since $\binom{a}{b} \le (\frac{ea}{b})^b$, we have

$$
\begin{aligned}
&\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}\\
=&\frac{(n'-w)!}{(n'-w-d)!d!} \cdot \frac{(n'-w-d)!}{(n'-w-d-r+s)!(r-s)!} \cdot \binom{w}{s} \cdot \frac{(d+r-s)!}{(d+r-s)!}\\
=&\binom{n'-w}{d+r-s}\binom{w}{s}\binom{d+r-s}{d}\\
=&\binom{n'-w}{k-s}\binom{w}{s}\binom{k-s}{d}\\
\le&(\frac{e(n'-\frac{n's}{k})}{k-s})^{k-s} \cdot (\frac{e(\frac{n's}{k})}{s})^s \cdot \binom{k-s}{d}\\
=&(e \cdot \frac{n'}{k})^{k-s} \cdot (e\frac{n'}{k})^s \cdot \binom{k-s}{d}\\
=&e^k(\frac{n'}{k})^k \cdot \binom{k-s}{d}\\
<&e^k(\frac{n+k}{k})^k \cdot \binom{k-s}{d}\\
=&e^k(\frac{n}{k}+1)^k \cdot \binom{k-s}{d}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s})\\
<&\ln(e^k(\frac{n}{k}+1)^k \cdot \binom{k-s}{d}) = k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-s}{d}.
\end{aligned}
$$

$\square$

The substitutions of w for various subclasses are summarized in the following table:

types	parameters		
d-disjunct	$s = r = 1,\ z = 1$	$w = \dfrac{n-d}{k}$	$w = \dfrac{n'}{k}$
$(d; z]$-disjunct	$s = r = 1$	$w = \dfrac{n-d}{k}$	$w = \dfrac{n'}{k}$
$(d, r]$-disjunct	$s = r,\ z = 1$	$w = \dfrac{nr-d}{k}$	$w = \dfrac{n'r}{k}$
$(d, r; z]$-disjunct	$s = r$	$w = \dfrac{nr-d}{k}$	$w = \dfrac{n'r}{k}$
(d, r)-disjunct	$s = 1,\ z = 1$	$w = \dfrac{n-(k-1)}{k}$	$w = \dfrac{n'}{k}$
$(d, r; z)$-disjunct	$s = 1$	$w = \dfrac{n-(k-1)}{k}$	$w = \dfrac{n'}{k}$
$(d, s$ out of $r]$-disjunct	$z = 1$	$w = \dfrac{ns-(k-s)}{k}$	$w = \dfrac{n's}{k}$
$(d, s$ out of $r; z]$-disjunct		$w = \dfrac{ns-(k-s)}{k}$	$w = \dfrac{n's}{k}$

types	parameters		
(k, m, n)-selectors	$c = 1,\ z = 1$	$w = \dfrac{n-(k-1)}{k}$	$w = \dfrac{n'}{k}$
$(k, m, n; z)$-selectors	$c = 1$	$w = \dfrac{n-(k-1)}{k}$	$w = \dfrac{n'}{k}$
(k, m, c, n)-selectors	$z = 1$	$w = \dfrac{nc-(k-c)}{k}$	$w = \dfrac{n'c}{k}$
$(k, m, c, n; z)$-selectors		$w = \dfrac{nc-(k-c)}{k}$	$w = \dfrac{n'c}{k}$

Chapter 3

The Stein-Lovász Theorem and its extension

3.1 The Stein-Lovász Theorem

We now introduce the Stein-Lovász theorem as follows. The Stein-Lovász theorem was first used by Stein [11] and Lovász [10] in studying some combinatorial covering problems. In [6], the authors applied this theorem to some problems in coding theory. The Stein-Lovász theorem is now stated and the proof is included for completeness [8], with a minor modification.

Theorem 3.1.1. [8] Let A be a (0,1) matrix with N rows and M columns. Assume that each row contains at least v ones, and each column at most a ones. Then there exists an $N \times K$ submatrix C with

$$K < (\frac{N}{a}) + (\frac{M}{v}) \ln a \leq (\frac{M}{v})(1 + \ln a),$$

such that C does not contain an all-zero row.

Proof. A constructive approach for producing C is presented. Let $A_a = A$. We begin by picking the maximal number K_a of columns from A_a, whose supports are pairwise disjoint and each column having a ones (perhaps, $K_a = 0$). Discarding these columns and all rows incident to one of them, we are left with a $k_a \times (M - K_a)$ matrix A_{a-1}, where $k_a = N - aK_a$. Clearly, the columns of A_{a-1} have at most $a-1$ ones (indeed, otherwise such a column could be added to the previously discarded set, contradicting its maximality). Now we remove from A_{a-1} a maximal number K_{a-1} of columns having $a-1$ ones and whose supports are pairwise disjoint, thus getting a $k_{a-1} \times (M - K_a - K_{a-1})$ matrix A_{a-2}, where $k_{a-1} = N - aK_a - (a-1)K_{a-1}$.

The process will terminate after at most a steps. The union of the columns of the discarded sets form the desired submatrix C with

$$K = \sum_{i=1}^{a} K_i.$$

The first step of the algorithm gives

$$k_a = N - aK_a,$$

which we rewrite, setting $k_{a+1} = N$, as

$$K_a = \frac{k_{a+1} - k_a}{a}.$$

Analogously,

$$K_i = \frac{k_{i+1} - k_i}{i}, i = 1, ..., a.$$

Now we derive an upper bound for k_i by counting the number of ones in A_{i-1} in two ways: every row of A_{i-1} contains at least v ones, and every column at most $i-1$ ones, thus

$$vk_i \leq (i-1)(M - K_a - \cdots - K_i) \leq (i-1)M.$$

Furthermore,

$$K = \sum_{i=1}^{a} K_i = \sum_{i=1}^{a} \frac{k_{i+1} - k_i}{i} = \frac{k_{a+1}}{a} + \frac{k_a}{a(a-1)} + \frac{k_{a-1}}{(a-1)(a-2)} + \cdots + \frac{k_2}{2 \times 1} - k_1$$
$$\leq (N/a) + (M/v)(1/a + 1/(a-1) + \cdots + 1/2),$$

thus giving the result.

$\square$

The greedy procedure as shown in the proof constructs the desired submatrix one column at a time, and hence the algorithm below follows [8].

Algorithm: STEIN-LOVÁSZ(A)

input: A is an $N \times M$ matrix, each column has at most a ones, each row has at least v ones

$C \leftarrow \emptyset$

while A has at least one row

$$\textbf{do} \begin{cases} \text{find a column } c \text{ in } A \text{ having maximum weight} \\ \text{delete all rows of } A \text{ that contain a ``1'' in column } c \\ \text{delete column } c \text{ from } A \end{cases}$$

output: Returns a submatrix of A with no all-zero row

At each state, a new column is added to the submatrix that maximizes the number of "new" rows that are yet uncovered. When all rows are covered, the algorithm stops. It seems quite interesting that we can use the Stein-Lovász theorem to derive bounds for some combinatorial array [8].

3.2 Extension of The Stein-Lovász Theorem

The Stein-Lovász theorem can be further extended from rows of the resulting submatrix with weight at least 1 to the case of rows of the resulting submatrix with weight at least $z \geq 1$. The bound can be further improved when A is a matrix with constant row weight and column weight as well, i.e., in the language of hypergraphs, uniform and regular.

Theorem 3.2.1. Let A be a (0,1) matrix of order $N \times M$, and let v, a, z be positive integers. Assume that each row contains at least v ones, and each column at most a ones. Then there exists a submatrix C of order $N \times K$ with

$$K < \frac{v}{v-(z-1)}\, z(\frac{M}{v})(1+\ln a),$$

such that each row of C has weight at least z.

More specifically, if the matrix is v-uniform and a-regular, the upper bound can then be reduced to

$$K < z(\frac{M}{v})(1+\ln a).$$

The strategy for the proof of Theorem 3.2.1 is as follows:

1. Use the Stein-Lovász theorem to obtain a submatrix C_1 with each row has weight at least 1.

2. Choose some columns in the matrix $A \backslash C_1$ to combine with the submatrix C_1 to form a submatrix C_2 with each row has weight at least 2.

