國　立　交　通　大　學

應　用　數　學　系

碩　士　論　文

N相黎曼面上的路徑積分及微分方程上之應用

Path Integrals on Riemann Surfaces of Genus N and Its Applications on Differential Equations

研　究　生：施　建　興

指導教授：李　榮　耀

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碩士論文

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摘要

假設 $P_N(u)$ 是一個 u 的多項式函數且 $f(u)=\sqrt{P_N(u)}$ 。 f 在 complex plane $\mathbb{C}$ 上是一個多值函數。在 extended complex plane 上我們利用適合的 cut-structure 建立 f 的 Riemann surface $\mathcal{R}$。則 f 是一個定義在 $\mathcal{R}$ 上的單值函數。接著我們在 f 的代數結構上面做積分的運算。特別地，我們主要針對兩種特別的路徑來積分，分別為 $a-cycle$ 及 $b-cycle$。運用 principle of deformation of paths 來計算這些積分。此外，我們將以上的方法應用在微分方程上。

Path Integrals on Riemann Surfaces of Genus N and Its Applications on Differential Equations

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Abstract

Let $P_N(u)$ be a polynomial of u and let $f(u) = \sqrt{P_N(u)}$. f is a 2-valued function defined on the complex plane $\mathbb{C}$. We construct the Riemann surface $\mathcal{R}$ by a proper cut-structure on the extended complex plane. Then f is a single-valued function on $\mathcal{R}$. Then we do evaluations of path integrals on $\mathcal{R}$ with its algebraic structure for f. In particular, we evaluate integrals along two special paths, $a - cycle$ and $b - cycle$, respectively. We apply the principle of deformation of paths to evaluate those integrals. Furthermore, we apply the above argument to differential equations.

誌　　謝

終於卸下心中那重重的大石頭了，能完成此論文比什麼都好。當然，首要感謝我的指導教授李榮耀教授，從撰寫論文到學位考試期間給了很大的協助以及鼓勵，在最後口試前的緊要關頭，製作投影片上，給了對我非常有用的意見，口試結束後也不忘再次向我提醒往後人生旅途路上需要改進、需要注意的事項。還有也要感謝同學們的關心，在研究所求學期間能夠交到你們這群好同學，也是未來一生中的好朋友，有你們的陪伴真的很好。最後，一定要感謝我偉大的父親及母親，從小辛苦的栽培，雖然他們所擁有學歷不高，但卻辛苦的賺錢，提供了對我而言很好的環境，從小時候用心，細心的照顧，到長大後的關心，都讓我感到溫暖的愛。謹此獻給世上最愛我、最關心我的父母親。

施建興
國立交通大學
2011 年 7 月 8 日

Contents

1 Introduction

1.1 Motivation

Let u be a twice differentiable function of t. Consider the following differential equation

$$u'' + f(u) = 0, \tag{1}$$

where f is a polynomial of u. Multiply the equation by u',

$$u''u' + f(u)u' = 0, \tag{2}$$

and integrate it using change of variables, we obtain

$$\int u''(t)u'(t)\,dt + \int f(u)u'(t)\,dt = E, \text{ where E is a constant.} \tag{3}$$

$$\Longrightarrow \int u'(t)\,du'(t) + \int f(u)\,du(t) = E, \tag{4}$$

$$\Longrightarrow \frac{1}{2}[u'(t)]^2 + F(u(t)) = E, \text{ where } F \text{ is an antiderivative of } f. \tag{5}$$

$$\Longrightarrow u'(t) = \pm\sqrt{2[E - F(u)]} \tag{6}$$

We obtain the first order differential equation

$$\frac{du}{dt} = u'(t) = \sqrt{2[E - F(u)]}, \tag{7}$$

or

$$\int \frac{1}{\sqrt{2[E - F(u)]}}\,du = \int dt. \tag{8}$$

Since $2[E - F(u)]$ is a polynomial of u, it can be written as

$$\begin{aligned} 2[E - F(u)] &= (u - u_1)(u - u_2)\ldots(u - u_n) \\ &= \prod_{k=1}^{n}(u - u_k), \end{aligned}$$

where $u_1, \ldots, u_n$ are the complex roots of the equation $2[E - F(u)] = 0$. Thus, equation (8) can be written as

$$\int \frac{1}{\sqrt{\prod_{k=1}^{n}(u - u_k)}}\,du = \int dt \tag{9}$$

In order to solve for u, we need to evaluate the term

$$\int \frac{1}{\sqrt{\prod_{k=1}^{n}(u - u_k)}}\,du.$$

In the denominator, the u_k's are possibly complex numbers, and they are also the branch points or poles. Here, $u : \mathbb{C} \to \mathbb{C}$, we will see later that the integrand is in fact a multiple-valued function. It is not so easy to evaluate the integral.

In this thesis, we will use the Riemann's approach to evaluate the integrals of this kind. In addition, we will also discuss how to compute the integrals using the computer software "*Mathematica*".

1.2 Stereographic Projection

In this section, we give a short introduction to the concept of *stereographic projection.*

The complex plane together with the point at infinity ∞ is called the *extended complex plane* or the *extended* z*-plane.* One can think of the complex plane as passing through the equator of a unit sphere centered at the point $z = 0$. To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is determined by the intersection of the line through the point z and the north pole N of the sphere with that surface. In like manner, to each point P on the surface of the sphere, other than the north pole N, there correspondings exactly one point z in the plane. By letting the point N of the sphere corresponds to the point ∞, we obtain a one to one corespondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere*, and the correspondence is called a *stereographic projection.*

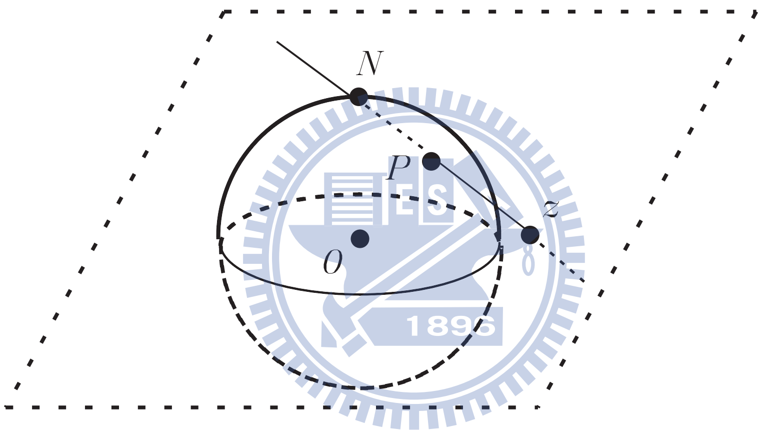


Figure 1

1.3 Some Basic Definitions

Definition 1. *A function f of the complex variable z is **analytic** in an open set if it has a derivative at each point in that set.*

Definition 2. *A **branch** of a multiple-valued function f is any single-valued funtion F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values $f(z)$.*

Definition 3. *A **branch cut** is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Any point that is common to all branch cuts of f is called a **branch point**.*

Definition 4. *A set of points $z = (x, y)$ in the complex plane is said to be an **arc** if*

$$x = x(t), \quad y = y(t), \qquad a \le t \le b,$$

where $x(t)$ and $y(t)$ are continous functions of the real parameter t.

It is convenient to describe the points of an arc C by means of the equation

$$z = z(t), \quad a \leq t \leq b,$$

where

$$z(t) = x(t) + iy(t).$$

Definition 5. *An arc C is a* ***simple arc*** *if it does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a* ***simple closed curve****.*

Definition 6. *A* ***contour****, or* ***piecewise smooth arc****, is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values of $z(t)$ are the same, a contour C is called a* ***simple closed contour****.*

Definition 7. *An analytic function $w = w(z)$ is called an* ***algebraic function*** *if it satisfies a functional equation*

$$a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0, \quad a_0(z) \neq 0, \tag{10}$$

in which the $a_i(z)$ are polynomials in z with complex numbers as coefficients.

One simple example is the algebraic function, $w = \sqrt{z}$, defined by $w^2 - z = 0$. It is not single-valued in the extended z-plane. In the next chapter, we will introduce a new surface on which to consider the algebraic function defined, and on which it is a single-valued function. This surface is called a *Riemann surface*.

2 Riemann Surfaces and Cut Structures

2.1 The Riemann Surface for $f(z) = \sqrt{z}$

We begin with the algebraic function $f(z) = \sqrt{z}$ to explain how to construct the Riemann surface for $f(z)$ such that f is a single-valued function on it.

Let $z = re^{i(\theta+2k\pi)}$, $r \neq 0$, $k \in \mathbb{Z}$. Then

$$f(z) = \sqrt{r}e^{\frac{1}{2}i(\theta+2k\pi)} \tag{11}$$

$$= \sqrt{r}e^{\frac{1}{2}i\theta}e^{ik\pi} \tag{12}$$

$$= \begin{cases} \sqrt{r}e^{\frac{1}{2}i\theta} & \text{if } k \text{ is even,} \\ -\sqrt{r}e^{\frac{1}{2}i\theta} & \text{if } k \text{ is odd.} \end{cases} \tag{13}$$

Thus, f is a two-valued function in the extended z-plane. We use the following way to construct the Riemann surface for $f(z)$.

If we cut the extended z-plane along the negative real axis (the branch cut is drawn using bold dashed line as in Figure 2) and restrict ourselves so as never to continue $f(z)$ over this cut, we get two single-valued branches of $f(z)$, namely,

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad -\pi \leq \theta < \pi,$$

and

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \pi \leq \theta < 3\pi.$$

To build the Riemann surface for $f(z)$, we take two replicas of the z-plane cut along the negative real axis and call them sheet I and sheet II. The cut on each sheet has two edges. We label the edge of the third quadrant with a $+$ and the edge of the second quadrant with a $-$. Then attach the $+$ edge of the cut on sheet I to the $-$ edge of the cut on sheet II, and attach the $-$ edge of the cut on sheet I to the $+$ edge of the cut on sheet II. Thus, whenever we cross the cut, we pass from one sheet to the other.

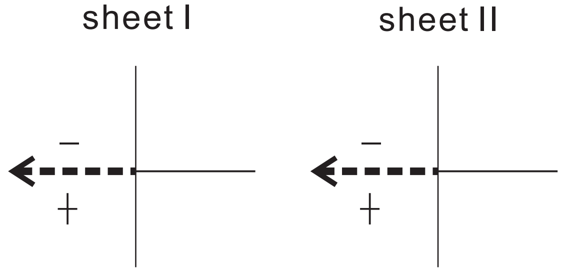


Figure 2

We imagine that the surface as two sheets lying over the extended z-plane, each cut along the negative axis. Using stereographic projection, we can consider the two sheets to be spheres. There is one cut on the surfaces of each sphere.

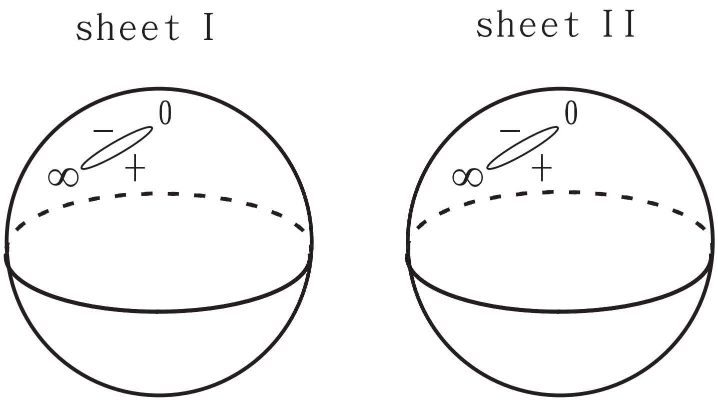


Figure 3

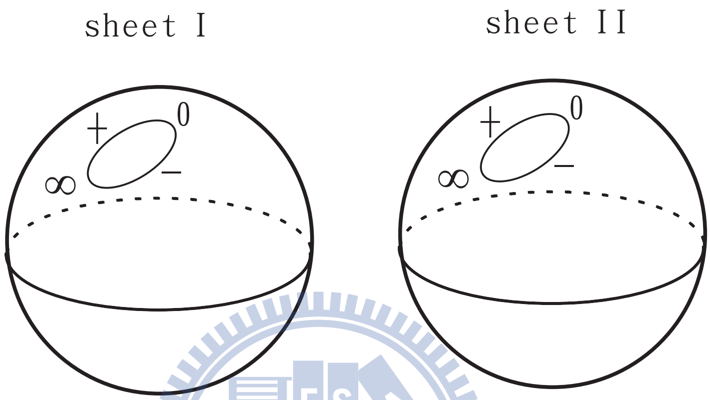


Figure 4

Now imagine that the spheres are made of rubber. By spreading the edges of the cuts, we can deform each sheet into a hemisphere. When each sheet is rotated so that the openings of the hemispheres face each other, the edges marked + and − face each other and the two hemispheres may be pasted together to give us a sphere. We call this surface the ***Riemann surface of genus 0*** for $f(z) = \sqrt{z}$, denoted by $\mathcal{R}_0$ (Figure 6).

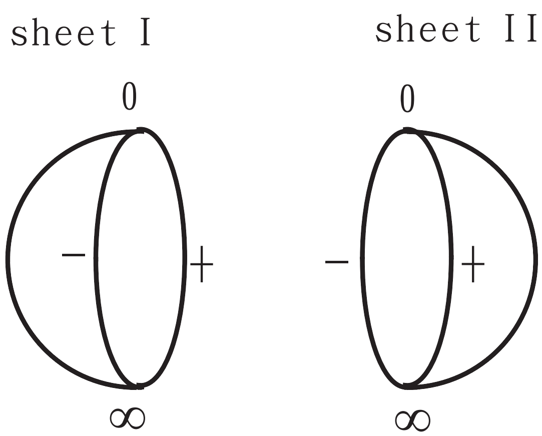


Figure 5

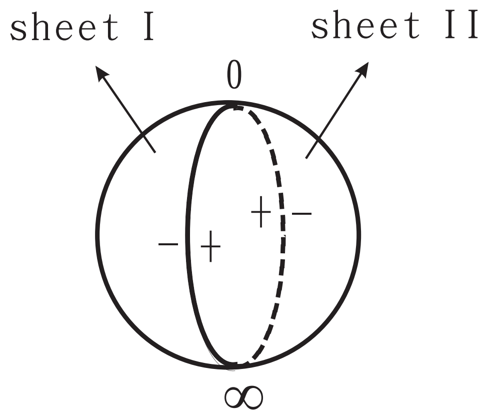


Figure 6

2.2 The Riemann Surface for $f(z) = \sqrt{(z - r_1)(z - r_2)}$

In this section, we discuss how to construct the Riemann surface for the function $f(z) = \sqrt{(z - r_1)(z - r_2)}$, $r_1 \neq r_2$. We will find that this is essentially the same as the situation for $f(z) = \sqrt{z}$.

The two point $z = r_1$ and $z = r_2$ are branch points of $f(z) = \sqrt{(z - r_1)(z - r_2)}$. We obtain two single-valued branches of $f(z)$ by cutting the z-plane along the line segment joining r_1 and r_2. As in section 2.1, we have two replicas of the z-plane along this cut. Joining them, we obtain a two-sheeted Riemann surface on which $f(z)$ is single-valued.

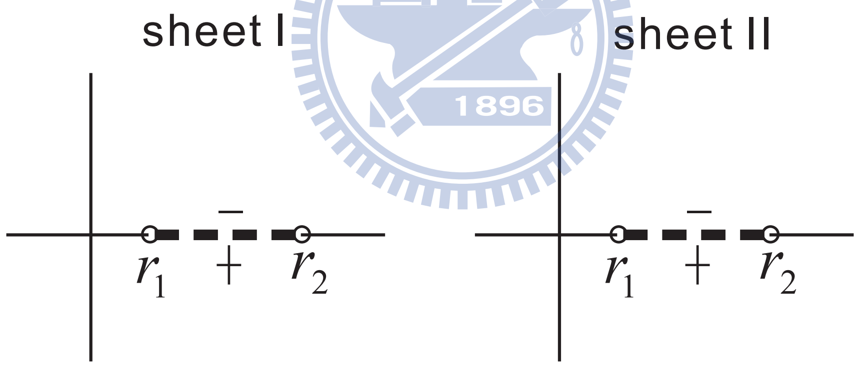


Figure 7

If the surface were made of rubber, it could be deformed continuously into that of $f(z) = \sqrt{z}$ by moving r_1 to ∞ and r_2 to 0 and deforming the cut into the negative real axis. Thus this new surface may also be mapped topologically into a sphere.

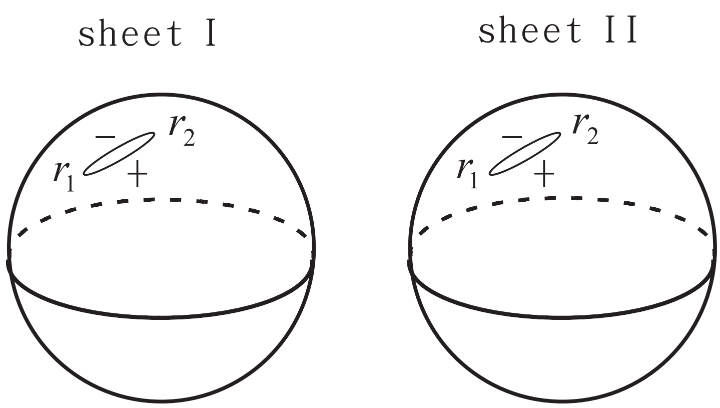


Figure 8

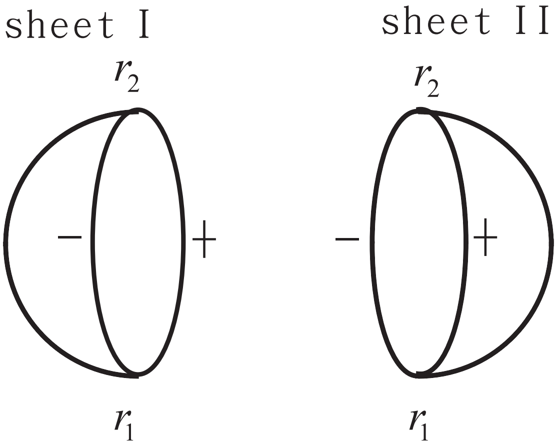


Figure 9

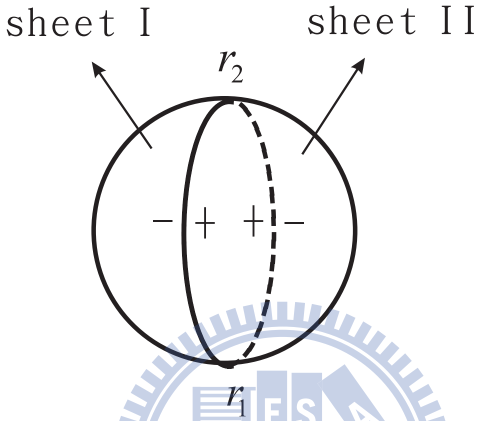


Figure 10

Also, this surface is the Riemann surface of genus 0 for $f(z) = \sqrt{(z - r_1)(z - r_2)}$.

2.3 The Riemann Surface for $f(z) = \sqrt{(z - r_1)(z - r_2)(z - r_3)}$

Now, we look another example whose Riemann surface is defferent from the ones in the earlier examples.

Let $f(z) = \sqrt{(z - r_1)(z - r_2)(z - r_3)}$ be the algebraic function defined by $w^2 = (z - r_1)(z - r_2)(z - r_3)$, where r_1, r_2, r_3 are distinct. For each i, let $z = r_i + re^{i(\theta + 2k\pi)}$, $r \neq 0$, $k \in \mathbb{Z}$ be in the cut plane. We have

$$\sqrt{z - r_i} = \sqrt{(r_i + re^{i(\theta+2k\pi)}) - r_i} \tag{14}$$

$$= \sqrt{re^{i(\theta+2k\pi)}} \tag{15}$$

$$= \begin{cases} \sqrt{r}e^{\frac{1}{2}i\theta} & \text{if } k \text{ is even,} \\ -\sqrt{r}e^{\frac{1}{2}i\theta} & \text{if } k \text{ is odd.} \end{cases} \tag{16}$$

Thus, we go from one point to the other by continuing $f(z)$ over any closed path winding once around one of the roots r_1, r_2, r_3, $\sqrt{z - r_i}$ changes sign when the argument $\theta = \arg(z - r_i)$ changes by 2π.

We cut the z-palne from r_1 to ∞ and from r_2 to r_3. Then we take two copies of the cut z-plane and connect them crosswise over the cuts as before, we obtain a two-sheeted Riemann surface on which $f(z)$ is single-valued.

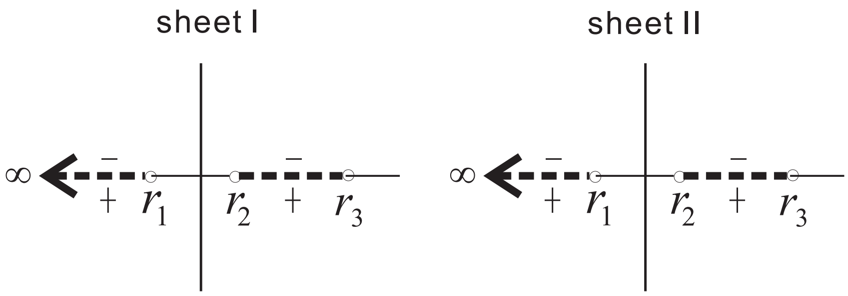


Figure 11

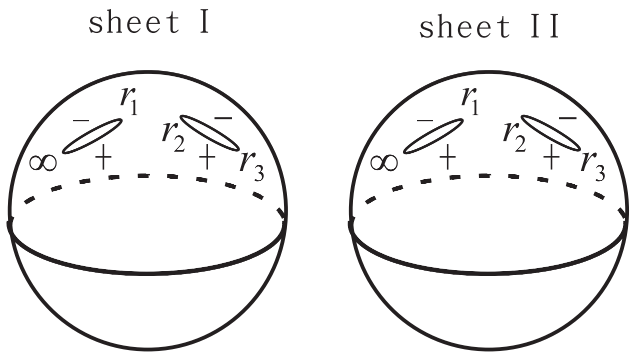


Figure 12

Stretch each cut into a circular hole and rotate the spheres until the holes face each other, as in Figure 13.

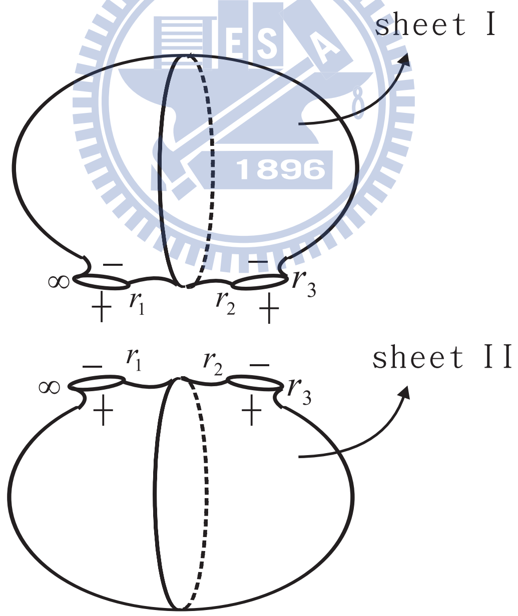


Figure 13

We may join them together so that each + edges is attached to the − edge of the corresponding cut on the other sphere, as in Figure 14.

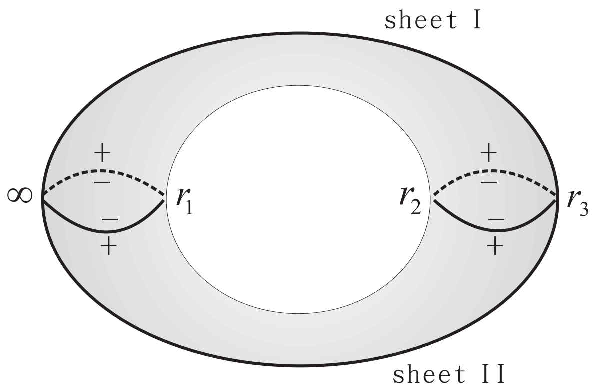


Figure 14

Thus, The two-sheeted Riemann surface can be mapped topologically onto a torus. This surface is called the ***Riemann surface of genus 1*** for $f(z)$, denoted by $\mathcal{R}_1$.

2.4 Riemann Surfaces of Genus N

We now generalize the results from section 2.1 to section 2.3. Let

$$f(z) = \sqrt{P(z)} = \sqrt{(z - r_1)(z - r_2) \cdots (z - r_n)},$$

where $r_1, r_2, \ldots, r_n$ are the roots of the polynomial $P(z)$ of order n.

If the number of roots is even, say $n = 2N + 2$, we can separate the branch points into pairs, $(r_1, r_2), (r_3, r_4), \ldots, (r_{2N+1}, r_{2N+2})$. This gives us $\frac{n}{2} = N + 1$ cuts in the cut plane drawn in Figure 15.

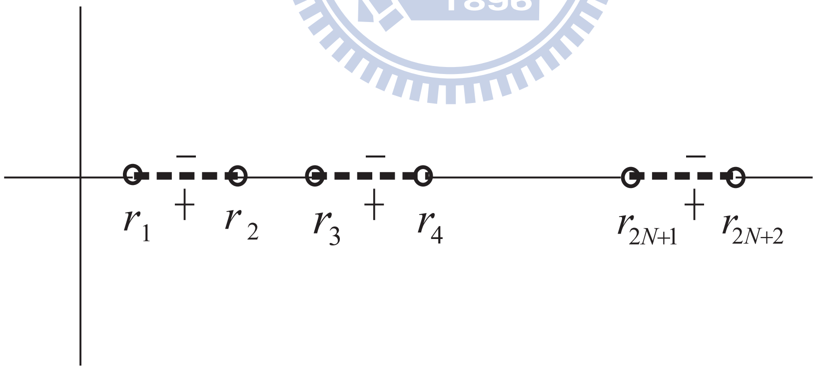


Figure 15

There are N holes in the Riemann surface drawn in Figure 16.

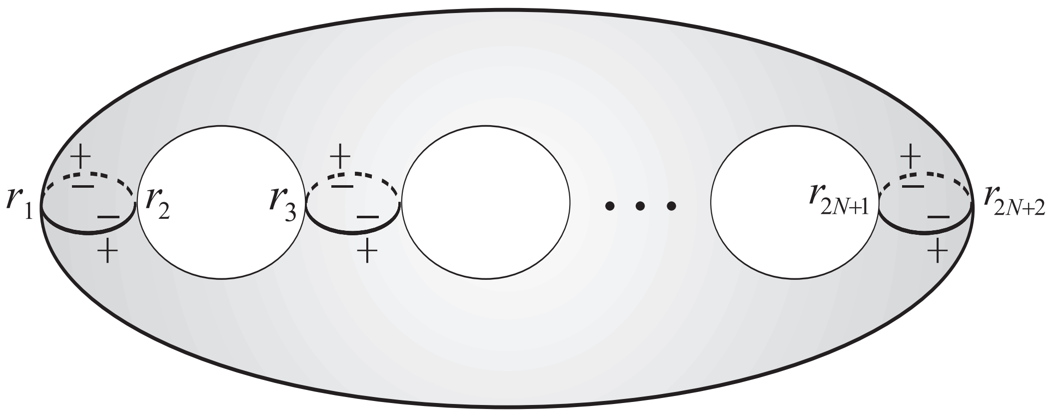


Figure 16

If the number of roots is odd, say $n = 2N+1$, there must be a cut from ∞ to r_1. The remaining branch points $r_2, \ldots, r_{2N+1}$ can be separated into pairs, $(r_2, r_3), \ldots, (r_{2N}, r_{2N+1})$. This will give us $\frac{n+1}{2} = N + 1$ cuts in the cut plane dranw in Figure 17.

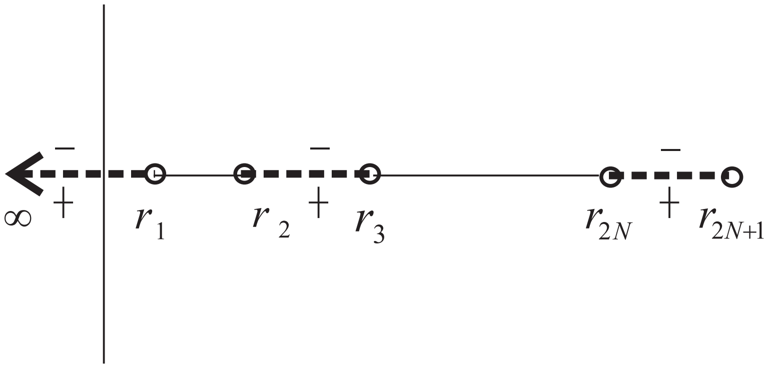


Figure 17

There are also N holes in the Riemann surface drawn in Figure 18.

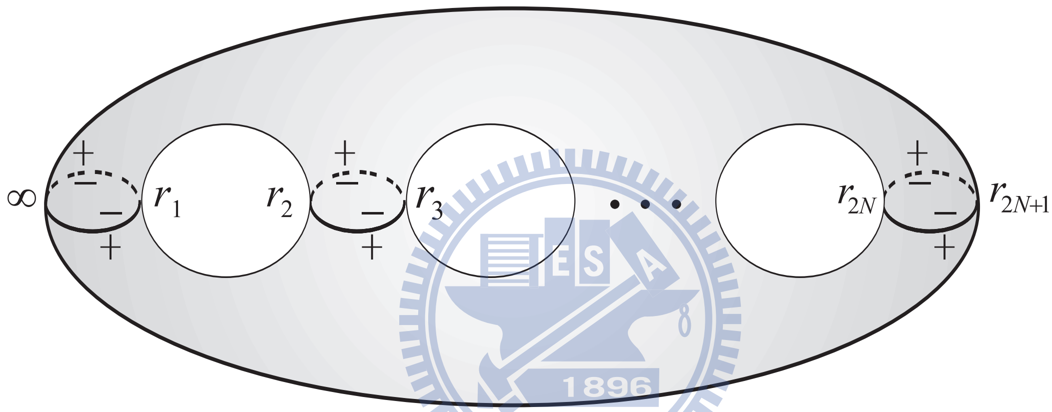


Figure 18

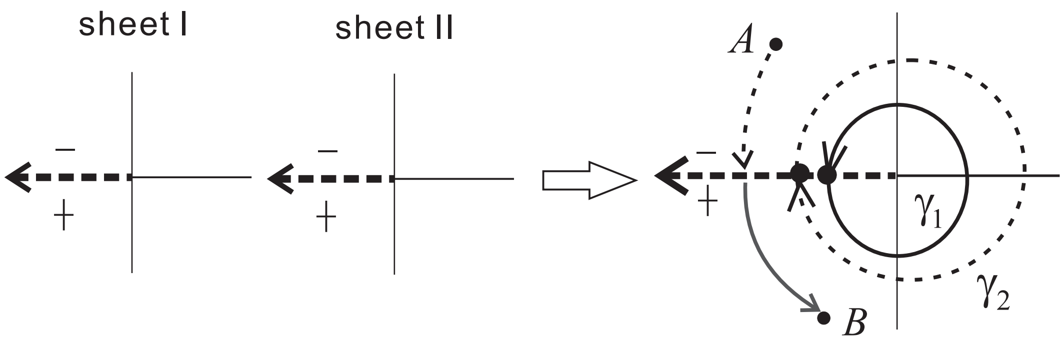
The surface in which there are N holes is called the ***Riemann surface of genus*** N, denoted by $\mathcal{R}_N$, as in Figure 16 and Figure 18.

2.5 To Draw Paths on Cut Planes and on Riemann Surfaces

In this section, we explain how to draw the paths on cut planes and on Riemann surfaces.

In the cut planes, we use solid lines to draw a path on sheet I and use dash lines to draw a path on sheet II.

In the Riemann surfaces, we use dash lines to draw a path on the back of the surfaces and use solid lines to draw a path on the front of the surfaces. Let $f(z) = \sqrt{z}$.



Figure

Let $(I,+)$ denote the $+$ edge of sheet I, $(I,-)$ denote the $-$ edge of sheet I, $(II,+)$ denote the $+$ edge of sheet II, and $(II,-)$ denote the $-$ edge of sheet II. The path from point A to point B denotes that start from A in sheet II, cross through the cut from $(II,-)$ to $(I,+)$, to B. γ_1 denotes a path in sheet I, from a point in $(I,+)$ to a point in $(I,-)$. γ_2 denotes a path in sheet II, from a point in $(II,-)$ to a point in $(II,+)$. The corresponding paths in Riemann surface is drawn in following figure.

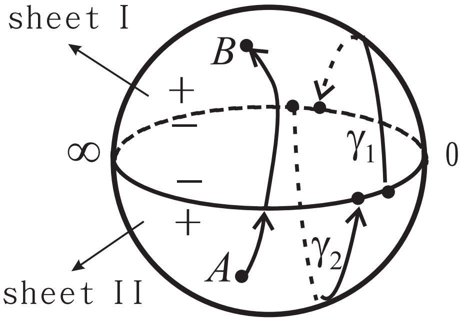


Figure 20

$a-cycle$ is a closed path that encloses a finite cut (the endpoint of cut is a finite number). $b-cycle$ is a closed path that starts from $+$ edge of a cut (it maybe finite cut or infinte cut) without encloed by any $a-cycle$, to $+$ edge of another cut encloed by a $a-cycle$. Then the path crosses through $-$ edge of this cut and goes into sheet II, and finally arrives to the $-$ edge of the starting cut.

Let $f(z)=\sqrt{z(z-1)(z-2)(z-3)(z-4)}$. The $a-cycles$ and $b-cycles$ in cut plane and their corresponding paths in Riemann surface are drawn in the following figures.

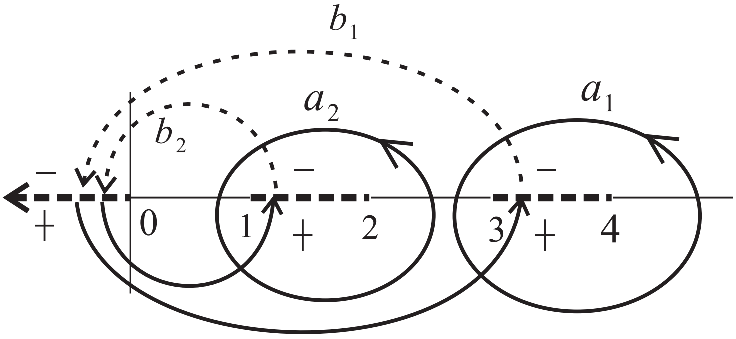


Figure 21

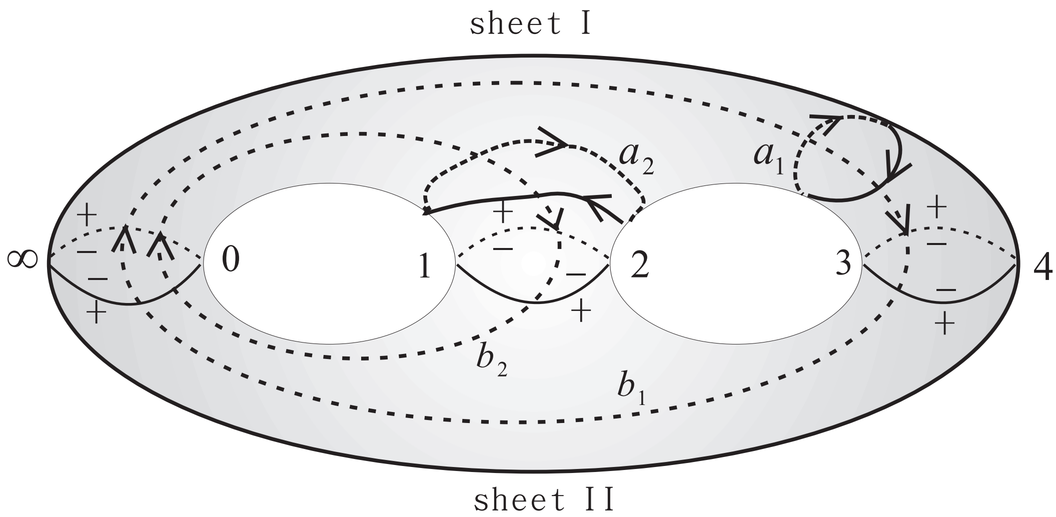


Figure 22

The numbers of $a-cycles$ and $b-cycles$ must be the same. In next few chapters, we aim to evaluate the integrals along $a-cycles$ and $b-cycles$.

3 Integrals for Horizontal Cuts

3.1 Two Examples

We give a few of examples to explain how to evaluate path integrals. We will use the *principle of deformation of paths* (Theorem 1). It tells us that if a simple closed contour(piecewise smooth arc) C_1 is continously deformed into another simple closed coutour C_2, always passing through points at which a function f is analytic, then the value of the integral of f over C_1 never changes.

The circle

$$z = z_0 + Re^{i\theta}, \quad -\pi \leq \theta < \pi,$$

is a circle centered at the point z_0 and with radius R. It is a simple closed curve, oriented in the counterclockwise direction.

Cauchy-Goursat Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\, dz = 0.$$

Theorem 1. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

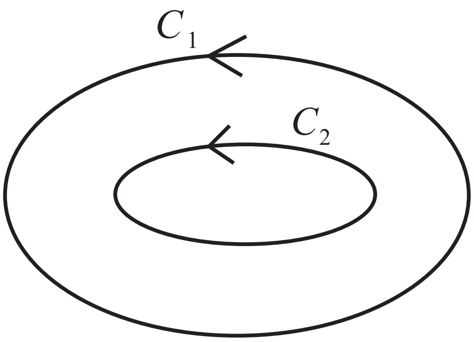


Figure 23

Example 1. *Let $f(z) = \sqrt{z}$ and let γ be the positively oriented (counterclockwise oriented) circular path $z = e^{i\theta}$, $-\pi \leq \theta < \pi$. This is a path looked like a circle centered at the point 0 with radius 1. Evaluate the integral $\int_\gamma f(z)\, dz$.*

Solution.

(1) Integral along the circular path

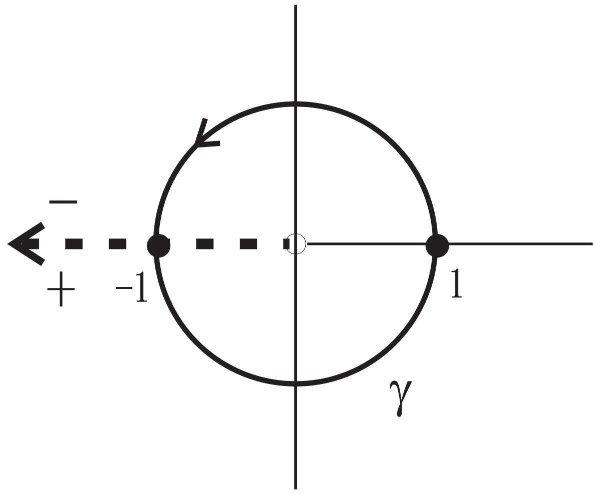


Figure 24

$$
\begin{aligned}
z \in \gamma &\Longrightarrow z = e^{i\theta},\ -\pi \le \theta < \pi \\
&\Longrightarrow \sqrt{z} = e^{\frac{1}{2}i\theta},\ dz = ie^{i\theta}d\theta.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_\gamma f(z)\,dz &= \int_{-\pi}^{\pi} f(e^{i\theta})ie^{i\theta}\,d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{e^{i\theta}}\,ie^{i\theta}\,d\theta \\
&= -\frac{4}{3}i.
\end{aligned}
$$

(2) Deformation of path

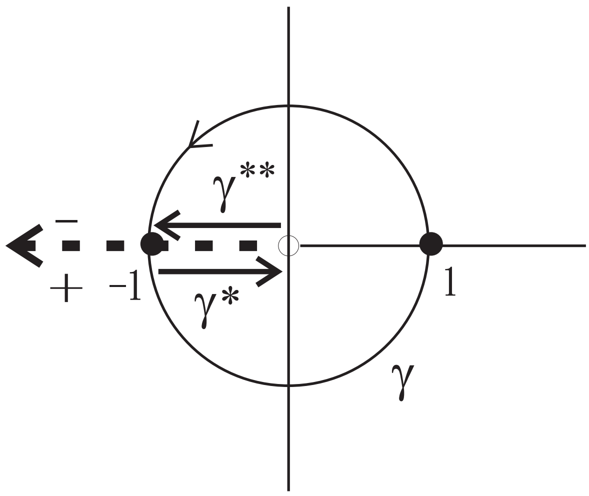


Figure 25

In Figure 25, γ^* is a line segment from -1 to 0 and γ^{**} is a line segment from 0 to -1.

Let $C = \gamma \cup -\gamma^{**} \cup -\gamma^{*}$. Since C is a simple closed contour,

$$
\begin{aligned}
&\int_C f(z)\,dz = 0 \\
\Longrightarrow &\int_\gamma f(z)\,dz + \int_{-\gamma^{**}} f(z)\,dz + \int_{-\gamma^{*}} f(z)\,dz = 0 \\
\Longrightarrow &\int_\gamma f(z)\,dz + \left(-\int_{\gamma^{**}} f(z)\,dz\right) + \left(-\int_{\gamma^{*}} f(z)\,dz\right) = 0 \\
\Longrightarrow &\int_\gamma f(z)\,dz = \int_{\gamma^{*}} f(z)\,dz + \int_{\gamma^{**}} f(z)\,dz. \qquad (17)
\end{aligned}
$$

In (1), we have evaluated $\int_\gamma f(z)dz$. Now we evaluate the value of the right hand side of equation (17).

Since the points along the path γ^* is in the + edge of the cut plane, the points on γ^* has the angle $-\pi$.

$$
\begin{aligned}
z \in \gamma^* &\Longrightarrow z = re^{i(-\pi)}, \quad r : 1 \to 0 \\
&\Longrightarrow \sqrt{z} = \sqrt{r}e^{\frac{1}{2}i(-\pi)} = -i\sqrt{r},\ dz = -dr.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_{\gamma^*} f(z)\,dz &= \int_1^0 (-i\sqrt{r})(-dr) \\
&= i\int_1^0 \sqrt{r}\,dr \\
&= -\frac{2}{3}i\,.
\end{aligned}
$$

Similarly, the points along the path γ^{**} is in the $-$ edge of the cut plane. So the points on γ^{**} has the angle π.

$$
\begin{aligned}
z \in \gamma^{**} &\Longrightarrow z = re^{i\pi}, \quad r : 0 \to 1 \\
&\Longrightarrow \sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\pi} = i\sqrt{r},\ dz = -dr.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_{\gamma^{**}} f(z)\,dz &= \int_0^1 i\sqrt{r}(-dr) \\
&= -i\int_0^1 \sqrt{r}\,dr \\
&= -\frac{2}{3}i\,.
\end{aligned}
$$

Finally, we obtain the result

$$\int_{\gamma^*} f(z)\,dz + \int_{\gamma^{**}} f(z)\,dz = (-\frac{2}{3}i) + (-\frac{2}{3}i)$$
$$= -\frac{4}{3}i$$
$$= \int_{\gamma} f(z)\,dz.$$

From (1) and (2), we have verified equation (17).