3. Choose some columns in the matrix $A \backslash C_2$ to combine with the submatrix C_2 to form a submatrix C_3 with each row has weight at least 3.

4. Step by step, and finally we obtain the desired submatrix $C = C_z$ with each row has weight at least z.

Note that this upper bound makes sense only if

$$\frac{v}{v-(z-1)}\, z(\frac{M}{v})(1+\ln a) < M,$$

i.e.,

$$z < \frac{v+1}{2+\ln a}$$

in general, or if

$$z(\frac{M}{v})(1+\ln a) < M,$$

i.e.,

$$z < \frac{v}{1+\ln a}$$

for the case of uniform and regular.

Proof. A constructive approach for producing C is presented. Let $A_1 = A$. By the Stein-Lovász theorem, there exists an $N \times M_1$ submatrix $C_1 (= B_1^{'} = B_1)$ of A_1 with $M_1 < \frac{M}{v}(1+\ln a)$ such that each row of C_1 has weight at least 1.

The algorithm used in the proof of the Stein-Lovász theorem shows that some rows of C_1 have weight exactly 1. Let R_1 be the set of indices of those rows and $|R_1| = r_1$. Let A_2 be the submatrix of order $r_1 \times (M - M_1)$ obtained from A_1 by deleting the submatrix C_1 and the i-th row, $i \notin R_1$ as well. Then each row of A_2 contains at least $v-1$ ones, and each column at most a ones. Again, by the Stein-Lovász theorem, there exists an $r_1 \times M_2$ submatrix $B_2^{'}$ with $M_2 < \frac{M-M_1}{v-1}(1+\ln a)$ such that each row of $B_2^{'}$ has weight at least 1. Let B_2 be the matrix of order $N \times M_2$ obtained from $B_2^{'}$ by adding the i-th row, $i \notin R_1$. Let C_2 be the matrix of order $N \times (M_1 + M_2)$ obtained by the union of B_1 and B_2. Then C_2 is a submatrix of A with each row weight at least 2.

Similarly, there exist some rows of C_2 that have weight exactly 2. Let R_2 be the set of indices of those rows and $|R_2| = r_2$. Continue in this way, we have:
For $2 \le i \le z$,

1. A_i is a matrix of order $r_{i-1} \times (M - \sum_{j=1}^{i-1} M_j)$, and each row contains at least $v-(i-1)$ ones, and each column at most a ones.
2. $B_i^{'}$ is an $r_{i-1} \times M_i$ submatrix of A_i with $M_i < \dfrac{M - \sum_{j=1}^{i-1} M_j}{v-(i-1)}(1+\ln a)$, and each row has

weight at least 1.

For $1 \le i \le z$,

3. B_i is a matrix of order $N \times M_i$.

4. C_i is an $N \times \sum_{j=1}^{i} M_j$ submatrix of A, and each row has weight at least i.

Hence, $C = C_z$ is the submatrix required, that is,

$$
\begin{aligned}
K &= \sum_{j=1}^{z} M_j = M_1 + M_2 + \cdots + M_z \\
&< \frac{M}{v}(1+\ln a) + \frac{M - M_1}{v-1}(1+\ln a) + \cdots + \frac{M - \sum_{j=1}^{z-1} M_j}{v-(z-1)}(1+\ln a) \\
&< \frac{M}{v}(1+\ln a) + \frac{M}{v-1}(1+\ln a) + \cdots + \frac{M}{v-(z-1)}(1+\ln a) \\
&= M(1+\ln a)\left(\frac{1}{v} + \frac{1}{v-1} + \cdots + \frac{1}{v-(z-1)}\right) \\
&< M(1+\ln a)\left(\frac{1}{v-(z-1)} + \frac{1}{v-(z-1)} + \cdots + \frac{1}{v-(z-1)}\right) \\
&= \frac{z}{v-(z-1)} M(1+\ln a) \\
&= \frac{v}{v-(z-1)} z\left(\frac{M}{v}\right)(1+\ln a),
\end{aligned}
$$

thus gives the result.

More specifically, for the case of uniform and regular, using similar argument as above with a minor modification. First we note that $Nv = Ma$ by counting the weight of A in two ways. For $2 \le i \le z$, A_i is a matrix of order $r_{i-1} \times (M - \sum_{j=1}^{i-1} M_j)$, and each row contains exactly $v-(i-1)$ ones, and each column at most a ones. Moreover, a lower bound for $\sum_{j=1}^{i-1} M_j$ is derived by counting the weight of the submatrix C_{i-1} in two ways; each row of C_{i-1} contains at least $i-1$ ones, and each column exactly a ones, thus $N(i-1) \le (\sum_{j=1}^{i-1} M_j)a$, and hence $\frac{M}{v}(i-1) \le \sum_{j=1}^{i-1} M_j$ for $2 \le i \le z$. Furthermore,

$$
\begin{aligned}
K &= \sum_{j=1}^{z} M_j = M_1 + M_2 + \cdots + M_z \\
&< \frac{M}{v}(1+\ln a) + \frac{M-M_1}{v-1}(1+\ln a) + \cdots + \frac{M - \sum_{j=1}^{z-1} M_j}{v-(z-1)}(1+\ln a) \\
&\leq \frac{M}{v}(1+\ln a) + \frac{M-\frac{M}{v}}{v-1}(1+\ln a) + \cdots + \frac{M-(z-1)\cdot\frac{M}{v}}{v-(z-1)}(1+\ln a) \\
&= \frac{M}{v}(1+\ln a) + \frac{M}{v}(1+\ln a) + \cdots + \frac{M}{v}(1+\ln a) \\
&= z(\frac{M}{v})(1+\ln a),
\end{aligned}
$$

thus gives the result.

□

Remark 3.2.1. Since this upper bound makes sense only if

$$z < \frac{v+1}{2+\ln a}$$

in general,

$$z < \frac{v+1}{2+\ln a} < \frac{v+1}{2} < \frac{v+2}{2} = \frac{v}{2} + 1.$$

Then

$$z - 1 < \frac{v}{2}$$

and thus

$$\frac{v}{v-(z-1)} = 1 + \frac{z-1}{v-(z-1)} < 1 + 1 = 2.$$

Hence

$$K < \frac{v}{v-(z-1)} z \frac{M}{v}(1+\ln a) < 2z\frac{M}{v}(1+\ln a)$$

for general case.

Similarly, Theorem 3.2.1 can be restated in the language of hypergraphs in the following corollary. Recall that a subset $T \subseteq X$ such that $|T \bigcap E| \geq z$ for any hyperedge E is called a *z-cover* of the hypergarph H, and the minimum size of a z-cover of the hypergraph H is denoted by $\tau_z(H)$.

Corollary 3.2.1. For a hypergraph $H = (X, \Gamma)$ and a positive integer $z \geq 2$,

$$\tau_z(H) < \frac{2z|X|}{\min_{E\in\Gamma}|E|}(1 + \ln \triangle),$$

where $\triangle = \max_{x\in X} |\{E : E \in \Gamma \text{ with } x \in E\}|$.

More specifically, for the case of uniform and regular, we have the following corollary.

Corollary 3.2.2. Let $H = (X, \Gamma)$ be a v-uniform and a-regular hypergraph with vertex set X and edge set Γ, then

$$\tau_z(H) < z\frac{|X|}{v}(1 + \ln a).$$

We conjecture that $\tau_z(H) \leq z\tau_1(H)$ holds for hypergraphs which are uniform and regular. However, it is not true in general as shown in the following example. For the hypergraph H with $X = \{1, 2, 3, ..., 8\}$ and $\Gamma = \{\{1, 2, 3\}, \{4, 5, 6\}, \{1, 7, 8\}\}$. It is easy to see that $\{1, 4\}$ is a 1-cover with minimum size, hence $\tau_1(H) = 2$. Similarly, $\{1, 2, 4, 5, 7\}$ is a 2-cover with minimum size, hence $\tau_2(H) = 5$. This shows that $\tau_2(H) = 5 > 2 \cdot 2 = 2\tau_1(H)$.