Example 2. *Let* $f(z) = \sqrt{(z-1)(z-2)}$ *and let* γ *be the positively oriented circular path* $z = \frac{3}{2} + e^{i\theta}$, $-\pi \le \theta < \pi$. *Evaluate the integral* $\int_\gamma f(z)\,dz$.

Solution.

Step 1. Draw the cut plane

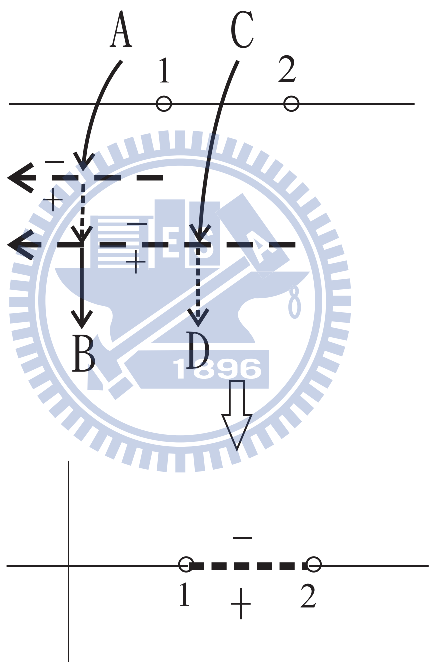


Figure 26

If a point goes from point A , crossing through the left part of 1 on real axis ($\{x \in \mathbb{R} | x < 1\}$), to point B, it crosses two branch cuts. Then $\sqrt{z-1}$ changes sign one time and $\sqrt{z-2}$ also changes sign one time. So $f(z)$ totally changes sign two times. If a point goes from point C, crossing through the line segment between 1 and 2 ($\{x \in \mathbb{R} | 1 < x < 2\}$), to point D, it crosses only one branch cut. Then $\sqrt{z-1}$ does not change sign but $\sqrt{z-2}$ changes sign one time. So $f(z)$ totally changes sign only one times. Thus, there is only one cut between 1 and 2 (Figure 25).

Step 2. Evaluate the integrals

(1) Integral along the circle

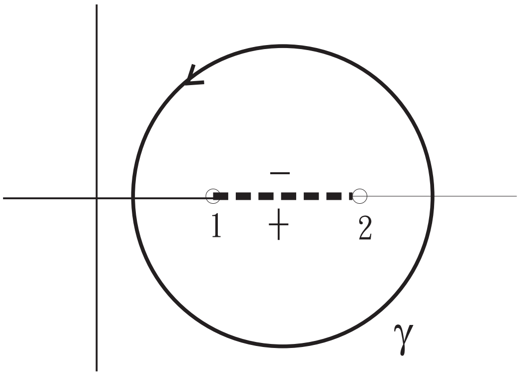


Figure 27

$$z \in \gamma \Longrightarrow z = \frac{3}{2} + e^{i\theta}, \; -\pi \leq \theta < \pi$$
$$\Longrightarrow dz = ie^{i\theta} d\theta.$$

Then,

$$\int_\gamma f(z)\, dz = \int_{-\pi}^{\pi} \sqrt{(\frac{3}{2} + e^{i\theta}) - 1}\sqrt{(\frac{3}{2} + e^{i\theta}) - 2}\; ie^{i\theta}\, d\theta$$
$$= -0.785395i\,.$$

(2) Deformation of path

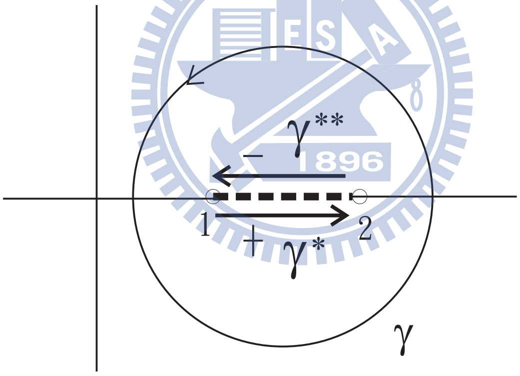


Figure 28

Since the points along the path γ^* is in the + edge of the cut plane, the points on γ^* has the angle $-\pi$.

$$z \in \gamma^* \Longrightarrow z = 2 + re^{i(-\pi)} = 2 - r, \quad r : 1 \to 0$$
$$\Longrightarrow \sqrt{z-2} = \sqrt{r}e^{\frac{1}{2}i(-\pi)} = -i\sqrt{r}, \; dz = -dr.$$

Then,

$$\int_{\gamma^*} f(z)\, dz = \int_1^0 \sqrt{1 + re^{i(-\pi)}}(-i\sqrt{r})(-dr)$$
$$= \int_1^0 \sqrt{1-r}(-i\sqrt{r})(-dr)$$
$$= i\int_1^0 \sqrt{1-r}\sqrt{r}\; dr$$
$$= -0.392699i\,.$$

Similarly, the points along the path γ^{**} is in the $-$ edge of the cut plane. So the points on γ^{**} has the angle π.

$$z \in \gamma^{**} \Longrightarrow z = 2 + re^{i\pi} = 2 - r, \quad r : 0 \to 1$$
$$\Longrightarrow \sqrt{z-2} = \sqrt{r}e^{\frac{1}{2}i\pi} = i\sqrt{r}, \; dz = -dr.$$

Then,

$$\begin{aligned}\int_{\gamma^{**}} f(z)\,dz &= \int_0^1 \sqrt{1-r}(i\sqrt{r})(-dr)\\ &= -i\int_0^1 \sqrt{1-r}\sqrt{r}\,dr\\ &= -0.392699i\,.\end{aligned}$$

We obtain

$$\begin{aligned}\int_{\gamma^{*}} f(z)\,dz + \int_{\gamma^{**}} f(z)\,dz &= (-0.392699i) + (-0.392699i)\\ &= -0.785398i\,.\end{aligned}$$

Again, we verify equation (17).
Note that,

$$\begin{aligned}\int_{\gamma} f(z)\,dz &= \int_{\gamma^{*}} f(z)\,dz + \int_{\gamma^{**}} f(z)\,dz\\ &= i\int_1^0 \sqrt{1-r}\sqrt{r}\,dr + \left(-i\int_0^1 \sqrt{1-r}\sqrt{r}\,dr\right)\\ &= i\int_1^0 \sqrt{1-r}\sqrt{r}\,dr + \left(i\int_1^0 \sqrt{1-r}\sqrt{r}\,dr\right)\\ &= 2i\int_1^0 \sqrt{1-r}\sqrt{r}\,dr\\ &= 2\int_{\gamma^{*}} f(z)\,dz\,.\end{aligned}$$

That is,

$$\int_{\gamma} f(z)\,dz = 2\int_{1\xrightarrow{+}2} f(z)\,dz \tag{18}$$

3.2 The Problem in Using Mathematica

Before we use Mathematica to compute the integrals, we need to know what phenomena will happens. Let $z = re^{i\theta}$. We use the notation, arg z to denote the argument of the complex number z. So, arg $z = \theta$. Let (I) denote sheet I and let (II) denote sheet II. Let $w = f(z) = \sqrt{z}$. In theoretical aspect,

$$z \in (I) \Longrightarrow -\pi \le \text{arg } z < \pi \Longrightarrow -\frac{\pi}{2} \le \frac{1}{2}\text{arg } z < \frac{\pi}{2}.$$

f maps the points on sheet I into the right-half plane $\{z \in \mathbb{C} | -\frac{\pi}{2} \le \text{arg } z < \frac{\pi}{2}\}$.

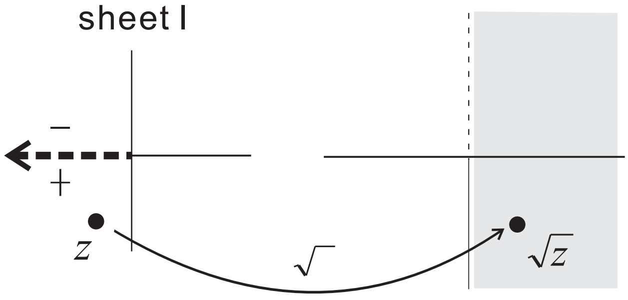


Figure 29

And,

$$z \in (II) \Longrightarrow \pi \leq \arg\, z < 3\pi \Longrightarrow \frac{\pi}{2} \leq \frac{1}{2}\arg\, z < \frac{3\pi}{2}.$$

f maps the points on sheet II into the left-half plane $\{z \in \mathbb{C} | \frac{\pi}{2} \leq \arg\, z < \frac{3\pi}{2}\}$.

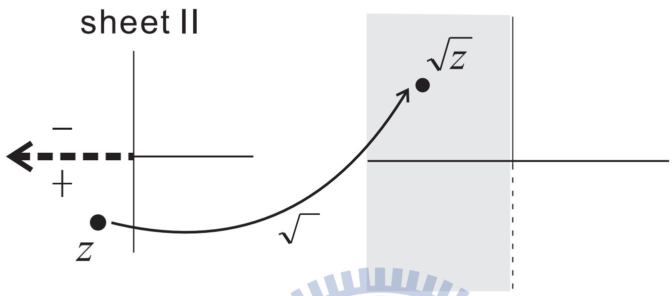


Figure 30

If you compute $\sqrt{z}$ using Mathematica, you can discover that the range of $f(z) = \sqrt{z}$ are as same as the range described above except the points along the + edge of sheet I ,that is, $\{z \in \mathbb{C} | \arg\, z = -\pi\}$. For example, suppose that $z = -2 \in (I, +)$, where $(I, +)$ denotes the + edge of sheet I.

$$\begin{aligned} z \in (I, +) &\Longrightarrow \arg\, z = -\pi \\ &\Longrightarrow \arg\, \sqrt{z} = -\frac{\pi}{2} \\ &\Longrightarrow -2 = 2e^{i(-\pi)} \\ &\Longrightarrow f(z) = \sqrt{-2} = \left(2e^{i(-\pi)}\right)^{\frac{1}{2}} = \sqrt{2}e^{i(-\frac{\pi}{2})} = -\sqrt{2}i. \end{aligned}$$

But in Mathematica, $\sqrt{-2} = \sqrt{2}i$. This value needs to time -1 to obtain the correct value. Therefore,

$$z \in (I, +) \Longrightarrow \sqrt{z} = (-1) \cdot \text{MATH}\left(\sqrt{z}\right),$$

where $\text{MATH}\left(\sqrt{z}\right)$ means the value of $\sqrt{z}$ computed by Mathematica. We use the notation $\text{MATH}\left(\cdot\right)$ to denote the value of " · " computed by Mathematica.

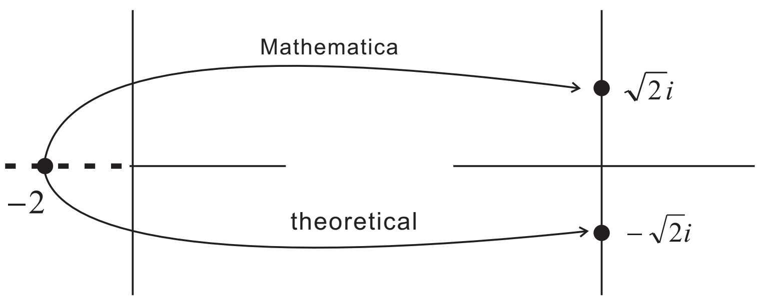


Figure 31

Thus, if you want to compute an integral along the + edge in sheet I, e.g., $\int_{-1\xrightarrow{+}0} \sqrt{z}dz$ (integration along the line segment $\{z \in (I,+)| -1 < z < 0\}$ from -1 to 0), you must multiply the result value computed in Mathematica by -1 to obtain the correct value. That is,

$$\int_{-1\xrightarrow{+}0} \sqrt{z}\,dz = (-1)\cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\,dz\right) = -0.666667i.$$

Suppose that $f(z) = \sqrt{z}$. Let $\theta_2 = \theta_1 + 2\pi$ and let $z_1 = re^{i\theta_1}, -\pi \le \theta_1 < \pi$ and $z_2 = re^{i\theta_2}, \pi \le \theta_2 < 3\pi$. Then z_1 and z_2 are the same points in the complex plane $\mathbb{C}$, but in the cut plane, $z_1 \in (I)$ and $z_2 \in (II)$ (Figure 32).

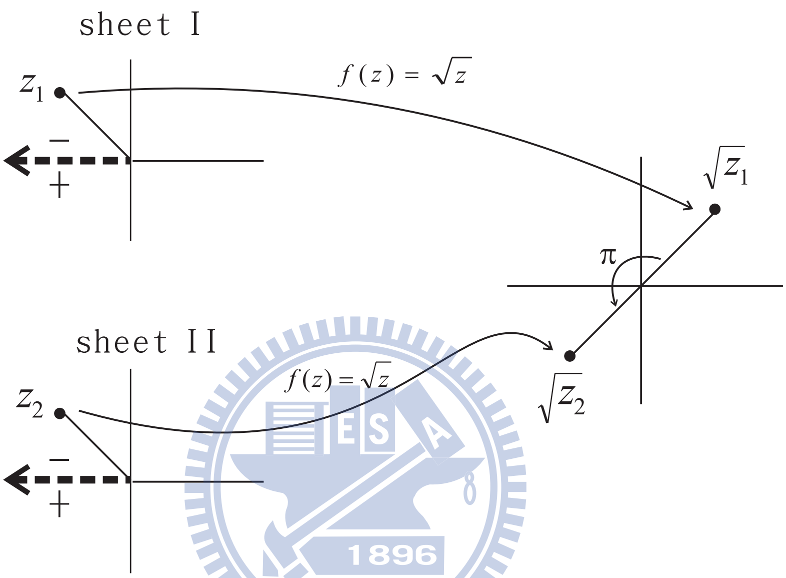


Figure 32

$$\begin{aligned} f(z_2) = \sqrt{z_2} = \sqrt{r}e^{\frac{1}{2}i\theta_2} = \sqrt{r}e^{\frac{1}{2}i(\theta_1+2\pi)} = \sqrt{r}e^{\frac{1}{2}i\theta_1 + i\pi} \\ = \sqrt{r}e^{\frac{1}{2}i\theta_1}e^{i\pi} = \sqrt{r}e^{\frac{1}{2}i\theta_1}\cdot(-1) = -\sqrt{z_1} = -f(z_1) \end{aligned} \tag{19}$$

This tells us that $\sqrt{z}|_{II} = -\sqrt{z}|_I$, the value of $\sqrt{z}$ in sheet II is the value of $\sqrt{z}$ in sheet I multiplied by -1. Thus, if $g(z) = \sqrt{h(z)}$ where $h(z) = (z - z_1)(z - z_2)\cdots(z - z_k) = \prod_{j=1}^{k}(z - z_j)$, we can assume that $h(z) = Re^{i\theta}$ for some positive real number R and θ. Let $h(z)|_I = Re^{i\theta_1}, -\pi \le \theta_1 < \pi$ and $h(z)|_{II} = Re^{i\theta_2}, \pi \le \theta_2 < 3\pi$, where $\theta_2 = \theta_1 + 2\pi$.

$$\begin{aligned} g(z)|_{II} = \sqrt{h(z)|_{II}} = \sqrt{Re^{i\theta_2}} = \sqrt{Re^{i(\theta_1+2\pi)}} = \sqrt{Re^{i\theta_1}}e^{i\pi} \\ = \sqrt{Re^{i\theta_1}}\cdot(-1) = (-1)\cdot\sqrt{h(z)|_I} = (-1)\cdot g(z)|_I \end{aligned} \tag{20}$$

3.3 Evaluating Integrals Using Mathematica

In Example 1 and Example 2, we use the analytic method to evaluate the integral. In this section, we will explain how to modify the value computed in Mathematica.

Example 3. *Evaluate the integral in Example 1 Using Mathematica .*

Solution.
(1)Along $-1 \xrightarrow{+} 0 \ \ (z \in \gamma^*)$(Figure 25)

$$\arg z = -\pi \Longrightarrow \sqrt{z} = (-1) \cdot \text{MATH}\left(\sqrt{z}\right)$$

$$\begin{aligned}\int_{\gamma^*} f(z)\, dz &= \int_{-1 \xrightarrow{+} 0} f(z)\, dz \\ &= \int_{-1}^{0} (-1) \cdot \text{MATH}\left(\sqrt{z}\right)\, dz \\ &= (-1) \cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\, dz\right).\end{aligned}$$

(2)Along $-1 \xleftarrow{-} 0 \ \ (z \in \gamma^{**})$(Figure 25)

$$\arg z = \pi \Longrightarrow \sqrt{z} = \text{MATH}\left(\sqrt{z}\right)$$

$$\begin{aligned}\int_{\gamma^{**}} f(z)\, dz &= \int_{-1 \xleftarrow{-} 0} f(z)\, dz \\ &= \int_{0}^{-1} \text{MATH}\left(\sqrt{z}\right)\, dz \\ &= \text{MATH}\left(\int_{0}^{-1} \sqrt{z}\, dz\right).\end{aligned}$$

Then,

$$\begin{aligned}\int_{\gamma} f(z)\, dz &= \int_{\gamma^*} f(z)\, dz + \int_{\gamma^{**}} f(z)\, dz \\ &= (-1) \cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\, dz\right) + \text{MATH}\left(\int_{0}^{-1} \sqrt{z}\, dz\right) \\ &= (-1) \cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\, dz\right) + (-1) \cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\, dz\right) \\ &= (-2) \cdot \text{MATH}\left(\int_{-1}^{0} \sqrt{z}\, dz\right) \qquad (21) \\ &= -1.33333i\,.\end{aligned}$$

This agrees with the value $\int_{\gamma} f(z)dz = -\frac{4}{3}i = -1.33333i$ in Example 1.

Example 4. *Evaluate the integral in Example 2 Using Mathematica .*

Solution.
(1)Along $1 \xrightarrow{+} 2$

$$\begin{aligned}\arg(z-1) = 0 \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\ \arg(z-2) = -\pi \quad &\Longrightarrow \quad \sqrt{z-2} = (-1) \cdot \text{MATH}\left(\sqrt{z-2}\right).\end{aligned}$$

Thus,

$$f(z) = \sqrt{z-1}\sqrt{z-2} = (-1) \cdot \text{MATH}\left(\sqrt{z-1}\sqrt{z-2}\right).$$

Then,

$$\int_{\gamma^*} f(z)\, dz = (-1) \cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\right).$$

(2)Along $1 \xleftarrow{-} 2$

$$\begin{aligned} \arg(z-1) = 0 \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\ \arg(z-2) = \pi \quad &\Longrightarrow \quad \sqrt{z-2} = \text{MATH}\left(\sqrt{z-2}\right). \end{aligned}$$

Thus,

$$f(z) = \sqrt{z-1}\sqrt{z-2} = \text{MATH}\left(\sqrt{z-1}\sqrt{z-2}\right).$$

Then,

$$\begin{aligned} \int_{\gamma^{**}} f(z)\, dz &= \text{MATH}\left(\int_2^1 \sqrt{z-1}\sqrt{z-2}\right) \\ &= (-1) \cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\right) \\ &= \int_{\gamma^*} f(z)\, dz. \end{aligned}$$

So,

$$\begin{aligned} \int_{\gamma} f(z)\, dz &= \int_{\gamma^*} f(z)\, dz + \int_{\gamma^{**}} f(z)\, dz \\ &= 2\int_{\gamma^*} f(z)\, dz \\ &= (-2) \cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\right) \\ &= -0.785398i. \end{aligned}$$

This value also agrees with the answer in Example 2.

In the next example, we evaluate an integral along a positively oriented simple closed curve in which there are two branch cuts.

Example 5. *Suppose that* $f(z) = \sqrt{(z-1)(z-2)(z-3)(z-4)}$ *and* γ *is a positively oriented simple closed curve that encloses all cuts* (Figure 33). *Evaluate the integral* $\int_\gamma f(z)dz$.

Solution.

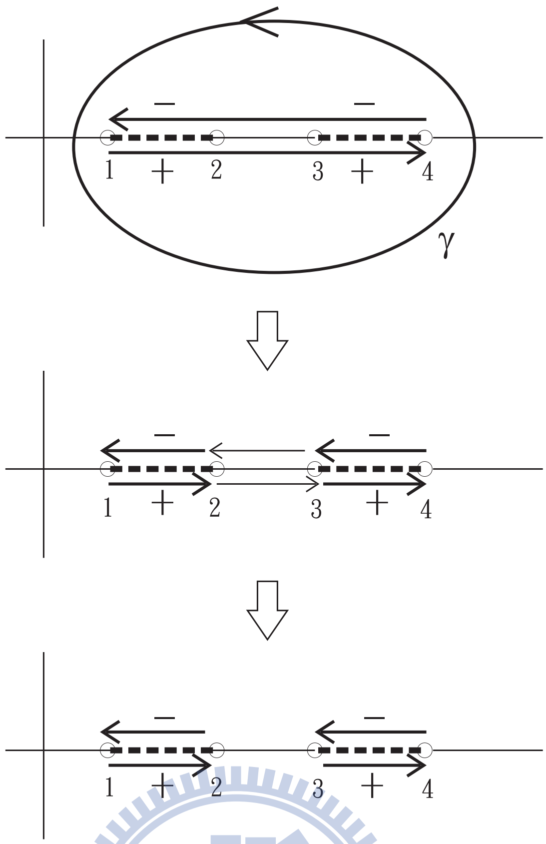


Figure 33

According to the deformation of path, we have

$$\int_\gamma f(z)\,dz = \int_{1\overset{+}{\longrightarrow}4} f(z)\,dz + \int_{1\overset{-}{\longleftarrow}4} f(z)\,dz \tag{22}$$

$$\begin{aligned} &= \left(\int_{1\overset{+}{\longrightarrow}2} f(z)\,dz + \int_{2\overset{+}{\longrightarrow}3} f(z)\,dz + \int_{3\overset{+}{\longrightarrow}4} f(z)\,dz\right) \\ &\quad + \left(\int_{3\overset{-}{\longleftarrow}4} f(z)\,dz + \int_{2\overset{-}{\longleftarrow}3} f(z)\,dz + \int_{1\overset{-}{\longleftarrow}2} f(z)\,dz\right) \end{aligned} \tag{23}$$

$$\begin{aligned} &= \left(\int_{1\overset{+}{\longrightarrow}2} f(z)\,dz + \int_{3\overset{+}{\longrightarrow}4} f(z)\,dz\right) \\ &\quad + \left(\int_{3\overset{-}{\longleftarrow}4} f(z)\,dz + \int_{1\overset{-}{\longleftarrow}2} f(z)\,dz\right). \end{aligned} \tag{24}$$

Because of the two paths $2 \overset{+}{\longrightarrow} 3$ and $2 \overset{-}{\longleftarrow} 3$ are not along any branch cut, the two integrals $\int_{2\overset{+}{\longrightarrow}3} f(z)dz$ and $\int_{2\overset{-}{\longleftarrow}3} f(z)dz$ in equation (23) are canceled by each other. Thus, we only investigate the four integrals in equation (24).

<u>Theoretical Evaluation</u>

(1)Along $1 \overset{+}{\longrightarrow} 2$

$$z = 2 + re^{i(-\pi)} = 2 - r, \quad r : 1 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$
\begin{aligned}
z-1>0 &\implies \arg(z-1)=0 \implies z-1=(1-r)e^{i0}\\
&\implies \sqrt{z-1}=\sqrt{1-r}e^{\frac{1}{2}i0}=\sqrt{1-r}\\
z-2<0 &\implies \arg(z-2)=-\pi \implies z-2=re^{i(-\pi)}\\
&\implies \sqrt{z-2}=\sqrt{r}e^{\frac{1}{2}i(-\pi)}=-i\sqrt{r}\\
z-3<0 &\implies \arg(z-3)=-\pi \implies z-3=(1+r)e^{i(-\pi)}\\
&\implies \sqrt{z-3}=\sqrt{1+r}e^{\frac{1}{2}i(-\pi)}=-i\sqrt{1+r}\\
z-4<0 &\implies \arg(z-4)=-\pi \implies z-4=(2+r)e^{i(-\pi)}\\
&\implies \sqrt{z-4}=\sqrt{2+r}e^{\frac{1}{2}i(-\pi)}=-i\sqrt{2+r}
\end{aligned}
$$

$$
\begin{aligned}
\int_{1\xrightarrow{+}2} f(z)\,dz &= \int_1^0 \sqrt{1-r}(-i\sqrt{r})(-i\sqrt{1+r})(-i\sqrt{2+r})(-dr)\\
&= i^3\int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\sqrt{2+r}\,dr\\
&= -i\int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\sqrt{2+r}\,dr\\
&= 0.76002i\,.
\end{aligned}
$$

From the procedure above, we find that we can simplify the representation of $\sqrt{z-k}$, $k = 1,2,3,4$. We only substitute $z = 2-r$ directly into $\sqrt{z-k}$ for each k and remember the following rules :

$$
z\in(I,+) \implies \begin{cases} z-k>0 & \implies \sqrt{z-k}=\sqrt{(2-r)-k}\\ z-k<0 & \implies \sqrt{z-k}=\sqrt{-(k-z)}=(-i)\sqrt{k-z}=-i\sqrt{k-(2-r)} \end{cases}
$$

The minus sign, "−", is necessary because $z-k\in(I,+)$ and $\arg(z-k)=-\pi$. It is the cause of the factor $(-i)$ appearing.

$$
z\in(I,-) \implies \begin{cases} z-k>0 & \implies \sqrt{z-k}=\sqrt{(2-r)-k}\\ z-k<0 & \implies \sqrt{z-k}=\sqrt{-(k-z)}=i\sqrt{k-z}=i\sqrt{k-(2-r)} \end{cases}
$$

It is important that we must make the number inside square roots to be positive. Thus, we can also write

$$
\begin{aligned}
z-1>0 &\implies \arg(z-1)=0\\
&\implies \sqrt{z-1}=\sqrt{(2-r)-1}=\sqrt{1-r}\\
z-2<0 &\implies \arg(z-2)=-\pi\\
&\implies \sqrt{z-2}=\sqrt{(2-r)-2}=\sqrt{-r}=-i\sqrt{r}\\
z-3<0 &\implies \arg(z-3)=-\pi\\
&\implies \sqrt{z-3}=\sqrt{(2-r)-3}=\sqrt{-(1+r)}=-i\sqrt{1+r}\\
z-4<0 &\implies \arg(z-4)=-\pi\\
&\implies \sqrt{z-4}=\sqrt{(2-r)-4}=\sqrt{-(2+r)}=-i\sqrt{2+r}
\end{aligned}
$$

(2)Along $3 \xrightarrow{+} 4$

$$z = 4 + re^{i(-\pi)} = 4 - r, \quad r : 1 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z - 1 > 0 \quad &\Longrightarrow \arg(z-1) = 0 \\
&\Longrightarrow \sqrt{z-1} = \sqrt{(4-r)-1} = \sqrt{3-r} \\
z - 2 > 0 \quad &\Longrightarrow \arg(z-2) = 0 \\
&\Longrightarrow \sqrt{z-2} = \sqrt{(4-r)-2} = \sqrt{2-r} \\
z - 3 > 0 \quad &\Longrightarrow \arg(z-3) = 0 \\
&\Longrightarrow \sqrt{z-3} = \sqrt{(4-r)-3} = \sqrt{1-r} \\
z - 4 < 0 \quad &\Longrightarrow \arg(z-4) = -\pi \\
&\Longrightarrow \sqrt{z-4} = \sqrt{(4-r)-4} = -i\sqrt{r}
\end{aligned}$$

$$\begin{aligned}
\int_{3 \xrightarrow{+} 4} f(z)\, dz &= \int_1^0 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}(-i\sqrt{r})(-dr) \\
&= i \int_1^0 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}\sqrt{r}\, dr \\
&= -0.76002i\,.
\end{aligned}$$

(3)Along $3 \xleftarrow{-} 4$

$$z = 4 + re^{i(-\pi)} = 4 - r, \quad r : 0 \longrightarrow 1 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z - 1 > 0 \quad &\Longrightarrow \arg(z-1) = 0 \\
&\Longrightarrow \sqrt{z-1} = \sqrt{(4-r)-1} = \sqrt{3-r} \\
z - 2 < 0 \quad &\Longrightarrow \arg(z-2) = 0 \\
&\Longrightarrow \sqrt{z-2} = \sqrt{(4-r)-2} = \sqrt{2-r} \\
z - 3 < 0 \quad &\Longrightarrow \arg(z-3) = 0 \\
&\Longrightarrow \sqrt{z-3} = \sqrt{(4-r)-3} = \sqrt{1-r} \\
z - 4 < 0 \quad &\Longrightarrow \arg(z-4) = \pi \\
&\Longrightarrow \sqrt{z-4} = \sqrt{(4-r)-4} = \sqrt{-r} = i\sqrt{r}
\end{aligned}$$

$$\begin{aligned}
\int_{3 \xleftarrow{-} 4} f(z)\, dz &= \int_0^1 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}(i\sqrt{r})(-dr) \\
&= -i \int_0^1 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}\sqrt{r}\, dr \\
&= -i \left(- \int_1^0 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}\sqrt{r}\, dr \right) \\
&= i \int_1^0 \sqrt{3-r}\sqrt{2-r}\sqrt{1-r}\sqrt{r}\, dr \\
&= \int_{3 \xrightarrow{+} 4} f(z)\, dz \\
&= -0.76002i\,.
\end{aligned}$$

(4)Along $1 \overset{-}{\longleftarrow} 2$

$$z = 2 + re^{i(-\pi)} = 2 - r, \quad r : 0 \longrightarrow 1 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z - 1 > 0 \quad &\Longrightarrow \arg(z-1) = 0 \\
&\Longrightarrow \sqrt{z-1} = \sqrt{(2-r)-1} = \sqrt{1-r} \\
z - 2 < 0 \quad &\Longrightarrow \arg(z-2) = \pi \\
&\Longrightarrow \sqrt{z-2} = \sqrt{(2-r)-2} = \sqrt{-r} = i\sqrt{r} \\
z - 3 < 0 \quad &\Longrightarrow \arg(z-3) = \pi \\
&\Longrightarrow \sqrt{z-3} = \sqrt{(2-r)-3} = \sqrt{-(1+r)} = i\sqrt{1+r} \\
z - 4 < 0 \quad &\Longrightarrow \arg(z-4) = \pi \\
&\Longrightarrow \sqrt{z-4} = \sqrt{(2-r)-4} = \sqrt{-(2+r)} = i\sqrt{2+r}
\end{aligned}$$

$$\begin{aligned}
\int_{1 \overset{-}{\longleftarrow} 2} f(z)\, dz &= \int_0^1 \sqrt{1-r}(i\sqrt{r})(i\sqrt{1+r})(i\sqrt{2+r})(-dr) \\
&= -i^3 \int_0^1 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\sqrt{2+r}\, dr \\
&= i \int_0^1 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\sqrt{2+r}\, dr \\
&= -i \int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\sqrt{2+r}\, dr \\
&= \int_{1 \overset{+}{\longrightarrow} 2} f(z)\, dz \\
&= 0.76002i\,.
\end{aligned}$$

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_{1 \overset{+}{\longrightarrow} 2} f(z)\, dz + \int_{3 \overset{+}{\longrightarrow} 4} f(z)\, dz + \int_{3 \overset{-}{\longleftarrow} 4} f(z)\, dz + \int_{1 \overset{-}{\longleftarrow} 2} f(z)\, dz \\
&= 2\int_{1 \overset{+}{\longrightarrow} 2} f(z)\, dz + 2\int_{3 \overset{+}{\longrightarrow} 4} f(z)\, dz \\
&= 2\left(\int_{1 \overset{+}{\longrightarrow} 2} f(z)\, dz + \int_{3 \overset{+}{\longrightarrow} 4} f(z)\, dz\right) \\
&= 2\left((0.76002i) + (-0.76002i)\right) \\
&= 0\,.
\end{aligned}$$

Using Mathematica

(1)Along $1 \overset{+}{\longrightarrow} 2$

$$\begin{aligned}
\arg(z-1) = 0 \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-2) = -\pi \quad &\Longrightarrow \quad \sqrt{z-2} = (-1)\cdot\text{MATH}\left(\sqrt{z-2}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = -\pi \quad &\Longrightarrow \quad \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right).
\end{aligned}$$

$$\begin{aligned}\int_{1\xrightarrow{+}2} f(z)\,dz &= (-1)^3 \cdot \operatorname{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right)\\ &= (-1)\cdot \operatorname{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right).\end{aligned}$$

(2)Along $3 \xrightarrow{+} 4$

$$\begin{aligned}&\arg(z-1)=0 && \Longrightarrow && \sqrt{z-1}=\operatorname{MATH}\left(\sqrt{z-1}\right)\\ &\arg(z-2)=0 && \Longrightarrow && \sqrt{z-2}=\operatorname{MATH}\left(\sqrt{z-2}\right)\\ &\arg(z-3)=0 && \Longrightarrow && \sqrt{z-3}=\operatorname{MATH}\left(\sqrt{z-3}\right)\\ &\arg(z-4)=-\pi && \Longrightarrow && \sqrt{z-4}=(-1)\cdot\operatorname{MATH}\left(\sqrt{z-4}\right).\end{aligned}$$

$$\int_{3\xrightarrow{+}4} f(z)\,dz = (-1)\cdot \operatorname{MATH}\left(\int_3^4 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right).$$

(3)Along $3 \xleftarrow{-} 4$

$$\begin{aligned}&\arg(z-1)=0 && \Longrightarrow && \sqrt{z-1}=\operatorname{MATH}\left(\sqrt{z-1}\right)\\ &\arg(z-2)=0 && \Longrightarrow && \sqrt{z-2}=\operatorname{MATH}\left(\sqrt{z-2}\right)\\ &\arg(z-3)=0 && \Longrightarrow && \sqrt{z-3}=\operatorname{MATH}\left(\sqrt{z-3}\right)\\ &\arg(z-4)=\pi && \Longrightarrow && \sqrt{z-4}=\operatorname{MATH}\left(\sqrt{z-4}\right).\end{aligned}$$

$$\begin{aligned}\int_{3\xleftarrow{-}4} f(z)\,dz &= \operatorname{MATH}\left(\int_4^3 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right)\\ &= (-1)\cdot \operatorname{MATH}\left(\int_3^4 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right)\\ &= \int_{3\xrightarrow{+}4} f(z)\,dz.\end{aligned}$$

(4)Along $1 \xleftarrow{-} 2$

$$\begin{aligned}&\arg(z-1)=0 && \Longrightarrow && \sqrt{z-1}=\operatorname{MATH}\left(\sqrt{z-1}\right)\\ &\arg(z-2)=\pi && \Longrightarrow && \sqrt{z-2}=\operatorname{MATH}\left(\sqrt{z-2}\right)\\ &\arg(z-3)=\pi && \Longrightarrow && \sqrt{z-3}=\operatorname{MATH}\left(\sqrt{z-3}\right)\\ &\arg(z-4)=\pi && \Longrightarrow && \sqrt{z-4}=\operatorname{MATH}\left(\sqrt{z-4}\right).\end{aligned}$$

$$\begin{aligned}\int_{1\xleftarrow{-}2} f(z)\,dz &= \operatorname{MATH}\left(\int_2^1 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right)\\ &= (-1)\cdot \operatorname{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right)\\ &= \int_{1\xrightarrow{+}2} f(z)\,dz.\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\gamma f(z)\,dz &= 2\int_{1\xrightarrow{+}2} f(z)\,dz + 2\int_{3\xrightarrow{+}4} f(z)\,dz \\
&= (-2)\cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right) \\
&\quad + (-2)\cdot \text{MATH}\left(\int_3^4 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right) \\
&= (-2)\cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right. \\
&\quad \left. + \int_3^4 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}\,dz\right) \\
&= -3.15797\times 10^{-15} + 1.77636\times 10^{-15} i\,. \qquad (25)
\end{aligned}$$

To compare the values in equation (24) and equation (25), the two values in fact are the same. Note that -3.15797×10^{-15} and 1.77636×10^{-15} are very small numbers, so we can say them to be 0.

In the next example, we evaluate an integral along one $b-cycle$.

Example 6. *Let* $f(z)=\sqrt{(z-1)(z-2)(z-3)}$ *and let* γ *be the oriented positively circular path*

$$z=\begin{cases} \frac{3}{2}+e^{i\theta} & \text{if } -\pi\le\theta<0, \quad \text{(in sheet I)} \\ \frac{3}{2}+e^{i\theta} & \text{if } 2\pi\le\theta<3\pi. \quad \text{(in sheet II)} \end{cases}$$

Solution.

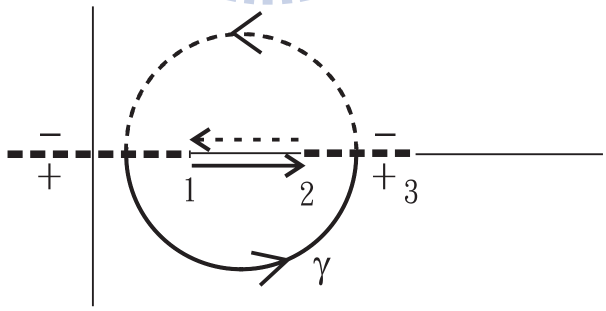


Figure 34

1. Integral along the circle

   Since $f(z)|_{II}=-f(z)|_I$, we have

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_{-\pi}^{0} f(z)\,dz + \int_{2\pi}^{3\pi} f(z)\,dz \\
&= \int_{-\pi}^{0} f(z)\,dz + (-1)\int_0^{\pi} f(z)\,dz \\
&= \int_{-\pi}^{0} f(z)\,dz - \int_0^{\pi} f(z)\,dz.
\end{aligned}$$

$$z = \frac{3}{2} + e^{i\theta} \Longrightarrow dz = ie^{i\theta} d\theta$$

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_{-\pi}^{0} \sqrt{(\frac{3}{2}+e^{i\theta})-1}\sqrt{(\frac{3}{2}+e^{i\theta})-2}\sqrt{(\frac{3}{2}+e^{i\theta})-3}\,ie^{i\theta}\,dz \\ &\quad - \int_{0}^{\pi} \sqrt{(\frac{3}{2}+e^{i\theta})-1}\sqrt{(\frac{3}{2}+e^{i\theta})-2}\sqrt{(\frac{3}{2}+e^{i\theta})-3}\,ie^{i\theta}\,dz \\ &= -0.958512\,.\end{aligned}$$

2. Deformation of path

Theoretical Evaluation

(1) Along 1⟶2

$$z = 2 + re^{i(-\pi)} = 2 - r, \quad r : 1 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned} z-1>0 &\Longrightarrow \arg(z-1) = 0 \\ &\Longrightarrow \sqrt{z-1} = \sqrt{(2-r)-1} = \sqrt{1-r} \\ z-2<0 &\Longrightarrow \arg(z-2) = -\pi \\ &\Longrightarrow \sqrt{z-2} = \sqrt{(2-r)-2} = \sqrt{-r} = -i\sqrt{r} \\ z-3<0 &\Longrightarrow \arg(z-3) = -\pi \\ &\Longrightarrow \sqrt{z-3} = \sqrt{(2-r)-3} = \sqrt{-(1+r)} = -i\sqrt{1+r} \end{aligned}$$

$$\begin{aligned}\int_{1\longrightarrow 2} f(z)\,dz &= \int_1^0 \sqrt{1-r}(-i\sqrt{r})(-i\sqrt{1+r})(-dr) \\ &= -i^2 \int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr \\ &= \int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr.\end{aligned}$$

(2) Along 1⇠2

Since $f(z)|_{II} = -f(z)|_I$, we have $\int_{1\dashleftarrow 2} f(z)\,dz = -\int_{1\longleftarrow 2} f(z)\,dz$. Therefore, we first evaluate the value of the integral in sheet I then we multiply the value by -1.

$$z = 2 + re^{i(-\pi)} = 2 - r, \quad r : 0 \longrightarrow 1 \Longrightarrow dz = -dr$$

$$\begin{aligned} z-1>0 &\Longrightarrow \arg(z-1) = 0 \\ &\Longrightarrow \sqrt{z-1} = \sqrt{(2-r)-1} = \sqrt{1-r} \\ z-2<0 &\Longrightarrow \arg(z-2) = \pi \\ &\Longrightarrow \sqrt{z-2} = \sqrt{(2-r)-2} = \sqrt{-r} = i\sqrt{r} \\ z-3<0 &\Longrightarrow \arg(z-3) = \pi \\ &\Longrightarrow \sqrt{z-3} = \sqrt{(2-r)-3} = \sqrt{-(1+r)} = i\sqrt{1+r} \end{aligned}$$

$$
\begin{aligned}
\int_{1\longleftarrow 2} f(z)\,dz &= \int_0^1 \sqrt{1-r}(i\sqrt{r})(i\sqrt{1+r})(-dr) \\
&= -i^2 \int_0^1 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr \\
&= \int_0^1 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{1\dashleftarrow 2} f(z)\,dz &= -\int_{1\longleftarrow 2} f(z)\,dz \\
&= -\int_0^1 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr \\
&= \int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr \\
&= \int_{1\longrightarrow 2} f(z)\,dz
\end{aligned}
$$

From (1) and (2), we obtain

$$
\begin{aligned}
\int_\gamma f(z)\,dz &= \int_{1\longrightarrow 2} f(z)\,dz + \int_{1\dashleftarrow 2} f(z)\,dz && (26)\\
&= \int_{1\longrightarrow 2} f(z)\,dz + \int_{1\longrightarrow 2} f(z)\,dz && (27)\\
&= 2\int_{1\longrightarrow 2} f(z)\,dz && (28)\\
&= 2\int_1^0 \sqrt{1-r}\sqrt{r}\sqrt{1+r}\,dr && (29)\\
&= -0.958512\,. && (30)
\end{aligned}
$$

Using Mathematica

According to equation (26) to equation (28), we have

$$
\int_\gamma f(z)\,dz = 2\int_{1\longrightarrow 2} f(z)\,dz. \tag{31}
$$

Thus we only evaluate $\int_{1\longrightarrow 2} f(z)\,dz$.

Along $1\longrightarrow 2$:

$$
\begin{aligned}
\arg(z-1) = 0 \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-2) = -\pi \quad &\Longrightarrow \quad \sqrt{z-2} = (-1)\cdot\text{MATH}\left(\sqrt{z-2}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right).
\end{aligned}
$$

$$
\begin{aligned}
\int_{1\longrightarrow 2} f(z)\,dz &= (-1)^2\cdot\text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\,dz\right) \\
&= \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\,dz\right).
\end{aligned}
$$

So,

$$\int_\gamma f(z)\,dz = 2\int_{1\longrightarrow 2} f(z)\,dz$$
$$= 2\cdot \text{MATH}\left(\int_1^2 \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\,dz\right)$$
$$= -0.958512\,.$$

We give the final example to evaluate integrals along all a-cycles and b-cycles.

Example 7. *Suppose that*

$$f(z) = \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}$$
$$= \sqrt{(z-(-5))(z-(-3))(z-(-1))(z-1)(z-3)(z-4)(z-6)}\,.$$

Let a_1, a_2, a_3 be three $a-$cycles and let b_1, b_2, b_3 be three $b-$cycles drawing in Figure 35. *Evaluate the six integrals $\int_{a_k} f(z)dz$ and $\int_{b_k} f(z)dz$, $k = 1, 2, 3$ using the method of deformation of path.*

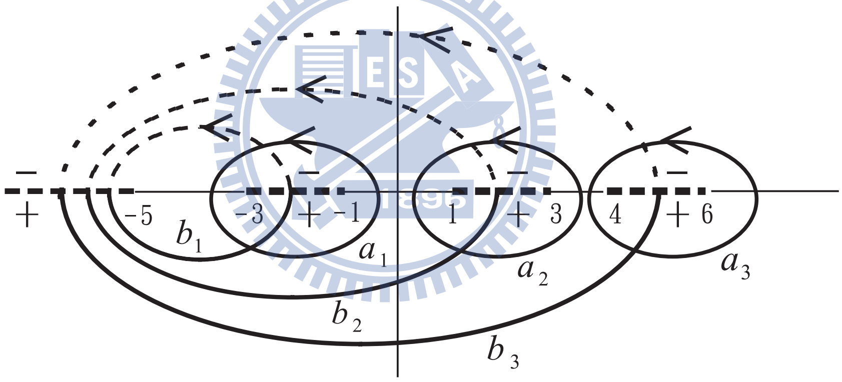


Figure 35

Solution.

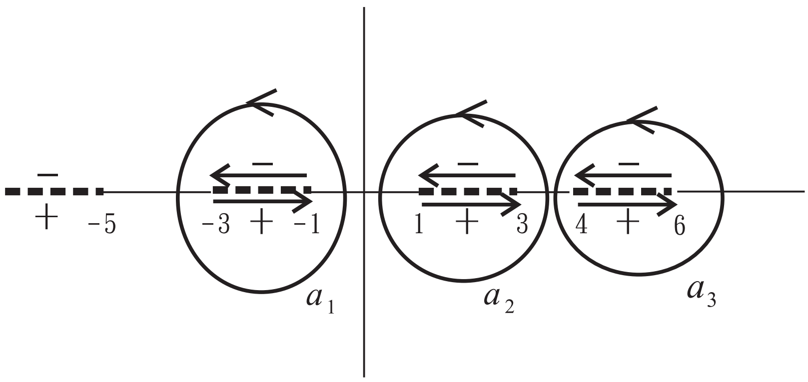


Figure 36

Suppose that there is an $a-cycle$ encloses a cut which is between z_1 to z_2. From equation (18) in example 2, we have the result

$$\int_{a-cycle} f(z)\, dz = 2\int_{z_1 \xrightarrow{+} z_2} f(z)\, dz \tag{32}$$

From equation (21) in example 3, we also have

$$\int_{a-cycle} f(z)\, dz = (-2)\cdot \text{MATH}\left(\int_{z_1}^{z_2} f(z)\, dz\right) \tag{33}$$

1. To evaluate $\int_{a_1} f(z)dz$

Theoretical Evaluation

Along $-3 \xrightarrow{+} -1$:

$$z = -1 + re^{i(-\pi)} = -1 - r,\quad r: 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{4-r}\\
z+3>0 &\Longrightarrow \arg(z+3)=0 \Longrightarrow \sqrt{z+3}=\sqrt{2-r}\\
z+1<0 &\Longrightarrow \arg(z+1)=-\pi \Longrightarrow \sqrt{z+1}=\sqrt{-r}=-i\sqrt{r}\\
z-1<0 &\Longrightarrow \arg(z-1)=-\pi \Longrightarrow \sqrt{z-1}=\sqrt{-(2+r)}=-i\sqrt{2+r}\\
z-3<0 &\Longrightarrow \arg(z-3)=-\pi \Longrightarrow \sqrt{z-3}=\sqrt{-(4+r)}=-i\sqrt{4+r}\\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=\sqrt{-(5+r)}=-i\sqrt{5+r}\\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=\sqrt{-(7+r)}=-i\sqrt{7+r}
\end{aligned}$$

$$\begin{aligned}
\int_{-3 \xrightarrow{+} -1} f(z)\, dz &= -(-i)^5 \int_2^0 \sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{5+r}\sqrt{7+r}\, dr\\
&= i\int_2^0 \sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{5+r}\sqrt{7+r}\, dr.
\end{aligned}$$

$$\begin{aligned}
\int_{a_1} f(z)dz &= 2\int_{-3 \xrightarrow{+} -1} f(z)\, dz\\
&= 2i\int_2^0 \sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{5+r}\sqrt{7+r}\, dr\\
&= -144.283i\,.
\end{aligned}$$

Using Mathematica

Along $-3 \xrightarrow{+} -1$:

$$\begin{aligned}
\arg(z+5)=0 &\Longrightarrow \sqrt{z+5}=\text{MATH}\left(\sqrt{z+5}\right)\\
\arg(z+3)=0 &\Longrightarrow \sqrt{z+3}=\text{MATH}\left(\sqrt{z+3}\right)\\
\arg(z+1)=-\pi &\Longrightarrow \sqrt{z+1}=(-1)\cdot\text{MATH}\left(\sqrt{z+1}\right)\\
\arg(z-1)=-\pi &\Longrightarrow \sqrt{z-1}=(-1)\cdot\text{MATH}\left(\sqrt{z-1}\right)\\
\arg(z-3)=-\pi &\Longrightarrow \sqrt{z-3}=(-1)\cdot\text{MATH}\left(\sqrt{z-3}\right)\\
\arg(z-4)=-\pi &\Longrightarrow \sqrt{z-4}=(-1)\cdot\text{MATH}\left(\sqrt{z-4}\right)\\
\arg(z-6)=-\pi &\Longrightarrow \sqrt{z-6}=(-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{-3 \xrightarrow{+} 1} f(z)\, dz = (-1)^5 \cdot \text{MATH}\left(\int_{-3}^{-1} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right)$$
$$= (-1) \cdot \text{MATH}\left(\int_{-3}^{-1} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right)$$
$$= -72.1417i\,.$$

$$\int_{a_1} f(z)dz = 2\int_{-3 \xrightarrow{+} -1} f(z)\, dz$$
$$= (-2) \cdot \text{MATH}\left(\int_{-3}^{-1} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right)$$
$$= -144.283i\,.$$

2. To evaluate $\int_{a_2} f(z)dz$

Theoretical Evaluation

Along $1 \xrightarrow{+} 3$:

$$z = 3 + re^{i(-\pi)} = 3 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$z+5>0 \Longrightarrow \arg(z+5) = 0 \Longrightarrow \sqrt{z+5} = \sqrt{8-r}$$
$$z+3>0 \Longrightarrow \arg(z+3) = 0 \Longrightarrow \sqrt{z+3} = \sqrt{6-r}$$
$$z+1>0 \Longrightarrow \arg(z+1) = 0 \Longrightarrow \sqrt{z+1} = \sqrt{4-r}$$
$$z-1>0 \Longrightarrow \arg(z-1) = 0 \Longrightarrow \sqrt{z-1} = \sqrt{2-r}$$
$$z-3<0 \Longrightarrow \arg(z-3) = -\pi \Longrightarrow \sqrt{z-3} = \sqrt{-r} = -i\sqrt{r}$$
$$z-4<0 \Longrightarrow \arg(z-4) = -\pi \Longrightarrow \sqrt{z-4} = \sqrt{-(1+r)} = -i\sqrt{1+r}$$
$$z-6<0 \Longrightarrow \arg(z-6) = -\pi \Longrightarrow \sqrt{z-6} = \sqrt{-(3+r)} = -i\sqrt{3+r}$$

$$\int_{1 \xrightarrow{+} 3} f(z)\, dz = -(-i)^3 \int_2^0 \sqrt{8-r}\sqrt{6-r}\sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{1+r}\sqrt{3+r}\, dr$$
$$= -i \int_2^0 \sqrt{8-r}\sqrt{6-r}\sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{1+r}\sqrt{3+r}\, dr.$$

$$\int_{a_2} f(z)dz = 2\int_{1 \xrightarrow{+} 3} f(z)\, dz$$
$$= -2i \int_2^0 \sqrt{8-r}\sqrt{6-r}\sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{1+r}\sqrt{3+r}\, dr$$
$$= 88.2841i\,.$$

Using Mathematica

Along $1 \xrightarrow{+} 3$:

$$
\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = 0 \quad &\Longrightarrow \quad \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = 0 \quad &\Longrightarrow \quad \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = 0 \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1) \cdot \text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = -\pi \quad &\Longrightarrow \quad \sqrt{z-4} = (-1) \cdot \text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = -\pi \quad &\Longrightarrow \quad \sqrt{z-6} = (-1) \cdot \text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}
$$

$$
\begin{aligned}
\int_{1 \xrightarrow{+} 3} f(z)\, dz &= (-1)^3 \cdot \text{MATH}\left(\int_1^3 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right) \\
&= (-1) \cdot \text{MATH}\left(\int_1^3 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right) \\
&= 44.142i\,.
\end{aligned}
$$

$$
\begin{aligned}
\int_{a_2} f(z)dz &= 2\int_{1 \xrightarrow{+} 3} f(z)\, dz \\
&= (-2) \cdot \text{MATH}\left(\int_1^3 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right) \\
&= 88.2841i\,.
\end{aligned}
$$

3. To evaluate $\int_{a_3} f(z)dz$

Theoretical Evaluation

Along $4 \xrightarrow{+} 6$:

$$
z = 6 + re^{i(-\pi)} = 6 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr
$$

$$
\begin{aligned}
z+5 > 0 \quad &\Longrightarrow \arg(z+5) = 0 \Longrightarrow \sqrt{z+5} = \sqrt{11-r} \\
z+3 > 0 \quad &\Longrightarrow \arg(z+3) = 0 \Longrightarrow \sqrt{z+3} = \sqrt{9-r} \\
z+1 > 0 \quad &\Longrightarrow \arg(z+1) = 0 \Longrightarrow \sqrt{z+1} = \sqrt{7-r} \\
z-1 > 0 \quad &\Longrightarrow \arg(z-1) = 0 \Longrightarrow \sqrt{z-1} = \sqrt{5-r} \\
z-3 > 0 \quad &\Longrightarrow \arg(z-3) = 0 \Longrightarrow \sqrt{z-3} = \sqrt{3-r} \\
z-4 > 0 \quad &\Longrightarrow \arg(z-4) = 0 \Longrightarrow \sqrt{z-4} = \sqrt{2-r} \\
z-6 < 0 \quad &\Longrightarrow \arg(z-6) = -\pi \Longrightarrow \sqrt{z-6} = \sqrt{-r} = -i\sqrt{r}
\end{aligned}
$$

$$
\begin{aligned}
\int_{4 \xrightarrow{+} 6} f(z)\, dz &= -(-i)\int_2^0 \sqrt{11-r}\sqrt{9-r}\sqrt{7-r}\sqrt{5-r}\sqrt{3-r}\sqrt{2-r}\sqrt{r}\, dr \\
&= i\int_2^0 \sqrt{11-r}\sqrt{9-r}\sqrt{7-r}\sqrt{5-r}\sqrt{3-r}\sqrt{2-r}\sqrt{r}\, dr.
\end{aligned}
$$

$$\begin{aligned}\int_{a_3} f(z)dz &= 2\int_{4\xrightarrow{+}6} f(z)\,dz \\ &= 2i\int_2^0 \sqrt{11-r}\sqrt{9-r}\sqrt{7-r}\sqrt{5-r}\sqrt{3-r}\sqrt{2-r}\sqrt{r}\,dr \\ &= -198.138i\,.\end{aligned}$$

Using Mathematica

Along $4 \xrightarrow{+} 6$:

$$\begin{aligned}
\arg(z+5) = 0 &\implies \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right)\\
\arg(z+3) = 0 &\implies \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right)\\
\arg(z+1) = 0 &\implies \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right)\\
\arg(z-1) = 0 &\implies \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right)\\
\arg(z-3) = 0 &\implies \sqrt{z-3} = \text{MATH}\left(\sqrt{z-3}\right)\\
\arg(z-4) = 0 &\implies \sqrt{z-4} = \text{MATH}\left(\sqrt{z-4}\right)\\
\arg(z-6) = -\pi &\implies \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\begin{aligned}\int_{4\xrightarrow{+}6} f(z)\,dz &= (-1)\cdot\text{MATH}\left(\int_4^6 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right)\\ &= -99.0688i\,.\end{aligned}$$

$$\begin{aligned}\int_{a_3} f(z)dz &= 2\int_{4\xrightarrow{+}6} f(z)\,dz\\ &= (-2)\cdot\text{MATH}\left(\int_4^6 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right)\\ &= -198.138i\,.\end{aligned}$$

From example 6 and equation (31), we have

$$\int_{b-cycle} f(z)\,dz = 2\int_{z_1\xrightarrow{+}z_2} f(z)\,dz, \tag{34}$$

and

$$\int_{b-cycle} f(z)\,dz = 2\cdot\text{MATH}\left(\int_{z_1}^{z_2} f(z)\,dz\right). \tag{35}$$

4. To evaluate $\int_{b_1} f(z)dz$

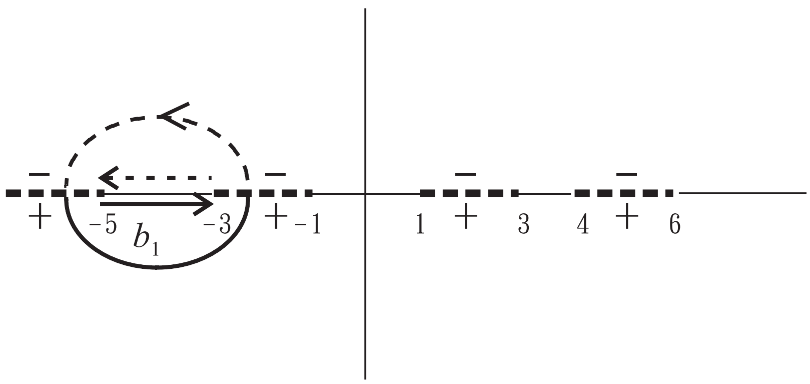


Figure 37

Theoretical Evaluation

Along $-5 \xrightarrow{+} -3$:

$$z = -3 + re^{i(-\pi)} = -3 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{2-r} \\
z+3<0 &\Longrightarrow \arg(z+3)=-\pi \Longrightarrow \sqrt{z+3}=\sqrt{-r}=-i\sqrt{r} \\
z+1<0 &\Longrightarrow \arg(z+1)=-\pi \Longrightarrow \sqrt{z+1}=\sqrt{-(2+r)}=-i\sqrt{2+r} \\
z-1<0 &\Longrightarrow \arg(z-1)=-\pi \Longrightarrow \sqrt{z-1}=\sqrt{-(4+r)}=-i\sqrt{4+r} \\
z-3<0 &\Longrightarrow \arg(z-3)=-\pi \Longrightarrow \sqrt{z-3}=\sqrt{-(6+r)}=-i\sqrt{6+r} \\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=\sqrt{-(7+r)}=-i\sqrt{7+r} \\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=\sqrt{-(9+r)}=-i\sqrt{9+r}
\end{aligned}$$

$$\begin{aligned}
\int_{-5\xrightarrow{+}-3} f(z)\,dz &= -(-i)^6 \int_2^0 \sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{6+r}\sqrt{7+r}\sqrt{9+r}\,dr \\
&= \int_2^0 \sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{6+r}\sqrt{7+r}\sqrt{9+r}\,dr
\end{aligned}$$

$$\begin{aligned}
\int_{b_1} f(z)dz &= 2\int_{-5\xrightarrow{+}-3} f(z)\,dz \\
&= 2\int_2^0 \sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{6+r}\sqrt{7+r}\sqrt{9+r}\,dr \\
&= -291.688\,.
\end{aligned}$$

Using Mathematica

Along $-5 \xrightarrow{+} -3$:

$$\begin{aligned}
\arg(z+5)=0 &\Longrightarrow \sqrt{z+5}=\text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3)=-\pi &\Longrightarrow \sqrt{z+3}=(-1)\cdot\text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1)=-\pi &\Longrightarrow \sqrt{z+1}=(-1)\cdot\text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1)=-\pi &\Longrightarrow \sqrt{z-1}=(-1)\cdot\text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3)=-\pi &\Longrightarrow \sqrt{z-3}=(-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4)=-\pi &\Longrightarrow \sqrt{z-4}=(-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6)=-\pi &\Longrightarrow \sqrt{z-6}=(-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\begin{aligned}
\int_{-5\xrightarrow{+}-3} f(z)\,dz &= (-1)^6\cdot\text{MATH}\left(\int_{-5}^{-3}\sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right) \\
&= \text{MATH}\left(\int_{-5}^{-3}\sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right) \\
&= -145.844\,.
\end{aligned}$$

$$\int_{b_1} f(z)dz = 2\int_{-5\xrightarrow{+}-3} f(z)\,dz$$
$$= 2\cdot \text{MATH}\left(\int_{-5}^{-3} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right)$$
$$= -291.688\,.$$

5. To evaluate $\int_{b_2} f(z)dz$

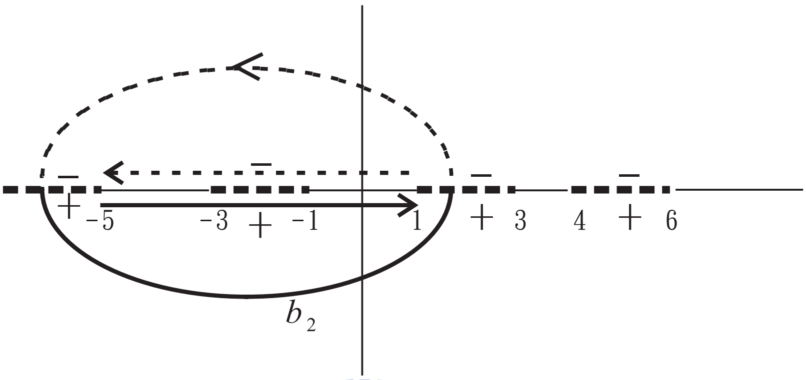


Figure 38

$$\int_{b_2} f(z)\,dz = \int_{-5\xrightarrow{+}1} f(z)\,dz + \int_{-5\xleftarrow{-}1} f(z)\,dz$$
$$= \int_{-5\xrightarrow{+}-3} f(z)\,dz + \int_{-3\xrightarrow{+}-1} f(z)\,dz + \int_{-1\xrightarrow{+}1} f(z)\,dz$$
$$+ \int_{-1\xleftarrow{-}1} f(z)\,dz + \int_{-3\xleftarrow{-}-1} f(z)\,dz + \int_{-5\xleftarrow{-}-3} f(z)\,dz$$

<u>**Theoretical Evaluation**</u>

(1)Along $-5 \xrightarrow{+} -3$

$$z = -3 + re^{i(-\pi)} = -3 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{2-r}\\
z+3<0 &\Longrightarrow \arg(z+3)=-\pi \Longrightarrow \sqrt{z+3}=\sqrt{-r}=-i\sqrt{r}\\
z+1<0 &\Longrightarrow \arg(z+1)=-\pi \Longrightarrow \sqrt{z+1}=\sqrt{-(2+r)}=-i\sqrt{2+r}\\
z-1<0 &\Longrightarrow \arg(z-1)=-\pi \Longrightarrow \sqrt{z-1}=\sqrt{-(4+r)}=-i\sqrt{4+r}\\
z-3<0 &\Longrightarrow \arg(z-3)=-\pi \Longrightarrow \sqrt{z-3}=\sqrt{-(6+r)}=-i\sqrt{6+r}\\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=\sqrt{-(7+r)}=-i\sqrt{7+r}\\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=\sqrt{-(9+r)}=-i\sqrt{9+r}
\end{aligned}$$

Let $u_1(r) = \sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{6+r}\sqrt{7+r}\sqrt{9+r}$. Then

$$\int_{-5\xrightarrow{+}-3} f(z)\,dz = -(-i)^6\int_2^0 u_1(r)\,dr = \int_2^0 u_1(r)\,dr$$

(2)Along $-5 \overset{-}{\dashleftarrow} -3$

The same procedure in example 6, we first evaluate $-5 \xleftarrow{-} -3$.

$$\int_{-5\xleftarrow{-}-3} f(z)\,dz = \int_0^2 i^6 u_1(r)\,(-dr) = -i^6 \int_0^2 u_1(r)\,dr = \int_0^2 u_1(r)\,dr.$$

Then

$$\begin{aligned}\int_{-5\overset{-}{\dashleftarrow}-3} f(z)\,dz &= -\int_{-5\xleftarrow{-}-3} f(z)\,dz = -\int_0^2 u_1(r)\,dr \\ &= \int_2^0 u_1(r)\,dr = \int_{-5\xrightarrow{+}-3} f(z)\,dz.\end{aligned}$$

From (1) and (2), we have

$$\begin{aligned}\int_{-5\xrightarrow{+}-3} f(z)\,dz + \int_{-5\overset{-}{\dashleftarrow}-3} f(z)\,dz &= 2\int_{-5\xrightarrow{+}-3} f(z)\,dz && (36)\\ &= 2\int_2^0 u_1(r)\,dr. && (37)\end{aligned}$$

(3)Along $-3 \xrightarrow{+} -1$

$$z = -1 + re^{i(-\pi)} = -1 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{4-r}\\
z+3>0 &\Longrightarrow \arg(z+3)=0 \Longrightarrow \sqrt{z+3}=\sqrt{2-r}\\
z+1<0 &\Longrightarrow \arg(z+1)=-\pi \Longrightarrow \sqrt{z+1}=-i\sqrt{r}\\
z-1<0 &\Longrightarrow \arg(z-1)=-\pi \Longrightarrow \sqrt{z-1}=-i\sqrt{2+r}\\
z-3<0 &\Longrightarrow \arg(z-3)=-\pi \Longrightarrow \sqrt{z-3}=-i\sqrt{4+r}\\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=-i\sqrt{5+r}\\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=-i\sqrt{7+r}
\end{aligned}$$

Let $u_2(r) = \sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{4+r}\sqrt{5+r}\sqrt{7+r}$. Then

$$\int_{-3\xrightarrow{+}-1} f(z)\,dz = -(-i)^5 \int_2^0 u_2(r)\,dr = i\int_2^0 u_2(r)\,dr.$$

(4)Along $-3 \overset{-}{\dashleftarrow} -1$

$$\int_{-3\xleftarrow{-}-1} f(z)\,dz = -i^5 \int_0^2 u_2(r)\,dr = -i\int_0^2 u_2(r)\,dr.$$

Then

$$\begin{aligned}\int_{-3\overset{-}{\dashleftarrow}-1} f(z)\,dz &= -\int_{-3\xleftarrow{-}-1} f(z)\,dz = i\int_0^2 u_2(r)\,dr \\ &= -i\int_2^0 u_2(r)\,dr = -\int_{-3\xrightarrow{+}-1} f(z)\,dz.\end{aligned}$$

From (3) and (4), we have

$$\int_{-3 \xrightarrow{+} -1} f(z)\, dz + \int_{-3 \overset{-}{\dashleftarrow} -1} f(z)\, dz \tag{38}$$

$$= \int_{-3 \xrightarrow{+} -1} f(z)\, dz + \left(- \int_{-3 \xrightarrow{+} -1} f(z)\, dz \right) \tag{39}$$

$$= 0. \tag{40}$$

(5)Along $-1 \xrightarrow{+} 1$

$$z = 1 + re^{i(-\pi)} = 1 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{6-r}\\
z+3>0 &\Longrightarrow \arg(z+3)=0 \Longrightarrow \sqrt{z+3}=\sqrt{4-r}\\
z+1>0 &\Longrightarrow \arg(z+1)=0 \Longrightarrow \sqrt{z+1}=\sqrt{2-r}\\
z-1<0 &\Longrightarrow \arg(z-1)=-\pi \Longrightarrow \sqrt{z-1}=-i\sqrt{r}\\
z-3<0 &\Longrightarrow \arg(z-3)=-\pi \Longrightarrow \sqrt{z-3}=-i\sqrt{2+r}\\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=-i\sqrt{3+r}\\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=-i\sqrt{5+r}
\end{aligned}$$

Let $u_3(r) = \sqrt{6-r}\sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{2+r}\sqrt{3+r}\sqrt{5+r}$. Then

$$\int_{-1 \xrightarrow{+} 1} f(z)\, dz = -(-i)^4 \int_2^0 u_3(r)\, dr = -\int_2^0 u_3(r)\, dr.$$

(6)Along $-1 \overset{-}{\dashleftarrow} 1$

$$\int_{-1 \xleftarrow{-} 1} f(z)\, dz = -i^4 \int_0^2 u_3(r)\, dr = -\int_0^2 u_3(r)\, dr.$$

Then

$$\int_{-1 \overset{-}{\dashleftarrow} 1} f(z)\, dz = -\int_{-1 \xleftarrow{-} 1} f(z)\, dz = \int_0^2 u_3(r)\, dr$$

$$= -\int_2^0 u_3(r)\, dr = \int_{-1 \xrightarrow{+} 1} f(z)\, dz.$$

From (5) and (6), we have

$$\int_{-1 \xrightarrow{+} 1} f(z)\, dz + \int_{-1 \overset{-}{\dashleftarrow} 1} f(z)\, dz = 2 \int_{-1 \xrightarrow{+} 1} f(z)\, dz \tag{41}$$

$$= -2 \int_2^0 u_3(r)\, dr. \tag{42}$$

According to equation (36), (37), (38), (39), (40), (41), and (42),

$$\begin{aligned}
\int_{b_2} f(z)\, dz &= 2 \int_{-5 \xrightarrow{+} -3} f(z)\, dz + 2 \int_{-1 \xrightarrow{+} 1} f(z)\, dz\\
&= 2 \int_2^0 u_1(r)\, dr + \left(-2 \int_2^0 u_3(r)\, dr \right)\\
&= -291.688 + 101.116\\
&= -190.572\,.
\end{aligned}$$

Using Mathematica

(1)Along $-5 \xrightarrow{+} -3$

$$
\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = -\pi \quad &\Longrightarrow \quad \sqrt{z+3} = (-1)\cdot\text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = -\pi \quad &\Longrightarrow \quad \sqrt{z+1} = (-1)\cdot\text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = -\pi \quad &\Longrightarrow \quad \sqrt{z-1} = (-1)\cdot\text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = -\pi \quad &\Longrightarrow \quad \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = -\pi \quad &\Longrightarrow \quad \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}
$$

$$
\int_{-5 \xrightarrow{+} -3} f(z)\,dz = (-1)^6 \cdot \text{MATH}\left(\int_{-5}^{-3} f(z)\,dz\right) = \text{MATH}\left(\int_{-5}^{-3} f(z)\,dz\right)
$$

(2)Along $-5 \overset{-}{\dashleftarrow} -3$

$$
\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = \pi \quad &\Longrightarrow \quad \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = \pi \quad &\Longrightarrow \quad \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = \pi \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = \pi \quad &\Longrightarrow \quad \sqrt{z-3} = \text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = \pi \quad &\Longrightarrow \quad \sqrt{z-4} = \text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = \pi \quad &\Longrightarrow \quad \sqrt{z-6} = \text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}
$$

$$
\int_{-5 \xleftarrow{-} -3} f(z)\,dz = \text{MATH}\left(\int_{-3}^{-5} f(z)\,dz\right) = -\text{MATH}\left(\int_{-5}^{-3} f(z)\,dz\right) = -\int_{-5 \xrightarrow{+} -3} f(z)\,dz
$$

$$
\int_{-5 \overset{-}{\dashleftarrow} -3} f(z)\,dz = -\int_{-5 \xleftarrow{-} -3} f(z)\,dz = \int_{-5 \xrightarrow{+} -3} f(z)\,dz
$$

From (1) and (2),

$$
\begin{aligned}
&\int_{-5 \xrightarrow{+} -3} f(z)\,dz + \int_{-5 \overset{-}{\dashleftarrow} -3} f(z)\,dz \\
&= \int_{-5 \xrightarrow{+} -3} f(z)\,dz + \int_{-5 \xrightarrow{+} -3} f(z)\,dz \\
&= 2\int_{-5 \xrightarrow{+} -3} f(z)\,dz \\
&= 2\cdot\text{MATH}\left(\int_{-5}^{-3} f(z)\,dz\right) \\
&= 2\cdot\text{MATH}\left(\int_{-5}^{-3} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right) \\
&= -291.688\,.
\end{aligned}
$$

(3)Along $-3 \xrightarrow{+} -1$

$$\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = 0 \quad &\Longrightarrow \quad \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = -\pi \quad &\Longrightarrow \quad \sqrt{z+1} = (-1)\cdot\text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = -\pi \quad &\Longrightarrow \quad \sqrt{z-1} = (-1)\cdot\text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = -\pi \quad &\Longrightarrow \quad \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = -\pi \quad &\Longrightarrow \quad \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{-3\xrightarrow{+}-1} f(z)\,dz = (-1)^5\cdot\text{MATH}\left(\int_{-3}^{-1} f(z)\,dz\right) = -\text{MATH}\left(\int_{-3}^{-1} f(z)\,dz\right)$$

(4)Along $-3 \overset{-}{\dashleftarrow} -1$

$$\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = 0 \quad &\Longrightarrow \quad \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = \pi \quad &\Longrightarrow \quad \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = \pi \quad &\Longrightarrow \quad \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = \pi \quad &\Longrightarrow \quad \sqrt{z-3} = \text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = \pi \quad &\Longrightarrow \quad \sqrt{z-4} = \text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = \pi \quad &\Longrightarrow \quad \sqrt{z-6} = \text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{-3\xleftarrow{-}-1} f(z)\,dz = \text{MATH}\left(\int_{-1}^{-3} f(z)\,dz\right) = -\text{MATH}\left(\int_{-3}^{-1} f(z)\,dz\right) = \int_{-3\xrightarrow{+}-1} f(z)\,dz$$

$$\int_{-3\overset{-}{\dashleftarrow}-1} f(z)\,dz = -\int_{-3\xleftarrow{-}-1} f(z)\,dz = -\int_{-3\xrightarrow{+}-1} f(z)\,dz$$

From (3) and (4),

$$\begin{aligned}
\int_{-3\xrightarrow{+}-1} f(z)\,dz + \int_{-3\overset{-}{\dashleftarrow}-1} f(z)\,dz &= \int_{-3\xrightarrow{+}-1} f(z)\,dz - \int_{-3\xrightarrow{+}-1} f(z)\,dz \\
&= 0\,.
\end{aligned}$$

(5)Along $-1 \xrightarrow{+} 1$

$$\begin{aligned}
\arg(z+5) = 0 \quad &\Longrightarrow \quad \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = 0 \quad &\Longrightarrow \quad \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = 0 \quad &\Longrightarrow \quad \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = -\pi \quad &\Longrightarrow \quad \sqrt{z-1} = (-1)\cdot\text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = -\pi \quad &\Longrightarrow \quad \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = -\pi \quad &\Longrightarrow \quad \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = -\pi \quad &\Longrightarrow \quad \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{-1\xrightarrow{+}1} f(z)\,dz = (-1)^4 \cdot \mathrm{MATH}\left(\int_{-1}^{1} f(z)\,dz\right) = \mathrm{MATH}\left(\int_{-1}^{1} f(z)\,dz\right)$$

(6)Along $-1 \overset{-}{\dashleftarrow} 1$

$$\begin{aligned}
\arg(z+5) = 0 &\implies \sqrt{z+5} = \mathrm{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) = 0 &\implies \sqrt{z+3} = \mathrm{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) = 0 &\implies \sqrt{z+1} = \mathrm{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) = \pi &\implies \sqrt{z-1} = \mathrm{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) = \pi &\implies \sqrt{z-3} = \mathrm{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) = \pi &\implies \sqrt{z-4} = \mathrm{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) = \pi &\implies \sqrt{z-6} = \mathrm{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{-1\xleftarrow{-}1} f(z)\,dz = \mathrm{MATH}\left(\int_{1}^{-1} f(z)\,dz\right) = -\mathrm{MATH}\left(\int_{-1}^{1} f(z)\,dz\right) = -\int_{-1\xrightarrow{+}1} f(z)\,dz$$

$$\int_{-1\overset{-}{\dashleftarrow}1} f(z)\,dz = -\int_{-1\xleftarrow{-}1} f(z)\,dz = \int_{-1\xrightarrow{+}1} f(z)\,dz$$

From (5) and (6),

$$\begin{aligned}
&\int_{-1\xrightarrow{+}1} f(z)\,dz + \int_{-1\overset{-}{\dashleftarrow}1} f(z)\,dz \\
&= \int_{-1\xrightarrow{+}1} f(z)\,dz + \int_{-1\xrightarrow{+}1} f(z)\,dz \\
&= 2\int_{-1\xrightarrow{+}1} f(z)\,dz \\
&= 2\cdot \mathrm{MATH}\left(\int_{-1}^{1} f(z)\,dz\right) \\
&= 2\cdot \mathrm{MATH}\left(\int_{-1}^{1} \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\,dz\right) \\
&= -101.116\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{b_2} f(z)\,dz &= \left(\int_{-5\xrightarrow{+}-3} f(z)\,dz + \int_{-5\overset{-}{\dashleftarrow}-3} f(z)\,dz\right) + \left(\int_{-1\xrightarrow{+}1} f(z)\,dz + \int_{-1\overset{-}{\dashleftarrow}-1} f(z)\,dz\right) \\
&= -291.688 + 101.116 \\
&= -190.572\,.
\end{aligned}$$

6. To evaluate $\int_{b_3} f(z)dz$

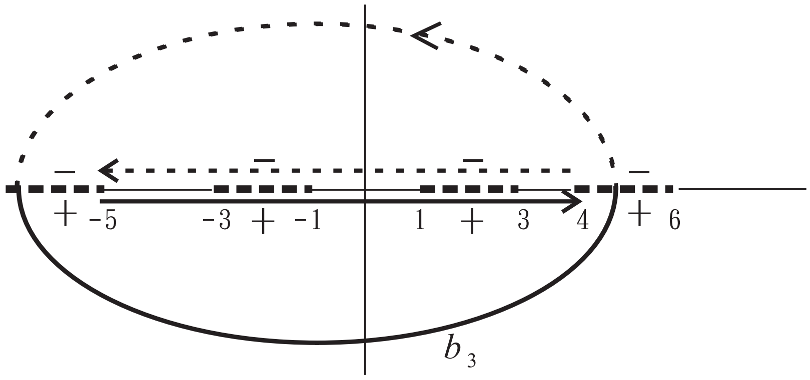


Figure 39

$$
\begin{aligned}
\int_{b_3} f(z)\,dz &= \int_{-5\xrightarrow{+}4} f(z)\,dz + \int_{-5\overset{-}{\dashleftarrow}-4} f(z)\,dz \\
&= \int_{-5\xrightarrow{+}-3} f(z)\,dz + \int_{-3\xrightarrow{+}-1} f(z)\,dz + \int_{-1\xrightarrow{+}1} f(z)\,dz + \int_{1\xrightarrow{+}3} f(z)\,dz \\
&\quad + \int_{3\xrightarrow{+}4} f(z)\,dz + \int_{3\overset{-}{\dashleftarrow}4} f(z)\,dz + \int_{1\overset{-}{\dashleftarrow}3} f(z)\,dz + \int_{-1\overset{-}{\dashleftarrow}1} f(z)\,dz \\
&\quad + \int_{-3\overset{-}{\dashleftarrow}-1} f(z)\,dz + \int_{-5\overset{-}{\dashleftarrow}-3} f(z)\,dz
\end{aligned}
$$

We only need to evaluate the four integrals, $\int_{1\xrightarrow{+}3} f(z)dz$, $\int_{1\overset{-}{\dashleftarrow}3} f(z)dz$, $\int_{3\xrightarrow{+}4} f(z)dz$, and $\int_{3\overset{-}{\dashleftarrow}4} f(z)\,dz$. The other six integrals are as same as the six integrals evaluated in 5.

Theoretical Evaluation

(1)Along $1 \xrightarrow{+} 3$

$$z = 3 + re^{i(-\pi)} = 3 - r, \quad r : 2 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$
\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5) = 0 \Longrightarrow \sqrt{z+5} = \sqrt{8-r} \\
z+3>0 &\Longrightarrow \arg(z+3) = 0 \Longrightarrow \sqrt{z+3} = \sqrt{6-r} \\
z+1>0 &\Longrightarrow \arg(z+1) = 0 \Longrightarrow \sqrt{z+1} = \sqrt{4-r} \\
z-1>0 &\Longrightarrow \arg(z-1) = 0 \Longrightarrow \sqrt{z-1} = \sqrt{2-r} \\
z-3<0 &\Longrightarrow \arg(z-3) = -\pi \Longrightarrow \sqrt{z-3} = -i\sqrt{r} \\
z-4<0 &\Longrightarrow \arg(z-4) = -\pi \Longrightarrow \sqrt{z-4} = -i\sqrt{1+r} \\
z-6<0 &\Longrightarrow \arg(z-6) = -\pi \Longrightarrow \sqrt{z-6} = -i\sqrt{3+r}
\end{aligned}
$$

Let $u_4(r) = \sqrt{8-r}\sqrt{6-r}\sqrt{4-r}\sqrt{2-r}\sqrt{r}\sqrt{1+r}\sqrt{3+r}$. Then

$$\int_{1\xrightarrow{+}3} f(z)\,dz = -(-i)^3 \int_2^0 u_4(r)\,dr = -i\int_2^0 u_4(r)\,dr.$$

(2)Along $1 \overset{-}{\dashleftarrow} 3$

$$\int_{1\overset{-}{\longleftarrow}3} f(z)\,dz = -i^3 \int_0^2 u_4(r)\,dr = i\int_0^2 u_4(r)\,dr.$$

Then

$$\begin{aligned}\int_{1 \overset{-}{\dashleftarrow} 3} f(z)\,dz &= -\int_{1 \overset{-}{\longleftarrow} 3} f(z)\,dz = -i\int_0^2 u_4(r)\,dr \\ &= i\int_2^0 u_4(r)\,dr = -\int_{1 \overset{+}{\longrightarrow} 3} f(z)\,dz.\end{aligned}$$

From (1) and (2), we have

$$\begin{aligned}&\int_{1 \overset{+}{\longrightarrow} 3} f(z)\,dz + \int_{1 \overset{-}{\dashleftarrow} 3} f(z)\,dz \\ &= \int_{1 \overset{+}{\longrightarrow} 3} f(z)\,dz + \left(-\int_{1 \overset{+}{\longrightarrow} 3} f(z)\,dz\right) \\ &= 0. \end{aligned} \tag{43}$$

(3)Along $3 \overset{+}{\longrightarrow} 4$

$$z = 4 + re^{i(-\pi)} = 4 - r, \quad r : 1 \longrightarrow 0 \Longrightarrow dz = -dr$$

$$\begin{aligned}
z+5>0 &\Longrightarrow \arg(z+5)=0 \Longrightarrow \sqrt{z+5}=\sqrt{9-r}\\
z+3>0 &\Longrightarrow \arg(z+3)=0 \Longrightarrow \sqrt{z+3}=\sqrt{7-r}\\
z+1>0 &\Longrightarrow \arg(z+1)=0 \Longrightarrow \sqrt{z+1}=\sqrt{5-r}\\
z-1>0 &\Longrightarrow \arg(z-1)=0 \Longrightarrow \sqrt{z-1}=\sqrt{3-r}\\
z-3>0 &\Longrightarrow \arg(z-3)=0 \Longrightarrow \sqrt{z-3}=\sqrt{1-r}\\
z-4<0 &\Longrightarrow \arg(z-4)=-\pi \Longrightarrow \sqrt{z-4}=-i\sqrt{r}\\
z-6<0 &\Longrightarrow \arg(z-6)=-\pi \Longrightarrow \sqrt{z-6}=-i\sqrt{2+r}
\end{aligned}$$

Let $u_5(r) = \sqrt{9-r}\sqrt{7-r}\sqrt{5-r}\sqrt{3-r}\sqrt{1-r}\sqrt{r}\sqrt{2+r}$. Then

$$\int_{3 \overset{+}{\longrightarrow} 4} f(z)\,dz = -(-i)^2\int_1^0 u_5(r)\,dr = \int_1^0 u_5(r)\,dr.$$

(4)Along $3 \overset{-}{\dashleftarrow} 4$

$$\int_{3 \overset{-}{\longleftarrow} 4} f(z)\,dz = -i^2\int_0^1 u_5(r)\,dr = \int_0^1 u_5(r)\,dr.$$

Then

$$\begin{aligned}\int_{3 \overset{-}{\dashleftarrow} 4} f(z)\,dz &= -\int_{3 \overset{-}{\longleftarrow} 4} f(z)\,dz = -\int_0^1 u_5(r)\,dr \\ &= \int_1^0 u_5(r)\,dr = \int_{3 \overset{+}{\longrightarrow} 4} f(z)\,dz.\end{aligned}$$

From (3) and (4), we have

$$\begin{aligned}\int_{3 \overset{+}{\longrightarrow} 4} f(z)\,dz + \int_{3 \overset{-}{\dashleftarrow} 4} f(z)\,dz &= 2\int_{3 \overset{+}{\longrightarrow} 4} f(z)\,dz \\ &= 2\int_1^0 u_5(r)\,dr. \end{aligned} \tag{44}$$

So, we have

$$\begin{aligned}
\int_{b_3} f(z)\,dz &= 2\int_{-5\xrightarrow{+}-3} f(z)\,dz + 2\int_{-1\xrightarrow{+}1} f(z)\,dz + 2\int_{3\xrightarrow{+}4} f(z)\,dz \\
&= 2\int_2^0 u_1(r)\,dr + \left(-2\int_2^0 u_3(r)\,dr\right) + 2\int_1^0 u_5(r)\,dr \\
&= -291.688 + 101.116 + (-30.8213) \\
&= -221.393\,.
\end{aligned}$$

<u>**Using Mathematica**</u>

(1)Along $1 \xrightarrow{+} 3$

$$\begin{aligned}
\arg(z+5) &= 0 &&\Longrightarrow \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) &= 0 &&\Longrightarrow \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) &= 0 &&\Longrightarrow \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) &= 0 &&\Longrightarrow \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) &= -\pi &&\Longrightarrow \sqrt{z-3} = (-1)\cdot\text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) &= -\pi &&\Longrightarrow \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) &= -\pi &&\Longrightarrow \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{1\xrightarrow{+}3} f(z)\,dz = (-1)^3\cdot\text{MATH}\left(\int_1^3 f(z)\,dz\right) = -\text{MATH}\left(\int_1^3 f(z)\,dz\right)$$

(2)Along $1 \overset{-}{\dashleftarrow} 3$

$$\int_{1\xleftarrow{-}3} f(z)\,dz = \text{MATH}\left(\int_3^1 f(z)\,dz\right) = -\text{MATH}\left(\int_1^3 f(z)\,dz\right) = \int_{1\xrightarrow{+}3} f(z)\,dz$$

$$\int_{1\overset{-}{\dashleftarrow}3} f(z)\,dz = -\int_{1\xleftarrow{-}3} f(z)\,dz = -\int_{1\xrightarrow{+}3} f(z)\,dz$$

From (1) and (2),

$$\begin{aligned}
\int_{1\xrightarrow{+}3} f(z)\,dz + \int_{1\overset{-}{\dashleftarrow}3} f(z)\,dz &= \int_{1\xrightarrow{+}3} f(z)\,dz - \int_{1\xrightarrow{+}3} f(z)\,dz \\
&= 0\,.
\end{aligned}$$

(3)Along $3 \xrightarrow{+} 4$

$$\begin{aligned}
\arg(z+5) &= 0 &&\Longrightarrow \sqrt{z+5} = \text{MATH}\left(\sqrt{z+5}\right) \\
\arg(z+3) &= 0 &&\Longrightarrow \sqrt{z+3} = \text{MATH}\left(\sqrt{z+3}\right) \\
\arg(z+1) &= 0 &&\Longrightarrow \sqrt{z+1} = \text{MATH}\left(\sqrt{z+1}\right) \\
\arg(z-1) &= 0 &&\Longrightarrow \sqrt{z-1} = \text{MATH}\left(\sqrt{z-1}\right) \\
\arg(z-3) &= 0 &&\Longrightarrow \sqrt{z-3} = \text{MATH}\left(\sqrt{z-3}\right) \\
\arg(z-4) &= -\pi &&\Longrightarrow \sqrt{z-4} = (-1)\cdot\text{MATH}\left(\sqrt{z-4}\right) \\
\arg(z-6) &= -\pi &&\Longrightarrow \sqrt{z-6} = (-1)\cdot\text{MATH}\left(\sqrt{z-6}\right).
\end{aligned}$$

$$\int_{3 \xrightarrow{+} 4} f(z)\, dz = (-1)^2 \cdot \textsc{math}\left(\int_3^4 f(z)\, dz\right) = \textsc{math}\left(\int_3^4 f(z)\, dz\right)$$

(4)Along $3 \overset{-}{\dashleftarrow} 4$

$$\int_{3 \xleftarrow{-} 4} f(z)\, dz = \textsc{math}\left(\int_4^3 f(z)\, dz\right) = -\textsc{math}\left(\int_3^4 f(z)\, dz\right) = -\int_{3 \xrightarrow{+} 4} f(z)\, dz$$

$$\int_{3 \overset{-}{\dashleftarrow} 4} f(z)\, dz = -\int_{3 \xleftarrow{-} 4} f(z)\, dz = \int_{3 \xrightarrow{+} 4} f(z)\, dz$$

From (3) and (4),

$$\begin{aligned}
&\int_{3 \xrightarrow{+} 4} f(z)\, dz + \int_{3 \overset{-}{\dashleftarrow} 4} f(z)\, dz \\
&= \int_{3 \xrightarrow{+} 4} f(z)\, dz + \int_{3 \xrightarrow{+} 4} f(z)\, dz \\
&= 2\int_{3 \xrightarrow{+} 4} f(z)\, dz \\
&= 2 \cdot \textsc{math}\left(\int_3^4 f(z)\, dz\right) \\
&= 2 \cdot \textsc{math}\left(\int_3^4 \sqrt{(z+5)(z+3)(z+1)(z-1)(z-3)(z-4)(z-6)}\, dz\right) \\
&= -30.8213\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{b_3} f(z)\, dz &= \left(\int_{-5 \xrightarrow{+} -3} f(z)\, dz + \int_{-5 \overset{-}{\dashleftarrow} -3} f(z)\, dz\right) + \left(\int_{-1 \xrightarrow{+} 1} f(z)\, dz + \int_{-1 \overset{-}{\dashleftarrow} 1} f(z)\, dz\right) \\
&= \quad + \left(\int_{3 \xrightarrow{+} 4} f(z)\, dz + \int_{3 \overset{-}{\dashleftarrow} 4} f(z)\, dz\right) \\
&= -291.688 + 101.116 + (-30.8213) \\
&= -221.393\,.
\end{aligned}$$

3.4 Generalization of Integrals Along Horizontal Cuts

We evaluate the integrals on the Riemann surface of genus N. If the Riemann surface is of genus N, then there are $2N+1$ or $2N+2$ branch points.