Chapter 4

Some Applications of the extended Stein-Lovász Theorem

The Stein-Lovász theorem was first used in dealing with the upper bounds for the sizes in the model of (k, m, n)-selecters [2]. Inspired by this work, it was also used in dealing with the upper bounds for the sizes of $(d, r; z]$-disjunct matrices [5]. In Section 4.1 and Section 4.2, the extended Stein-Lovász theorem will be used in dealing the upper bounds for the sizes of several disjunct matrices (Theorem 4.1.1$\sim$ 4.1.8) and selectors (Theorem 4.2.1 $\sim$ 4.2.4), respectively. Those upper bounds for the sizes of uniform splitting systems, uniform separating systems, covering designs and lotto designs are given in Section 4.3 (Theorem 4.3.1 $\sim$ 4.3.8) respectively.

4.1 Bounds for several disjunct matrices

Note that upper bounds for the sizes of d-disjunct matrices, $(d, r]$-disjunct matrices, (d, r)-disjunct matrices and $(d, s$ out of $r]$-disjunct matrices were given in [13, 14] by the Lovász Local Lemma.

Recall that $t(n, d)$ is the minimum size over all d-disjunct matrices with n columns.

Theorem 4.1.1. For any positive integers n and d, if $k = d + 1 \leq n$, then

$$t(n, d) < k(\frac{k}{d})^d\{1 + k[1 + \ln(\frac{n}{k} + 1)]\}.$$

Proof. For $1 \leq w \leq n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{1}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,s) \mid D \in \binom{[n]}{d}, s \in [n] \setminus D\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D,s) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and the entry of v at s is one; and 0 otherwise.

Observe that each row of A has weight $\binom{n-(d+1)}{w-1}$, and each column of A has weight $\binom{n-w}{d}\binom{w}{1}$. By the Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{1}] \times t$ having no zero rows, where

$$t < \frac{\binom{n}{w}}{\binom{n-(d+1)}{w-1}}\{1 + \ln[\binom{n-w}{d}\binom{w}{1}]\} = \frac{\binom{n}{d}\binom{n-d}{1}}{\binom{n-w}{d}\binom{w}{1}}\{1 + \ln[\binom{n-w}{d}\binom{w}{1}]\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a d-disjunct matrix of order $t \times n$. We then have

$$t(n,d) < \frac{\binom{n}{d}\binom{n-d}{1}}{\binom{n-w}{d}\binom{w}{1}}\{1 + \ln[\binom{n-w}{d}\binom{w}{1}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \frac{n'}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{1}}{\binom{n'-w}{d}\binom{w}{1}} \leq k(\frac{k}{d})^d$$

by Lemma 2.4.5 (taking $s = r = 1$), and

$$\ln(\binom{n'-w}{d}\binom{w}{1}) < k[1 + \ln(\frac{n}{k} + 1)]$$

by Lemma 2.4.6 (taking $s = r = 1$). Therefore,

$$t(n,d) \le t(n',d) < \frac{\binom{n'}{d}\binom{n'-d}{1}}{\binom{n'-w}{d}\binom{w}{1}}\{1+\ln[\binom{n'-w}{d}\binom{w}{1}]\}$$

$$< k(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}$$

as required.

$\square$

Recall that $t(n,d;z]$ is the minimum size over all $(d;z]$-disjunct matrices with n columns.

Theorem 4.1.2. For any positive integers n, d and z, if $k = d+1 \le n$, then

$$t(n,d;z] < zk(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}.$$

Proof. For $1 \le w \le n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{1}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,s) \mid D \in \binom{[n]}{d}, s \in [n] \setminus D\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D,s) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and the entry of v at s is one; and 0 otherwise.

Observe that each row of A has weight $\binom{n-(d+1)}{w-1}$, and each column of A has weight $\binom{n-w}{d}\binom{w}{1}$. By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{1}] \times t$ with each row weight at least z, where

$$t < \frac{z\binom{n}{w}}{\binom{n-(d+1)}{w-1}}\{1+\ln[\binom{n-w}{d}\binom{w}{1}]\} = \frac{z\binom{n}{d}\binom{n-d}{1}}{\binom{n-w}{d}\binom{w}{1}}\{1+\ln[\binom{n-w}{d}\binom{w}{1}]\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d;z]$-disjunct matrix of order $t \times n$. We then have

$$t(n,d;z] < \frac{z\binom{n}{d}\binom{n-d}{1}}{\binom{n-w}{d}\binom{w}{1}}\{1+\ln[\binom{n-w}{d}\binom{w}{1}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \frac{n'}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{1}}{\binom{n'-w}{d}\binom{w}{1}} \leq k(\frac{k}{d})^d$$

by Lemma 2.4.5 (taking $s = r = 1$), and

$$\ln(\binom{n'-w}{d}\binom{w}{1}) < k[1+\ln(\frac{n}{k}+1)]$$

by Lemma 2.4.6 (taking $s = r = 1$). Therefore,

$$\begin{aligned} t(n,d;z] \leq t(n',d;z] &< \frac{z\binom{n'}{d}\binom{n'-d}{1}}{\binom{n'-w}{d}\binom{w}{1}}\{1+\ln[\binom{n'-w}{d}\binom{w}{1}]\} \\ &< zk(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\} \end{aligned}$$

as required.

$\square$

Recall that $t(n,d,r]$ is the minimum size over all $(d,r]$-disjunct matrices with n columns.

Theorem 4.1.3. For any positive integers n, d and r, if $k = d + r \leq n$, then

$$t(n,d,r] < (\frac{k}{r})^r(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}.$$

Proof. For $r \leq w \leq n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and

$V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D, R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and the entries of v over R are all one; and 0 otherwise.

Observe that each row of A has weight $\binom{n-(d+r)}{w-r}$, and each column of A has weight $\binom{n-w}{d}\binom{w}{r}$. By the Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}] \times t$ having no zero rows, where

$$t < \frac{\binom{n}{w}}{\binom{n-(d+r)}{w-r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\} = \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{w}{r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d, r]$-disjunct matrix of order $t \times n$. We then have

$$t(n, d, r] < \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{w}{r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \frac{n'r}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{w}{r}} \leq (\frac{k}{r})^r(\frac{k}{d})^d$$

by Lemma 2.4.5 (taking $s = r$), and

$$\ln(\binom{n'-w}{d}\binom{w}{r}) < k[1+\ln(\frac{n}{k}+1)]$$

by Lemma 2.4.6 (taking $s = r$). Therefore,

$$t(n,d,r] \le t(n',d,r] < \frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{w}{r}}\{1+\ln[\binom{n'-w}{d}\binom{w}{r}]\}$$

$$< (\frac{k}{r})^r(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}$$

as required.

$\square$

Recall that $t(n,d,r;z]$ is the minimum size over all $(d,r;z]$-disjunct matrices with n columns.

Theorem 4.1.4. [5] For any positive integers n, d, r and z, if $k = d + r \le n$, then

$$t(n,d,r;z] < z(\frac{k}{r})^r(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}.$$

Proof. For $r \le w \le n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D,R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and the entries of v over R are all one; and 0 otherwise.

Observe that each row of A has weight $\binom{n-(d+r)}{w-r}$, and each column of A has weight $\binom{n-w}{d}\binom{w}{r}$. By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}] \times t$ with each row weight at least z, where

$$t < \frac{z\binom{n}{w}}{\binom{n-(d+r)}{w-r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\} = \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{w}{r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d,r;z]$-disjunct matrix of order $t \times n$. We then have

$$t(n,d,r] < \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{w}{r}}\{1+\ln[\binom{n-w}{d}\binom{w}{r}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \dfrac{n'r}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{w}{r}} \leq (\frac{k}{r})^r(\frac{k}{d})^d$$

by Lemma 2.4.5 (taking $s = r$), and

$$\ln(\binom{n'-w}{d}\binom{w}{r}) < k[1+\ln(\frac{n}{k}+1)]$$

by Lemma 2.4.6 (taking $s = r$). Therefore,

$$\begin{aligned} t(n,d,r;z] \leq t(n',d,r;z] &< \frac{z\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{w}{r}}\{1+\ln[\binom{n'-w}{d}\binom{w}{r}]\} \\ &< z(\frac{k}{r})^r(\frac{k}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\} \end{aligned}$$

as required.