Case 1. The number of branch points is odd ($2N+1$ branch points)

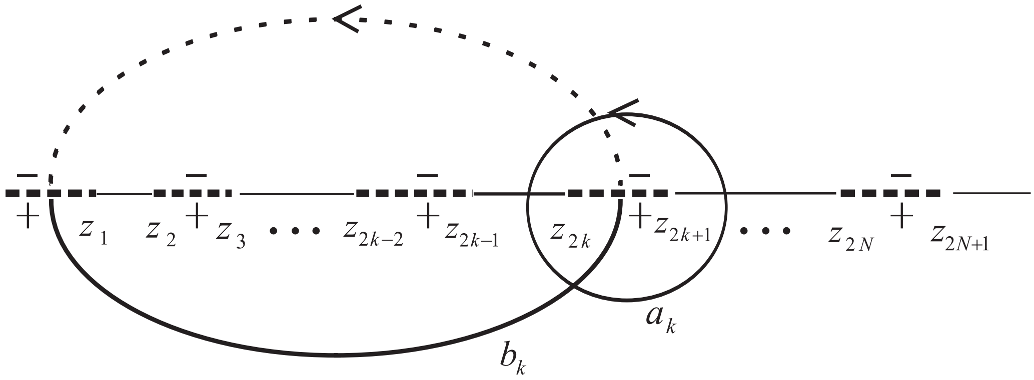


Figure 40

$1 \leq k \leq N$ in Figure 40. Let

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+1})} = \prod_{j=1}^{2N+1} \sqrt{z-z_j}.$$

(1) To evaluate $\int_{a_k} f(z)dz$

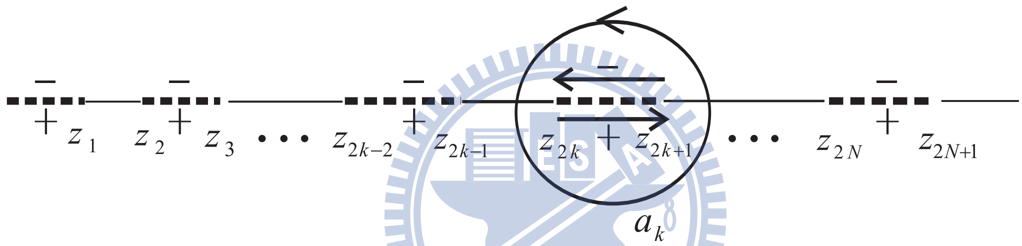


Figure 41

__**Theoretical Evaluation**__

Along $z_{2k} \xrightarrow{+} z_{2k+1}$, let $d = |z_{2k+1} - z_{2k}|$.

$$z = z_{2k+1} + re^{i(-\pi)} = z_{2k+1} - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 1, 2, \cdots, 2k$,

$$\arg(z-z_j) = 0 \Longrightarrow \sqrt{z-z_j} = \sqrt{(z_{2k+1}-r)-z_j}$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z-z_j) = -\pi \Longrightarrow \sqrt{z-z_j} = -i\sqrt{z_j-(z_{2k+1}-r)}$$

Let

$$u(r) = \left(\prod_{j=1}^{2k} \sqrt{(z_{2k+1}-r)-z_j}\right)\left(\prod_{j=2k+1}^{2N+1} \sqrt{z_j-(z_{2k+1}-r)}\right).$$

Then

$$\int_{z_{2k}\xrightarrow{+}z_{2k+1}} f(z)\,dz = -(-i)^{(2N+1)-(2k+1)+1}\int_d^0 u(r)\,dr$$
$$= -(-i)^{2N-2k+1}\int_d^0 u(r)\,dr$$
$$= i^{2N-2k+1}\int_d^0 u(r)\,dr$$
$$= i^{2N-2k}\cdot i\int_d^0 u(r)\,dr$$
$$= (i^2)^{N-k}\cdot i\int_d^0 u(r)\,dr$$
$$= (-1)^{N-k}\cdot i\int_d^0 u(r)\,dr$$

Thus,

$$\int_{a_k} f(z)\,dz = 2\int_{z_{2k}\xrightarrow{+}z_{2k+1}} f(z)\,dz \quad (45)$$
$$= (-1)^{N-k}\cdot 2i\int_d^0 u(r)\,dr \quad (46)$$

Note that the value of the integral is a pure imaginary number.

Using Mathematica

For $j = 1, 2, \cdots, 2k$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z - z_j}\right)$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = (-1)\cdot\text{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\int_{z_{2k}\xrightarrow{+}z_{2k+1}} f(z)\,dz = (-1)^{(2N+1)-(2k+1)+1}\cdot\text{MATH}\left(\int_{z_{2k}}^{z_{2k+1}} f(z)\,dz\right)$$
$$= (-1)\cdot\text{MATH}\left(\int_{z_{2k}}^{z_{2k+1}} f(z)\,dz\right)$$

Thus,

$$\int_{a_k} f(z)\,dz = 2\int_{z_{2k}\xrightarrow{+}z_{2k+1}} f(z)\,dz \quad (47)$$
$$= (-2)\cdot\text{MATH}\left(\int_{z_{2k}}^{z_{2k+1}} f(z)\,dz\right) \quad (48)$$

(2) To evaluate $\int_{b_k} f(z)dz$

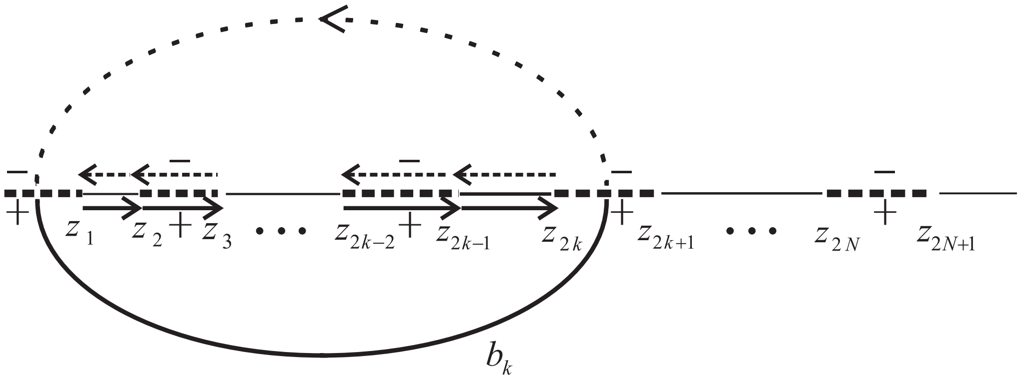


Figure 42

Before our compuation, we first discuss the integrals of the two kinds of path drawn in Figure 43 and Figure 44.

Class 1. Along a path that there is a cut on it

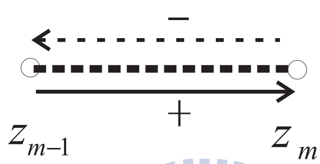


Figure 43

Since $f(z)|_{II} = -f(z)|_I$,

$$\begin{aligned}\int_{z_{m-1}\overset{-}{\dashleftarrow} z_m} f(z)\,dz &= -\int_{z_{m-1}\xleftarrow{-} z_m} f(z)\,dz \\ &= -\int_{z_{m-1}\xrightarrow{+} z_m} f(z)\,dz\end{aligned}$$

So, we have

$$\int_{z_{m-1}\xrightarrow{+} z_m} f(z)\,dz + \int_{z_{m-1}\overset{-}{\dashleftarrow} z_m} f(z)\,dz = 0. \tag{49}$$

Class 2. Along a path that there is no cut on it

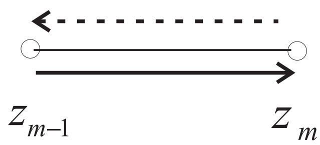


Figure 44

Since $f(z)|_{II} = -f(z)|_I$,

$$\begin{aligned}\int_{z_{m-1}\overset{-}{\dashleftarrow} z_m} f(z)\,dz &= -\int_{z_{m-1}\xleftarrow{-} z_m} f(z)\,dz \\ &= -\left(-\int_{z_{m-1}\xrightarrow{+} z_m} f(z)\,dz\right) \\ &= \int_{z_{m-1}\xrightarrow{+} z_m} f(z)\,dz\end{aligned}$$

So, we have

$$\int_{z_{m-1}\longrightarrow z_m} f(z)\,dz + \int_{z_{m-1}\dashleftarrow z_m} f(z)\,dz = 2\int_{z_{m-1}\longrightarrow z_m} f(z)\,dz. \tag{50}$$

Thus, we only need to evaluate the integrals $\int_{z_j \overset{+}{\longrightarrow} z_{j+1}} f(z)\,dz$ for $j = 1, 2, \cdots, 2k-1$ and add them together. That is,

$$\begin{aligned}\int_{b_k} f(z)\,dz &= 2\left(\int_{z_1\longrightarrow z_2} f(z)\,dz + \int_{z_3\longrightarrow z_4} f(z)\,dz + \cdots + \int_{z_{2k-1}\longrightarrow z_{2k}} f(z)\,dz\right)\\ &= 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz \end{aligned} \tag{51}$$

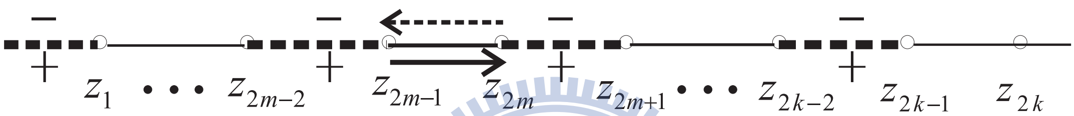


Figure 45

<u>Theoretical Evaluation</u>

Along $z_{2m-1}\longrightarrow z_{2m}$, let $d = |z_{2m} - z_{2m-1}|$.

$$z = z_{2m} + re^{i(-\pi)} = z_{2m} - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \sqrt{(z_{2m} - r) - z_j}$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = -i\sqrt{z_j - (z_{2m} - r)}$$

Let

$$u(r) = \left(\prod_{j=1}^{2m-1}\sqrt{(z_{2m} - r) - z_j}\right)\left(\prod_{j=2m}^{2N+1}\sqrt{z_j - (z_{2m} - r)}\right).$$

Then

$$\begin{aligned}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz &= -(-i)^{(2N+1)-2m+1}\int_d^0 u(r)\,dr\\ &= -(-i)^{2N-2m+2}\int_d^0 u(r)\,dr\\ &= (-1)^{N-m}\int_d^0 u(r)\,dr\end{aligned}$$

Thus,

$$\int_{b_k} f(z)\,dz = 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz \tag{52}$$

$$= 2\sum_{m=1}^{k/2}(-1)^{N-m}\int_{d}^{0} u(r)\,dr \tag{53}$$

Note that the value of the integral is a real number.

<u>Using Mathematica</u>

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \mathrm{MATH}\left(\sqrt{z - z_j}\right)$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = (-1)\cdot\mathrm{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\int_{z_{2m-1}\xrightarrow{+} z_{2m}} f(z)\,dz = (-1)^{(2N+1)-2m+1}\cdot\mathrm{MATH}\left(\int_{z_{2m-1}}^{z_{2m}} f(z)\,dz\right)$$

$$= \mathrm{MATH}\left(\int_{z_{2m-1}}^{z_{2m}} f(z)\,dz\right)$$

Thus,

$$\int_{b_k} f(z)\,dz = 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz \tag{54}$$

$$= 2\sum_{m=1}^{k}\mathrm{MATH}\left(\int_{z_{2m-1}}^{z_{2m}} f(z)\,dz\right) \tag{55}$$

Case 2. The number of branch points is even ($2N+2$ branch points)

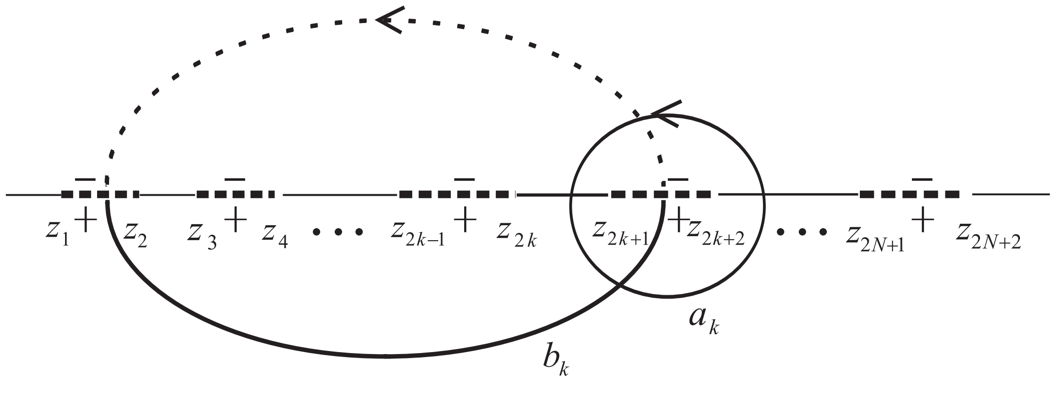


Figure 46

$1 \leq k \leq N$ in Figure 46. Let

$$f(z) = \sqrt{(z - z_1)(z - z_2) \cdots (z - z_{2N+2})} = \prod_{j=1}^{2N+2} \sqrt{z - z_j}.$$

(1) To evaluate $\int_{a_k} f(z)dz$

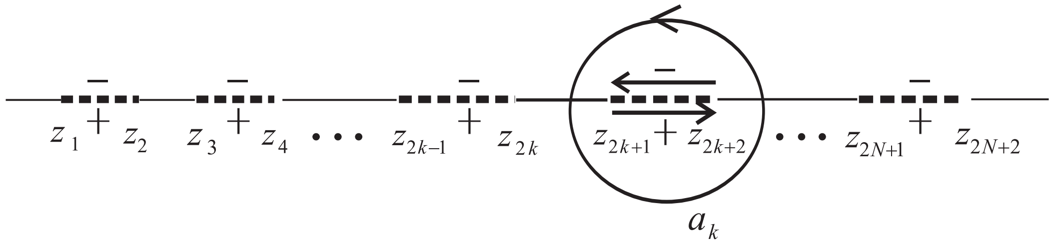


Figure 47

Theoretical Evaluation

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$, let $d = |z_{2k+2} - z_{2k+1}|$.

$$z = z_{2k+2} + re^{i(-\pi)} = z_{2k+2} - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \sqrt{(z_{2k+2} - r) - z_j}$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = -i\sqrt{z_j - (z_{2k+2} - r)}$$

Let

$$u(r) = \left(\prod_{j=1}^{2k+1} \sqrt{(z_{2k+2} - r) - z_j} \right) \left(\prod_{j=2k+2}^{2N+2} \sqrt{z_j - (z_{2k+2} - r)} \right).$$

Then

$$\begin{aligned} \int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz &= -(-i)^{(2N+2)-(2k+2)+1} \int_d^0 u(r)\, dr \\ &= -(-i)^{2N-2k+1} \int_d^0 u(r)\, dr \\ &= (-1)^{N-k} \cdot i \int_d^0 u(r)\, dr \end{aligned}$$

Thus,

$$\int_{a_k} f(z)\, dz = 2 \int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz \tag{56}$$

$$= (-1)^{N-k} \cdot 2i \int_d^0 u(r)\, dr \tag{57}$$

Note that the value of the integral is a pure imaginary number.

Using Mathematica

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z - z_j}\right)$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\begin{aligned}\int_{z_{2k+1} \overset{+}{\longrightarrow} z_{2k+2}} f(z)\, dz &= (-1)^{(2N+2)-(2k+2)+1} \cdot \text{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right) \\ &= (-1) \cdot \text{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right)\end{aligned}$$

Thus,

$$\int_{a_k} f(z)\, dz = 2\int_{z_{2k+1} \overset{+}{\longrightarrow} z_{2k+2}} f(z)\, dz \quad (58)$$

$$= (-2) \cdot \text{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right) \quad (59)$$

(2) To evaluate $\int_{b_k} f(z)dz$

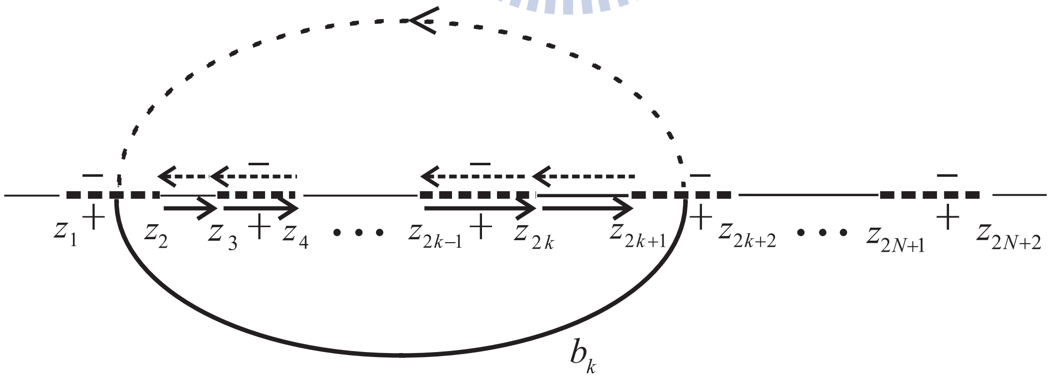


Figure 48

Using similar arguments in class 1 and class 2 of Case 1, we also can see that

$$\begin{aligned}\int_{b_k} f(z)\, dz &= 2\left(\int_{z_2 \longrightarrow z_3} f(z)\, dz + \int_{z_4 \longrightarrow z_5} f(z)\, dz + \cdots + \int_{z_{2k} \longrightarrow z_{2k+1}} f(z)\, dz\right) \\ &= 2\sum_{m=1}^{k} \int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \quad (60)\end{aligned}$$

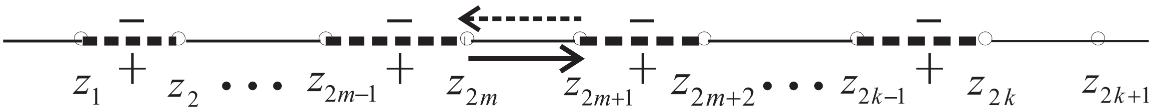


Figure 49

Theoretical Evaluation

Along $z_{2m} \longrightarrow z_{2m+1}$, let $d = |z_{2m+1} - z_{2m}|$.

$$z = z_{2m+1} + re^{i(-\pi)} = z_{2m+1} - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \sqrt{(z_{2m} - r) - z_j}$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = -i\sqrt{z_j - (z_{2m} - r)}$$

Let

$$u(r) = \left(\prod_{j=1}^{2m} \sqrt{(z_{2m} - r) - z_j}\right)\left(\prod_{j=2m+1}^{2N+2} \sqrt{z_j - (z_{2m} - r)}\right).$$

Then

$$\begin{aligned}\int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz &= -(-i)^{(2N+2)-(2m+1)+1} \int_d^0 u(r)\, dr \\ &= -(-i)^{2N-2m+2} \int_d^0 u(r)\, dr \\ &= (-1)^{N-m} \int_d^0 u(r)\, dr\end{aligned}$$

Thus,

$$\int_{b_k} f(z)\, dz = 2\sum_{m=1}^{k} \int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \tag{61}$$

$$= 2\sum_{m=1}^{k} (-1)^{N-m} \int_d^0 u(r)\, dr \tag{62}$$

Note that the value of the integral is a real number.

Using Mathematica

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = 0 \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z - z_j}\right)$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\int_{z_{2m}\longrightarrow z_{2m+1}} f(z)\,dz = (-1)^{(2N+2)-(2m+1)+1}\cdot \text{MATH}\left(\int_{z_{2m}}^{z_{2m+1}} f(z)\,dz\right)$$
$$= \text{MATH}\left(\int_{z_{2m}}^{z_{2m+1}} f(z)\,dz\right)$$

Thus,

$$\int_{b_k} f(z)\,dz = 2\sum_{m=1}^{k}\int_{z_{2m}\longrightarrow z_{2m+1}} f(z)\,dz \tag{63}$$
$$= 2\sum_{m=1}^{k}\text{MATH}\left(\int_{z_{2m}}^{z_{2m+1}} f(z)\,dz\right) \tag{64}$$

Example 8. *Let*

$$f(z) = \prod_{j=1}^{2N+2}\sqrt{z-z_j}.$$

Suppose that $Im(z_{2j-1}) = Im(z_{2j}), j = 1, 2, \cdots, N+1$. *The cuts are drawn in Figure 50. Let* $Re(z_j) = x_j, j = 1, 2, \cdots, 2N+2$ *and* $Im(z_{2j-1}) = Im(z_{2j}) = y_j, j = 1, 2, \cdots, N+1$. *Evaluate* $\int_{a_k} f(z)dz$ *and* $\int_{b_k} f(z)dz$.

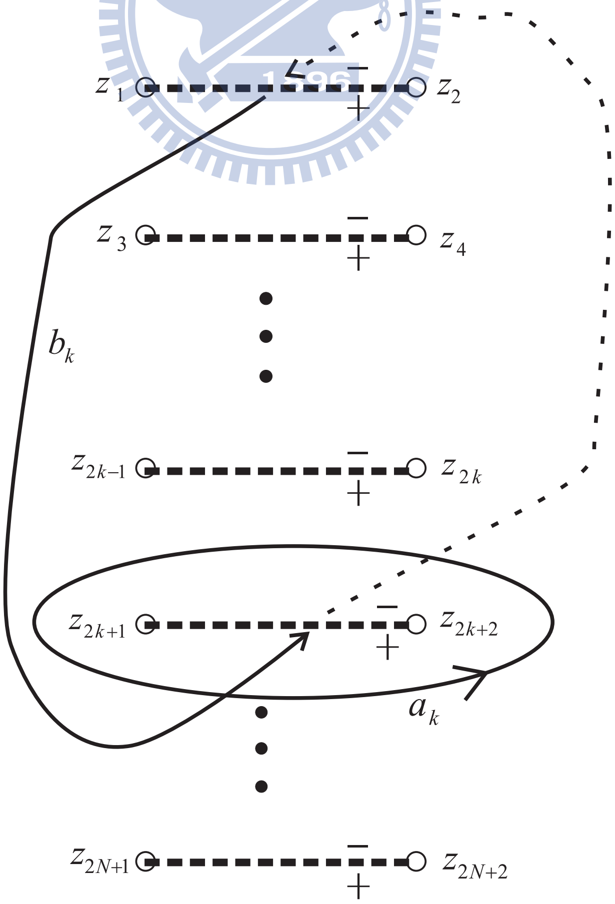


Figure 50

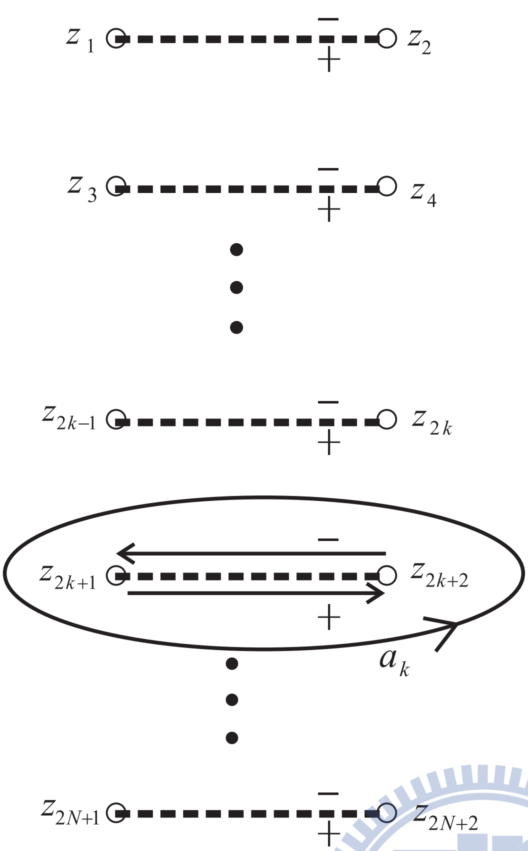


Figure 51

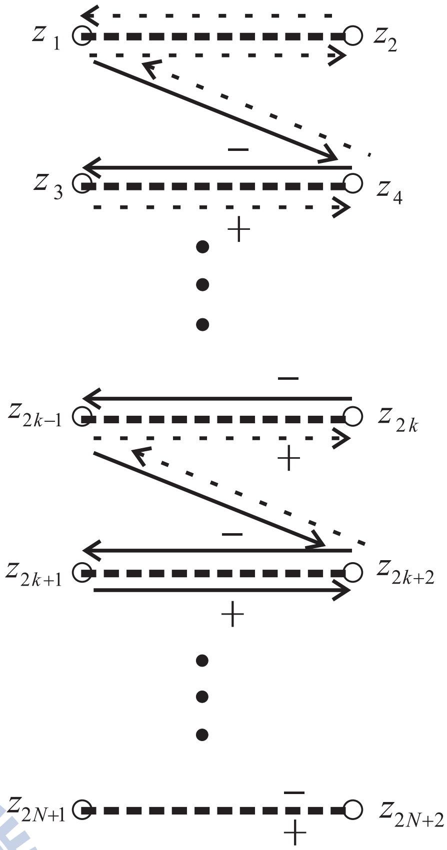


Figure 52

Solution.

(1) To evaluate $\int_{a_k} f(z)dz$ (Figure 51)

<u>Theoretical Evaluation</u>

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$, let $d = |z_{2k+2} - z_{2k+1}|$.

$$z = z_{2k+2} + re^{i(-\pi)} = z_{2k+2} - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 2k+2$,

$$\arg(z - z_{2k+2}) = -\pi \Longrightarrow \sqrt{z - z_{2k+2}} = -i\sqrt{r}$$

For other j,

$$\arg(z - z_j) \neq -\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{(z_{2k+2} - r) - z_j}$$

Let

$$u(r) = \left(\prod_{j=1}^{2k+1} \sqrt{(z_{2k+2} - r) - z_j} \right) (\sqrt{r}) \left(\prod_{j=2k+3}^{2N+2} \sqrt{z_j - (z_{2k+2} - r)} \right).$$

Then

$$\int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz = -(-i) \int_d^0 u(r)\, dr = i \int_d^0 u(r)\, dr$$

Thus,

$$\int_{a_k} f(z)\, dz = 2i \int_d^0 u(r)\, dr\,.$$

Using Mathematica

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$.

For $j = 2k+2$,

$$\arg(z - z_{2k+2}) = -\pi \Longrightarrow \sqrt{z - z_{2k+2}} = (-1) \cdot \operatorname{MATH}\left(\sqrt{z - z_{2k+2}}\right)$$

For other j,

$$\arg(z - z_j) \neq -\pi \Longrightarrow \sqrt{z - z_j} = \operatorname{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz = (-1) \cdot \operatorname{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right)$$

Thus,

$$\begin{aligned}
\int_{a_k} f(z)\, dz &= 2 \int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz \\
&= (-2) \cdot \operatorname{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right)
\end{aligned}$$

(2) To evaluate $\int_{b_k} f(z)dz$ (Figure 52)

For $j = 2, 3, \cdots, k$,

$$\begin{aligned}
&\int_{z_{2j-1} \overset{+}{\dashrightarrow} z_{2j}} f(z)\, dz + \int_{z_{2j-1} \overset{-}{\longleftarrow} z_{2j}} f(z)\, dz \\
&= -\int_{z_{2j-1} \xrightarrow{+} z_{2j}} f(z)\, dz + \int_{z_{2j-1} \xrightarrow{+} z_{2j}} f(z)\, dz \\
&= 0\,.
\end{aligned}$$

So,

$$\begin{aligned}
\int_{b_k} f(z)\, dz = &\left(\int_{z_1 \overset{+}{\dashrightarrow} z_2} f(z)\, dz + \int_{z_1 \overset{-}{\dashleftarrow} z_2} f(z)\, dz\right) \\
&+ \sum_{j=1}^{k} \left(\int_{z_{2j-1} \longrightarrow z_{2j+2}} f(z)\, dz + \int_{z_{2j-1} \dashleftarrow z_{2j+2}} f(z)\, dz\right) \\
&+ \left(\int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz + \int_{z_{2k+1} \overset{-}{\longleftarrow} z_{2k+2}} f(z)\, dz\right) \\
= &\ (-1) \cdot 2 \int_{z_1 \xrightarrow{+} z_2} f(z)\, dz + 2 \sum_{j=1}^{k} \int_{z_{2j-1} \longrightarrow z_{2j+2}} f(z)\, dz \\
= &\ 2i \int_{d}^{0} \left(\prod_{j=1}^{2k+1} \sqrt{(z_{2k+2} - r) - z_j}\right) \left(\sqrt{r}\right) \left(\prod_{j=2k+3}^{2N+2} \sqrt{z_j - (z_{2k+2} - r)}\right)\, dr\,.
\end{aligned}$$

Theoretical Evaluation

Along $z_1 \xrightarrow{+} z_2$, let $d = |z_2 - z_1|$.

$$z = z_2 + re^{i(-\pi)} = z_2 - r, \quad r : d \longrightarrow 0 \Longrightarrow dz = -dr$$

For $j = 2$,

$$\arg(z - z_2) = -\pi \Longrightarrow \sqrt{z - z_2} = -i\sqrt{r}$$

For other j,

$$\arg(z - z_j) \neq -\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{(z_2 - r) - z_j}$$

Let

$$u(r) = \left(\sqrt{(z_2 - r) - z_1}\right)\left(\sqrt{r}\right)\left(\prod_{j=3}^{2N+2}\sqrt{z_j - (z_2 - r)}\right).$$

Then

$$\int_{z_1 \xrightarrow{+} z_2} f(z)\, dz = -(-i)\int_d^0 u(r)\, dr = i\int_d^0 u(r)\, dr$$

Along $z_{2j-1} \xrightarrow{+} z_{2j+2}$, let

$$\theta = -\arctan\frac{|z_{2j-1} - z_{2j+1}|}{|z_{2j+1} - z_{2j+2}|}, \qquad d = |z_{2j-1} - z_{2j+2}|.$$

$$z = z(r) = z_{2j-1} + re^{i\theta}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i\theta}dr$$

Then,

$$\int_{z_{2j-1} \longrightarrow z_{2j+2}} f(z)\, dz = \int_0^d \prod_{j=1}^{2N+2}\sqrt{|z(r) - z_j|}\; e^{i\theta}dr\,.$$

Thus,

$$\begin{aligned}\int_{b_k} f(z)\,dz &= (-1)\cdot 2\int_{z_1 \xrightarrow{+} z_2} f(z)\, dz + 2\sum_{j=1}^{k}\int_{z_{2j-1}\longrightarrow z_{2j+2}} f(z)\, dz + \int_{a_k} f(z)\,dz \\ &= (-1)\cdot 2i\int_d^0 u(r)\, dr + 2\sum_{j=1}^{k}\int_0^d \prod_{j=1}^{2N+2}\sqrt{|z(r) - z_j|}\; e^{i\theta}dr + \int_{a_k} f(z)\,dz\,.\end{aligned}$$

Using Mathematica

Along $z_1 \xrightarrow{+} z_2$, let $d = |z_2 - z_1|$.

For $j = 2$,

$$\arg(z - z_2) = -\pi \Longrightarrow \sqrt{z - z_2} = (-1)\cdot \text{MATH}\left(\sqrt{z - z_2}\right)$$

For other j,

$$\arg(z - z_j) \neq -\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z - z_j}\right)$$

Then,

$$\int_{z_1 \xrightarrow{+} z_2} f(z)\, dz = (-1) \cdot \text{MATH}\left(\int_{z_1}^{z_2} f(z)\, dz\right)$$

Along $z_{2j-1} \xrightarrow{+} z_{2j+2}$, let

$$\theta = -\arctan \frac{|z_{2j-1} - z_{2j+1}|}{|z_{2j+1} - z_{2j+2}|}, \qquad d = |z_{2j-1} - z_{2j+2}|.$$

$$z = z_{2j-1} + re^{i\theta}, \;\; r : d \longrightarrow 0 \Longrightarrow dz = e^{i\theta} dr$$

Then,

$$\int_{z_{2j-1} \longrightarrow z_{2j+2}} f(z)\, dz = \text{MATH}\left(\int_0^d f(z_{2j-1} + re^{i\theta}) e^{i\theta} dr\right).$$

Thus,

$$\begin{aligned}
\int_{b_k} f(z)\, dz &= (-1) \cdot 2 \int_{z_1 \xrightarrow{+} z_2} f(z)\, dz + 2 \sum_{j=1}^{k} \int_{z_{2j-1} \longrightarrow z_{2j+2}} f(z)\, dz + \int_{a_k} f(z)\, dz \\
&= (-1) \cdot 2 \cdot (-1) \cdot \text{MATH}\left(\int_{z_1}^{z_2} f(z)\, dz\right) + 2 \sum_{j=1}^{k} \text{MATH}\left(\int_0^d f(z_{2j-1} + re^{i\theta}) e^{i\theta} dr\right) \\
&\quad + (-2) \cdot \text{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{z_1}^{z_2} f(z)\, dz\right) + 2 \sum_{j=1}^{k} \text{MATH}\left(\int_0^d f(z_{2j-1} + re^{i\theta}) e^{i\theta} dr\right) \\
&\quad + (-2) \cdot \text{MATH}\left(\int_{z_{2k+1}}^{z_{2k+2}} f(z)\, dz\right).
\end{aligned}$$

4 Integrals for Vertical Cuts

4.1 Cut Structures for Vertical Cuts

We first define the branches for vertical cuts. Let $f(z) = \sqrt{z}$ and let $z = re^{i\theta}$, where $\theta = \arg\, z$. We define two single-valued branches of f as

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad -\frac{3\pi}{2} \leq \theta < \frac{\pi}{2},$$

and

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \frac{\pi}{2} \leq \theta < \frac{5\pi}{2}.$$

And we define sheet I and sheet II as

$$\text{sheet I} = \{z \in \mathbb{C} | -\frac{3\pi}{2} \leq \arg\, z < \frac{\pi}{2}\},$$

and

$$\text{sheet II} = \{z \in \mathbb{C} | \frac{\pi}{2} \leq \arg\, z < \frac{5\pi}{2}\}.$$

To Label the second quadrant with a $+$ and label the first quadrant with a $-$.

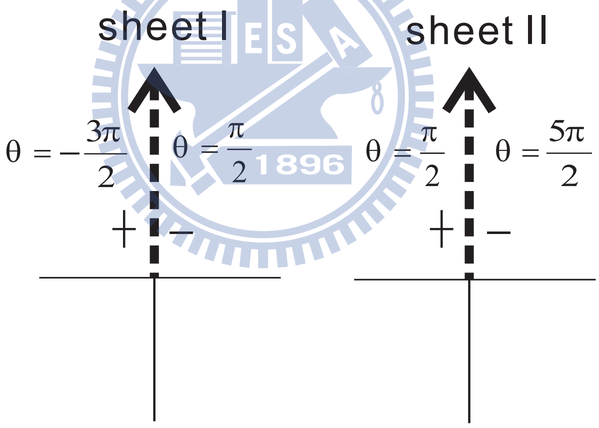


Figure 53

Then we can use the same method used in section 2.1 to construct the Riemann surface for f. (see Figure 6)

4.2 The Problem in Using Mathematica

We use (I) to denote sheet I and (II) to denote sheet II. We can see

$$z \in (I) \Longrightarrow -\frac{3\pi}{2} \leq \arg\, z < \frac{\pi}{2} \Longrightarrow -\frac{3\pi}{4} \leq \frac{1}{2}\arg\, z < \frac{\pi}{4}.$$

f maps the points on sheet I into the region $\{z \in \mathbb{C} | -\frac{3\pi}{4} \leq \arg\, z < \frac{\pi}{4}\}$.

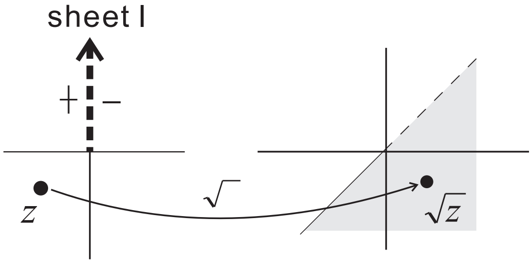


Figure 54

And,

$$z \in (II) \Longrightarrow \frac{\pi}{2} \le \arg z < \frac{5\pi}{2} \Longrightarrow \frac{\pi}{4} \le \frac{1}{2}\arg z < \frac{5\pi}{4}.$$

f maps the points on sheet II into the region $\{z \in \mathbb{C} | \frac{\pi}{4} \le \arg z < \frac{5\pi}{4}\}$.

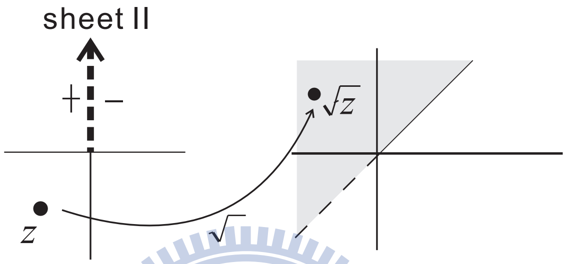


Figure 55

Let $z \in I_c = [-\frac{3\pi}{2}, -\pi] \subseteq (I)$. For example, suppose that $z = -1 + i \in (I)$. Then $\arg z = \frac{-5\pi}{4}$ and $z = e^{i(\frac{-5\pi}{4})}$.

$$\arg z = -\frac{5\pi}{4} \in I_c \Longrightarrow \arg \sqrt{z} = -\frac{5\pi}{8}$$

$$\Longrightarrow f(z) = \sqrt{-1+i} = (e^{i(\frac{-5\pi}{4})})^{\frac{1}{2}} = e^{i(\frac{-5\pi}{8})}$$

But in Mathematica,

$$-1 + i = e^{i(\frac{3\pi}{4})} \Longrightarrow \sqrt{-1+i} = e^{i(\frac{3\pi}{8})}$$

We find that $e^{i(\frac{3\pi}{8})} = (-1) \cdot e^{i(\frac{-5\pi}{8})}$.

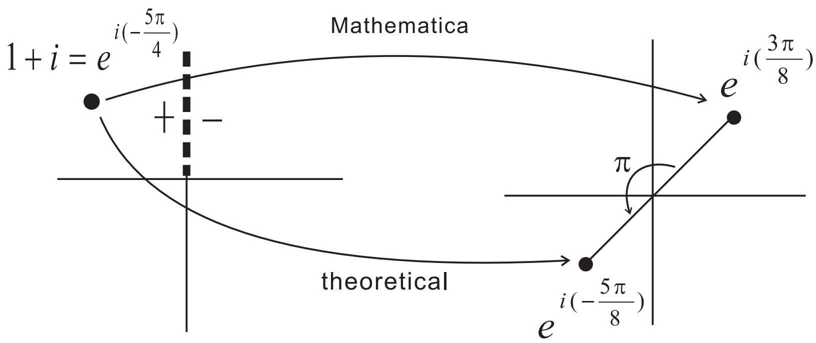


Figure 56

Thus we have the result

$$z \in \text{sheet I and } -\frac{3\pi}{2} \le \arg z \le -\pi \Longrightarrow \sqrt{z} = (-1)\cdot \text{MATH}\left(\sqrt{z}\right)$$

Let $\theta = \arg\, z$, and let

$$A = \{z \in \mathbb{C} | -\frac{\pi}{2} < \theta < \frac{\pi}{4}\},$$
$$B_T = \{z \in \mathbb{C} | -\frac{3\pi}{4} \le \theta \le -\frac{\pi}{2}\},$$
$$B_M = \{z \in \mathbb{C} | \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}.$$

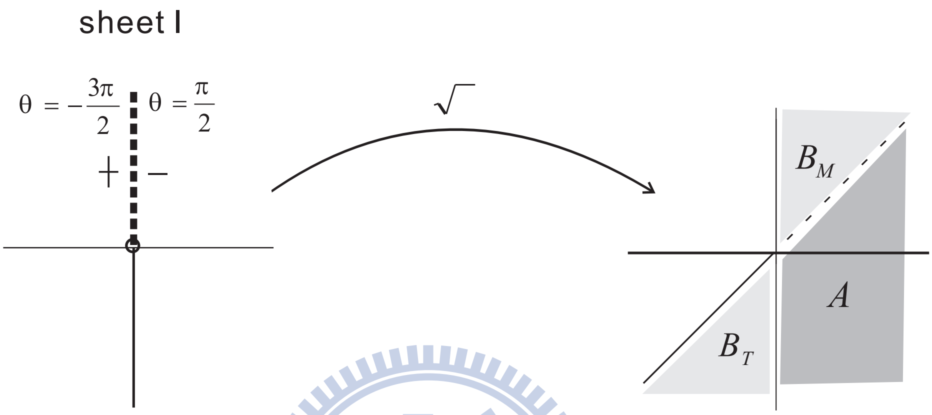


Figure 57

Theoretically,

$$f(\text{sheet I}) = A \cup B_T.$$

In Mathematica,

$$f(\text{sheet I}) = A \cup B_M.$$

4.3 Evaluating Integrals Using Mathematica

Example 9. *Let $f(z) = \sqrt{z}$ and let γ be the positively oriented (counterclockwise oriented) circular path $z = e^{i\theta}$, $-\frac{3\pi}{2} \le \theta < \frac{\pi}{2}$. Evaluate the integral $\int_\gamma f(z)\, dz$.*

Solution.

(1) Integral along the circular path

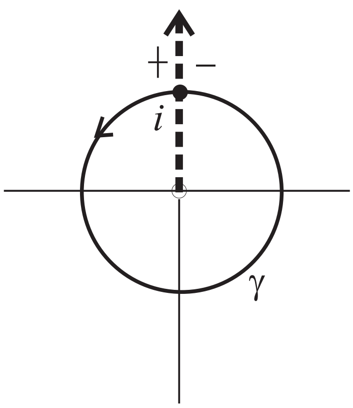


Figure 58

$$z \in \gamma \Longrightarrow z = e^{i\theta},\ -\frac{3\pi}{2} \le \theta < \frac{\pi}{2}$$
$$\Longrightarrow dz = ie^{i\theta} d\theta.$$

Then,

$$\begin{aligned}\int_{\gamma} f(z)\, dz &= \text{MATH}\left((-1) \cdot \int_{-\frac{3\pi}{2}}^{-\pi} f(e^{i\theta}) ie^{i\theta}\, d\theta + \int_{-\pi}^{\frac{\pi}{2}} f(e^{i\theta}) ie^{i\theta}\, d\theta\right) \\ &= \text{MATH}\left((-1) \cdot \int_{-\frac{3\pi}{2}}^{-\pi} \sqrt{e^{i\theta}}\, ie^{i\theta}\, d\theta + \int_{-\pi}^{\frac{\pi}{2}} \sqrt{e^{i\theta}}\, ie^{i\theta}\, d\theta\right) \\ &= -0.942809 + 0.942809i\,.\end{aligned}$$

(2) Deformation of path

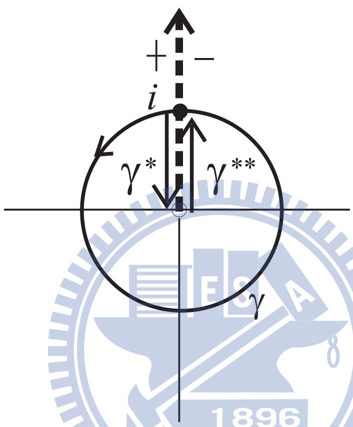


Figure 59

Theoretical Evaluation

Along $i \xrightarrow{+} 0 (z \in \gamma^*)$

$$z = ri,\quad r : 1 \longrightarrow 0 \Longrightarrow dz = idr$$
$$\arg z = -\frac{3}{2}\pi \Longrightarrow \sqrt{z} = \sqrt{|ri|}\, e^{i(\frac{-3\pi}{4})}$$

$$\begin{aligned}\int_{\gamma^*} f(z)\, dz &= \int_1^0 \sqrt{|ri|}\, e^{i(\frac{-3\pi}{4})} i\, dr \\ &= -0.471405 + 0.471405i\,.\end{aligned}$$

To use the similar method of deriving Equation(18), we can know that

$$\begin{aligned}\int_{\gamma} f(z)\, dz &= 2 \int_{\gamma^*} f(z)\, dz \\ &= -0.942809 + 0.942809i\,.\end{aligned}$$

Using Mathematica

$$\arg z = -\frac{3}{2}\pi \Longrightarrow \sqrt{z} = (-1) \cdot \text{MATH}\left(\sqrt{z}\right)$$

$$\int_{\gamma^*} f(z)\,dz = (-1)\cdot \text{MATH}\left(\int_1^0 f(ri)\,i\,dr\right)$$
$$= (-1)\cdot \text{MATH}\left(\int_1^0 \sqrt{ri}\,i\,dr\right).$$

Then,

$$\int_{\gamma} f(z)\,dz = 2\int_{\gamma^*} f(z)\,dz$$
$$= -0.942809 + 0.942809i\,.$$

Example 10. *Suppose that* $f(z) = \sqrt{(z-i)(z-2i)(z-3i)(z-4i)}$ *and* γ *is a positively oriented simple closed curve that encloses all cuts. Evaluate the integral* $\int_\gamma f(z)dz$.

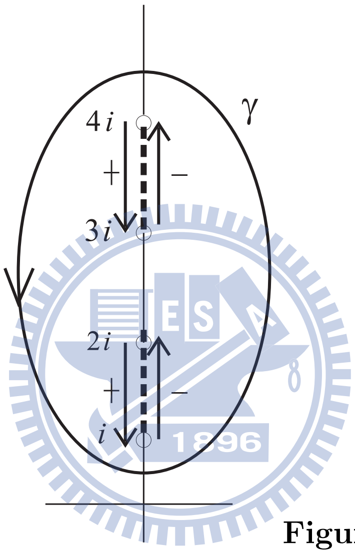


Figure 60

Solution.

<u>**Theoretical Evaluation**</u>

(1)Along $4i \xrightarrow{+} 3i$

$$z = ri,\quad r: 4 \longrightarrow 3 \Longrightarrow dz = idr$$

For $k = 1, 2, 3$,

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{3\pi}{4})},$$

and

$$\arg(z - 4i) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - 4i} = \sqrt{|ri - 4i|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\int_{4i \xrightarrow{+} 3i} f(z)\,dz = \int_4^3 \left(\prod_{k=1}^{4} \sqrt{|ri - ki|}\right)\left(e^{i(\frac{-3\pi}{4})}\right)^3 e^{i(\frac{-\pi}{4})}\, i\,dr$$
$$= -0.76002\,.$$

(2)Along $2i \xrightarrow{+} i$

$$z = ri, \quad r : 2 \longrightarrow 1 \Longrightarrow dz = idr$$

$$\arg(z - i) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - i} = \sqrt{|ri - i|}\, e^{i(-\frac{3\pi}{4})}$$

For $k = 2, 3, 4$,

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\begin{aligned}\int_{2i \xrightarrow{+} i} f(z)\, dz &= \int_2^1 \left(\prod_{k=1}^{4} \sqrt{|ri - ki|} \right) e^{i(\frac{-3\pi}{4})} \left(e^{i(\frac{-\pi}{4})} \right)^3 i\, dr \\ &= 0.76002\,.\end{aligned}$$

Thus,

$$\begin{aligned}\int_\gamma f(z)\, dz &= \left(\int_{4i \xrightarrow{+} 3i} f(z)\, dz + \int_{3i \xrightarrow{-} 4i} f(z)\, dz \right) \\ &\quad + \left(\int_{2i \xleftarrow{+} i} f(z)\, dz + \int_{i \xleftarrow{-} 2i} f(z)\, dz \right) \\ &= 2 \int_{4i \xrightarrow{+} 3i} f(z)\, dz + 2 \int_{2i \xleftarrow{+} i} f(z)\, dz \\ &= 2 \left(\int_{4i \xrightarrow{+} 3i} f(z)\, dz + \int_{2i \xleftarrow{+} i} f(z)\, dz \right) \\ &= 0.\end{aligned}$$

Using Mathematica

(1)Along $4i \xrightarrow{+} 3i$

$$z = ri, \quad r : 4 \longrightarrow 3 \Longrightarrow dz = idr$$

For $k = 1, 2, 3$,

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = (-1) \cdot \text{MATH} \left(\sqrt{z - ki} \right),$$

and

$$\arg(z - 4i) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - 4i} = \text{MATH} \left(\sqrt{z - 4i} \right)$$

Then,

$$\begin{aligned}\int_{4i \xrightarrow{+} 3i} f(z)\, dz &= (-1)^3 \cdot \text{MATH} \left(\int_4^3 \left(\prod_{k=1}^{4} \sqrt{|ri - ki|} \right) i\, dr \right) \\ &= -0.76002\,.\end{aligned}$$

(2)Along $2i \xrightarrow{+} i$

$$z = ri, \quad r : 2 \longrightarrow 1 \Longrightarrow dz = idr$$

$$\arg(z-i) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-i} = (-1)\cdot \text{MATH}\left(\sqrt{z-i}\right)$$

For $k = 2, 3, 4$,

$$\arg(z-ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-ki} = \text{MATH}\left(\sqrt{z-ki}\right)$$

Then,

$$\int_{2i \xrightarrow{+} i} f(z)\, dz = (-1)\cdot \text{MATH}\left(\int_2^1 \left(\prod_{k=1}^{4}\sqrt{|ri-ki|}\right) i\, dr\right)$$
$$= 0.76002\,.$$

Thus,

$$\int_\gamma f(z)\, dz = 2\left(\int_{4i \xrightarrow{+} 3i} f(z)\, dz + \int_{2i \xleftarrow{+} i} f(z)\, dz\right)$$
$$= 0\,.$$

Example 11. *Suppose that*

$$f(z) = \sqrt{(z-i)(z-2i)(z-3i))(z-4i)(z-5i)(z-6i)}\,.$$

Let a_1, a_2 be two $a-$cycles and let b_1, b_2 be two $b-$cycles drawing in Figure 61. *Evaluate the four integrals $\int_{a_k} f(z)dz$ and $\int_{b_k} f(z)dz$, $k = 1, 2$ using the method of deformation of path.*

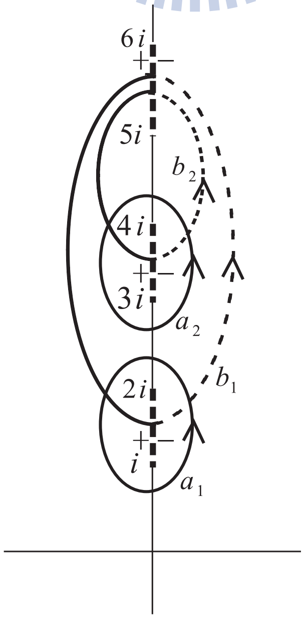


Figure 61

Solution.

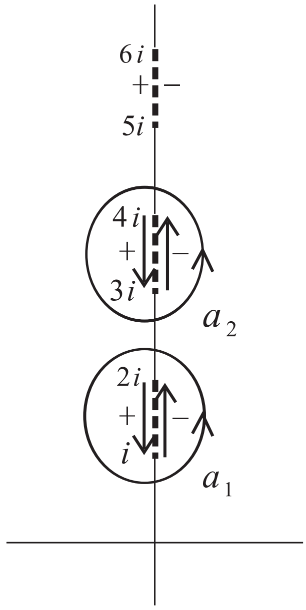


Figure 62

1. To evaluate $\int_{a_1} f(z)dz$

Theoretical Evaluation

Along $2i \xrightarrow{+} i$,

$$z = ri, \quad r: 2 \longrightarrow 1 \Longrightarrow dz = idr$$

$$\arg(z-i) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-i} = \sqrt{|ri-i|}\, e^{i(-\frac{3\pi}{4})},$$

For $k = 2, 3, 4, 5, 6$,

$$\arg(z-ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-ki} = \sqrt{|ri-ki|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\begin{aligned}
\int_{a_1} f(z)\, dz &= 2\int_{2i \xrightarrow{+} i} f(z)\, dz \\
&= 2\int_2^1 \left(\prod_{k=1}^{6} \sqrt{|ri-ki|}\right) e^{i(\frac{-3\pi}{4})} \left(e^{i(\frac{-\pi}{4})}\right)^5 i\, dr \\
&= 2\int_2^1 \left(\prod_{k=1}^{6} \sqrt{|ri-ki|}\right) i\, dr \\
&= -6.08344i\,.
\end{aligned}$$

Using Mathematica

Along $2i \xrightarrow{+} i$,

$$z = ri, \quad r: 2 \longrightarrow 1 \Longrightarrow dz = idr$$

$$\arg(z-i) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-i} = (-1)\cdot \text{MATH}\left(\sqrt{z-i}\right),$$

For $k = 2, 3, 4, 5, 6,$

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \text{MATH}\left(\sqrt{z - ki}\right)$$

Then,

$$\begin{aligned}\int_{a_1} f(z)\, dz &= 2\int_{2i \xrightarrow{+} i} f(z)\, dz \\ &= 2 \cdot (-1) \cdot \text{MATH}\left(\int_2^1 \left(\prod_{k=1}^{6} \sqrt{ri - ki}\right) i\, dr\right) \\ &= -6.08344i\,.\end{aligned}$$

2. To evaluate $\int_{a_2} f(z)dz$

Theoretical Evaluation

Along $4i \xrightarrow{+} 3i$,

$$z = ri, \quad r : 4 \longrightarrow 3 \Longrightarrow dz = idr$$

For $k = 1, 2, 3,$

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{3\pi}{4})}$$

For $k = 4, 5, 6,$

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\begin{aligned}\int_{a_2} f(z)\, dz &= 2\int_{4i \xrightarrow{+} 3i} f(z)\, dz \\ &= 2\int_4^3 \left(\prod_{k=1}^{6} \sqrt{|ri - ki|}\right) \left(e^{i(\frac{-3\pi}{4})}\right)^3 \left(e^{i(\frac{-\pi}{4})}\right)^3 i\, dr \\ &= 2 \cdot (-1)\int_4^3 \left(\prod_{k=1}^{6} \sqrt{|ri - ki|}\right) i\, dr \\ &= 2.88937i\,.\end{aligned}$$

Using Mathematica

Along $4i \xrightarrow{+} 3i$,

$$z = ri, \quad r : 4 \longrightarrow 3 \Longrightarrow dz = idr$$

For $k = 1, 2, 3,$

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = (-1) \cdot \text{MATH}\left(\sqrt{z - ki}\right)$$

For $k = 4, 5, 6,$

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \text{MATH}\left(\sqrt{z - ki}\right)$$

Then,

$$\begin{aligned}\int_{a_2} f(z)\,dz &= 2\int_{4i \xrightarrow{+} 3i} f(z)\,dz \\ &= 2 \cdot (-1)^3 \cdot \text{MATH}\left(\int_4^3 \left(\prod_{k=1}^{6} \sqrt{ri - ki}\right) i\,dr\right) \\ &= 2.88937i\,.\end{aligned}$$

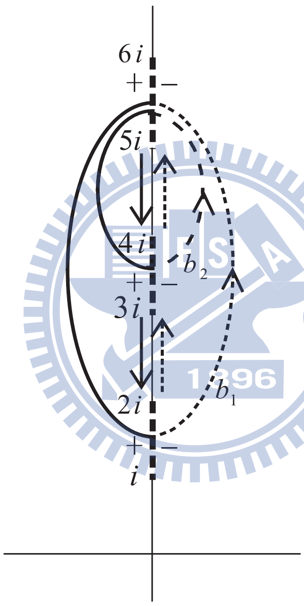


Figure 63

3. To evaluate $\int_{b_1} f(z)dz$

Theoretical Evaluation

(1)Along $5i \longrightarrow 4i$

$$z = ri, \quad r : 5 \longrightarrow 4 \Longrightarrow dz = idr$$

For $k = 1, 2, 3, 4,$

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{3\pi}{4})}$$

For $k = 5, 6,$

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \sqrt{|ri - ki|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\begin{aligned}
&\int_{5i\longrightarrow 4i} f(z)\,dz + \int_{5i\leftarrow -- 4i} f(z)\,dz \\
&= 2\int_{5i\longrightarrow 4i} f(z)\,dz \\
&= 2\int_{5}^{4}\left(\prod_{k=1}^{6}\sqrt{|ri-ki|}\right)\left(e^{i(\frac{-3\pi}{4})}\right)^{4}\left(e^{i(\frac{-\pi}{4})}\right)^{2}\, i\,dr \\
&= 2\cdot i\int_{5}^{4}\left(\prod_{k=1}^{6}\sqrt{|ri-ki|}\right) i\,dr \\
&= 3.43541\,.
\end{aligned}$$

(2)Along $3i \longrightarrow 2i$

$$z = ri,\quad r: 3 \longrightarrow 2 \Longrightarrow dz = idr$$

For $k = 1, 2,$

$$\arg(z-ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-ki} = \sqrt{|ri-ki|}\, e^{i(-\frac{3\pi}{4})}$$

For $k = 3, 4, 5, 6,$

$$\arg(z-ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-ki} = \sqrt{|ri-ki|}\, e^{i(-\frac{\pi}{4})}$$

Then,

$$\begin{aligned}
&\int_{3i\longrightarrow 2i} f(z)\,dz + \int_{3i\leftarrow -- 2i} f(z)\,dz \\
&= 2\int_{3i\longrightarrow 2i} f(z)\,dz \\
&= 2\int_{3}^{2}\left(\prod_{k=1}^{6}\sqrt{|ri-ki|}\right)\left(e^{i(\frac{-3\pi}{4})}\right)^{2}\left(e^{i(\frac{-\pi}{4})}\right)^{4}\, i\,dr \\
&= 2\cdot i\int_{3}^{2}\left(\prod_{k=1}^{6}\sqrt{|ri-ki|}\right) i\,dr \\
&= -3.43541\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{b_1} f(z)\,dz &= 2\int_{5i\longrightarrow 4i} f(z)\,dz + 2\int_{3i\longrightarrow 2i} f(z)\,dz \\
&= 0\,.
\end{aligned}$$

Using Mathematica

(1)Along $5i \longrightarrow 4i$

$$z = ri,\quad r: 5 \longrightarrow 4 \Longrightarrow dz = idr$$

For $k = 1, 2, 3, 4$,

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = (-1) \cdot \text{MATH}\left(\sqrt{z - ki}\right)$$

For $k = 5, 6$,

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \text{MATH}\left(\sqrt{z - ki}\right)$$

Then,

$$\begin{aligned}
&\int_{5i \longrightarrow 4i} f(z)\, dz + \int_{5i \longleftarrow -4i} f(z)\, dz \\
&= 2 \int_{5i \longrightarrow 4i} f(z)\, dz \\
&= 2 \cdot (-1)^4 \int_5^4 \left(\prod_{k=1}^{6} \sqrt{ri - ki} \right) i\, dr \\
&= 3.43541\,.
\end{aligned}$$

(2)Along $3i \longrightarrow 2i$

$$z = ri, \quad r : 3 \longrightarrow 2 \Longrightarrow dz = idr$$

For $k = 1, 2$,

$$\arg(z - ki) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - ki} = (-1) \cdot \text{MATH}\left(\sqrt{z - ki}\right)$$

For $k = 3, 4, 5, 6$,

$$\arg(z - ki) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - ki} = \text{MATH}\left(\sqrt{z - ki}\right)$$

Then,

$$\begin{aligned}
&\int_{3i \longrightarrow 2i} f(z)\, dz + \int_{3i \longleftarrow -2i} f(z)\, dz \\
&= 2 \int_{3i \longrightarrow 2i} f(z)\, dz \\
&= 2 \cdot (-1)^2 \int_3^2 \left(\prod_{k=1}^{6} \sqrt{ri - ki} \right) i\, dr \\
&= -3.43541\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{b_1} f(z)\, dz &= 2 \int_{5i \longrightarrow 4i} f(z)\, dz + 2 \int_{3i \longrightarrow 2i} f(z)\, dz \\
&= 0\,.
\end{aligned}$$

4. To evaluate $\int_{b_2} f(z) dz$: We have done in 3.

$$\int_{b_2} f(z)\, dz = 2 \int_{5i \longrightarrow 4i} f(z)\, dz = 3.43541\,.$$

Next, we discuss how to determine the region needed to change the sign of a given function f.

Example 12. *Let* $f(z) = \sqrt{z - z_1}\sqrt{z - z_2}$*. Determine the region needed to change the sign in sheet I of the cut plane for* f*.*

Solution.

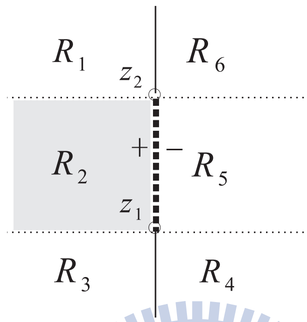


Figure 64

We separate the cut plane to six region, $R_1, R_2, R_3, R_4, R_5, R_6$ (Figure 64). Let $I_c = [-\frac{3\pi}{2}, -\pi]$. We investigate the sign of each $\sqrt{z - z_k}$ for all $z \in$ sheet I.

If $z \in R_1$,

$$\arg(z - z_1) \in I_c \Longrightarrow \sqrt{z - z_1} = (-1) \cdot \text{MATH}\left(\sqrt{z - z_1}\right)$$
$$\arg(z - z_2) \in I_c \Longrightarrow \sqrt{z - z_2} = (-1) \cdot \text{MATH}\left(\sqrt{z - z_2}\right).$$

Then,

$$f(z) = (-1)^2 \cdot \text{MATH}\left(f(z)\right) = \text{MATH}\left(f(z)\right).$$

If $z \in R_2$,

$$\arg(z - z_1) \in I_c \Longrightarrow \sqrt{z - z_1} = (-1) \cdot \text{MATH}\left(\sqrt{z - z_1}\right)$$
$$\arg(z - z_2) \notin I_c \Longrightarrow \sqrt{z - z_2} = \text{MATH}\left(\sqrt{z - z_2}\right).$$

Then,

$$f(z) = (-1) \cdot \text{MATH}\left(f(z)\right).$$

If $z \in R_3$,

$$\arg(z - z_1) \notin I_c \Longrightarrow \sqrt{z - z_1} = \text{MATH}\left(\sqrt{z - z_1}\right)$$
$$\arg(z - z_2) \notin I_c \Longrightarrow \sqrt{z - z_2} = \text{MATH}\left(\sqrt{z - z_2}\right).$$

Then,

$$f(z) = \mathrm{MATH}\left(f(z)\right).$$

If $z \in R_4 \cup R_5 \cup R_6$,

$$\begin{aligned} \arg(z - z_1) \notin I_c &\Longrightarrow \sqrt{z - z_1} = \mathrm{MATH}\left(\sqrt{z - z_1}\right) \\ \arg(z - z_2) \notin I_c &\Longrightarrow \sqrt{z - z_2} = \mathrm{MATH}\left(\sqrt{z - z_2}\right). \end{aligned}$$

Then,

$$f(z) = \mathrm{MATH}\left(f(z)\right).$$

Thus, the region needed to change the sign is R_2. We call such region the *sign-region* of f.

Example 13. *Let* $f(z) = \sqrt{z - z_1}\sqrt{z - z_2}\sqrt{z - z_3}$. *Determine the region needed to change the sign in sheet I of the cut plane for* f.

Solution.

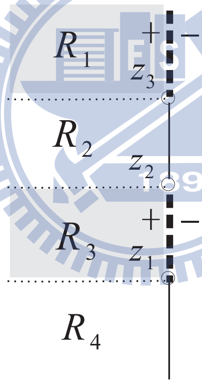


Figure 65

From example 11, we know that it does not to change sign on the right-half side of sheet I. So we only discuss the left-half side.

We separate the left-half side to four region, R_1, R_2, R_3, R_4 (Figure 65). Let $I_c = [-\frac{3\pi}{2}, -\pi]$.

If $z \in R_1$,

$$\begin{aligned} \arg(z - z_1) \in I_c &\Longrightarrow \sqrt{z - z_1} = (-1) \cdot \mathrm{MATH}\left(\sqrt{z - z_1}\right) \\ \arg(z - z_2) \in I_c &\Longrightarrow \sqrt{z - z_2} = (-1) \cdot \mathrm{MATH}\left(\sqrt{z - z_2}\right) \\ \arg(z - z_3) \in I_c &\Longrightarrow \sqrt{z - z_3} = (-1) \cdot \mathrm{MATH}\left(\sqrt{z - z_3}\right). \end{aligned}$$

Then,

$$f(z) = (-1)^3 \cdot \mathrm{MATH}\left(f(z)\right) = (-1) \cdot \mathrm{MATH}\left(f(z)\right).$$

If $z \in R_2$,

$$\arg(z-z_1) \in I_c \Longrightarrow \sqrt{z-z_1} = (-1)\cdot \text{MATH}\left(\sqrt{z-z_1}\right)$$
$$\arg(z-z_2) \in I_c \Longrightarrow \sqrt{z-z_2} = (-1)\cdot \text{MATH}\left(\sqrt{z-z_2}\right)$$
$$\arg(z-z_3) \in I_c \Longrightarrow \sqrt{z-z_3} = \text{MATH}\left(\sqrt{z-z_3}\right).$$

Then,

$$f(z) = (-1)^2 \cdot \text{MATH}\,(f(z)) = \text{MATH}\,(f(z)).$$

If $z \in R_3$,

$$\arg(z-z_1) \in I_c \Longrightarrow \sqrt{z-z_1} = (-1)\cdot \text{MATH}\left(\sqrt{z-z_1}\right)$$
$$\arg(z-z_2) \in I_c \Longrightarrow \sqrt{z-z_2} = \text{MATH}\left(\sqrt{z-z_2}\right)$$
$$\arg(z-z_3) \in I_c \Longrightarrow \sqrt{z-z_3} = \text{MATH}\left(\sqrt{z-z_3}\right).$$

Then,

$$f(z) = (-1)\text{MATH}\,(f(z)).$$

If $z \in R_4$,

$$\arg(z-z_1) \in I_c \Longrightarrow \sqrt{z-z_1} = \text{MATH}\left(\sqrt{z-z_1}\right)$$
$$\arg(z-z_2) \in I_c \Longrightarrow \sqrt{z-z_2} = \text{MATH}\left(\sqrt{z-z_2}\right)$$
$$\arg(z-z_3) \in I_c \Longrightarrow \sqrt{z-z_3} = \text{MATH}\left(\sqrt{z-z_3}\right).$$

Then,

$$f(z) = \text{MATH}\,(f(z)).$$

Thus, the sign-region are R_1 and R_3.

We generalize the result of example 11 and example 12 in next two examples.

Example 14. (There are odd branch points)
Let

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+1})} = \prod_{j=1}^{2N+1} \sqrt{z-z_j}\,,$$

where $Re(z_k), k = 1, 2, \cdots, 2N+1,$ *are all the same (Figure 62). Determine the region needed to change the sign in sheet I of the cut plane for* f*.*

Solution.

We separate the left-half side to the $2N+1$ region, $R_1, R_2, \cdots, R_{2N+1}$. Let $I_c = [-\frac{3\pi}{2}, -\pi]$.

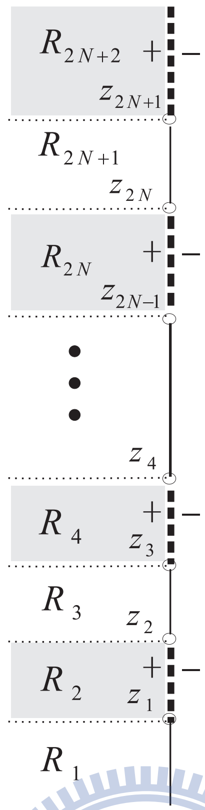


Figure 66

(1) $z \in R_{2j-1}$, $j = 1, 2, \cdots, N+1$.
For $k = 1, 2, \cdots, 2j-2$,

$$\arg(z - z_k) \in I_c \Longrightarrow \sqrt{z - z_k} = (-1) \cdot \mathrm{MATH}\left(\sqrt{z - z_k}\right)$$

For $k = 2j-1, 2j, \cdots, 2N+1$,

$$\arg(z - z_k) \notin I_c \Longrightarrow \sqrt{z - z_k} = \mathrm{MATH}\left(\sqrt{z - z_k}\right)$$

Then,

$$f(z) = (-1)^{2j-2} \cdot \mathrm{MATH}\left(f(z)\right) = \mathrm{MATH}\left(f(z)\right).$$

(2) $z \in R_{2j}$, $j = 1, 2, \cdots, N+1$.
For $k = 1, 2, \cdots, 2j-1$,

$$\arg(z - z_k) \in I_c \Longrightarrow \sqrt{z - z_k} = (-1) \cdot \mathrm{MATH}\left(\sqrt{z - z_k}\right)$$

For $k = 2j, 2j+1, \cdots, 2N+2$,

$$\arg(z - z_k) \notin I_c \Longrightarrow \sqrt{z - z_k} = \mathrm{MATH}\left(\sqrt{z - z_k}\right)$$

Then,

$$f(z) = (-1)^{2j-1} \cdot \mathrm{MATH}\left(f(z)\right) = (-1) \cdot \mathrm{MATH}\left(f(z)\right).$$

Thus, the sign-region are $R_2, R_4, \cdots, R_{2N+2}$.

Example 15. (There are even branch points)
Let

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+2})} = \prod_{j=1}^{2N+2} \sqrt{z-z_j}\,,$$

where $Re(z_j), j = 1, 2, \cdots, 2N+2$, *are all the same (Figure 67). Determine the region needed to change the sign in sheet I of the cut plane for* f.

Solution.

We separate the left-half side to the $2N+2$ region, $R_1, R_2, \cdots, R_{2N+2}$. Let $I_c = [-\frac{3\pi}{2}, -\pi]$.

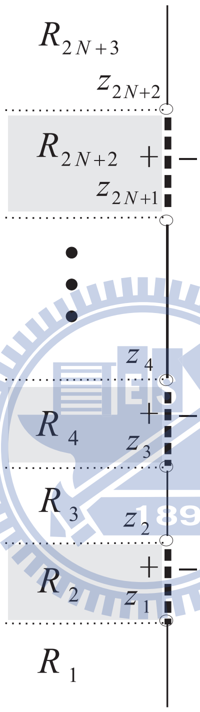


Figure 67

(1) $z \in R_{2j-1}$, $j = 1, 2, \cdots, N+2$.
For $k = 1, 2, \cdots, 2j-2$,

$$\arg(z-z_k) \in I_c \Longrightarrow \sqrt{z-z_k} = (-1)\cdot \text{MATH}\left(\sqrt{z-z_k}\right)$$

For $k = 2j-1, 2j, \cdots, 2N+1$,

$$\arg(z-z_k) \notin I_c \Longrightarrow \sqrt{z-z_k} = \text{MATH}\left(\sqrt{z-z_k}\right)$$

Then,

$$f(z) = (-1)^{2j-2}\cdot \text{MATH}\,(f(z)) = \text{MATH}\,(f(z))\,.$$

(2) $z \in R_{2j}$, $j = 1, 2, \cdots, N+1$.
For $k = 1, 2, \cdots, 2j-1$,

$$\arg(z-z_k) \in I_c \Longrightarrow \sqrt{z-z_k} = (-1)\cdot \text{MATH}\left(\sqrt{z-z_k}\right)$$

For $k = 2j, 2j+1, \cdots, 2N+2$,

$$\arg(z - z_k) \notin I_c \Longrightarrow \sqrt{z - z_k} = \text{MATH}\left(\sqrt{z - z_k}\right)$$

Then,

$$f(z) = (-1)^{2j-1} \cdot \text{MATH}\left(f(z)\right) = (-1) \cdot \text{MATH}\left(f(z)\right).$$

Thus, the sign-region are $R_2, R_4, \cdots, R_{2N+2}$.

4.4 Generalization of Integrals Along Vertical Cuts

Case 1. The number of branch points is odd ($2N+1$ branch points)

Let

$$f(z) = \sqrt{(z - z_1)(z - z_2)\cdots(z - z_{2N+1})} = \prod_{j=1}^{2N+1} \sqrt{z - z_j}\,.$$

Aussume that $Im(z_j) > Im(z_{j+1})$ for $j = 1, 2, \cdots, 2N$. Let y_j to denote the imaginary part of z_j for all j. That is, $y_j = Im(z_j)$.

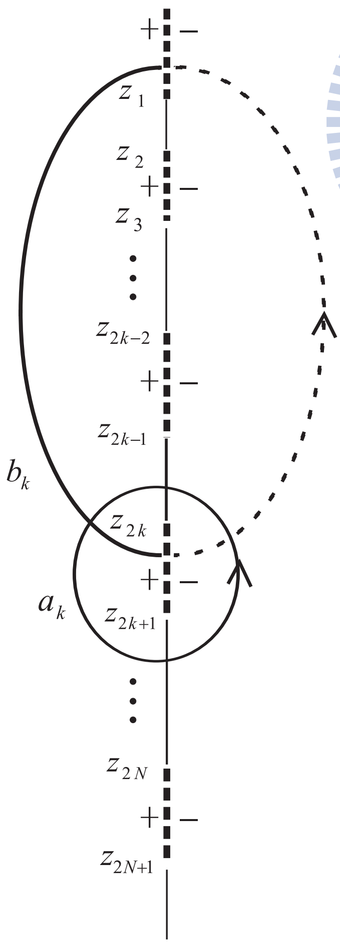


Figure 68

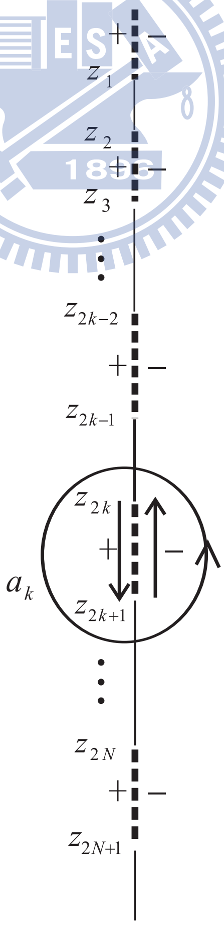


Figure 69

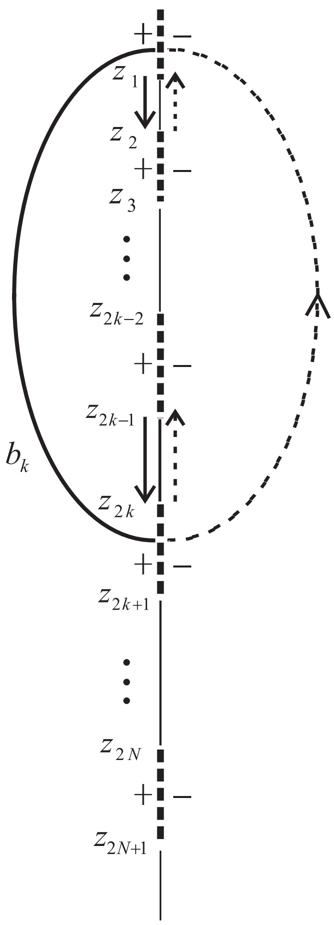


Figure 70

(1) To evaluate $\int_{a_k} f(z)dz$ (Figure 69)

Theoretical Evaluation

Along $z_{2k} \xrightarrow{+} z_{2k+1}$,

$$z = ri, \quad r : y_{2k} \longrightarrow y_{2k+1} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2k$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{\pi}{4})}$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{3\pi}{4})}$$

Then,

$$\begin{aligned}
\int_{a_k} f(z)\, dz &= 2\int_{z_{2k} \xrightarrow{+} z_{2k+1}} f(z)\, dz \\
&= 2\left(e^{i(\frac{-3\pi}{4})}\right)^{(2N+1)-2k}\left(e^{i(\frac{-\pi}{4})}\right)^{2k}\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri - z_j|}\right) i\, dr \\
&= 2\cdot\left(e^{i(\frac{\pi}{4})}\right)^{-3[(2N+1)-2k]-2k}\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri - z_j|}\right) i\, dr \\
&= 2\cdot\left(e^{i(\frac{\pi}{4})}\right)^{-4(N-k)-(2N+3)}\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri - z_j|}\right) i\, dr \\
&= 2\cdot\left(e^{i(-\pi)}\right)^{N-k}\left(e^{i(\frac{\pi}{4})}\right)^{-(2N+3)}\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri - z_j|}\right) i\, dr \\
&= 2\cdot(-1)^{N-k} e^{i(-\frac{2N+3}{4}\pi)}\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri - z_j|}\right) i\, dr\,.
\end{aligned}$$

Note that $|Re(\int_{a_k} f(z)dz)| = |Im(\int_{a_k} f(z)dz)|$. It is due to $e^{i(-\frac{2N+3}{4}\pi)}$.

Using Mathematica

Along $z_{2k} \xrightarrow{+} z_{2k+1}$,

$$z = ri, \quad r : y_{2k} \longrightarrow y_{2k+1} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2k$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{ri - z_j}\right)$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = (-1)\cdot\text{MATH}\left(\sqrt{ri - z_j}\right)$$

Then,

$$\int_{a_k} f(z)\,dz = 2\int_{z_{2k}\overset{+}{\longrightarrow}z_{2k+1}} f(z)\,dz$$
$$= 2\cdot(-1)^{(2N+1)-2k}\cdot \text{MATH}\left(\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{ri-z_j}\right)\,i\,dr\right)$$
$$= 2\cdot(-1)\cdot \text{MATH}\left(\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+1}\sqrt{ri-z_j}\right)\,i\,dr\right).$$

(2) To evaluate $\int_{b_k} f(z)dz$ (Figure 70)

To use the similar method of deriving Equation (49) in case 1 of (2) in section 3.4, we obtain

$$\int_{b_k} f(z)\,dz = 2\left(\int_{z_1\longrightarrow z_2} f(z)\,dz + \int_{z_3\longrightarrow z_4} f(z)\,dz + \cdots + \int_{z_{2k-1}\longrightarrow z_{2k}} f(z)\,dz\right)$$
$$= 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz \tag{65}$$

Theoretical Evaluation

Along $z_{2m-1}\longrightarrow z_{2m}$,

$$z = ri,\quad r: y_{2m-1}\longrightarrow y_{2m} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z-z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-z_j} = \sqrt{|ri-z_j|}\,e^{i(-\frac{\pi}{4})}$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z-z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-z_j} = \sqrt{|ri-z_j|}\,e^{i(-\frac{3\pi}{4})}$$

Then,

$$\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz = \left(e^{i(\frac{-3\pi}{4})}\right)^{(2N+1)-(2m-1)}\left(e^{i(\frac{-\pi}{4})}\right)^{2m-1}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right)\,i\,dr$$
$$= \left(e^{i(\frac{\pi}{4})}\right)^{-3[(2N+1)-(2m-1)]-(2m-1)}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right)\,i\,dr$$
$$= \left(e^{i(\frac{\pi}{4})}\right)^{-4(N-m+1)-(2N+1)}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right)\,i\,dr$$
$$= \left(e^{i(-\pi)}\right)^{N-m+1}\left(e^{i(\frac{\pi}{4})}\right)^{-(2N+1)}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right)\,i\,dr$$
$$= (-1)^{N-m+1}e^{i(-\frac{2N+1}{4}\pi)}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right)\,i\,dr\,.$$

Thus,

$$\int_{b_k} f(z)\,dz = 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz$$
$$= 2\cdot(-1)^{N-m+1}e^{i(-\frac{2N+1}{4}\pi)}\sum_{m=1}^{k}\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{|ri-z_j|}\right) i\,dr\,.$$

Note that $|Re(\int_{a_k} f(z)dz)| = |Im(\int_{a_k} f(z)dz)|$.

Using Mathematica

Along $z_{2m-1}\longrightarrow z_{2m}$,

$$z = ri,\quad r: y_{2m-1}\longrightarrow y_{2m} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{ri - z_j}\right)$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = (-1)\cdot\text{MATH}\left(\sqrt{ri - z_j}\right)$$

Then,

$$\int_{b_k} f(z)\,dz = 2\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz$$
$$= 2\cdot(-1)^{(2N+1)-(2m-1)}\cdot\text{MATH}\left(\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{ri-z_j}\right) i\,dr\right)$$
$$= 2\cdot\text{MATH}\left(\int_{y_{2m-1}}^{y_{2m}}\left(\prod_{j=1}^{2N+1}\sqrt{ri-z_j}\right) i\,dr\right).$$

Case 2. The number of branch points is even ($2N+2$ branch points)

Let

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+2})} = \prod_{j=1}^{2N+2}\sqrt{z-z_j}\,.$$

Aussume that $Im(z_j) > Im(z_{j+1})$ for $j = 1, 2, \cdots, 2N+1$. Let y_j to denote the imaginary part of z_j for all j. That is, $y_j = Im(z_j)$.