$\square$

Recall that $t(n,d,r)$ is the minimum size over all (d,r)-disjunct matrices with n columns.

Theorem 4.1.5. For any positive integers n, d and r, if $k = d + r \leq n$, then

$$t(n,d,r) < \frac{k}{r}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{d}\}.$$

Proof. For $1 \leq w \leq n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and

$V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D, R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and at least one entry of v over R is one; and 0 otherwise.

Observe that each row of A has weight $\sum_{j=1}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}$, and each column of A has weight $\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}$. By the Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}] \times t$ having no zero rows, where

$$
\begin{aligned}
t &< \frac{\binom{n}{w}}{\sum_{j=1}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}} \{1 + \ln[\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}]\} \\
&= \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}} \{1 + \ln[\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}]\}.
\end{aligned}
$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a (d, r)-disjunct matrix of order $t \times n$. We then have

$$
\begin{aligned}
t(n,d,r) &< \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}} \{1 + \ln[\binom{n-w}{d} \sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}]\} \\
&< \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{n-w-d}{r-1}\binom{w}{1}} \{1 + \ln[\binom{n-w}{d}\binom{n-w-d}{r-1}\binom{w}{1}]\},
\end{aligned}
$$

by Lemma 2.4.1. Let $n' \geq n$ be the smallest positive integer such that $w = \frac{n'}{k}$ is an integer.

We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}} \leq \frac{k}{r}(1+\frac{1}{k-1})^{k-1}$$

by Lemma 2.4.5 (taking $s=1$), and

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}) < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-1}{d}$$

by Lemma 2.4.6 (taking $s=1$). Therefore,

$$\begin{aligned} t(n,d,r) \leq t(n',d,r) &< \frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}}\{1+\ln[\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}]\} \\ &< \frac{k}{r}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{d}\} \end{aligned}$$

as required.

□

Recall that $t(n,d,r;z)$ is the minimum size over all $(d,r;z)$-disjunct matrices with n columns.

Theorem 4.1.6. For any positive integers n, d, r and z, if $k = d + r \leq n$, then

$$t(n,d,r;z) < z\frac{k}{r}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{d}\}.$$

Proof. For $1 \leq w \leq n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D,R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and at least one entry of v over R is one; and 0 otherwise.

Observe that each row of A has weight $\sum_{j=1}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}$, and each column of A has weight $\binom{n-w}{d}\sum_{j=1}^{\min(r,w)} \binom{n-w-d}{r-j}\binom{w}{j}$. By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}]\times t$ with each row weight at least z, where

$$\begin{aligned} t &< \frac{z\binom{n}{w}}{\sum_{j=1}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}}\{1+\ln[\binom{n-w}{d}\sum_{j=1}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\} \\ &= \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=1}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=1}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}. \end{aligned}$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d,r;z)$-disjunct matrix of order $t\times n$. We then have

$$\begin{aligned} t(n,d,r;z) &< \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=1}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=1}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\} \\ &< \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{n-w-d}{r-1}\binom{w}{1}}\{1+\ln[\binom{n-w}{d}\binom{n-w-d}{r-1}\binom{w}{1}]\}, \end{aligned}$$

by Lemma 2.4.1. Let $n'\geq n$ be the smallest positive integer such that $w=\frac{n'}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}} \leq \frac{k}{r}(1+\frac{1}{k-1})^{k-1}$$

by Lemma 2.4.5 (taking $s = 1$), and

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}) < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-1}{d}$$

by Lemma 2.4.6 (taking $s = 1$). Therefore,

$$\begin{aligned} t(n,d,r;z) &\le t(n',d,r;z) \\ &< \frac{z\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}}\{1+\ln[\binom{n'-w}{d}\binom{n'-w-d}{r-1}\binom{w}{1}]\} \\ &< z\frac{k}{r}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{d}\} \end{aligned}$$

as required.

□

Recall that $t(n, d, r, s]$ is the minimum size over all $(d, s$ out of $r]$-disjunct matrices with n columns.

Theorem 4.1.7. For any positive integers n, d, r and s, with $1 \le s \le r$, if $k = d + r \le n$, then

$$t(n,d,r,s] < \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}\}.$$

Proof. For $s \le w \le n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D, R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and at least s entries of v over R are one; and 0 otherwise.

Observe that each row of A has weight $\sum_{j=s}^{\min(r,w)}\binom{r}{j}\binom{n-(d+r)}{w-j}$, and each column of A

has weight $\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}$. By the Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}]\times t$ having no zero rows, where

$$\begin{aligned} t &< \frac{\binom{n}{w}}{\sum_{j=s}^{\min(r,w)}\binom{r}{j}\binom{n-(d+r)}{w-j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\} \\ &= \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}. \end{aligned}$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d, s$ out of $r]$-disjunct matrix of order $t\times n$. We then have

$$\begin{aligned} t(n,d,r,s] &< \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\} \\ &< \frac{\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}}\{1+\ln[\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}]\}, \end{aligned}$$

by Lemma 2.4.1. Let $n'\geq n$ be the smallest positive integer such that $w=\frac{n's}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}}\leq\frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}$$

by Lemma 2.4.5, and

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}) < k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}$$

by Lemma 2.4.6. Therefore,

$$\begin{aligned}
t(n,d,r,s] &\le t(n',d,r,s] \\
&< \frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}}\{1+\ln[\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}]\} \\
&< \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}\}
\end{aligned}$$

as required.

$\square$

Recall that $t(n,d,r,s;z]$ is the minimum size over all $(d, s$ out of $r;z]$-disjunct matrices with n columns.

Theorem 4.1.8. For any positive integers n, d, r, s and z, with $1 \le s \le r$, if $k = d + r \le n$, then

$$t(n,d,r,s;z] < \frac{z(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}\}.$$

Proof. For $s \le w \le n-d$, let A be the binary matrix of order $[\binom{n}{d}\binom{n-d}{r}] \times \binom{n}{w}$ with rows and columns indexed by $\{(D,R) \mid D \in \binom{[n]}{d}, R \in \binom{[n]}{r}$ with $D \bigcap R$ empty$\}$ and $V = \{v \mid v \in \{0,1\}^n, wt(v) = w\}$ respectively. The entry of A at the row indexed by the pair (D,R) and the column indexed by the vector $v \in V$ is 1 if the entries of v over D are all zero and at least s entries of v over R are one; and 0 otherwise.

Observe that each row of A has weight $\displaystyle\sum_{j=s}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}$, and each column of A has weight $\displaystyle\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}$. By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{n}{d}\binom{n-d}{r}] \times t$ with each row weight at least z, where

$$t < \frac{z\binom{n}{w}}{\sum_{j=s}^{\min(r,w)} \binom{r}{j}\binom{n-(d+r)}{w-j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}$$

$$= \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form a $(d, s$ out of $r; z]$-disjunct matrix of order $t \times n$. We then have

$$t(n,d,r,s;z] < \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}}\{1+\ln[\binom{n-w}{d}\sum_{j=s}^{\min(r,w)}\binom{n-w-d}{r-j}\binom{w}{j}]\}$$

$$< \frac{z\binom{n}{d}\binom{n-d}{r}}{\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}}\{1+\ln[\binom{n-w}{d}\binom{n-w-d}{r-s}\binom{w}{s}]\},$$

by Lemma 2.4.1. Let $n' \geq n$ be the smallest positive integer such that $w = \frac{n's}{k}$ is an integer. We have

$$\frac{\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}} \leq \frac{(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}$$

by Lemma 2.4.5, and

$$\ln(\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}) < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-s}{d}$$

by Lemma 2.4.6. Therefore,

$$\begin{aligned}
t(n,d,r,s;z] &\leq t(n',d,r,s;z] \\
&< \frac{z\binom{n'}{d}\binom{n'-d}{r}}{\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}}\{1+\ln[\binom{n'-w}{d}\binom{n'-w-d}{r-s}\binom{w}{s}]\} \\
&< \frac{z(\frac{k}{s})^s(\frac{k}{k-s})^{k-s}}{\binom{r}{s}}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-s}{d}\}
\end{aligned}$$

as required.