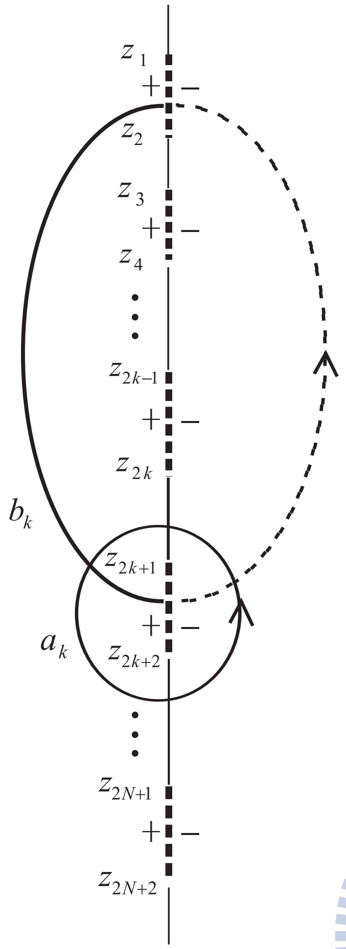


Figure 71

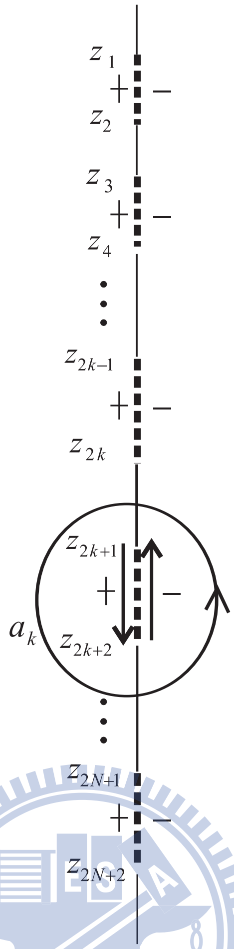


Figure 72

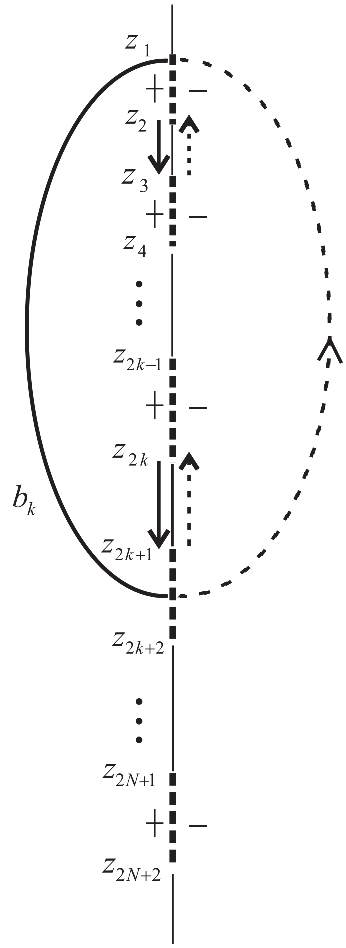


Figure 73

(1) To evaluate $\int_{a_k} f(z)dz$ (Figure 72)

Theoretical Evaluation

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$,

$$z = ri, \quad r : y_{2k+1} \longrightarrow y_{2k+2} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{\pi}{4})}$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{3\pi}{4})}$$

Then,

$$
\begin{aligned}
\int_{a_k} f(z)\,dz &= 2\int_{z_{2k+1}\xrightarrow{+}z_{2k+2}} f(z)\,dz \\
&= 2\left(e^{i(\frac{-3\pi}{4})}\right)^{(2N+1)-(2k+1)}\left(e^{i(\frac{-\pi}{4})}\right)^{2k+1}\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri-z_j|}\right) i\,dr \\
&= 2\cdot\left(e^{i(\frac{\pi}{4})}\right)^{-3[(2N+1)-(2k+1)]-(2k+1)}\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri-z_j|}\right) i\,dr \\
&= 2\cdot\left(e^{i(\frac{\pi}{4})}\right)^{-4(N-k+1)-2N}\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri-z_j|}\right) i\,dr \\
&= 2\cdot\left(e^{i(-\pi)}\right)^{N-k+1}\left(e^{i(\frac{\pi}{4})}\right)^{-2N}\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri-z_j|}\right) i\,dr \\
&= 2\cdot(-1)^{N-k+1}e^{i(-\frac{N}{2}\pi)}\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri-z_j|}\right) i\,dr\,.
\end{aligned}
$$

Note that this value is a pure imaginary number.

Using Mathematica

Along $z_{2k+1}\xrightarrow{+}z_{2k+2}$,

$$z = ri,\quad r: y_{2k+1}\longrightarrow y_{2k+2}\Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z-z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-z_j} = \text{MATH}\left(\sqrt{ri-z_j}\right)$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z-z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-z_j} = (-1)\cdot\text{MATH}\left(\sqrt{ri-z_j}\right)$$

Then,

$$
\begin{aligned}
\int_{a_k} f(z)\,dz &= 2\int_{z_{2k+1}\xrightarrow{+}z_{2k+2}} f(z)\,dz \\
&= 2\cdot(-1)^{(2N+1)-(2k+1)}\cdot\text{MATH}\left(\int_{y_{2k}}^{y_{2k+1}}\left(\prod_{j=1}^{2N+2}\sqrt{ri-z_j}\right) i\,dr\right) \\
&= 2\cdot\text{MATH}\left(\int_{y_{2k+1}}^{y_{2k+2}}\left(\prod_{j=1}^{2N+2}\sqrt{ri-z_j}\right) i\,dr\right).
\end{aligned}
$$

(2) To evaluate $\int_{b_k} f(z)dz$ (Figure 73)

$$\int_{b_k} f(z)\,dz = 2\left(\int_{z_2\longrightarrow z_3} f(z)\,dz + \int_{z_4\longrightarrow z_5} f(z)\,dz + \cdots + \int_{z_{2k}\longrightarrow z_{2k+1}} f(z)\,dz\right)$$
$$= 2\sum_{m=1}^{k}\int_{z_{2m}\longrightarrow z_{2m+1}} f(z)\,dz$$

Theoretical Evaluation

Along $z_{2m}\longrightarrow z_{2m+1}$,

$$z = ri, \quad r : y_{2m} \longrightarrow y_{2m+1} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{\pi}{4})}$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|ri - z_j|}\, e^{i(-\frac{3\pi}{4})}$$

Then,

$$\begin{aligned}
\int_{z_{2m}\longrightarrow z_{2m+1}} f(z)\,dz &= \left(e^{i(\frac{-3\pi}{4})}\right)^{(2N+2)-2m}\left(e^{i(\frac{-\pi}{4})}\right)^{2m}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr \\
&= \left(e^{i(\frac{\pi}{4})}\right)^{-3[(2N+2)-2m]-2m}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr \\
&= \left(e^{i(\frac{\pi}{4})}\right)^{-4(N-m+1)-2N}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr \\
&= \left(e^{i(-\pi)}\right)^{N-m+1}\left(e^{i(\frac{\pi}{4})}\right)^{-2N}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr \\
&= (-1)^{N-m+1} e^{i(-\frac{N}{2}\pi)}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{b_k} f(z)\,dz &= 2\sum_{m=1}^{k}\int_{z_{2m}\longrightarrow z_{2m+1}} f(z)\,dz \\
&= 2\cdot(-1)^{N-m+1} e^{i(-\frac{N}{2}\pi)}\sum_{m=1}^{k}\int_{y_{2m}}^{y_{2m+1}}\left(\prod_{j=1}^{2N+2}\sqrt{|ri - z_j|}\right) i\,dr\,.
\end{aligned}$$

Using Mathematica

Along $z_{2m}\longrightarrow z_{2m+1}$,

$$z = ri, \quad r : y_{2m} \longrightarrow y_{2m+1} \Longrightarrow dz = idr$$

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{ri - z_j}\right)$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{ri - z_j}\right)$$

Then,

$$\begin{aligned}\int_{b_k} f(z)\, dz &= 2\int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \\ &= 2 \cdot (-1)^{(2N+2)-2m} \cdot \text{MATH}\left(\int_{y_{2m}}^{y_{2m+1}} \left(\prod_{j=1}^{2N+2} \sqrt{ri - z_j}\right) i\, dr\right) \\ &= 2 \cdot \text{MATH}\left(\int_{y_{2m}}^{y_{2m+1}} \left(\prod_{j=1}^{2N+2} \sqrt{ri - z_j}\right) i\, dr\right).\end{aligned}$$

Next, we investigate the sign-regions for other complicated examples.

Example 16. *Suppose that*

$$f(z) = \prod_{j=1}^{7} \sqrt{z - z_j}$$

and the cutted plane (sheet I) is drawn below. To determine the sign-regions of f.

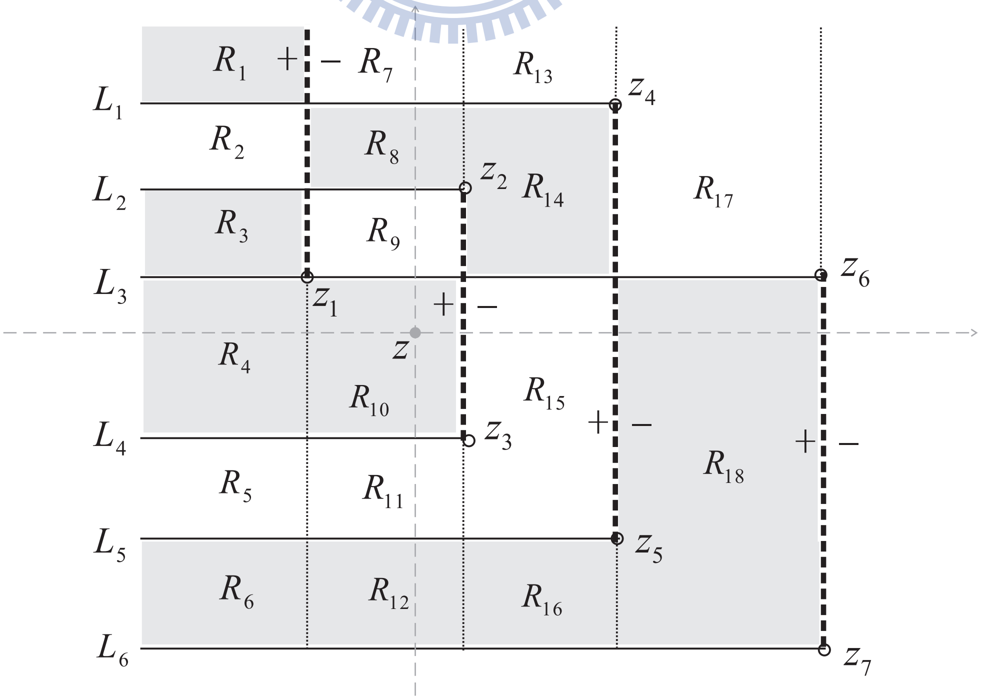


Figure 74

Solution.

First we draw a horizontal line from each end point of every cut to the direction of minus real asix, $L_1, L_2, L_3, L_4, L_5, L_6$. Then these horizontal lines and all cuts separate the complex plane to several regions, $R_1, \cdots, R_{18}$. Let $I_c = [-\frac{3\pi}{2}, -\pi]$.

$$
\begin{aligned}
& z \in R_1 \\
& \Longrightarrow \arg(z - z_k) \in I_c \text{ for all } k \\
& \Longrightarrow f(z) = (-1)^7 \cdot \text{MATH}\left(f(z)\right) = (-1) \cdot \text{MATH}\left(f(z)\right).
\end{aligned}
$$

$$
\begin{aligned}
& z \in R_8 \\
& \Longrightarrow \begin{cases} \arg(z - z_k) \in I_c & \text{if } k = 2, 3, 5, 6, 7, \\ \arg(z - z_k) \notin I_c & \text{otherwise.} \end{cases} \\
& \Longrightarrow f(z) = (-1)^5 \cdot \text{MATH}\left(f(z)\right) = (-1) \cdot \text{MATH}\left(f(z)\right).
\end{aligned}
$$

$$
\begin{aligned}
& z \in R_{15} \\
& \Longrightarrow \begin{cases} \arg(z - z_k) \in I_c & \text{if } k = 5, 7, \\ \arg(z - z_k) \notin I_c & \text{otherwise.} \end{cases} \\
& \Longrightarrow f(z) = (-1)^2 \cdot \text{MATH}\left(f(z)\right) = \text{MATH}\left(f(z)\right).
\end{aligned}
$$

To discuss the other regions using the above method, you can find that the sign-regions are the gray regions drawn in Figure 74.

We also give a simple method of finding the sign-regions. For $z \in R_k$, e,g, $z \in R_{10}$, we imgine that there is a coordinate with orgin z. If there are odd branch points in the forth quadrant, then $f(z) = (-1) \cdot \text{MATH}\left(f(z)\right)$. If there are even branch points in the forth quadrant, then $f(z) = \text{MATH}\left(f(z)\right)$. Finally the sign-regions is shown below.

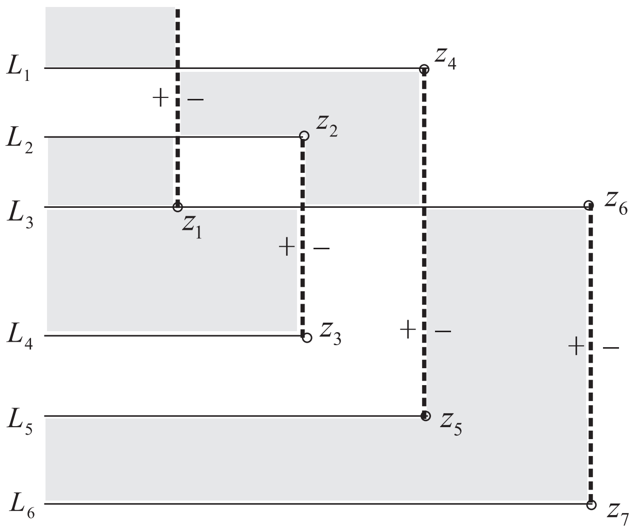


Figure 75

Example 17. *Suppose that*

$$f(z) = \sqrt{[z-(1+2i)][z-(1-2i)][z-(2+i)][z-(2-i)]}.$$

Evaluate the integral $\int_b f(z)dz$.

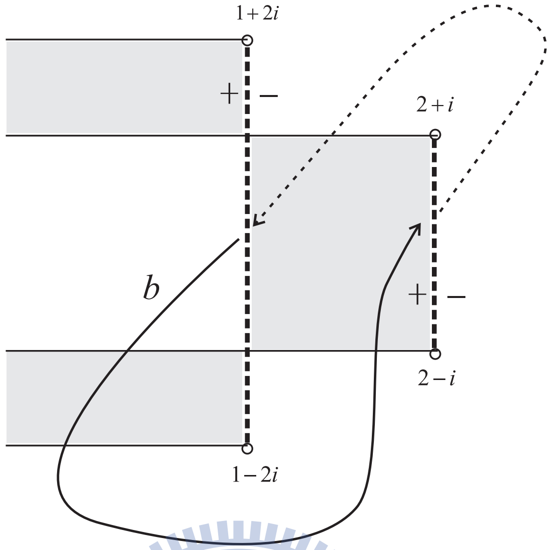


Figure 76

Solution.

We evaluate this integral along two different, but equivalent paths, respectively.

(1) Along the paths in Figure 77

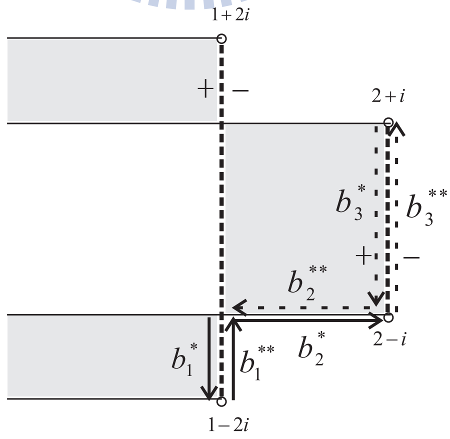


Figure 77

Since the path b_1^* lies in a sign-region, we obtain

$$\int_{b_1^*} f(z)\,dz + \int_{b_1^{**}} f(z)\,dz = 2\int_{b_1^*} f(z)\,dz$$
$$= 2\cdot(-1)\cdot \text{MATH}\left(\int_{1-i \xrightarrow{+} 1-2i} f(z)\,dz\right)$$

Since b_3^* and b_3^{**} are two paths in sheet II and $f(z)|_{II} = (-1) \cdot f(z)|_I$,

$$\int_{b_3^*} f(z)\,dz + \int_{b_3^{**}} f(z)\,dz = 2\int_{b_3^*} f(z)\,dz$$
$$= (-1) \cdot 2 \int_{2+i \xrightarrow{+} 2-i} f(z)\,dz$$

Furthermore, b_3^* lies in a sign-region, so

$$\int_{b_3^*} f(z)\,dz + \int_{b_3^{**}} f(z)\,dz = (-1) \cdot 2 \int_{2+i \xrightarrow{+} 2-i} f(z)\,dz$$
$$= (-1) \cdot 2 \cdot (-1) \cdot \text{MATH}\left(\int_{2+i \xrightarrow{+} 2-i} f(z)\,dz\right)$$
$$= 2 \cdot \text{MATH}\left(\int_{2+i \xrightarrow{+} 2-i} f(z)\,dz\right)$$

If $z \in b_2^*$, then

$$\arg(z-(2-i)) \in I_c \Longrightarrow \sqrt{z-(2-i)} = (-1) \cdot \text{MATH}\left(\sqrt{z-(2-i)}\right)$$
$$\arg(z-(2+i)) \notin I_c \Longrightarrow \sqrt{z-(1+i)} = \text{MATH}\left(\sqrt{z-(2+i)}\right)$$
$$\arg(z-(1-2i)) \notin I_c \Longrightarrow \sqrt{z-(1-2i)} = \text{MATH}\left(\sqrt{z-(1-2i)}\right)$$
$$\arg(z-(1+2i)) \notin I_c \Longrightarrow \sqrt{z-(1+2i)} = \text{MATH}\left(\sqrt{z-(1+2i)}\right),$$

where $I_c = [-\frac{3\pi}{2}, -\pi]$. Hence,

$$\int_{b_2^*} f(z)\,dz + \int_{b_2^{**}} f(z)\,dz = 2\int_{b_2^*} f(z)\,dz$$
$$= 2 \cdot (-1) \cdot \text{MATH}\left(\int_{1-i \xrightarrow{+} 2-i} f(z)\,dz\right)$$

We obtain

$$\int_b f(z)\,dz = \sum_{k=1}^{3}\left(\int_{b_k^*} f(z)\,dz + \int_{b_k^{**}} f(z)\,dz\right)$$
$$= 2 \cdot (-1) \cdot \text{MATH}\left(\int_{1-i \xrightarrow{+} 1-2i} f(z)\,dz\right) + 2 \cdot \text{MATH}\left(\int_{2+i \xrightarrow{+} 2-i} f(z)\,dz\right)$$
$$+ 2 \cdot (-1) \cdot \text{MATH}\left(\int_{1-i \xrightarrow{+} 2-i} f(z)\,dz\right)$$
$$= 2 \cdot (-1)\int_{-1}^{-2} f(1+ri)i\,dr + 2\int_{1}^{-1} f(2+ri)i\,dr + 2 \cdot (-1)\int_{1}^{2} f(x-i)\,dx$$
$$= -0.13095 - 11.5969i\,.$$

(2) Along the paths in Figure 78

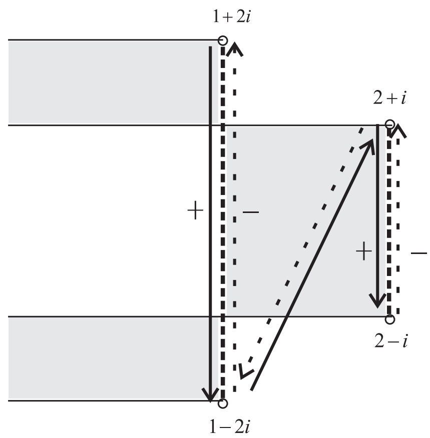


Figure 78

$$\int_{1+2i \xrightarrow{+} 1+i} f(z)\,dz = (-1)\cdot \text{MATH}\left(\int_2^1 f(1+ri)i\,dr\right)$$

and

$$\begin{aligned}\int_{1+2i \xleftarrow{-} 1+i} f(z)\,dz &= (-1)\int_{1+2i \xleftarrow{+} 1+i} f(z)\,dz \\ &= (-1)\cdot \text{MATH}\left(\int_1^2 f(1+ri)i\,dr\right) \\ &= (-1)\int_{1+2i \xrightarrow{+} 1+i} f(z)\,dz\end{aligned}$$

Thus,

$$\int_{1+2i \xrightarrow{+} 1+i} f(z)\,dz + \int_{1+2i \xleftarrow{-} 1+i} f(z)\,dz = 0\,.$$

Similarly,

$$\int_{1+i \xrightarrow{+} 1-i} f(z)\,dz + \int_{1-i \xleftarrow{-} 1+i} f(z)\,dz = 0$$

$$\int_{1-i \xrightarrow{+} 1-2i} f(z)\,dz + \int_{1-2i \xleftarrow{-} 1-i} f(z)\,dz = 0$$

So, we have

$$\int_{1+2i \xrightarrow{+} 1-2i} f(z)\,dz + \int_{1-2i \xleftarrow{-} 1+2i} f(z)\,dz = 0$$

and

$$\int_{2+i \xrightarrow{+} 2-i} f(z)\,dz + \int_{2-i \xleftarrow{-} 2+i} f(z)\,dz = 0$$

It remains to evaluate the integrals along the slant paths $\int_{1-2i\longrightarrow 2+i} f(z)\,dz+\int_{2+i\dashleftarrow 1-2i} f(z)\,dz$.

$$
\begin{aligned}
&\int_{1-2i\longrightarrow 2+i} f(z)\,dz+\int_{2+i\dashleftarrow 1-2i} f(z)\,dz\\
&=2\int_{1-2i\longrightarrow 2+i} f(z)\,dz\\
&=2\int_0^{\frac{\sqrt{10}}{3}} f(1-2i+re^{i\tan^{-1}3})e^{i\tan^{-1}3}\,dr+2\cdot(-1)\int_{\frac{\sqrt{10}}{3}}^{\sqrt{10}} f(1-2i+re^{i\tan^{-1}3})e^{i\tan^{-1}3}\,dr\\
&=-0.13095-11.5969i\,.
\end{aligned}
$$

Thus,

$$\int_b f(z)\,dz=2\int_{1-2i\longrightarrow 2+i} f(z)\,dz=-0.13095-11.5969i\,.$$

Example 18. *Suppose that*

$$f(z)=\prod_{j=1}^{N+1}\sqrt{z-z_j}\sqrt{z-\overline{z}_j}\,.$$

satisfying that $Re(z_j)<Re(z_{j+1})$ *and* $Im(z_j)=Im(z_{j+1})$ *for all* j *and suppose that* $N+1$ *is odd. The cut plane is drawn in Figure 79. Evaluate* $\int_{a_k} f(z)dz$ *and* $\int_{b_k} f(z)dz$.

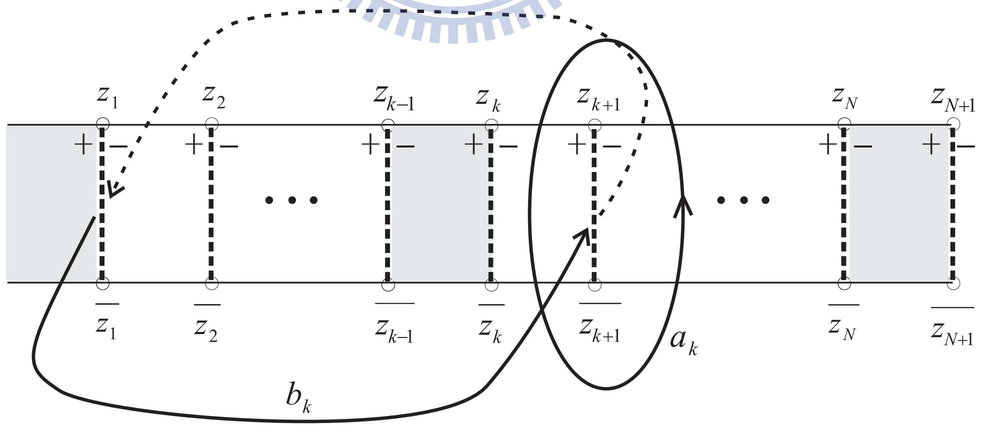


Figure 79

Solution.

1. Evaluate $\int_{a_k} f(z)dz$

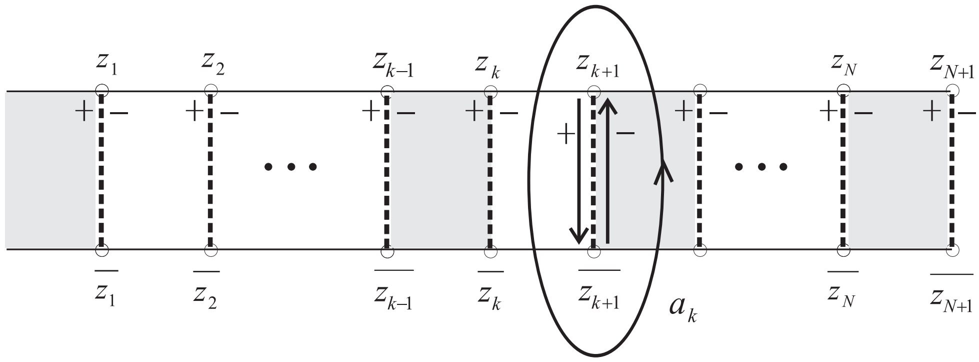


Figure 80

Let $z_j = x_j + iy_j$ for all j.

$$\int_{z_{k+1} \xrightarrow{+} \bar{z}_{k+1}} f(z)\, dz = \text{MATH}\left(\int_{y_{k+1}}^{-y_{k+1}} f(x_{k+1} + ri) i\, dr\right)$$

$$\begin{aligned}\int_{z_{k+1} \xleftarrow{-} \bar{z}_{k+1}} f(z)\, dz &= (-1) \cdot \text{MATH}\left(\int_{-y_{k+1}}^{y_{k+1}} f(x_{k+1} + ri) i\, dr\right) \\ &= \text{MATH}\left(\int_{y_{k+1}}^{-y_{k+1}} f(x_{k+1} + ri) i\, dr\right) \\ &= \int_{z_{k+1} \xrightarrow{+} \bar{z}_{k+1}} f(z)\, dz\end{aligned}$$

Thus,

$$\begin{aligned}\int_{a_k} f(z)\, dz &= 2 \int_{z_{k+1} \xrightarrow{+} \bar{z}_{k+1}} f(z)\, dz \\ &= 2 \cdot \text{MATH}\left(\int_{y_{k+1}}^{-y_{k+1}} f(x_{k+1} + ri) i\, dr\right).\end{aligned}$$

2. Evaluate $\int_{b_k} f(z)dz$

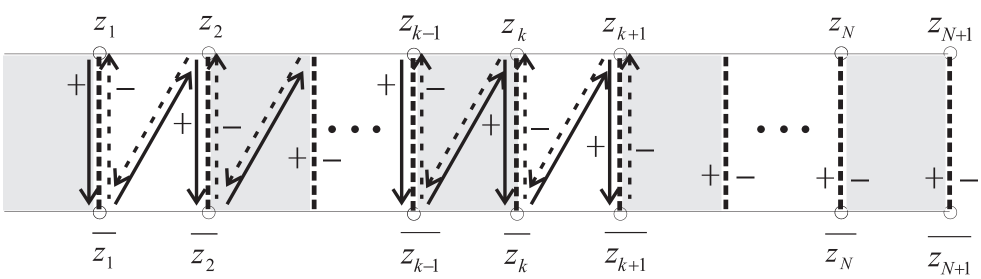


Figure 81

For $j = 1, 2, \cdots, k$,

$$\int_{z_{k+1} \xrightarrow{+} \bar{z}_{k+1}} f(z)\, dz + \int_{z_{k+1} \xleftarrow{-} \bar{z}_{k+1}} f(z)\, dz = 0$$

So,

$$\int_{b_k} f(z)dz = 2\sum_{j=1}^{k} \int_{\overline{z}_j \longrightarrow z_{j+1}} f(z)\,dz$$

Let $z_j = x_j + iy_j$ for all j and let

$$d = \frac{|z_{j+1} - \overline{z}_{j+1}|}{|\overline{z}_{j+1} - \overline{z}_j|} = \frac{2y_j}{x_{j+1} - x_j}$$

For $z \in \overline{z}_j \longrightarrow z_{j+1}$, let $z = \overline{z}_j + re^{i\tan^{-1}d}, r : 0 \longrightarrow |z_{j+1} - \overline{z}_j|$. Then

$$\int_{\overline{z}_j \longrightarrow z_{j+1}} f(z)\,dz = (-1)^j \cdot \text{MATH}\left(\int_0^{|z_{j+1}-\overline{z}_j|} f(\overline{z}_j + re^{i\tan^{-1}d})e^{i\tan^{-1}d}dr\right)$$

Therefore,

$$\int_{b_k} f(z)dz = \text{MATH}\left(2\sum_{j=1}^{k}(-1)^j \int_0^{|z_{j+1}-\overline{z}_j|} f(\overline{z}_j + re^{i\tan^{-1}d})e^{i\tan^{-1}d}dr\right).$$

Example 19. *Suppose that*

$$f(z) = \sqrt{[z-(-1+i)][z-0][z-2i][z-(1+3i)][z-(1+5i)][z-(2+2i)][z-(2-2i)]}.$$

The cut plane is drawn in Figure 82. Evaluate $\int_{b_k} f(z)dz$ *for* $k = 1, 2, 3$.

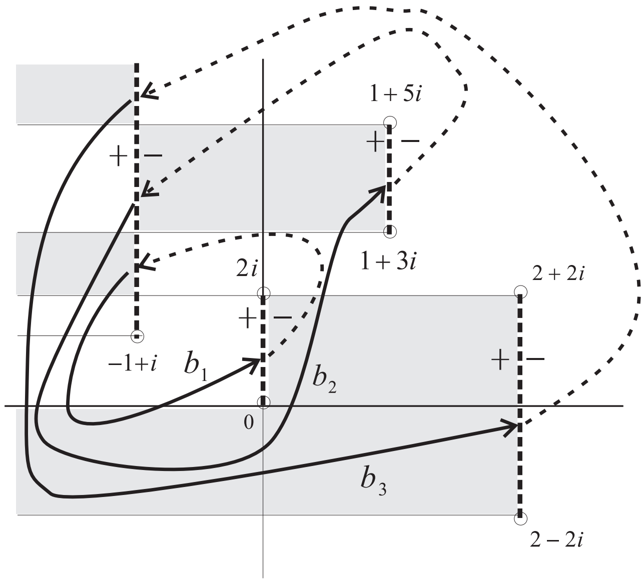


Figure 82

Solution.

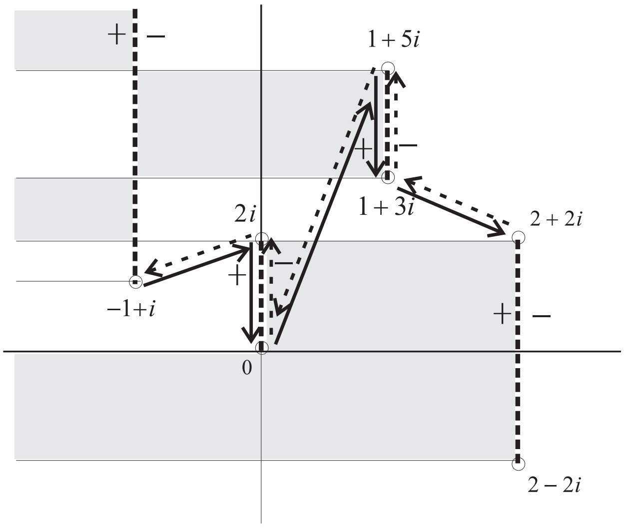


Figure 83

1. Evaluate $\int_{b_1} f(z)dz$

$$\int_{b_1} f(z)dz = \int_{-1+i \longrightarrow 2i} f(z)\,dz + \int_{-1+i \longleftarrow 2i} f(z)\,dz = 2\int_{-1+i \longrightarrow 2i} f(z)\,dz$$

Along $-1+i \longrightarrow 2i$, let $z = (-1+i) + re^{i(\frac{\pi}{4})}$, $r : 0 \longrightarrow \sqrt{2}$.

$$\begin{aligned}\int_{b_1} f(z)dz &= 2\int_{-1+i \longrightarrow 2i} f(z)\,dz \\ &= 2 \cdot \text{MATH}\left(\int_0^{\sqrt{2}} f((-1+i) + re^{i(\frac{\pi}{4})})e^{i(\frac{\pi}{4})}\,dr\right) \\ &= 0.0529343 - 18.4855i\,.\end{aligned}$$

2. Evaluate $\int_{b_2} f(z)dz$

$$\int_{b_2} f(z)dz = 2\int_{-1+i \longrightarrow 2i} f(z)\,dz + 2\int_{0 \longrightarrow 1+5i} f(z)\,dz$$

Along $0 \longrightarrow 1+5i$, let $z = re^{i\tan^{-1}3}$, $r : 0 \longrightarrow \sqrt{26}$.

$$\begin{aligned}\int_{0 \longrightarrow 1+5i} f(z)\,dz = 2 \cdot (-1) \cdot \text{MATH}&\left(\int_0^{\frac{2\sqrt{26}}{5}} f(re^{i\tan^{-1}3})e^{i\tan^{-1}3}\,dr\right) \\ &+ 2 \cdot \text{MATH}\left(\int_{\frac{2\sqrt{26}}{5}}^{\frac{3\sqrt{26}}{5}} f(re^{i\tan^{-1}3})e^{i\tan^{-1}3}\,dr\right) \\ &+ 2 \cdot (-1) \cdot \text{MATH}\left(\int_{\frac{3\sqrt{26}}{5}}^{\sqrt{26}} f(re^{i\tan^{-1}3})e^{i\tan^{-1}3}\,dr\right) \\ &= -67.2252 + 88.1591i\,.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{b_2} f(z)dz &= 2\int_{-1+i\longrightarrow 2i} f(z)\,dz + 2\int_{0\longrightarrow 1+5i} f(z)\,dz \\ &= (0.0529343 - 18.4855i) + (-67.2252 + 88.1591i) \\ &= -67.1723 + 69.6736i\,.\end{aligned}$$

3. Evaluate $\int_{b_3} f(z)dz$

$$\int_{b_3} f(z)dz = 2\int_{-1+i\longrightarrow 2i} f(z)\,dz + 2\int_{0\longrightarrow 1+5i} f(z)\,dz + 2\int_{1+3i\longrightarrow 2+2i} f(z)\,dz$$

Along $1+3i \longrightarrow 2+2i$, let $z = (1+3i) + re^{i(-\frac{\pi}{4})}$, $\ r : 0 \longrightarrow \sqrt{2}$.

$$\begin{aligned}\int_{1+3i\longrightarrow 2+2i} f(z)\,dz &= 2\cdot \text{MATH}\left(\int_0^{\sqrt{2}} f((1+3i) + re^{i(-\frac{\pi}{4})})e^{i(-\frac{\pi}{4})}\,dr\right) \\ &= 9.0209 + 17.2364i\,.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{b_3} f(z)dz &= 2\int_{-1+i\longrightarrow 2i} f(z)\,dz + 2\int_{0\longrightarrow 1+5i} f(z)\,dz + 2\int_{1+3i\longrightarrow 2+2i} f(z)\,dz \\ &= (0.0529343 - 18.4855i) + (-67.2252 + 88.1591i) + (9.0209 + 17.2364i) \\ &= -58.1514 + 86.91i\,.\end{aligned}$$

5 Integrals for Slant Cuts

5.1 Cut Structures for Slant Cuts

Let $f(z) = \sqrt{z}$ and let $z = re^{i\theta}$ where $\theta = \arg z$. Define two single-valued branches of f as

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad -\frac{7\pi}{4} \leq \theta < \frac{\pi}{4},$$

and

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \frac{\pi}{4} \leq \theta < \frac{9\pi}{4}.$$

Define sheet I and sheet II as

$$\text{sheet I} = \{z \in \mathbb{C} | -\frac{7\pi}{4} \leq \arg z < \frac{\pi}{4}\},$$

and

$$\text{sheet II} = \{z \in \mathbb{C} | \frac{\pi}{4} \leq \arg z < \frac{9\pi}{4}\}.$$

To Label a $+$ and label a $-$ as in Figure 84.

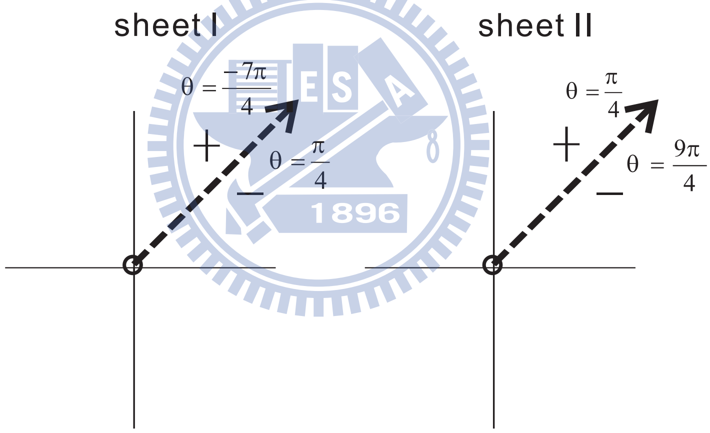


Figure 84

Then we can use the same method used in section 2.1 to build the Riemann surface for f. (see Figure 6)

More generally, suppose that $0 \leq \alpha \leq \pi$. we can define two single-valued branches of f as

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \alpha - 2\pi \leq \theta < \alpha,$$

and

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \alpha \leq \theta < \alpha + 2\pi.$$

Define sheet I and sheet II as

$$\text{sheet I} = \{z \in \mathbb{C} | \alpha - 2\pi \leq \arg z < \alpha\},$$

and

$$\text{sheet II} = \{z \in \mathbb{C} | \alpha \leq \arg\, z < \alpha + 2\pi\}.$$

To Label a $+$ and label a $-$ as in Figure 85.

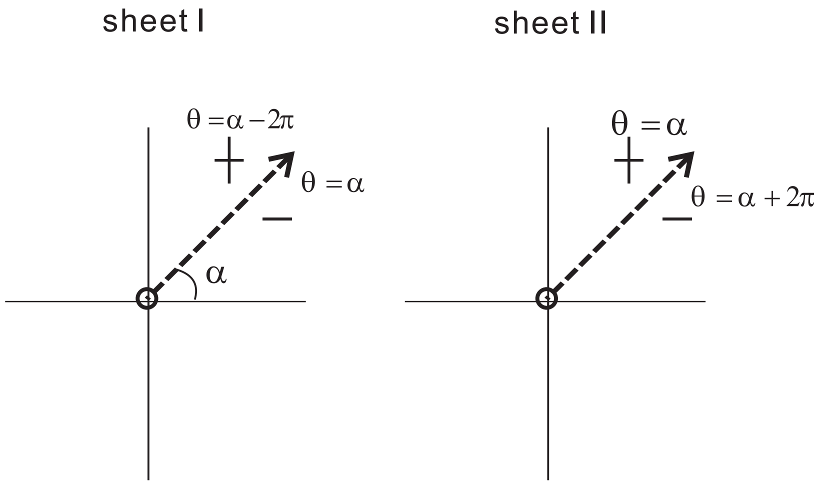


Figure 85

5.2 The Problem in Using Mathematica

Let (I) to denote sheet I and let (II) to denote sheet II. Then

$$z \in (I) \Longrightarrow -\frac{7\pi}{4} \leq \arg\, z < \frac{\pi}{4} \Longrightarrow -\frac{7\pi}{8} \leq \arg\, z < \frac{\pi}{8}.$$

f maps the points on sheet I into the region $\{z \in \mathbb{C}| -\frac{7\pi}{8} \leq \frac{1}{2}\arg\, z < \frac{\pi}{8}\}$.

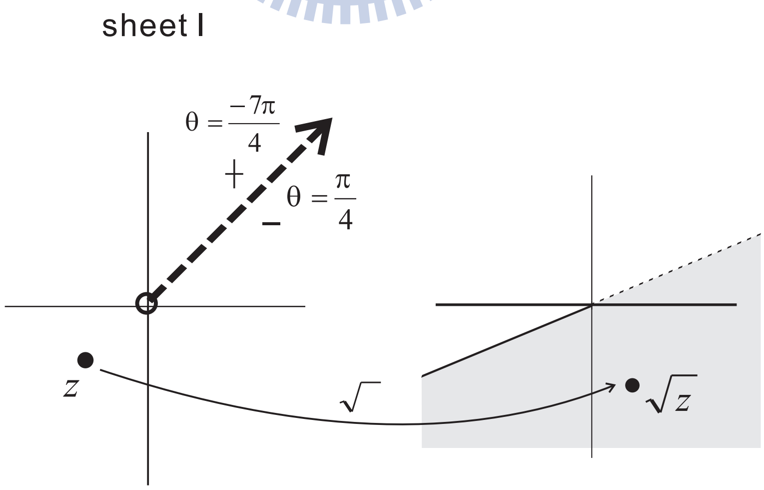


Figure 86

And,

$$z \in (II) \Longrightarrow \frac{\pi}{4} \leq \arg\, z < \frac{9\pi}{4} \Longrightarrow \frac{\pi}{8} \leq \frac{1}{2}\arg\, z < \frac{9\pi}{8}.$$

f maps the points on sheet II into the region $\{z \in \mathbb{C}| \frac{\pi}{8} \leq \arg\, z < \frac{9\pi}{8}\}$.

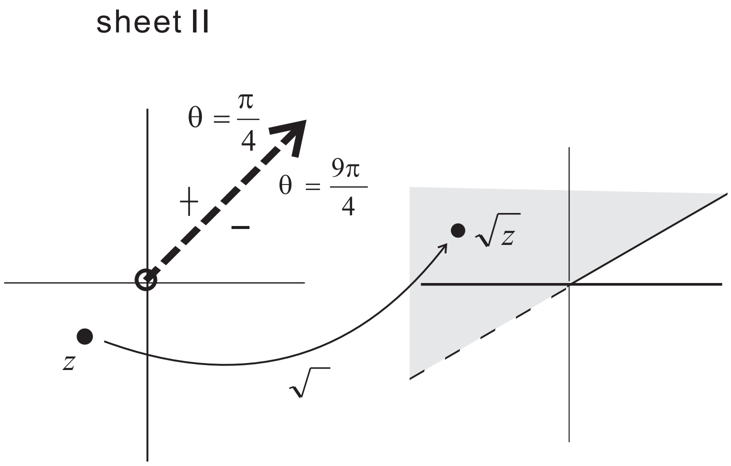


Figure 87

Let $z \in I_c = [-\frac{7\pi}{4}, -\pi] \subseteq (I)$. For example, suppose that $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i \in (I)$. Then $\arg z = \frac{-5\pi}{3}$ and $z = e^{i(\frac{-5\pi}{3})}$.

$$\arg z = -\frac{5\pi}{3} \in I_c \Longrightarrow \arg \sqrt{z} = -\frac{5\pi}{6}$$

$$\Longrightarrow f(z) = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}i} = (e^{i(\frac{-5\pi}{3})})^{\frac{1}{2}} = e^{i(\frac{-5\pi}{6})}$$

But in Mathematica,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i(\frac{\pi}{3})} \Longrightarrow \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}i} = e^{i(\frac{\pi}{6})}$$

Note that $e^{i(\frac{\pi}{6})} = (-1) \cdot e^{i(\frac{-5\pi}{6})}$.

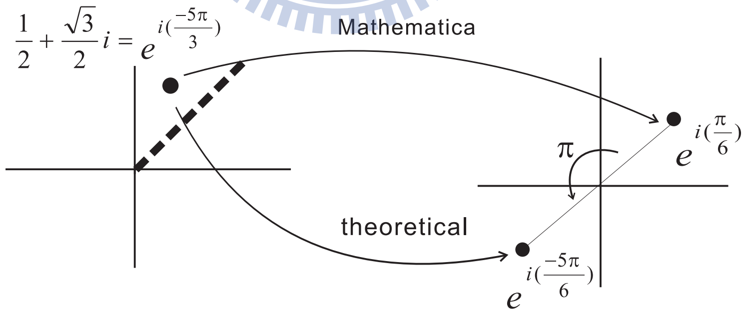


Figure 88

Thus we have the result

$$z \in \text{sheet I and } -\frac{7\pi}{4} \leq \text{arg z} \leq -\pi \Longrightarrow \sqrt{z} = (-1) \cdot \text{MATH}\left(\sqrt{z}\right)$$

Let $\theta = \arg z$, and let

$$A = \{z \in \mathbb{C} | -\frac{\pi}{2} < \theta < \frac{\pi}{4}\},$$

$$B_T = \{z \in \mathbb{C} | -\frac{3\pi}{4} \leq \theta \leq -\frac{\pi}{2}\},$$

$$B_M = \{z \in \mathbb{C} | \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}.$$

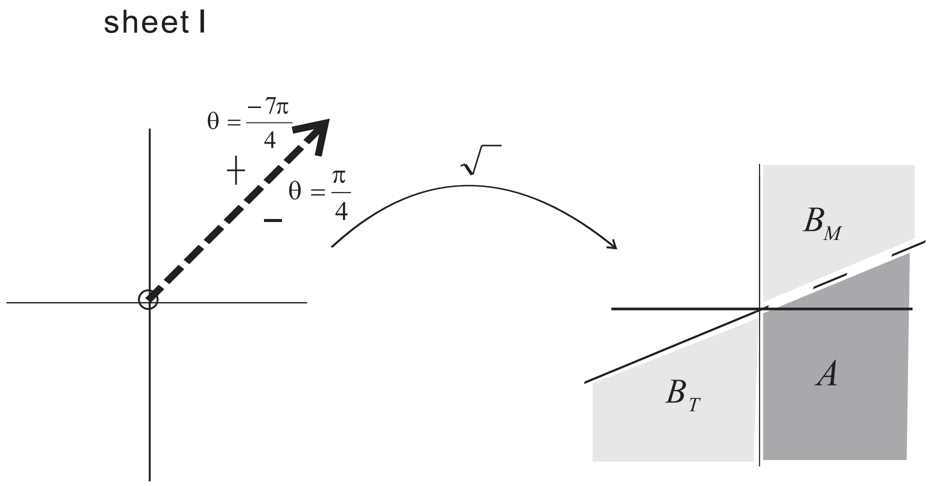


Figure 89

Theoretically,

$$f(\text{sheet I}) = A \cup B_T.$$

In Mathematica,

$$f(\text{sheet I}) = A \cup B_M.$$

5.3 Evaluating Integrals Using Mathematica

Example 20. *Let $f(z) = \sqrt{z}$ and let γ be the positively oriented (counterclockwise oriented) circular path $z = e^{i\theta}$, $-\frac{7\pi}{4} \le \theta < \frac{\pi}{4}$. Evaluate the integral $\int_\gamma f(z)\,dz$.*

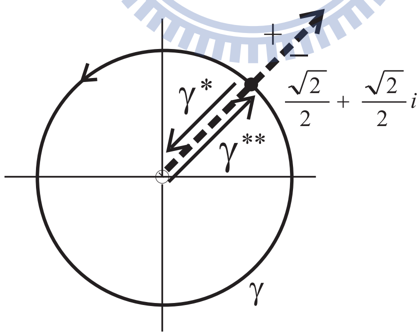


Figure 90

Solution.

(1) Integral along the circular path

$$z \in \gamma \Longrightarrow z = e^{i\theta},\ -\frac{7\pi}{4} \le \theta < \frac{\pi}{4} \Longrightarrow dz = ie^{i\theta}d\theta.$$

Then,

$$\begin{aligned}\int_\gamma f(z)\,dz &= \text{MATH}\left((-1)\cdot\int_{-\frac{7\pi}{4}}^{-\pi} f(e^{i\theta})ie^{i\theta}\,d\theta + \int_{-\pi}^{\frac{\pi}{4}} f(e^{i\theta})ie^{i\theta}\,d\theta\right)\\ &= \text{MATH}\left((-1)\cdot\int_{-\frac{7\pi}{4}}^{-\pi} \sqrt{e^{i\theta}}\,ie^{i\theta}\,d\theta + \int_{-\pi}^{\frac{\pi}{4}} \sqrt{e^{i\theta}}\,ie^{i\theta}\,d\theta\right)\\ &= 0.510245 + 1.23184i\,.\end{aligned}$$

(2) Deformation of path

Theoretical Evaluation

Along $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i \xrightarrow{+} 0 (z\in\gamma^*)$

$$z = re^{i(-\frac{7\pi}{4})},\quad r:1\longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

$$\arg z = -\frac{7}{4}\pi \Longrightarrow \sqrt{z} = \sqrt{r}\,e^{i(\frac{-7\pi}{8})}$$

$$\begin{aligned}\int_{\gamma^*} f(z)\,dz &= \int_1^0 \sqrt{r}\,e^{i(\frac{-7\pi}{8})}\,e^{i(-\frac{7\pi}{4})}\,dr\\ &= 0.255122 + 0.61592i\,.\end{aligned}$$

To use the similar method of deriving Equation(18), we can know that

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\int_{\gamma^*} f(z)\,dz\\ &= 0.510245 + 1.23184i\,.\end{aligned}$$

Using Mathematica

Along $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i \xrightarrow{+} 0 (z\in\gamma^*)$

$$z = re^{i(-\frac{7\pi}{4})},\quad r:1\longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

$$\arg z = -\frac{7}{4}\pi \Longrightarrow \sqrt{z} = (-1)\cdot\text{MATH}\left(\sqrt{z}\right)$$

$$\begin{aligned}\int_{\gamma^*} f(z)\,dz &= (-1)\cdot\text{MATH}\left(\int_1^0 f(re^{i(-\frac{7\pi}{4})})\,e^{i(-\frac{7\pi}{4})}\,dr\right)\\ &= (-1)\cdot\text{MATH}\left(\int_1^0 \sqrt{re^{i(-\frac{7\pi}{4})}}\,e^{i(-\frac{7\pi}{4})}\,dr\right).\end{aligned}$$

Then,

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\int_{\gamma^*} f(z)\,dz\\ &= 0.510245 + 1.23184i\,.\end{aligned}$$

Example 21. *Suppose that*

$$f(z) = \sqrt{z-(1+i)}\sqrt{z-(2+2i)}.$$

Let γ be the positively oriented circular path

$$\gamma : z = \frac{3}{2} + \frac{3}{2}i + e^{i\theta},\ -\frac{7\pi}{4} \le \theta < \frac{\pi}{4}.$$

Evaluate the integral $\int_\gamma f(z)\,dz$.

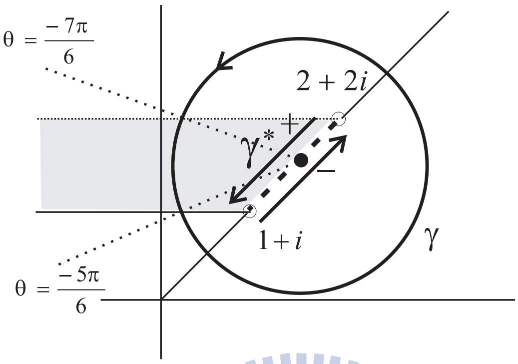


Figure 91

Solution.

Similar to the method of finding sign-regions for vertical cuts, the sign-region is shown in Figure 91.

(1) Integral along the circular path

$$z \in \gamma \Longrightarrow z = \frac{3}{2} + \frac{3}{2}i + e^{i\theta},\ -\frac{7\pi}{4} \le \theta < \frac{\pi}{4} \Longrightarrow dz = ie^{i\theta}d\theta.$$

$$\begin{aligned}\int_\gamma f(z)\,dz = {} & \text{MATH}\left(\int_{-\frac{7\pi}{4}}^{-\frac{7\pi}{6}} f(\frac{3}{2}+\frac{3}{2}i+e^{i\theta})ie^{i\theta}\,d\theta\right) \\ & + (-1)\cdot\text{MATH}\left(\int_{-\frac{7\pi}{6}}^{-\frac{5\pi}{6}} f(\frac{3}{2}+\frac{3}{2}i+e^{i\theta})ie^{i\theta}\,d\theta\right) \\ & + \text{MATH}\left(\int_{-\frac{5\pi}{6}}^{\frac{\pi}{4}} f(\frac{3}{2}+\frac{3}{2}i+e^{i\theta})ie^{i\theta}\,d\theta\right) \\ = {} & 1.5708\,.\end{aligned}$$

(2) Deformation of path

__**Theoretical Evaluation**__

Along $2+2i \xrightarrow{+} 1+i\ (z \in \gamma^*)$

$$z = 1+i+re^{i(-\frac{7\pi}{4})},\ \ r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

Then,

$$z-(1+i)=|1+i+re^{i(-\frac{7\pi}{4})}-(1+i)|e^{i(-\frac{7\pi}{4})}$$
$$\Longrightarrow \sqrt{z-(1+i)}=\sqrt{|1+i+re^{i(-\frac{7\pi}{4})}-(1+i)|}\,e^{i(-\frac{7\pi}{8})}$$

$$z-(2+2i)=|1+i+re^{i(-\frac{7\pi}{4})}-(2+2i)|e^{i(-\frac{3\pi}{4})}$$
$$\Longrightarrow \sqrt{z-(2+2i)}=\sqrt{|1+i+re^{i(-\frac{3\pi}{4})}-(2+2i)|}\,e^{i(-\frac{3\pi}{8})}$$

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\int_{\gamma^*} f(z)\,dz\\ &= 2\int_{\sqrt{2}}^{0}\sqrt{|1+i+re^{i(-\frac{7\pi}{4})}-(1+i)|}\sqrt{|1+i+re^{i(-\frac{3\pi}{4})}-(2+2i)|}\,e^{i(-\frac{7\pi}{8})}e^{i(-\frac{3\pi}{8})}e^{i(-\frac{7\pi}{4})}\,dr\\ &= 1.5708\,.\end{aligned}$$

Using Mathematica

Along $2+2i \xrightarrow{+} 1+i\,(z\in\gamma^*)$

$$z=1+i+re^{i(-\frac{7\pi}{4})},\quad r:\sqrt{2}\longrightarrow 0 \Longrightarrow dz=e^{i(-\frac{7\pi}{4})}dr$$

$$\arg\,(z-(1+i))=-\frac{7}{4}\pi \Longrightarrow \sqrt{z-(1+i)}=(-1)\cdot \text{MATH}\left(\sqrt{z-(1+i)}\right)$$
$$\arg\,(z-(2+2i))=-\frac{3}{4}\pi \Longrightarrow \sqrt{z-(2+2i)}=\text{MATH}\left(\sqrt{z-(2+2i)}\right)$$

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\int_{\gamma^*} f(z)\,dz\\ &= 2\cdot(-1)\cdot\text{MATH}\left(\int_{\sqrt{2}}^{0}\sqrt{1+i+re^{i(-\frac{7\pi}{4})}-(1+i)}\sqrt{1+i+re^{i(-\frac{3\pi}{4})}-(2+2i)}\,e^{i(-\frac{7\pi}{4})}\,dr\right)\\ &= 1.5708\,.\end{aligned}$$

Example 22. *Suppose that*

$$\begin{aligned}f(z)&=\sqrt{z-(1+i)}\sqrt{z-(2+2i)}\sqrt{z-(3+3i)}\sqrt{z-(4+4i)}\sqrt{z-(5+5i)}\\ &=\prod_{k=1}^{5}\sqrt{z-(k+ki)}.\end{aligned}$$

Let a_1, a_2 be two $a-$cycles and let b_1, b_2 be two $b-$cycles drawing in Figure 92. *Evaluate the four integrals $\int_{a_k} f(z)dz$ and $\int_{b_k} f(z)dz$, $k=1,2$ using the method of deformation of path.*

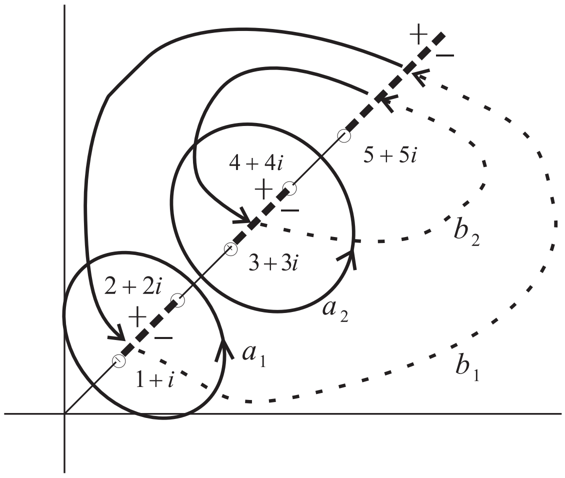


Figure 92

Solution.

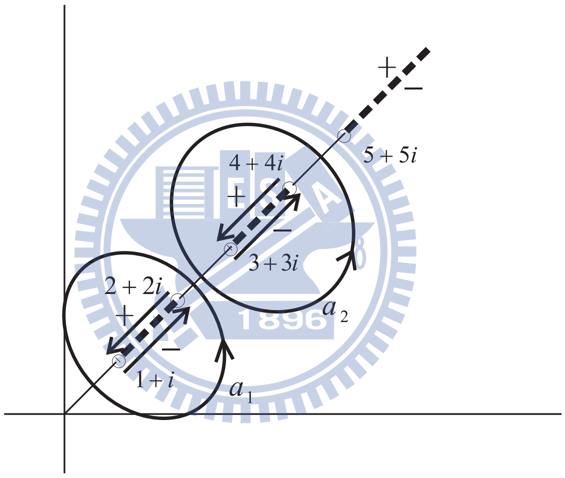


Figure 93

1. Evaluate $\int_{a_1} f(z)\,dz$

<u>Theoretical Evaluation</u>

Along $2+2i \xrightarrow{+} 1+i$,

$$z = 1+i+re^{i(-\frac{7\pi}{4})}, \quad r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k = 1$,

$$\arg(z-(1+i)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(1+i)} = \sqrt{\left|1+i+re^{i(-\frac{7\pi}{4})}-(1+i)\right|}\, e^{i(-\frac{7\pi}{8})}.$$

For $k = 2, 3, 4, 5$,

$$\arg(z-(k+ki)) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{\left|1+i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\, e^{i(-\frac{3\pi}{8})}$$

Then,

$$\begin{aligned}\int_{a_1} f(z)\, dz &= 2\int_{2+2i \xrightarrow{+} 1+i} f(z)\, dz \\ &= 2\int_{\sqrt{2}}^{0} \left(\prod_{k=1}^{5} \sqrt{|1+i+re^{i(-\frac{7\pi}{4})} - (k+ki)|} \right) e^{i(\frac{-7\pi}{8})} \left(e^{i(\frac{-3\pi}{8})} \right)^4 e^{i(-\frac{7\pi}{4})}\, dr \\ &= -8.8736 + 3.67557i\,.\end{aligned}$$

<u>**Using Mathematica**</u>

Along $2+2i \xrightarrow{+} 1+i$,

$$z = 1+i+re^{i(-\frac{7\pi}{4})}, \quad r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k = 1$,

$$\arg(z-(1+i)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(1+i)} = (-1)\cdot \text{MATH}\left(\sqrt{1+i+re^{i(-\frac{7\pi}{4})} - (1+i)} \right).$$

For $k = 2,3,4,5$,

$$\arg(z-(k+ki)) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \text{MATH}\left(\sqrt{1+i+re^{i(-\frac{7\pi}{4})} - (k+ki)} \right).$$

Then,

$$\begin{aligned}\int_{a_1} f(z)\, dz &= 2\int_{2+2i \xrightarrow{+} 1+i} f(z)\, dz \\ &= 2\cdot(-1)\cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} \left(\prod_{k=1}^{5} \sqrt{1+i+re^{i(-\frac{7\pi}{4})} - (k+ki)} \right) e^{i(-\frac{7\pi}{4})}\, dr \right) \\ &= -8.8736 + 3.67557i\,.\end{aligned}$$

2. Evaluate $\int_{a_2} f(z)\, dz$

<u>**Theoretical Evaluation**</u>

Along $4+4i \xrightarrow{+} 3+3i$,

$$z = 3+3i+re^{i(-\frac{7\pi}{4})}, \quad r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k = 1,2,3$,

$$\arg(z-(k+ki)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{|3+3i+re^{i(-\frac{7\pi}{4})} - (k+ki)|}\, e^{i(-\frac{7\pi}{8})}$$

For $k = 4,5$,

$$\arg(z-(k+ki)) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{|3+3i+re^{i(-\frac{7\pi}{4})} - (k+ki)|}\, e^{i(-\frac{3\pi}{8})}$$

Then,

$$\begin{aligned}\int_{a_2} f(z)\,dz &= 2\int_{4+4i\xrightarrow{+}3+3i} f(z)\,dz \\ &= 2\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{|3+3i+re^{i(-\frac{7\pi}{4})}-(k+ki)|}\right)\left(e^{i(\frac{-7\pi}{8})}\right)^3\left(e^{i(\frac{-3\pi}{8})}\right)^2 e^{i(-\frac{7\pi}{4})}\,dr \\ &= 5.69991-2.36098i\,.\end{aligned}$$

Using Mathematica

Along $4+4i\xrightarrow{+}3+3i$,

$$z=3+3i+re^{i(-\frac{7\pi}{4})},\quad r:\sqrt{2}\longrightarrow 0\Longrightarrow dz=e^{i(-\frac{7\pi}{4})}dr$$

For $k=1,2,3$,

$$\arg(z-(k+ki))=-\frac{7}{4}\pi\Longrightarrow\sqrt{z-(k+ki)}=(-1)\cdot\text{MATH}\left(\sqrt{3+3i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)$$

For $k=4,5$,

$$\arg(z-(k+ki))=-\frac{3}{4}\pi\Longrightarrow\sqrt{z-(k+ki)}=\text{MATH}\left(\sqrt{3+3i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)$$

Then,

$$\begin{aligned}\int_{a_2} f(z)\,dz &= 2\int_{4+4i\xrightarrow{+}3+3i} f(z)\,dz \\ &= 2\cdot(-1)^2\cdot\text{MATH}\left(\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{|3+3i+re^{i(-\frac{7\pi}{4})}-(k+ki)|}\right)e^{i(-\frac{7\pi}{4})}\,dr\right) \\ &= 5.69991-2.36098i\,.\end{aligned}$$

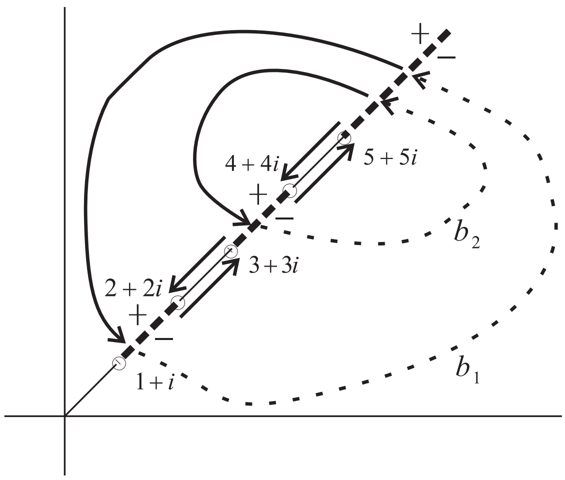


Figure 94

3. Evaluate $\int_{b_1} f(z)\,dz$

Theoretical Evaluation

(1)Along $5+5i \longrightarrow 4+4i$

$$z = 4+4i+re^{i(-\frac{7\pi}{4})}, \quad r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k = 1,2,3,4,$

$$\arg(z-(k+ki)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{\left|4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\,e^{i(-\frac{7\pi}{8})}$$

For $k = 5,$

$$\arg(z-(5+5i)) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z-(5+5i)} = \sqrt{\left|4+4i+re^{i(-\frac{7\pi}{4})}-(5+5i)\right|}\,e^{i(-\frac{3\pi}{8})}$$

Then,

$$\begin{aligned}
&\int_{5+5i\longrightarrow 4+4i} f(z)\,dz + \int_{5+5i \dashleftarrow 4+4i} f(z)\,dz \\
&= 2\int_{5+5i\longrightarrow 4+4i} f(z)\,dz \\
&= 2\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{\left|4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\right)\left(e^{i(\frac{-7\pi}{8})}\right)^4 e^{i(\frac{-3\pi}{8})}\,e^{i(-\frac{7\pi}{4})}\,dr \\
&= -3.67557-8.8736i\,.
\end{aligned}$$

(2)Along $3+3i \longrightarrow 2+2i$

$$z = 2+2i+re^{i(-\frac{7\pi}{4})}, \quad r : \sqrt{2} \longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k = 1,2,$

$$\arg(z-(k+ki)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{\left|2+2i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\,e^{i(-\frac{7\pi}{8})}$$

For $k = 3,4,5,$

$$\arg(z-(k+ki)) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-(k+ki)} = \sqrt{\left|2+2i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\,e^{i(-\frac{3\pi}{8})}$$

Then,

$$\begin{aligned}
&\int_{3+3i\longrightarrow 2+2i} f(z)\,dz + \int_{3+3i \dashleftarrow 2+2i} f(z)\,dz \\
&= 2\int_{3+3i\longrightarrow 2+2i} f(z)\,dz \\
&= 2\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{\left|4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)\right|}\right)\left(e^{i(\frac{-7\pi}{8})}\right)^2 \left(e^{i(\frac{-3\pi}{8})}\right)^3 e^{i(-\frac{7\pi}{4})}\,dr \\
&= -1.31458-3.17369i\,.
\end{aligned}$$

Thus,

$$\begin{aligned}\int_{b_1} f(z)\,dz &= 2\int_{5+5i\longrightarrow 4+4i} f(z)\,dz + 2\int_{3+3i\longrightarrow 2+2i} f(z)\,dz\\ &= -4.99015 - 12.0473i\,.\end{aligned}$$

Using Mathematica

(1)Along $5+5i \longrightarrow 4+4i$

$$z = 4+4i+re^{i(-\frac{7\pi}{4})},\quad r:\sqrt{2}\longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k=1,2,3,4$,

$$\arg(z-(k+ki)) = -\frac{3}{2}\pi \Longrightarrow \sqrt{z-(k+ki)} = (-1)\cdot\text{MATH}\left(\sqrt{4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)$$

For $k=5$,

$$\arg(z-(5+5i)) = -\frac{1}{2}\pi \Longrightarrow \sqrt{z-(5+5i)} = \text{MATH}\left(\sqrt{4+4i+re^{i(-\frac{7\pi}{4})}-(5+5i)}\right)$$

Then,

$$\begin{aligned}&\int_{5+5i\longrightarrow 4+4i} f(z)\,dz + \int_{5+5i\dashleftarrow 4+4i} f(z)\,dz\\ &= 2\int_{5+5i\longrightarrow 4+4i} f(z)\,dz\\ &= 2\cdot(-1)^4\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)\,e^{i(-\frac{7\pi}{4})}\,dr\\ &= -3.67557 - 8.8736i\,.\end{aligned}$$

(2)Along $3+3i \longrightarrow 2+2i$

$$z = 2+2i+re^{i(-\frac{7\pi}{4})},\quad r:\sqrt{2}\longrightarrow 0 \Longrightarrow dz = e^{i(-\frac{7\pi}{4})}dr$$

For $k=1,2$,

$$\arg(z-(k+ki)) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = (-1)\cdot\text{MATH}\left(\sqrt{2+2i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)$$

For $k=3,4,5$,

$$\arg(z-(k+ki)) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z-(k+ki)} = \text{MATH}\left(\sqrt{2+2i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)$$

Then,

$$\begin{aligned}&\int_{3+3i\longrightarrow 2+2i} f(z)\,dz + \int_{3+3i\dashleftarrow 2+2i} f(z)\,dz\\ &= 2\int_{3+3i\longrightarrow 2+2i} f(z)\,dz\\ &= 2\cdot(-1)^2\int_{\sqrt{2}}^{0}\left(\prod_{k=1}^{5}\sqrt{4+4i+re^{i(-\frac{7\pi}{4})}-(k+ki)}\right)\,e^{i(-\frac{7\pi}{4})}\,dr\\ &= -1.31458 - 3.17369i\,.\end{aligned}$$

Thus,

$$\int_{b_1} f(z)\,dz = 2\int_{5+5i\longrightarrow 4+4i} f(z)\,dz + 2\int_{3+3i\longrightarrow 2+2i} f(z)\,dz$$
$$= -4.99015 - 12.0473i\,.$$

4. Evaluate $\int_{b_2} f(z)\,dz$: We have done in 3.

$$\int_{b_2} f(z)\,dz = 2\int_{5+5i\longrightarrow 4+4i} f(z)\,dz = -3.67557 - 8.8736i\,.$$

Example 23. *Let*

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+1})} = \prod_{j=1}^{2N+1}\sqrt{z-z_j}$$

and

$$g(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+1})} = \prod_{j=1}^{2N+2}\sqrt{z-z_j}$$

where z_j is of the form $z_j = r_j e^{i(\alpha-2\pi)}$. That is, these z_j are lies on a slant cut of angle α. The cuts of f and g are drwan in Figure 95 and Figure 96, respectively.

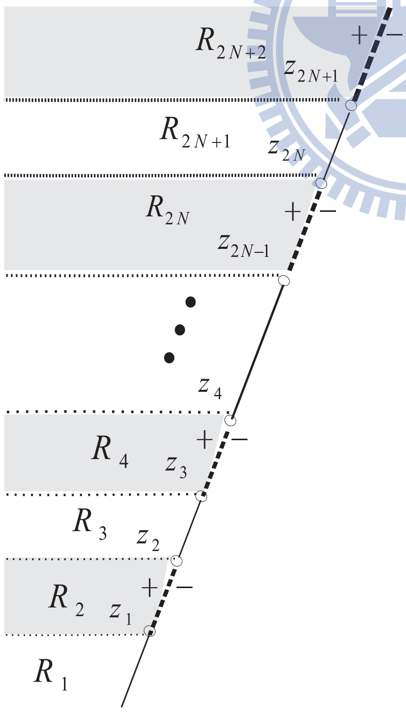


Figure 95

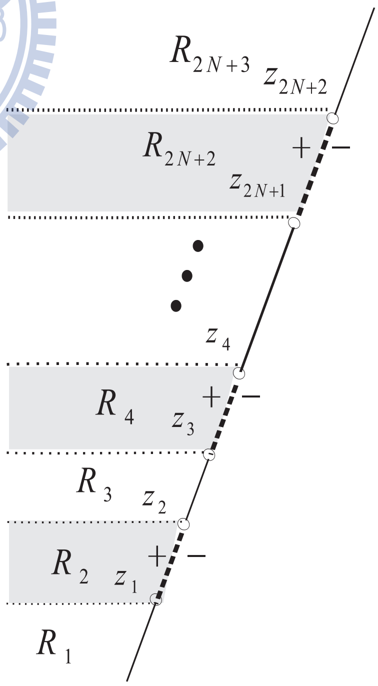


Figure 96

Let $I_c = [\alpha - 2\pi, -\pi]$. To Apply the similar method using in example 13 and example 14 for vertical cuts, we are able to easily determine the sign-regions for f and g. The sign-regions for f and for g are all $R_2, R_4, \cdots, R_{2N+2}$ which are shown in Figure 95 and Figure 96, respectively.

5.4 Generalization of Integrals Along Slant Cuts

Case 1. The number of branch points is odd ($2N+1$ branch points)

Let

$$f(z) = \sqrt{(z-z_1)(z-z_2)\cdots(z-z_{2N+1})} = \prod_{j=1}^{2N+1} \sqrt{z-z_j}\,.$$

Aussume that $Re(z_j) = Im(z_j)$ for $j = 1, 2, \cdots, 2N+1$.

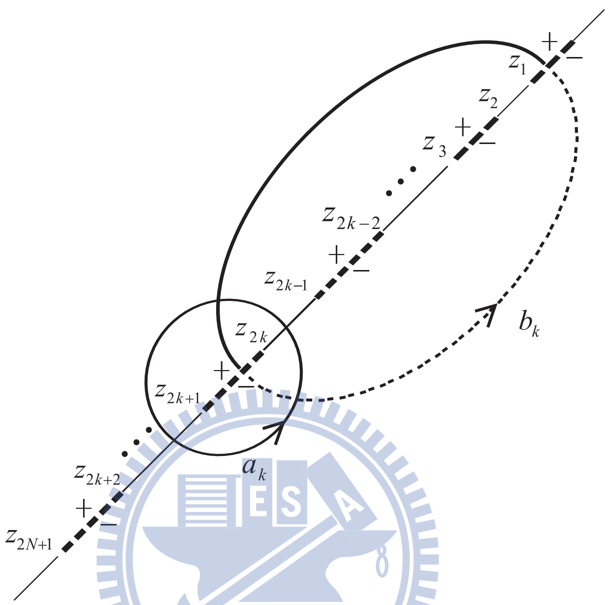


Figure 97

(1) To evaluate $\int_{a_k} f(z)dz$

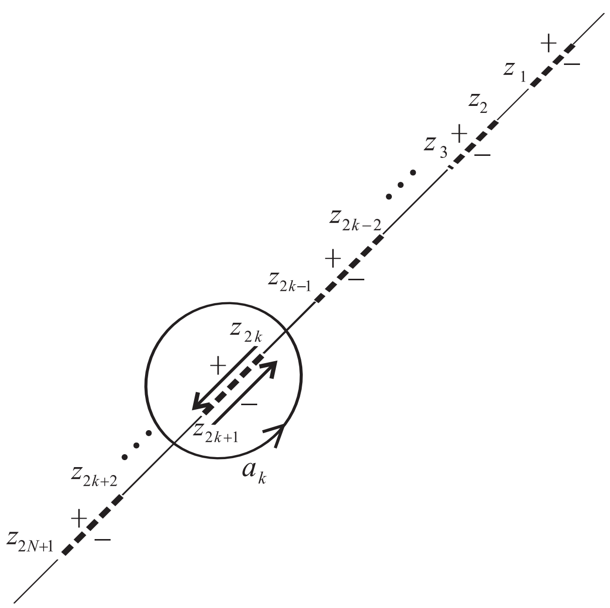


Figure 98

Theoretical Evaluation

Along $z_{2k} \xrightarrow{+} z_{2k+1}$, let $d = |z_{2k} - z_{2k+1}|$.

$$z = z(r) = z_{2k+1} + re^{i(\frac{-7\pi}{4})}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2k$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{3\pi}{8})}$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{7\pi}{8})}$$

Then,

$$\begin{aligned}\int_{a_k} f(z)\, dz &= 2\int_{z_{2k} \xrightarrow{+} z_{2k+1}} f(z)\, dz \\ &= 2\left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+1)-2k}\left(e^{i(\frac{-7\pi}{8})}\right)^{2k}\int_d^0 \left(\prod_{j=1}^{2N+1}\sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\, dr\end{aligned}$$

Using Mathematica

Along $z_{2k} \xrightarrow{+} z_{2k+1}$, let $d = |z_{2k} - z_{2k+1}|$.

$$z = z(r) = z_{2k+1} + re^{i(\frac{-7\pi}{4})}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2k$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

For $j = 2k+1, 2k+2, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = (-1)\cdot \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

Then,

$$\begin{aligned}\int_{a_k} f(z)\, dz &= 2\int_{z_{2k} \xrightarrow{+} z_{2k+1}} f(z)\, dz \\ &= 2\cdot(-1)^{(2N+1)-2k}\cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+1}\sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right) \\ &= 2\cdot(-1)\cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+1}\sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right)\end{aligned}$$

(2) To evaluate $\int_{b_k} f(z)dz$

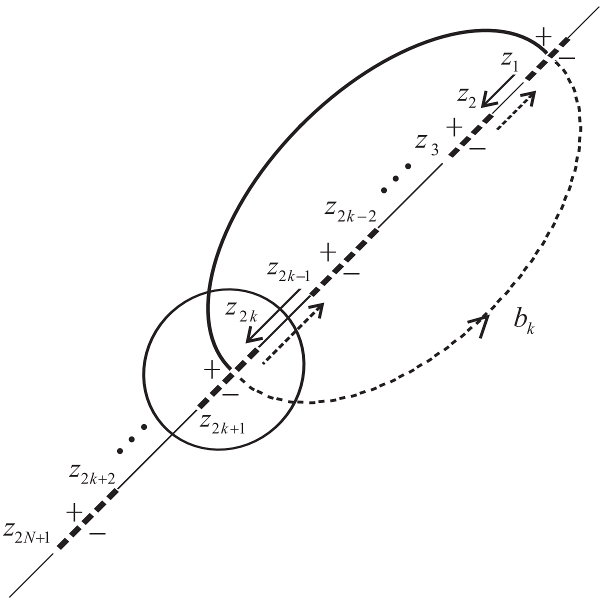


Figure 99

To use the similar method of deriving Equation (49) in case 1 of (2) in section 3.4, we obtain

$$\begin{aligned}\int_{b_k} f(z)\,dz &= 2\left(\int_{z_1\longrightarrow z_2} f(z)\,dz + \int_{z_3\longrightarrow z_4} f(z)\,dz + \cdots + \int_{z_{2k-1}\longrightarrow z_{2k}} f(z)\,dz\right)\\ &= 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz \end{aligned}\tag{66}$$

<u>Theoretical Evaluation</u>

Along $z_{2m-1} \longrightarrow z_{2m}$, let $d = |z_{2m-1} - z_{2m}|$.

$$z = z_{2m} + e^{i(\frac{-7\pi}{4})},\quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{3\pi}{8})}$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{7\pi}{8})}$$

Then,

$$\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz = \left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+1)-(2m-1)}\left(e^{i(\frac{-7\pi}{8})}\right)^{2m-1}\int_d^0\left(\prod_{j=1}^{2N+1}\sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\,dr\,.$$

Thus,

$$\begin{aligned}\int_{b_k} f(z)\,dz &= 2\sum_{m=1}^{k}\int_{z_{2m-1}\longrightarrow z_{2m}} f(z)\,dz\\ &= 2\cdot\left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+1)-(2m-1)}\left(e^{i(\frac{-7\pi}{8})}\right)^{2m-1}\sum_{m=1}^{k}\int_d^0\left(\prod_{j=1}^{2N+1}\sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\,dr\,.\end{aligned}$$

Using Mathematica

Along $z_{2m-1} \longrightarrow z_{2m}$, let $d = |z_{2m-1} - z_{2m}|$.

$$z = e^{i(\frac{-7\pi}{4})}, \quad r : y_d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})} dr$$

For $j = 1, 2, \cdots, 2m-1$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

For $j = 2m, 2m+1, \cdots, 2N+1$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

Then,

$$\begin{aligned}
\int_{b_k} f(z)\, dz &= 2 \int_{z_{2m-1} \longrightarrow z_{2m}} f(z)\, dz \\
&= 2 \cdot (-1)^{(2N+1)-(2m-1)} \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+1} \sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right) \\
&= 2 \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+1} \sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right).
\end{aligned}$$

Case 2. The number of branch points is even ($2N+2$ branch points)

Let

$$f(z) = \sqrt{(z - z_1)(z - z_2) \cdots (z - z_{2N+2})} = \prod_{j=1}^{2N+2} \sqrt{z - z_j}\,.$$

Aussume that $Re(z_j) = Im(z_j)$ for $j = 1, 2, \cdots, 2N+2$.

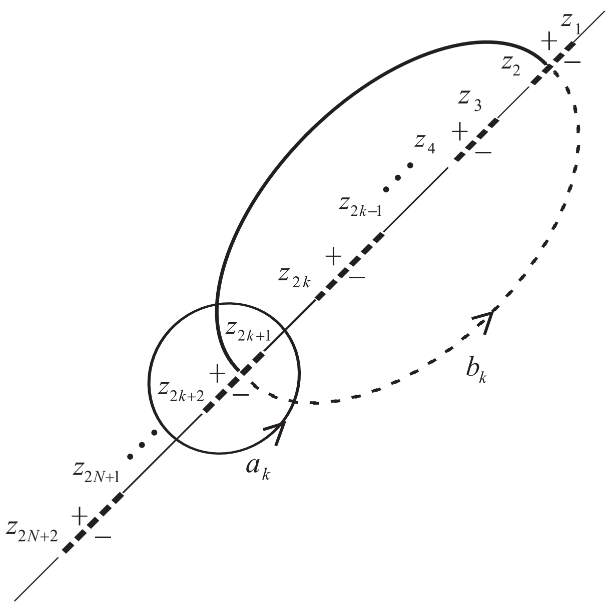


Figure 100

(1) To evaluate $\int_{a_k} f(z)dz$

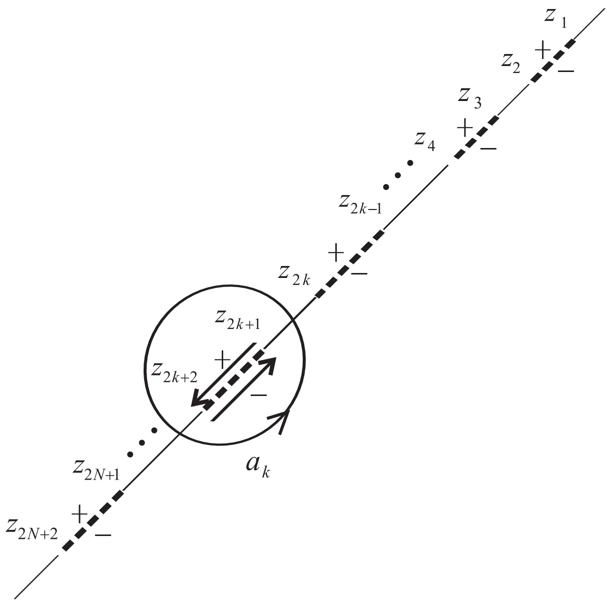


Figure 101

<u>**Theoretical Evaluation**</u>

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$, let $d = |z_{2k+1} - z_{2k+2}|$.

$$z = z(r) = z_{2k+2} + re^{i(\frac{-7\pi}{4})}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{3\pi}{8})}$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{7\pi}{8})}$$

Then,

$$\int_{a_k} f(z)\, dz = 2\int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz$$
$$= 2\left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+2)-(2k+1)} \left(e^{i(\frac{-7\pi}{8})}\right)^{2k+1} \int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\, dr$$

<u>**Using Mathematica**</u>

Along $z_{2k+1} \xrightarrow{+} z_{2k+2}$, let $d = |z_{2k} - z_{2k+1}|$.

$$z = z(r) = z_{2k+2} + re^{i(\frac{-7\pi}{4})}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2k+1$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

For $j = 2k+2, 2k+3, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

Then,

$$\begin{aligned}
\int_{a_k} f(z)\, dz &= 2 \int_{z_{2k+1} \xrightarrow{+} z_{2k+2}} f(z)\, dz \\
&= 2 \cdot (-1)^{(2N+2)-(2k+1)} \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{z(r) - z_j} \right) e^{i(\frac{-7\pi}{4})}\, dr \right) \\
&= 2 \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{z(r) - z_j} \right) e^{i(\frac{-7\pi}{4})}\, dr \right)
\end{aligned}$$

(2) To evaluate $\int_{b_k} f(z)dz$

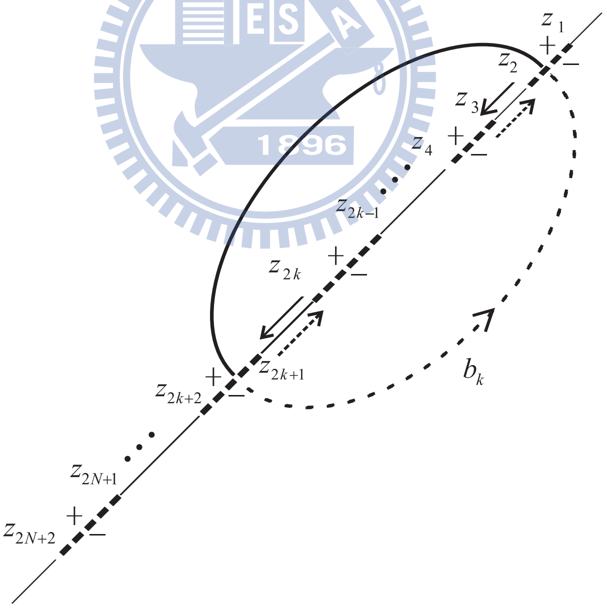


Figure 102

To use the similar method of deriving Equation (49) in case 1 of (2) in section 3.4, we obtain

$$\begin{aligned}
\int_{b_k} f(z)\, dz &= 2 \left(\int_{z_2 \longrightarrow z_3} f(z)\, dz + \int_{z_4 \longrightarrow z_5} f(z)\, dz + \cdots + \int_{z_{2k} \longrightarrow z_{2k+1}} f(z)\, dz \right) \\
&= 2 \sum_{m=1}^{k} \int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \qquad (67)
\end{aligned}$$

Theoretical Evaluation

Along $z_{2m} \longrightarrow z_{2m+1}$, let $d = |z_{2m} - z_{2m+1}|$.

$$z = z_{2m+1} + re^{i(\frac{-7\pi}{4})}, \quad r : d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{3\pi}{8})}$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = \sqrt{|z(r) - z_j|}\, e^{i(-\frac{7\pi}{8})}$$

Then,

$$\int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz = \left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+2)-2m} \left(e^{i(\frac{-7\pi}{8})}\right)^{2m} \int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\, dr\,.$$

Thus,

$$\begin{aligned}\int_{b_k} f(z)\, dz &= 2\sum_{m=1}^{k} \int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \\ &= 2 \cdot \left(e^{i(\frac{-3\pi}{8})}\right)^{(2N+2)-2m} \left(e^{i(\frac{-7\pi}{8})}\right)^{2m} \sum_{m=1}^{k} \int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{|z(r) - z_j|}\right) e^{i(\frac{-7\pi}{4})}\, dr\,.\end{aligned}$$

Using Mathematica

Along $z_{2m} \longrightarrow z_{2m+1}$, let $d = |z_{2m} - z_{2m+1}|$.

$$z = re^{i(\frac{-7\pi}{4})}, \quad r : y_d \longrightarrow 0 \Longrightarrow dz = e^{i(\frac{-7\pi}{4})}dr$$

For $j = 1, 2, \cdots, 2m$,

$$\arg(z - z_j) = -\frac{3}{4}\pi \Longrightarrow \sqrt{z - z_j} = \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

For $j = 2m+1, 2m+2, \cdots, 2N+2$,

$$\arg(z - z_j) = -\frac{7}{4}\pi \Longrightarrow \sqrt{z - z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z(r) - z_j}\right)$$

Then,

$$\begin{aligned}\int_{b_k} f(z)\, dz &= 2\int_{z_{2m} \longrightarrow z_{2m+1}} f(z)\, dz \\ &= 2 \cdot (-1)^{(2N+2)-2m} \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right) \\ &= 2 \cdot \text{MATH}\left(\int_d^0 \left(\prod_{j=1}^{2N+2} \sqrt{z(r) - z_j}\right) e^{i(\frac{-7\pi}{4})}\, dr\right).\end{aligned}$$

Next, we give two more complicated examples.