□

4.2 Bounds for several selectors

The notion of (k,m,n)-selectors was first introduced by De Bonis, Gasieniec and Vaccaro in [2] , and it was then generalized to the notion of (k,m,c,n)-selectors [1]. It is interesting to remark that the notions of (k,m,n)-selecters and (k,m,c,n)-selectors are equivalent to $(k,m,1,n;1)$-selectors and $(k,m,c,n;1)$-selectors respectively. Note that upper bounds for the sizes of (k,m,n)-selectors were also given in [14] by the Lovász Local Lemma.

Following similar arguments in [2] and [1] with a minor modification, upper bounds for the sizes of several selectors are given below.

Recall that $t_s(k,m,n)$ is the minimum size over all (k,m,n)-selectors.

Theorem 4.2.1.

$$t_s(k,m,n) < \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\}.$$

Proof. For $1 \leq w \leq n-k+1$, let $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ and $U = \{u \in \{0,1\}^k \mid wt(u) = 1\}$. Moreover, for any $A \subseteq U$ of size r, $r = 1, ..., k$, and any set $S \in \binom{[n]}{k}$, define $E_{A,S} = \{x \in X : x|_S \in A\}$.

Let M be the binary matrix of order $[\binom{k}{k-m+1}\binom{n}{k}] \times \binom{n}{w}$ with rows and columns indexed by $\Gamma = \{E_{A,S} \subseteq X \mid A \subseteq U$ with $|U| = k, |A| = k-m+1, S \in \binom{[n]}{k}\}$ and $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ respectively. The entry of M at the row indexed by the set $E_{A,S}$ and the column indexed by the vector $x \in X$ is 1 if $x \in E_{A,S}$; and 0 otherwise.

Observe that each row of M has weight $\binom{k-m+1}{1}\binom{n-k}{w-1}$, and each column of M has weight $\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{(k-m+1)-1}$. By the Stein-Lovász theorem, there exists a submatrix M' of M of order $[\binom{k}{k-m+1}\binom{n}{k}] \times t$ having no zero rows, where

$$\begin{aligned} t &< \frac{\binom{n}{w}}{(k-m+1)\binom{n-k}{w-1}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\} \\ &= \frac{\binom{k}{k-m+1}\binom{n}{k}}{\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\}. \end{aligned}$$

Note that the equality is obtained by counting the weight of M in two ways.

It suffices to show that the matrix M^* of order $t \times n$ formed by the columns of M' is a (k,m,n)-selector, that is, any submatrix of k arbitrary columns of M^* contains a submatrix with each row weight exactly one, with at least m distinct rows.

Let $x_1, x_2, ..., x_t$ be the t rows of M^* and let $T = \{x_1, x_2, ..., x_t\}$. Suppose contradictorily that there exists a set $S \in \binom{[n]}{k}$ such that the submatrix $M^*|_S$ of M^* contains a submatrix with each row weight exactly one, with at most $m-1$ distinct rows. Let $u_{j_1}, u_{j_2}, ..., u_{j_q}$ be such rows, with $q \leq m-1$; let A be any subset of $U \backslash \{u_{j_1}, u_{j_2}, ..., u_{j_q}\}$ of cardinality $|A| = k-m+1$, then we have $T \bigcap E_{A,S} = \emptyset$, contradicting the fact that M' is a matrix of order $[\binom{k}{k-m+1}\binom{n}{k}] \times t$ having no zero rows. Hence we have

$$t_s(k,m,n) < \frac{\binom{k}{k-m+1}\binom{n}{k}}{\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \dfrac{n'}{k}$ is an integer. We have

$$\begin{aligned} \frac{\binom{k}{k-m+1}\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}} &= \frac{\frac{k!}{(m-1)!(k-m+1)!}}{\frac{(k-1)!}{(m-1)!(k-m)!}} \cdot \frac{\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}} = \frac{k}{k-m+1} \cdot \frac{\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}} \\ &\leq \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1} \end{aligned}$$

by Lemma 2.4.5 (taking $s = r = 1$), and

$$\ln[\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}] < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-1}{k-m}$$

by Lemma 2.4.6 (taking $s = r = 1$). Therefore, we have

$$\begin{aligned} t_s(k,m,n) \leq & t_s(k,m,n') \\ < & \frac{\binom{k}{k-m+1}\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}]\} \\ < & \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\} \end{aligned}$$

as required. □

Recall that $t_s(k,m,n;z)$ is the minimum size over all $(k,m,n;z)$-selectors.

Theorem 4.2.2.

$$t_s(k,m,n;z) < \frac{(k-m+1)(z-1)+1}{k-m+1}k(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\}.$$

Proof. For $1 \leq w \leq n-k+1$, let $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ and $U = \{u \in \{0,1\}^k \mid wt(u) = 1\}$. Moreover, for any $A \subseteq U$ of size r, $r = 1, ..., k$, and any set $S \in \binom{[n]}{k}$, define $E_{A,S} = \{x \in X : x|_S \in A\}$.

Let M be the binary matrix of order $[\binom{k}{k-m+1}\binom{n}{k}] \times \binom{n}{w}$ with rows and columns indexed by $\Gamma = \{E_{A,S} \subseteq X \mid A \subseteq U \text{ with } |U| = k, |A| = k-m+1, S \in \binom{[n]}{k}\}$ and $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ respectively. The entry of M at the row indexed by the set $E_{A,S}$ and the column indexed by the vector $x \in X$ is 1 if $x \in E_{A,S}$; and 0 otherwise.

Observe that each row of M has weight $\binom{k-m+1}{1}\binom{n-k}{w-1}$, and each column of M has weight $\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{(k-m+1)-1}$. By the extended Stein-Lovász theorem, there exists a submatrix M' of M of order $[\binom{k}{k-m+1}\binom{n}{k}] \times t$ with each row weight at least $(k-m+1)(z-1)+1$, where

$$t < \frac{[(k-m+1)(z-1)+1]\binom{n}{w}}{(k-m+1)\binom{n-k}{w-1}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\}$$
$$= \frac{[(k-m+1)(z-1)+1]\binom{k}{k-m+1}\binom{n}{k}}{\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\}.$$

Note that the equality is obtained by counting the weight of M in two ways.

It suffices to show that the matrix M^* of order $t \times n$ formed by the columns of M' is a $(k, m, n; z)$-selector, that is, any submatrix of k arbitrary columns of M^* contains z disjiont submatrices with each row weight exactly one, with at least m distinct rows each.

Let $x_1, x_2, ..., x_t$ be the t rows of M^* and let $T = \{x_1, x_2, ..., x_t\}$. Suppose contradictorily that there exists a set $S \in \binom{[n]}{k}$ such that the submatrix $M^*|_S$ of M^* contains at most $z-1$ disjoint submatrices with each row weight exactly one, with at least m distinct rows. Moreover, $M^*|_S$ contains another disjoint submatrix with at most $m-1$ distinct rows with weight exactly one. Let $u_{j_1}, u_{j_2}, ..., u_{j_q}$ be such rows, with $q \le m-1$; let A be any subset of $U \backslash \{u_{j_1}, u_{j_2}, ..., u_{j_q}\}$ of cardinality $|A| = k-m+1$, then we have $|T \bigcap E_{A,S}| < (k-m+1)(z-1)+1$, contradicting the fact that M' is a matrix of order $[\binom{k}{k-m+1}\binom{n}{k}] \times t$ with each row weight at least $(k-m+1)(z-1)+1$. Hence we have

$$t_s(k,m,n;z) < \frac{[(k-m+1)(z-1)+1]\binom{k}{k-m+1}\binom{n}{k}}{\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n-w}{k-1}\binom{k-1}{k-m}]\}.$$

Let $n' \ge n$ be the smallest positive integer such that $w = \dfrac{n'}{k}$ is an integer. We have

$$\frac{\binom{k}{k-m+1}\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}} = \frac{\frac{k!}{(m-1)!(k-m+1)!}}{\frac{(k-1)!}{(m-1)!(k-m)!}} \cdot \frac{\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}} = \frac{k}{k-m+1} \cdot \frac{\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}}$$
$$\le \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}$$

by Lemma 2.4.5 (taking $s = r = 1$), and

$$\ln[\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}] < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{k-1}{k-m}$$

by Lemma 2.4.6 taking $s = r = 1$). Therefore, we have

$$\begin{aligned} t_s(k,m,n;z) \le & t_s(k,m,n';z) \\ < & \frac{[(k-m+1)(z-1)+1]\binom{k}{k-m+1}\binom{n'}{k}}{\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}}\{1+\ln[\binom{w}{1}\binom{n'-w}{k-1}\binom{k-1}{k-m}]\} \\ < & \frac{(k-m+1)(z-1)+1}{k-m+1}k(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{k-m}\} \end{aligned}$$

as required.

□

Moreover, we will also give better, but nonconstructive upper bounds for the sizes of $(k,m,n;z)$-selectors as follows.

Lemma 4.2.1. A $(k,m,n;z)$-selector M is $(m-1, k-m+1; z)$-disjunct.

Proof. Suppose not. Then there exist k columns $C_1, C_2, ..., C_k$ of M such that

$$|\bigcup_{i=m}^{k} C_i \setminus \bigcup_{i=1}^{m-1} C_i| \le z-1,$$

that is, there exist at most $z-1$ rows with row weight 1 such that each of them hits exactly one of the columns $C_m, C_{m+1}, ..., C_k$. Hence there exist at most $z-1$ disjoint $m \times k$ submatrices of the identity matrix I_k, it contradicts the fact that M is a $(k,m,n;z)$-selector.

□

By Lemma 4.2.1, upper bounds for the sizes of $(k,m,n;z)$-selectors can be obtained from that of $(m-1, k-m+1; z)$-disjunct matrices. Hence nonconstructive upper bounds for the sizes of $(k,m,n;z)$-selectors are given below.

Corollary 4.2.1.

$$t_s(k,m,n;z) < \frac{zk}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{m-1}\}.$$

Proof. Put

$$(d, r) = (m-1, k-(m-1))$$

in $t(n, d, r; z)$ as shown in Theorem 4.1.6, we then have

$$t_s(k, m, n; z) < \frac{zk}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{m-1}\}.$$

$\square$

Remark 4.2.1. For the case $z = 1$, i.e.,

$$t_s(k, m, n; 1) < \frac{k}{k-m+1}(1+\frac{1}{k-1})^{k-1}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{k-1}{m-1}\}$$

was also given in Theorem 4.2.1 followed by a constructive proof in term of the Stein-Lovász theorem.

Recall that $t_s(k, m, c, n)$ is the minimum size over all (k, m, c, n)-selectors.

Theorem 4.2.3. For $a = \binom{k}{c}$,

$$t_s(k, m, c, n) < \frac{1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\}.$$

Proof. For $c \le w \le n-k+c$, let $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ and $U = \{u \in \{0,1\}^k \mid wt(u) = c\}$. Moreover, for any $A \subseteq U$ of size r, $r = 1, ..., \binom{k}{c}$, and any set $S \in \binom{[n]}{k}$, define $E_{A,S} = \{x \in X : x|_S \in A\}$.

Let $a = \binom{k}{c}$ and M be the binary matrix of order $[\binom{a}{a-m+1}\binom{n}{k}] \times \binom{n}{w}$ with rows and columns indexed by $\Gamma = \{E_{A,S} \subseteq X \mid A \subseteq U \text{ with } |U| = a, |A| = a-m+1, S \in \binom{[n]}{k}\}$ and $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ respectively. The entry of M at the row indexed by the set $E_{A,S}$ and the column indexed by the vector $x \in X$ is 1 if $x \in E_{A,S}$; and 0 otherwise.

Observe that each row of M has weight $\binom{a-m+1}{1}\binom{n-k}{w-c}$, and each column of M has weight $\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{(a-m+1)-1}$. By the Stein-Lovász theorem, there exists a submatrix M' of M of order $[\binom{a}{a-m+1}\binom{n}{k}] \times t$ having no zero rows, where

$$t < \frac{\binom{n}{w}}{(a-m+1)\binom{n-k}{w-c}}\{1+\ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\}$$
$$= \frac{\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\}.$$

Note that the equality is obtained by counting the weight of M in two ways.

It suffices to show that the matrix M^* of order $t \times n$ formed by the columns of M' is a (k, m, c, n)-selector, that is, any submatrix of k arbitrary columns of M^* contains a submatrix with each row weight exactly c, with at least m distinct rows.

Let $x_1, x_2, ..., x_t$ be the t rows of M^* and let $T = \{x_1, x_2, ..., x_t\}$. Suppose contradictorily that there exists a set $S \in \binom{[n]}{k}$ such that the submatrix $M^*|_S$ of M^* contains a submatrix with each row weight exactly c, with at most $m-1$ distinct rows. Let $u_{j_1}, u_{j_2}, ..., u_{j_q}$ be such rows, with $q \leq m-1$; let A be any subset of $U \backslash \{u_{j_1}, u_{j_2}, ..., u_{j_q}\}$ of cardinality $|A| = a-m+1$, then we have $T \bigcap E_{A,S} = \emptyset$, contradicting the fact that M' is a matrix of order $[\binom{a}{a-m+1}\binom{n}{k}] \times t$ having no zero rows. Hence we have

$$t_s(k, m, c, n) < \frac{\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\}.$$

Let $n' \geq n$ be the smallest positive integer such that $w = \dfrac{n'c}{k}$ is an integer. We have

$$\frac{\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}} = \frac{\frac{a!}{(m-1)!(a-m+1)!}}{\frac{(a-1)!}{(m-1)!(a-m)!}} \cdot \frac{\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}} = \frac{1}{a-m+1} \cdot \frac{a\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}}$$
$$= \frac{1}{a-m+1} \cdot \frac{\binom{k}{c}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}} \leq \frac{1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}$$

by Lemma 2.4.5 (taking $s = r = c$), and

$$\ln[\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}] < k[1+\ln(\frac{n}{k}+1)] + \ln\binom{a-1}{a-m}$$

by Lemma 2.4.6 (taking $s = r = c$). Therefore, we have

$$\begin{aligned} t_s(k,m,c,n) \leq & t_s(k,m,c,n') \\ < & \frac{\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}]\} \\ < & \frac{1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\} \end{aligned}$$

as required.

$\square$

Recall that $t_s(k,m,c,n;z)$ is the minimum size over all $(k,m,c,n;z)$-selectors.

Theorem 4.2.4. For $a = \binom{k}{c}$,

$$t_s(k,m,c,n;z) < \frac{(a-m+1)(z-1)+1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\}.$$

Proof. For $c \leq w \leq n-k+c$, let $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ and $U = \{u \in \{0,1\}^k \mid wt(u) = c\}$. Moreover, for any $A \subseteq U$ of size r, $r = 1, ..., \binom{k}{c}$, and any set $S \in \binom{[n]}{k}$, define $E_{A,S} = \{x \in X : x|_S \in A\}$.

Let $a = \binom{k}{c}$ and M be the binary matrix of order $[\binom{a}{a-m+1}\binom{n}{k}] \times \binom{n}{w}$ with rows and columns indexed by $\Gamma = \{E_{A,S} \subseteq X \mid A \subseteq U$ with $|U| = a, |A| = a-m+1, S \in \binom{[n]}{k}\}$ and $X = \{x \in \{0,1\}^n \mid wt(x) = w\}$ respectively. The entry of M at the row indexed by the set $E_{A,S}$ and the column indexed by the vector $x \in X$ is 1 if $x \in E_{A,S}$; and 0 otherwise.

Observe that each row of M has weight $\binom{a-m+1}{1}\binom{n-k}{w-c}$, and each column of M has weight $\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{(a-m+1)-1}$. By the extended Stein-Lovász theorem, there exists a submatrix M' of M of order $[\binom{a}{a-m+1}\binom{n}{k}] \times t$ with each row weight at least $(a-m+1)(z-1)+1$, where

$$\begin{aligned} t < & \frac{[(a-m+1)(z-1)+1]\binom{n}{w}}{(a-m+1)\binom{n-k}{w-c}}\{1+\ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\} \\ = & \frac{[(a-m+1)(z-1)+1]\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\}. \end{aligned}$$

Note that the equality is obtained by counting the weight of M in two ways.

It suffices to show that the matrix M^* of order $t \times n$ formed by the columns of M' is a $(k, m, c, n; z)$-selector, that is, any submatrix of k arbitrary columns of M^* contains z disjoint submatrices with each row weight exactly c, with at least m distinct rows each.

Let $x_1, x_2, ..., x_t$ be the t rows of M^* and let $T = \{x_1, x_2, ..., x_t\}$. Suppose contradictorily that there exists a set $S \in \binom{[n]}{k}$ such that the submatrix $M^*|_S$ of M^* contains at most $z-1$ disjoint submatrices with each row weight exactly c, with at least m distinct rows each. Moreover, $M^*|_S$ contains another disjoint submatrix with at most $m-1$ distinct rows with weight exactly c. Let $u_{j_1}, u_{j_2}, ..., u_{j_q}$ be such rows, with $q \le m-1$; let A be any subset of $U \backslash \{u_{j_1}, u_{j_2}, ..., u_{j_q}\}$ of cardinality $|A| = a-m+1$, then we have $|T \bigcap E_{A,S}| < (a-m+1)(z-1)+1$, contradicting the fact that M' is a matrix of order $[\binom{a}{a-m+1}\binom{n}{k}] \times t$ with each row weight at least $(a-m+1)(z-1)+1$. Hence we have

$$t_s(k, m, c, n; z) < \frac{[(a-m+1)(z-1)+1]\binom{a}{a-m+1}\binom{n}{k}}{\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}}\{1 + \ln[\binom{w}{c}\binom{n-w}{k-c}\binom{a-1}{a-m}]\}.$$

Let $n' \ge n$ be the smallest positive integer such that $w = \dfrac{n'c}{k}$ is an integer. We have

$$\begin{aligned}\frac{\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}} &= \frac{\frac{a!}{(m-1)!(a-m+1)!}}{\frac{(a-1)!}{(m-1)!(a-m)!}} \cdot \frac{\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}} = \frac{1}{a-m+1} \cdot \frac{a\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}} \\ &= \frac{1}{a-m+1} \cdot \frac{\binom{k}{c}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}} \le \frac{1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}\end{aligned}$$

by Lemma 2.4.5 (taking $s = r = c$), and

$$\ln[\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}] < k[1 + \ln(\frac{n}{k}+1)] + \ln\binom{a-1}{a-m}$$

by Lemma 2.4.6 (taking $s = r = c$). Therefore, we have

$$
\begin{aligned}
t_s(k,m,c,n;z) \leq & t_s(k,m,c,n';z) \\
& < \frac{[(a-m+1)(z-1)+1]\binom{a}{a-m+1}\binom{n'}{k}}{\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}}\{1+\ln[\binom{w}{c}\binom{n'-w}{k-c}\binom{a-1}{a-m}]\} \\
& < \frac{(a-m+1)(z-1)+1}{a-m+1}(\frac{k}{c})^c(1+\frac{1}{k-c})^{k-c}\{1+k[1+\ln(\frac{n}{k}+1)]+\ln\binom{a-1}{a-m}\}
\end{aligned}
$$

as required.

$\square$

4.3 Bounds for some set systems

Note that upper bounds for the minimum number of blocks of uniform (m,t)-splitting systems were given in [7] by the Lovász Local Lemma. Note further that upper bounds for the minimum number of blocks of uniform (m,t)-splitting systems, (v,k,t)-covering designs, and (v,k,t,p)-lotto designs were given in [8] by the classical Stein-Lovász theorem, the proofs are included for completeness.

In this section, upper bounds for the minimum number of blocks of uniform $(m,t;z)$-splitting systems, uniform $(m,t_1,t_2;z)$-separating systems, $(v,k,t;z)$-covering designs and $(v,k,t,p;z)$-lotto designs will be derived by using the extended Stein-Lovász theorem.

Recall that $SP(m,t)$ is the minimum number of blocks of uniform (m,t)-splitting systems.

Theorem 4.3.1. [8]

$$SP(m,t) < \frac{\binom{m}{\frac{m}{2}}}{\binom{t}{\frac{t}{2}}\binom{m-t}{\frac{m}{2}-\frac{t}{2}}}\{1+\ln[\binom{\frac{m}{2}}{\frac{t}{2}}^2]\}.$$

Proof. Let A be the binary matrix of order $\binom{m}{t} \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{T \mid T \in \binom{[m]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if B splits T; and 0 otherwise.

Observe that each row of A has weight

$$v = \binom{t}{\frac{t}{2}} \binom{m-t}{\frac{m}{2} - \frac{t}{2}},$$

and each column of A has weight

$$a = \binom{\frac{m}{2}}{\frac{t}{2}} \binom{\frac{m}{2}}{\frac{t}{2}} = \binom{\frac{m}{2}}{\frac{t}{2}}^2.$$

By the Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{m}{t} \times N$ having no zero rows, where

$$N < \frac{\binom{m}{\frac{m}{2}}}{v} \{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform (m,t)-splitting system with N blocks, as required.

$\square$

Recall that $SP(m,t;z)$ is the minimum number of blocks of uniform $(m,t;z)$-splitting systems.

Theorem 4.3.2.

$$SP(m,t;z) < \frac{z\binom{m}{\frac{m}{2}}}{\binom{t}{\frac{t}{2}}\binom{m-t}{\frac{m}{2}-\frac{t}{2}}} \{1 + \ln[\binom{\frac{m}{2}}{\frac{t}{2}}^2]\}.$$

Proof. Let A be the binary matrix of order $\binom{m}{t} \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{T \mid T \in \binom{[m]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if B splits T; and 0 otherwise.

Observe that each row of A has weight

$$v = \binom{t}{\frac{t}{2}} \binom{m-t}{\frac{m}{2} - \frac{t}{2}},$$

and each column of A has weight

$$a = \binom{\frac{m}{2}}{\frac{t}{2}} \binom{\frac{m}{2}}{\frac{t}{2}} = \binom{\frac{m}{2}}{\frac{t}{2}}^2.$$

By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{m}{t} \times N$ with each row weight at least z, where

$$N < \frac{z\binom{m}{\frac{m}{2}}}{v}\{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform $(m, t; z)$-splitting system with N blocks, as required.

$\square$

For the sizes of uniform (m, t_1, t_2)-separating systems and uniform $(m, t_1, t_2; z)$-separating systems, we only discuss the case $t_1 \neq t_2$. Note that the case $t_1 = t_2$ can be handled in a similar way.

Recall that $SE(m, t_1, t_2)$ is the minimum number of blocks of uniform (m, t_1, t_2)-separating systems.

Theorem 4.3.3.

$$SE(m, t_1, t_2) < \frac{\binom{m}{\frac{m}{2}}}{2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1}}\{1 + \ln[2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}]\}.$$

Proof. Let A be the binary matrix of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{(T_1, T_2) \mid T_1 \in \binom{[m]}{t_1}, T_2 \in \binom{[m]}{t_2}$ with $T_1 \bigcap T_2$ empty $\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the pair (T_1, T_2) and the column indexed by the vector $B \in \Gamma$ is 1 if B separates the pair (T_1, T_2); and 0 otherwise.

Observe that each row of A has weight

$$v = \binom{m-(t_1+t_2)}{\frac{m}{2}-t_1} + \binom{m-(t_1+t_2)}{\frac{m}{2}-t_2} = 2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1},$$

and each column of A has weight

$$a = \binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2} + \binom{\frac{m}{2}}{t_2}\binom{\frac{m}{2}}{t_1} = 2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}.$$

By the Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times N$ having no zero rows, where

$$N < \frac{\binom{m}{\frac{m}{2}}}{v}\{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform (m, t_1, t_2)-splitting system with N blocks, as required.

□

Recall that $SE(m, t_1, t_2; z)$ is the minimum number of blocks of uniform $(m, t_1, t_2; z)$-separating systems.

Theorem 4.3.4.

$$SE(m, t_1, t_2; z) < \frac{z\binom{m}{\frac{m}{2}}}{2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1}}\{1 + \ln[2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}]\}.$$

Proof. Let A be the binary matrix of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times \binom{m}{\frac{m}{2}}$ with rows and columns indexed by $\{(T_1, T_2) \mid T_1 \in \binom{[m]}{t_1}, T_2 \in \binom{[m]}{t_2}$ with $T_1 \bigcap T_2$ empty $\}$ and $\Gamma = \{B \mid B \in \binom{[m]}{\frac{m}{2}}\}$ respectively. The entry of A at the row indexed by the pair (T_1, T_2) and the column indexed by the vector $B \in \Gamma$ is 1 if B separates the pair (T_1, T_2); and 0 otherwise.

Observe that each row of A has weight

$$v = \binom{m-(t_1+t_2)}{\frac{m}{2}-t_1} + \binom{m-(t_1+t_2)}{\frac{m}{2}-t_2} = 2\binom{m-(t_1+t_2)}{\frac{m}{2}-t_1},$$

and each column of A has weight

$$a = \binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2} + \binom{\frac{m}{2}}{t_2}\binom{\frac{m}{2}}{t_1} = 2\binom{\frac{m}{2}}{t_1}\binom{\frac{m}{2}}{t_2}.$$

By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $[\binom{m}{t_1}\binom{m-t_1}{t_2}] \times N$ with each row weight at least z, where

$$N < \frac{z\binom{m}{\frac{m}{2}}}{v}\{1 + \ln a\}.$$

It is straightforward to show that the columns of M form an uniform $(m, t_1, t_2; z)$-splitting system with N blocks, as required.

□

Recall that $C(v, k, t)$ is the minimum number of blocks of uniform (v, k, t)-covering designs.

Theorem 4.3.5. [8]

$$C(v,k,t) < \frac{\binom{v}{t}}{\binom{k}{t}}\{1 + \ln\binom{k}{t}\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $T \subseteq B$; and 0 otherwise.

Observe that each row of A has weight $\binom{v-t}{k-t}$, and each column of A has weight $\binom{k}{t}$. By the Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ having no zero rows, where

$$N < \frac{\binom{v}{k}}{\binom{v-t}{k-t}}\{1 + \ln\binom{k}{t}\} = \frac{\binom{v}{t}}{\binom{k}{t}}\{1 + \ln\binom{k}{t}\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form an (v, k, t)-covering design with N blocks, as required.

□

Recall that $C(v, k, t; z)$ is the minimum number of blocks of uniform $(v, k, t; z)$-covering designs.

Theorem 4.3.6.

$$C(v,k,t;z) < \frac{z\binom{v}{t}}{\binom{k}{t}}\{1 + \ln\binom{k}{t}\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $T \subseteq B$; and 0 otherwise.

Observe that each row of A has weight $\binom{v-t}{k-t}$, and each column of A has weight $\binom{k}{t}$. By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ with each row weight at least z, where

$$N < \frac{z\binom{v}{k}}{\binom{v-t}{k-t}}\{1 + \ln \binom{k}{t}\} = \frac{z\binom{v}{t}}{\binom{k}{t}}\{1 + \ln \binom{k}{t}\}.$$

Note that the equality is obtained by counting the weight of A in two ways. It is straightforward to show that the columns of M form an $(v, k, t; z)$-covering design with N blocks, as required.

□

Recall that $L(v, k, t, p)$ is the minimum number of blocks of uniform (v, k, t, p)-lotto designs.

Theorem 4.3.7. [8]

$$L(v, k, t, p) < \frac{\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}}\{1 + \ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $|T \bigcap B| \geq p$; and 0 otherwise.

Observe that each row of A has weight

$$\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i},$$

and each column of A has weight

$$\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}.$$

By the Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ having no zero rows, where

$$N < \frac{\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}}\{1+\ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

It is straightforward to show that the columns of M form an (v,k,t,p)-lotto design with N blocks, as required.

$\square$

Recall that $L(v,k,t,p;z)$ is the minimum number of blocks of uniform $(v,k,t,p;z)$-lotto designs.

Theorem 4.3.8.

$$L(v,k,t,p;z) < \frac{z\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}}\{1+\ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

Proof. Let A be the binary matrix of order $\binom{v}{t} \times \binom{v}{k}$ with rows and columns indexed by $\{T \mid T \in \binom{[v]}{t}\}$ and $\Gamma = \{B \mid B \in \binom{[v]}{k}\}$ respectively. The entry of A at the row indexed by the T and the column indexed by the vector $B \in \Gamma$ is 1 if $|T \bigcap B| \geq p$; and 0 otherwise.

Observe that each row of A has weight

$$\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i},$$

and each column of A has weight

$$\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}.$$

By the extended Stein-Lovász theorem, there exists a submatrix M of A of order $\binom{v}{t} \times N$ with each row weight at least z, where

$$N < \frac{z\binom{v}{k}}{\sum_{i=p}^{\min(t,k)} \binom{t}{i}\binom{v-t}{k-i}} \{1 + \ln[\sum_{i=p}^{\min(t,k)} \binom{k}{i}\binom{v-k}{t-i}]\}.$$

It is straightforward to show that the columns of M form an $(v, k, t, p; z)$-lotto design with N blocks, as required.

□

Chapter 5

Conclusion

In this thesis, we derive the extended Stein-Lovász theorem to deal with more combinatorial problems. From the strategy of the proof in Theorem 3.2.1, it is easy to see that the extended Stein-Lovász theorem also provides an algorithmic way to dealing with the existence of good coverings and then deriving some upper bounds related to some combinatorial structures in Chapter 4. Note that most of these upper bounds obtained in Chapter 4 are roughly the same as those derived by the basic probabilistic method including the Lovász Local Lemma (see Appendix). Thus, due to its constructive nature, the Stein-Lovász theorem can be regarded as a de-randomized algorithm for the probabilistic methods. The relationship between the (extended) Stein-Lovász theorem and the Lovász Local Lemma deserve further study.

Appendix

Some upper bounds for the sizes of several disjunct matrices and selectors obtained by the Lovász Local Lemma and the classical Stein-Lovász theorem are survey in the following.

d-disjunct matrices

$t(d,n) \leq (d+1)(1+\frac{1}{d})^d\{1+\ln[(d+1)(\binom{n}{d+1}-\binom{n-d-1}{d+1})]\}$ [13]

(by the Lovász Local Lemma)

$(d,r]$-disjunct matrices

$t(n,d,r] \leq (1+\frac{d}{r})^r(1+\frac{r}{d})^d\{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]\}$ [14]

(by the Lovász Local Lemma)

$(d,r;z]$-disjunct matrices

$t(n,d,r;z] < z(1+\frac{d}{r})^r(1+\frac{r}{d})^d\{1+k[1+\ln(\frac{n}{k}+1)]\}$, $k=d+r$ [5]

(by the classical Stein-Lovász theorem)

(d,r)-disjunct matrices

$t(n,d,r) \leq (1+\frac{d}{r})(1+\frac{r}{d})^{\frac{d}{r}}\{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]\}$ [14]

(by the Lovász Local Lemma)

$(d, s$ out of $r]$-disjunct matrices

$t(n,d,r,s] \leq \dfrac{1+\ln[\binom{n}{d}\binom{n-d}{r}-\binom{n-(d+r)}{d}\binom{n-(d+r)-d}{r}]}{f_{d,r,s}(p)}$ for all $0<p<1$,

where $f_{d,r,s}(p)=(1-p)^d[1-\sum_{i=0}^{s-1}\binom{r}{i}p^i(1-p)^{r-i}]$ [14]

(by the Lovász Local Lemma)

(k,m,n)-selectors

$t_s(k,m,n) < \frac{ek^2}{k-m+1}\ln\frac{n}{k} + \frac{ek(2k-1)}{k-m+1}$ [2]

(by the classical Stein-Lovász theorem)

$t_s(k,m,n) \leq \frac{m}{\binom{k}{m}m!}[k(1+\frac{1}{k-1})^{k-1}]\{1+\ln[\binom{n}{k}-\binom{n-k}{k}]\}$ [14]

(by the Lovász Local Lemma)

(k,m,c,n)-selectors

$t_s(k,m,c,n) < \frac{ek^{c+1}}{z}\ln\lceil\frac{n}{k}\rceil - \frac{ewk^c}{z}\ln c + \frac{ek^c}{z}(c+m+k+z-1)$,
where $z = \binom{k}{c} - m + 1$ [1]

(by the classical Stein-Lovász theorem)

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