Example 24. *Let* $z_1 = -1$, $z_2 = i$, $z_3 = 0$, $z_4 = 2+2i$, $z_5 = 2+i$, $z_6 = (2+\sqrt{3})+2i$, $z_7 = 3-i$, $z_8 = 4+(\sqrt{3}-1)i$. *Suppose that* $f(z) = \prod_{j=1}^{8} \sqrt{z - z_j}$. *Evaluate the integrals* $\int_{a_k} f(z)dz$ *and* $\int_{b_k} f(z)dz$, $k = 1, 2, 3$ *drawn in Figure 103.*

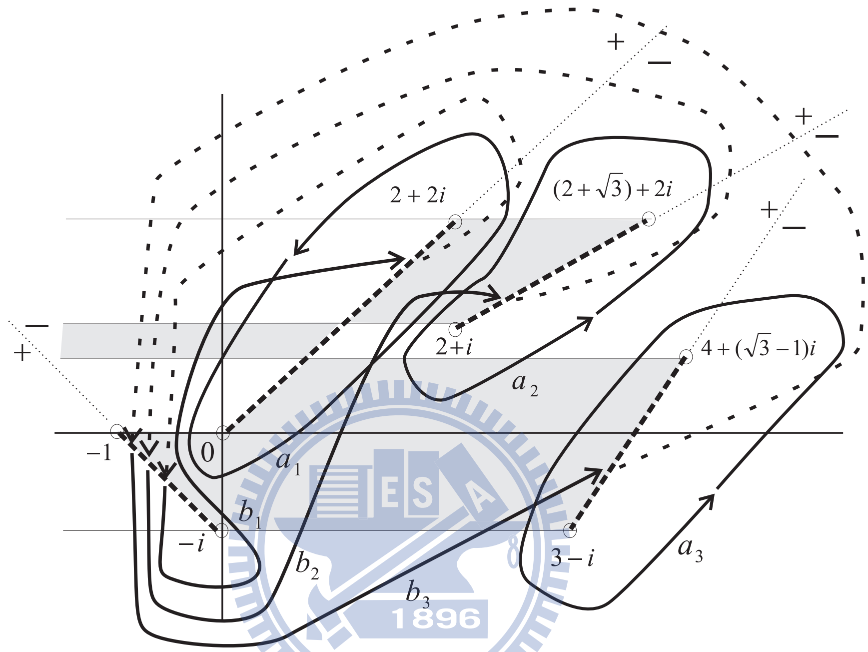


Figure 103

Solution.

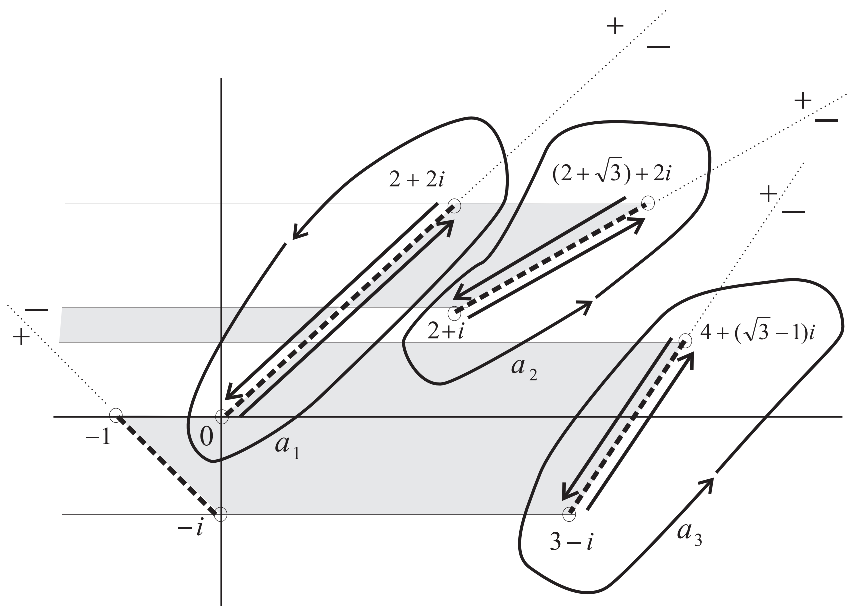


Figure 104

1. Evaluate $\int_{a_1} f(z)\, dz$

$$
\begin{aligned}
\int_{a_1} f(z)\, dz &= \left(\int_{(\sqrt{3}-1)+(\sqrt{3}-1)i \xrightarrow{+} 0} f(z)\, dz + \int_{(\sqrt{3}-1)+(\sqrt{3}-1)i \xleftarrow{-} 0} f(z)\, dz\right) \\
&\quad + \left(\int_{1+i \xrightarrow{+} (\sqrt{3}-1)+(\sqrt{3}-1)i} f(z)\, dz + \int_{1+i \xleftarrow{-} (\sqrt{3}-1)+(\sqrt{3}-1)i} f(z)\, dz\right) \\
&\quad + \left(\int_{2+2i \xrightarrow{+} 1+i} f(z)\, dz + \int_{2+2i \xleftarrow{-} +i} f(z)\, dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{(\sqrt{3}-1)+(\sqrt{3}-1)i \xrightarrow{+} 0} f(z)\, dz\right) \\
&\quad + 2 \cdot (-1) \cdot \text{MATH}\left(\int_{1+i \xrightarrow{+} (\sqrt{3}-1)+(\sqrt{3}-1)i} f(z)\, dz\right) \\
&\quad + 2 \cdot \text{MATH}\left(\int_{2+2i \xrightarrow{+} 1+i} f(z)\, dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{\sqrt{2}(\sqrt{3}-1)}^{0} f(e^{i(\frac{-7\pi}{4})})\, e^{i(\frac{-7\pi}{4})}\, dr\right) \\
&\quad + 2 \cdot (-1) \cdot \text{MATH}\left(\int_{\sqrt{2}}^{\sqrt{2}(\sqrt{3}-1)} f(e^{i(\frac{-7\pi}{4})})\, e^{i(\frac{-7\pi}{4})}\, dr\right) \\
&\quad + 2 \cdot \text{MATH}\left(\int_{2\sqrt{2}}^{\sqrt{2}} f(e^{i(\frac{-7\pi}{4})})\, e^{i(\frac{-7\pi}{4})}\, dr\right) \\
&= 54.6154 + 25.4057i\,.
\end{aligned}
$$

2. Evaluate $\int_{a_2} f(z)\, dz$

$$
\begin{aligned}
\int_{a_2} f(z)\, dz &= \int_{(2+\sqrt{3})+2i \xrightarrow{+} 2+i} f(z)\, dz + \int_{(2+\sqrt{3})+2i \xleftarrow{-} 2+i} f(z)\, dz \\
&= 2 \cdot (-1) \cdot \text{MATH}\left(\int_{(2+\sqrt{3})+2i \xrightarrow{+} 2+i} f(z)\, dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{2}^{0} f(2+i+re^{i(\frac{-11\pi}{6})})\, e^{i(\frac{-11\pi}{6})}\, dr\right) \\
&= 19.3388 - 40.8839i\,.
\end{aligned}
$$

3. Evaluate $\int_{a_3} f(z)\, dz$

$$
\begin{aligned}
\int_{a_3} f(z)\, dz &= \int_{3-i \xrightarrow{+} 4+(\sqrt{3}-1)i} f(z)\, dz + \int_{3-i \xleftarrow{-} 4+(\sqrt{3}-1)i} f(z)\, dz \\
&= 2 \cdot (-1) \cdot \text{MATH}\left(\int_{3-i \xrightarrow{+} 4+(\sqrt{3}-1)i} f(z)\, dz\right) \\
&= 2 \cdot (-1) \cdot \text{MATH}\left(\int_{2}^{0} f(3-i+re^{i(\frac{-5\pi}{3})})\, e^{i(\frac{-5\pi}{3})}\, dr\right) \\
&= 35.7104 - 61.8018i\,.
\end{aligned}
$$

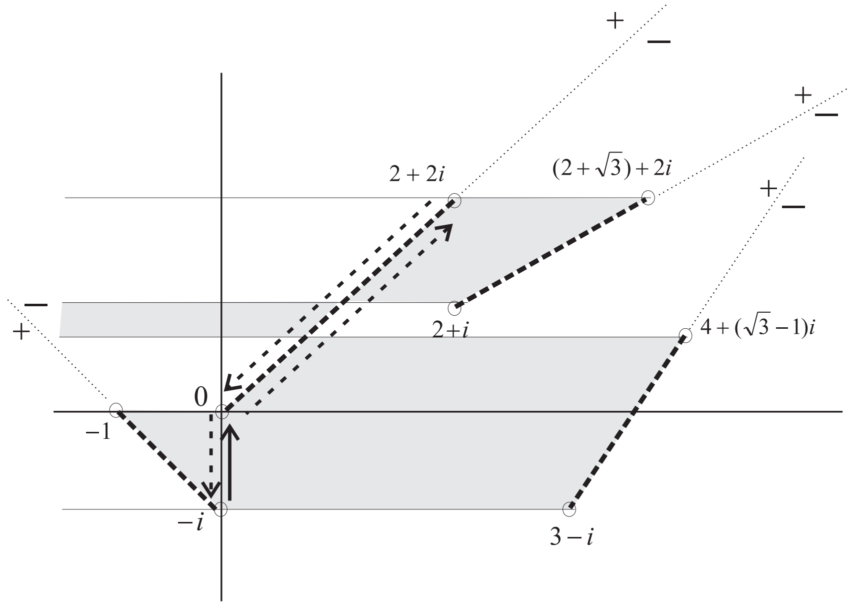


Figure 105

4. Evaluate $\int_{b_1} f(z)\, dz$

$$
\begin{aligned}
\int_{b_1} f(z)\, dz &= \left(\int_{0 \dashrightarrow -i} f(z)\, dz + \int_{0 \longleftarrow -i} f(z)\, dz \right) + (-1) \int_{a_1} f(z)\, dz \\
&= 2 \int_{0 \longleftarrow -i} f(z)\, dz + (-1) \int_{a_1} f(z)\, dz \\
&= 2 \cdot (-1) \cdot \mathrm{MATH} \left(\int_{0 \longleftarrow -i} f(z)\, dz \right) + (-1) \int_{a_1} f(z)\, dz \\
&= 2 \cdot (-1) \cdot \mathrm{MATH} \left(\int_{-1}^{0} f(ri)\, i\, dr \right) + (-1)(54.6154 + 25.4057i) \\
&= -43.5559 - 11.1194i\,.
\end{aligned}
$$

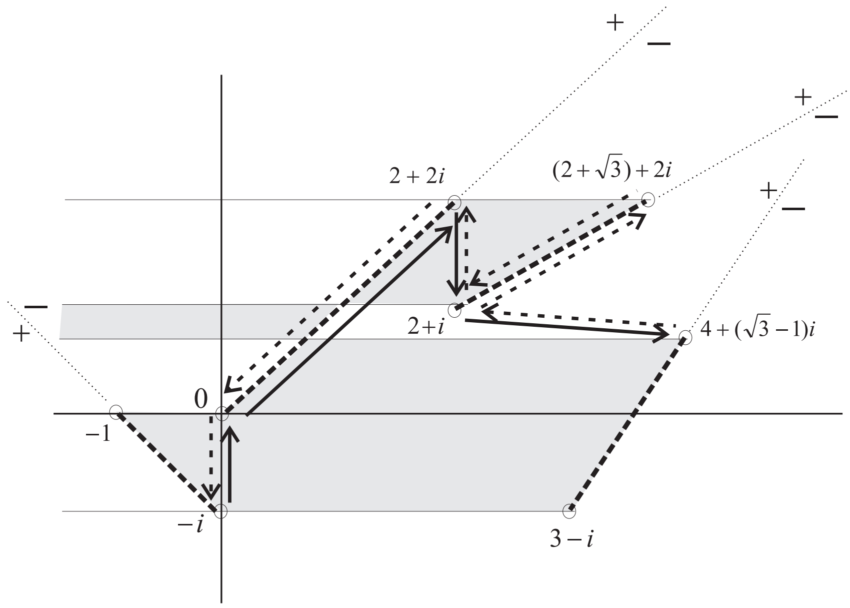


Figure 106

5. Evaluate $\int_{b_2} f(z)\, dz$

$$\int_{b_2} f(z)\, dz = \left(\int_{0\dashrightarrow -i} f(z)\, dz + \int_{0\longleftarrow -i} f(z)\, dz\right) + \left(\int_{2+2i\overset{+}{\dashrightarrow}0} f(z)\, dz + \int_{2+2i\overset{-}{\longleftarrow}0} f(z)\, dz\right)$$

$$+ \left(\int_{2+2i\longrightarrow 2+i} f(z)\, dz + \int_{2+2i\dashleftarrow 2+i} f(z)\, dz\right) + (-1)\int_{a_2} f(z)\, dz$$

$$= 2\cdot(-1)\cdot \text{MATH}\left(\int_{0\longleftarrow -i} f(z)\, dz\right) + 0 + 2\cdot(-1)\cdot \text{MATH}\left(\int_{2+2i\longrightarrow 2+i} f(z)\, dz\right)$$

$$+ (-1)\int_{a_2} f(z)\, dz$$

$$= 2\cdot(-1)\cdot \text{MATH}\left(\int_{-1}^{0} f(ri)\, i\, dr\right) + 2\cdot(-1)\cdot \text{MATH}\left(\int_{2}^{1} f(2+ri)\, i\, dr\right)$$

$$+ (-1)(19.3388 - 40.8839i)$$

$$= 17.9344 - 31.2184i\,.$$

6. Evaluate $\int_{b_3} f(z)\, dz$

$$\int_{b_3} f(z)\, dz$$

$$= \int_{b_2} f(z)\, dz + \left(\int_{2+i\longrightarrow 4+(\sqrt{3}-1)i} f(z)\, dz + \int_{2+i\dashleftarrow 4+(\sqrt{3}-1)i} f(z)\, dz\right)$$

$$= (17.9344 - 31.2184i) + 2\cdot \text{MATH}\left(\int_{0}^{\sqrt{11-4\sqrt{3}}} f\left(2+i+re^{i(-\tan^{-1}\frac{2-\sqrt{3}}{2})}\right) e^{i(-\tan^{-1}\frac{2-\sqrt{3}}{2})}\, dr\right)$$

$$= 36.6632 + 8.92194i\,.$$

Example 25. *Let* $z_1 = -1$, $z_2 = i$, $z_3 = 0$, $z_4 = 2+2i$, $z_5 = 2+i$, $z_6 = (2+\sqrt{3})+2i$, $z_7 = 3-i$, $z_8 = 4+(\sqrt{3}-1)i$*. Suppose that* $f(z) = \prod_{j=1}^{8}\sqrt{z-z_j}$*. Evaluate the integrals* $\int_{a_k} f(z)dz$ *and* $\int_{b_k} f(z)dz$, $k = 1,2,3$ *drawn in Figure 107.*

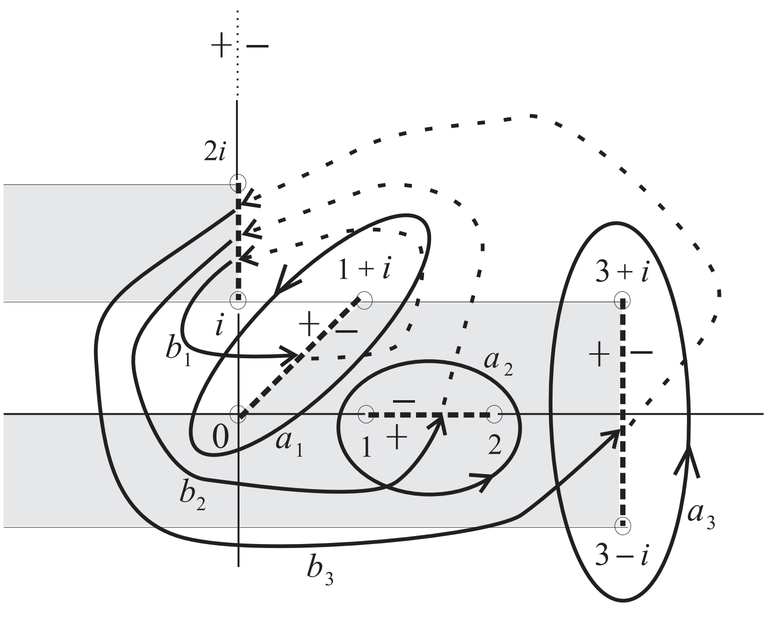


Figure 107

Solution.

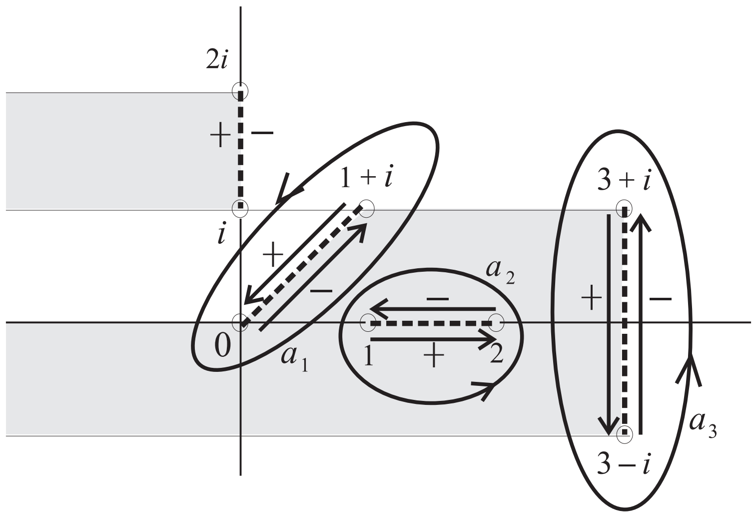


Figure 108

1. Evaluate $\int_{a_1} f(z)\,dz$

$$
\begin{aligned}
\int_{a_1} f(z)\,dz &= \int_{1+i \xrightarrow{+} 0} f(z)\,dz + \int_{1+i \xleftarrow{-} 0} f(z)\,dz \\
&= 2 \cdot \text{MATH}\left(\int_{1+i \xrightarrow{+} 0} f(z)\,dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} f(re^{i(\frac{-7\pi}{4})})\, e^{i(\frac{-7\pi}{4})}\,dr\right) \\
&= -0.876621 - 5.4111i\,.
\end{aligned}
$$

2. Evaluate $\int_{a_2} f(z)\,dz$

$$
\begin{aligned}
\int_{a_2} f(z)\,dz &= \int_{1 \xrightarrow{+} 2} f(z)\,dz + \int_{1 \xleftarrow{-} 2} f(z)\,dz \\
&= 2 \cdot \text{MATH}\left(\int_{1 \xrightarrow{+} 2} f(z)\,dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{1}^{2} f(x)\,dx\right) \\
&= 3.70585 + 0.993793i\,.
\end{aligned}
$$

3. Evaluate $\int_{a_3} f(z)\,dz$

$$
\begin{aligned}
\int_{a_3} f(z)\,dz &= \int_{3+i \xrightarrow{+} 3-i} f(z)\,dz + \int_{3+i \xleftarrow{-} 3-i} f(z)\,dz \\
&= 2 \cdot \text{MATH}\left(\int_{3+i \xrightarrow{+} 3-i} f(z)\,dz\right) \\
&= 2 \cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} f(3 - i + re^{i(\frac{-3\pi}{2})})\, e^{i(\frac{-3\pi}{2})}\,dr\right) \\
&= 23.1466 + 24.5632i\,.
\end{aligned}
$$

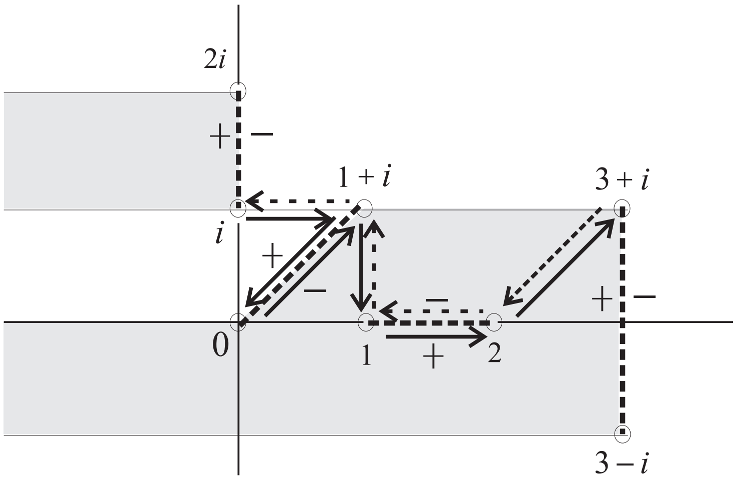


Figure 109

4. Evaluate $\int_{b_1} f(z)\, dz$

$$
\begin{aligned}
\int_{b_1} f(z)\, dz &= \int_{i \longrightarrow 1+i} f(z)\, dz + \int_{i \dashleftarrow 1+i} f(z)\, dz \\
&= 2 \cdot \textsc{math}\left(\int_0^1 f(x+i)\, dx \right) \\
&= 3.29859 + 1.33887i\,.
\end{aligned}
$$

5. Evaluate $\int_{b_2} f(z)\, dz$

$$
\begin{aligned}
\int_{b_2} f(z)\, dz &= \int_{a_1} f(z)\, dz + \int_{b_1} f(z)\, dz + \left(\int_{1+i \longrightarrow 1} f(z)\, dz + \int_{1+i \dashleftarrow 1} f(z)\, dz \right) \\
&= \int_{a_1} f(z)\, dz + \int_{b_1} f(z)\, dz + 2 \int_{1+i \longrightarrow 1} f(z)\, dz \\
&= (-0.876621 - 5.4111i) + (3.29859 + 1.33887i) + 2 \cdot (-1) \cdot \textsc{math}\left(\int_1^0 f(1+ri)\, i\, dr \right) \\
&= 0.63586 - 1.78728i\,.
\end{aligned}
$$

6. Evaluate $\int_{b_3} f(z)\, dz$

$$
\begin{aligned}
\int_{1 \overset{-}{\dashleftarrow} 2} f(z)\, dz &= (-1) \int_{1 \xleftarrow{-} 2} f(z)\, dz \\
&= (-1) \cdot (-1) \cdot \textsc{math}\left(\int_2^1 f(x)\, dx \right) \\
&= (-1) \cdot \textsc{math}\left(\int_1^2 f(x)\, dx \right) \\
&= (-1) \int_{1 \xrightarrow{+} 2} f(z)\, dz
\end{aligned}
$$

So,

$$
\int_{1 \xrightarrow{+} 2} f(z)\, dz + \int_{1 \overset{-}{\dashleftarrow} 2} f(z)\, dz = 0. \tag{68}
$$

Thus,

$$\begin{aligned}
\int_{b_3} f(z)\,dz &= \int_{b_2} f(z)\,dz + \left(\int_{1\xrightarrow{+}2} f(z)\,dz + \int_{1\xleftarrow{-}2} f(z)\,dz\right) + \left(\int_{2\xrightarrow{+}3+i} f(z)\,dz + \int_{2\xleftarrow{-}3+i} f(z)\,dz\right) \\
&= \int_{b_2} f(z)\,dz + 0 + 2\cdot(-1)\cdot \text{MATH}\left(\int_{2\xrightarrow{+}3+i} f(z)\,dz\right) \\
&= (0.63586 - 1.78728i) + 2\cdot(-1)\cdot \text{MATH}\left(\int_0^{\sqrt{2}} f(2+re^{i(\frac{\pi}{4})})\, e^{i(\frac{\pi}{4})}\,dr\right) \\
&= -23.3469 + 13.0985i\,.
\end{aligned}$$

Example 26. *Let* $z_1 = -1+2i, z_2 = -2+i, z_3 = -2-i, z_4 = -1-2i, z_5 = 1-2i, z_6 = 2-i, z_7 = 2+i, z_8 = 1+2i$. *Suppose that* $f(z) = \prod_{j=1}^{8}\sqrt{z-z_j}$. *Evaluate the integrals* $\int_{a_k} f(z)dz$ *and* $\int_{b_k} f(z)dz$, $k = 1, 2, 3$ *drawn in Figure 110.*

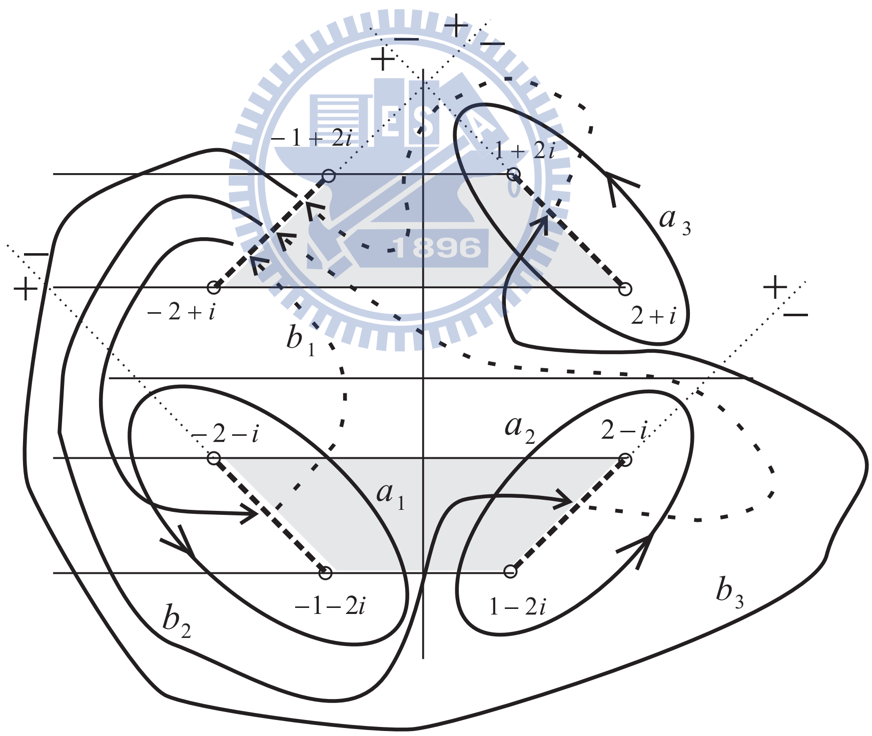


Figure 110

Solution.

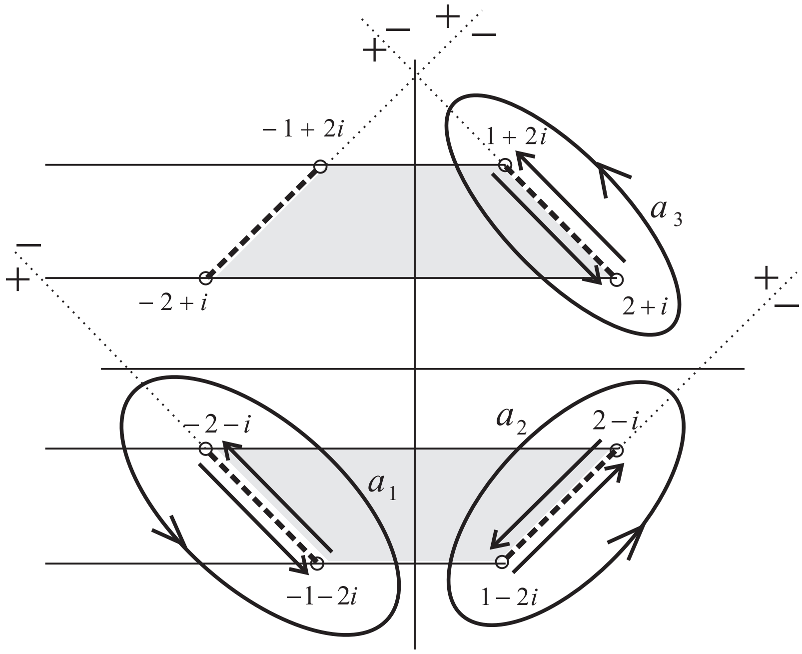


Figure 111

1. Evaluate $\int_{a_1} f(z)\,dz$

$$\begin{aligned}\int_{a_1} f(z)\,dz &= \int_{-2-i\xrightarrow{+}-1-2i} f(z)\,dz + \int_{-2-i\xleftarrow{-}-1-2i} f(z)\,dz \\ &= 2\cdot \text{MATH}\left(\int_{-2-i\xrightarrow{+}-1-2i} f(z)\,dz\right) \\ &= 2\cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} f(-1-2i+re^{i(\frac{-5\pi}{4})})\,e^{i(\frac{-5\pi}{4})}\,dr\right) \\ &= -41.8808+41.8808i\,.\end{aligned}$$

2. Evaluate $\int_{a_2} f(z)\,dz$

$$\begin{aligned}\int_{a_2} f(z)\,dz &= \int_{2-i\xrightarrow{+}1-2i} f(z)\,dz + \int_{2-i\xleftarrow{-}1-2i} f(z)\,dz \\ &= (-1)\cdot 2\cdot \text{MATH}\left(\int_{2-i\xrightarrow{+}1-2i} f(z)\,dz\right) \\ &= (-1)\cdot 2\cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} f(1-2i+re^{i(\frac{-7\pi}{4})})\,e^{i(\frac{-7\pi}{4})}\,dr\right) \\ &= -41.8808-41.8808i\,.\end{aligned}$$

3. Evaluate $\int_{a_3} f(z)\,dz$

$$\int_{a_3} f(z)\,dz = \int_{1+2i \xrightarrow{+} 2+i} f(z)\,dz + \int_{1+2i \xleftarrow{-} 2+i} f(z)\,dz$$
$$= (-1)\cdot 2 \cdot \text{MATH}\left(\int_{1+2i \xrightarrow{+} 2+i} f(z)\,dz\right)$$
$$= (-1)\cdot 2 \cdot \text{MATH}\left(\int_{\sqrt{2}}^{0} f(1+2i+re^{i(\frac{-5\pi}{4})})\,e^{i(\frac{-5\pi}{4})}\,dr\right)$$
$$= 49.0544 - 125.2i\,.$$

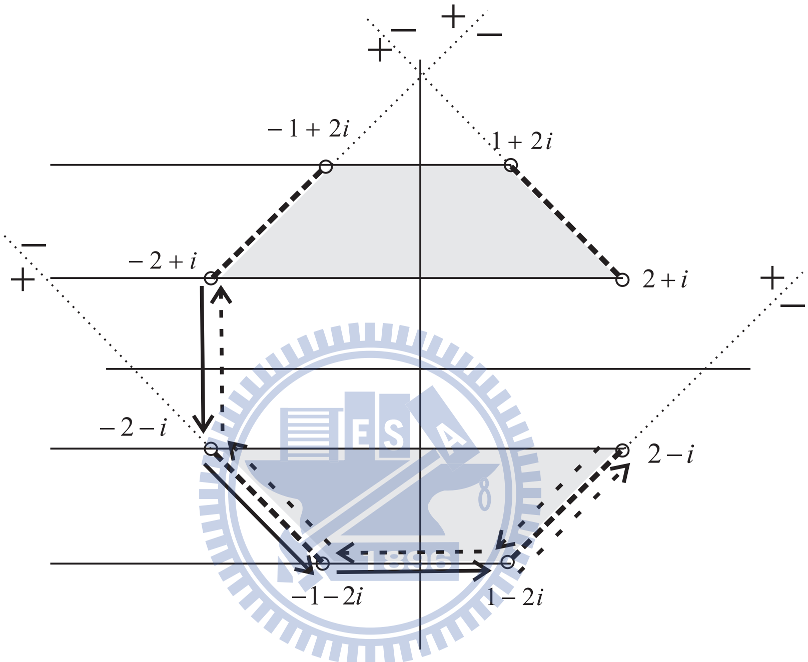


Figure 112

4. Evaluate $\int_{b_1} f(z)\,dz$

$$\int_{b_1} f(z)\,dz = \int_{-2+i \longrightarrow -2-i} f(z)\,dz + \int_{-2+i \dashleftarrow -2-i} f(z)\,dz$$
$$= 2 \cdot \text{MATH}\left(\int_{1}^{-1} f(-2+ri)\,idr\right)$$
$$= -98.1087i\,.$$

5. Evaluate $\int_{b_2} f(z)\,dz$

$$\begin{aligned}
\int_{b_2} f(z)\,dz &= \left(\int_{-2+i \longrightarrow -2-i} f(z)\,dz + \int_{-2+i \dashleftarrow\!-\!-\, -2-i} f(z)\,dz\right) \\
&\quad + \left(\int_{-2-i \overset{+}{\longrightarrow} -1-2i} f(z)\,dz + \int_{-2-i \overset{-}{\dashleftarrow}\!-\!- -1-2i} f(z)\,dz\right) \\
&\quad + \left(\int_{-1-2i \longrightarrow 1-2i} f(z)\,dz + \int_{-1-2i \dashleftarrow\!- 1-2i} f(z)\,dz\right) \\
&\quad + \left(\int_{2-i \overset{+}{\dashrightarrow} 1-2i} f(z)\,dz + \int_{2-i \overset{-}{\dashleftarrow} 1-2i} f(z)\,dz\right) \\
&= \int_{b_1} f(z)\,dz + 0 + 2\int_{-1-2i \longrightarrow 1-2i} f(z)\,dz + (-1)\cdot 2\int_{2-i \overset{+}{\longrightarrow} 1-2i} f(z)\,dz \\
&= -98.1087i + (-1)\cdot 2\cdot \textsc{math}\left(\int_{1}^{-1} f(-2+ri)\,i dr\right) \\
&\quad + (-1)\cdot 2\cdot(-1)\cdot \textsc{math}\left(\int_{\sqrt{2}}^{0} f(1-2i+re^{i(\frac{-7\pi}{4})})\,e^{i(\frac{-7\pi}{4})}\,dr\right) \\
&= 139.99 - 56.2279i\,.
\end{aligned}$$

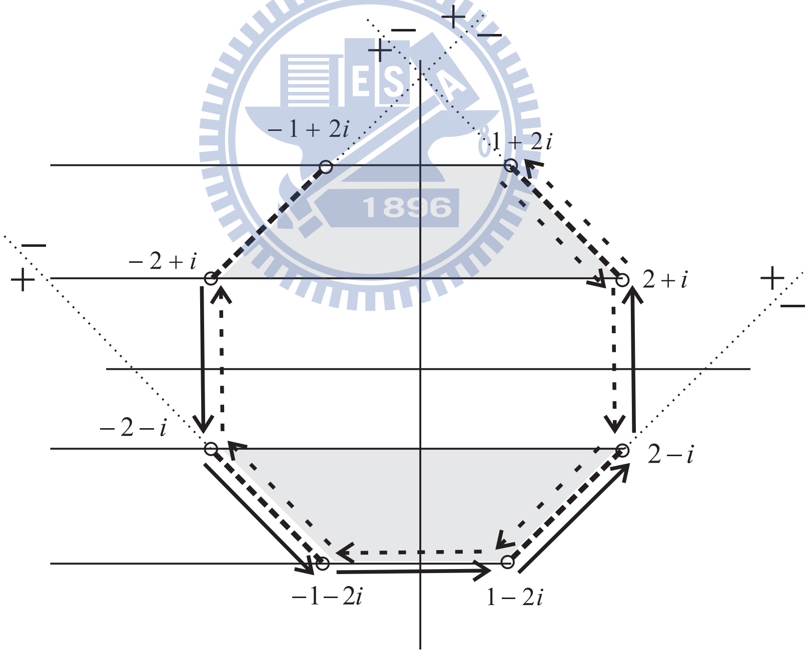


Figure 113

6. Evaluate $\int_{b_3} f(z)\, dz$

$$\int_{b_3} f(z)\, dz = \int_{b_1} f(z)\, dz + 0 + 2\int_{-1-2i\longrightarrow 1-2i} f(z)\, dz + 0 + 2\int_{2-i\longrightarrow 2+i} f(z)\, dz$$

$$+ (-1)\cdot 2\int_{1+2i\overset{+}{\longrightarrow}2+i} f(z)\, dz$$

$$= -98.1087i + (-1)\cdot 2\cdot \text{MATH}\left(\int_{1}^{-1} f(-2+ri)\, idr\right)$$

$$+ 2\cdot \text{MATH}\left(\int_{-1}^{1} f(2+ri)\, idr\right)$$

$$+ (-1)\cdot 2\cdot (-1)\cdot \text{MATH}\left(\int_{0}^{\sqrt{2}} f(1+2i+re^{i(\frac{-5\pi}{4})})\, e^{i(\frac{-5\pi}{4})}\, dr\right)$$

$$= -49.0544 + 223.309i\,.$$

6 An Application on Differential Equations

The umdamped pendulum equation can be written as

$$u'' + \cos u = 0. \tag{69}$$

We know that $\cos u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \frac{1}{6!}u^6 + \cdots + \frac{1}{(2n)!}u^{2n} + \cdots, -\infty < u < \infty$. We use the first three terms to be an estimation of $\cos u$.

$$\cos u \approx 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4.$$

Let

$$f(u) = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4.$$

Then, equation (69) becomes to

$$u'' + f(u) = 0.$$

In section1.1, we have derived that

$$\frac{1}{2}(u')^2 + F(u) = E. \tag{70}$$

This equation is the principle of conservation law $T + V = E$, where $T = \frac{1}{2}(u')^2$ is the kinetic energy, $V = F(u)$ is the potential energy, and E is the total energy. Equation (70) implies that

$$\int \frac{1}{\sqrt{2[E - F(u)]}}\, du = \int dt.$$

Here, the solution u is in the Riemann surface of $\sqrt{2[E - F(u)]}$.

Example 27. *Given that $E = 5$ and*

$$F(u) = \int_0^u f(s)ds = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5.$$

Let

$$h(u) = \frac{1}{\sqrt{2[E - F(u)]}}.$$

$$\begin{aligned}
\sqrt{2[E - F(u)]} &= \sqrt{2[5 - (u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5)]} \\
&= \sqrt{10 - 2u + \frac{2}{3!}u^3 - \frac{2}{5!}u^5} \\
&= \sqrt{\prod_{j=1}^{5}(u - u_j)},
\end{aligned}$$

where

$$u_1 = -3.86 - 1.63i\,,$$
$$u_2 = -3.86 + 1.63i\,,$$
$$u_3 = 1.59 - 2.24i\,,$$
$$u_4 = 1.59 - 2.24i\,,$$
$$u_5 = 4.54\,.$$

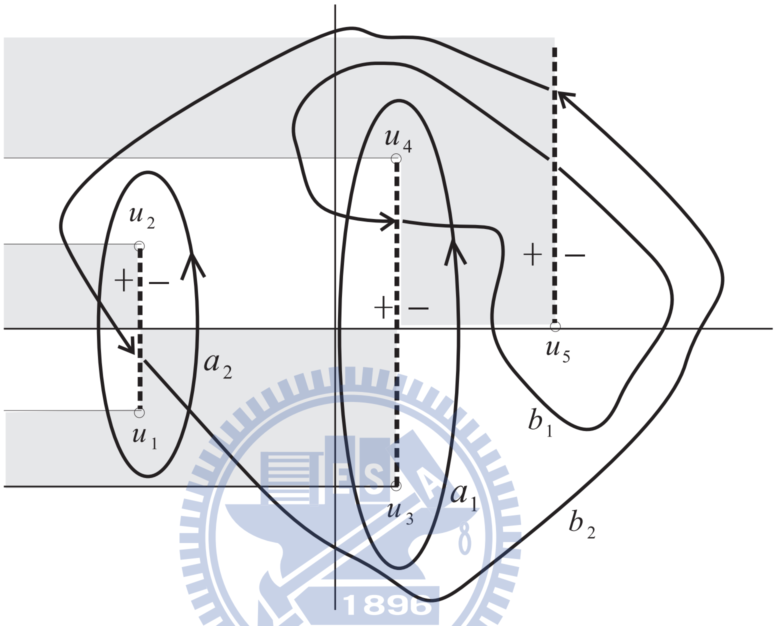


Figure 114

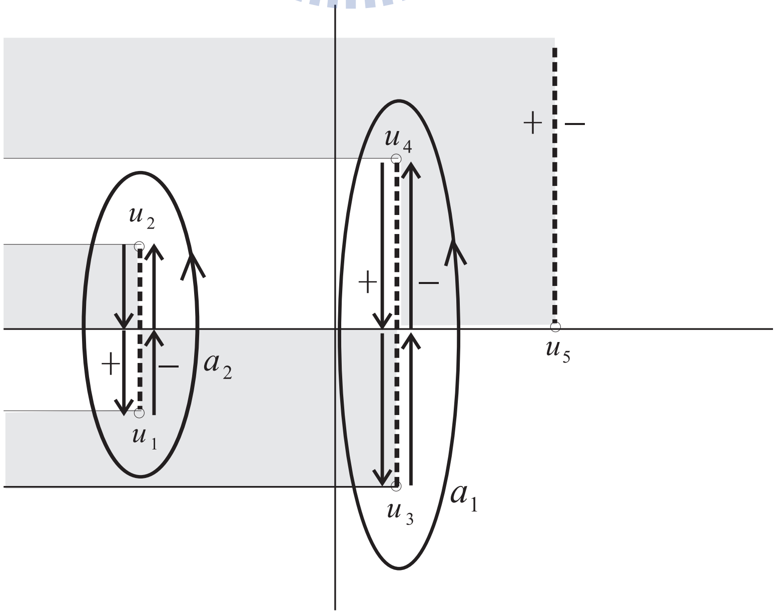


Figure 115

1. Evaluate $\int_{a_1} h(u)\,du$

$$\begin{aligned}\int_{a_1} h(u)\,du &= \left(\int_{u_4 \xrightarrow{+} 1.59} h(u)\,du + \int_{u_4 \xleftarrow{-} 1.59} h(u)\,du\right) + \left(\int_{1.59 \xrightarrow{+} u_3} h(u)\,du + \int_{1.59 \xleftarrow{-} u_3} h(u)\,du\right)\\ &= 2\int_{u_4 \xrightarrow{+} 1.59} h(u)\,du + 2\int_{1.59 \xrightarrow{+} u_3} h(u)\,du\\ &= 2\cdot \text{MATH}\left(\int_{2.24}^{0} h(1.59+ri)\,i\,dr\right) + 2\cdot(-1)\cdot\text{MATH}\left(\int_{0}^{-2.24} h(1.59+ri)\,i\,dr\right)\\ &= -0.587776\,.\end{aligned}$$

2. Evaluate $\int_{a_2} h(u)\,du$

$$\begin{aligned}&\int_{a_2} h(u)\,du\\ &= \left(\int_{u_2 \xrightarrow{+} -3.86} h(u)\,du + \int_{u_2 \xleftarrow{-} -3.86} h(u)\,du\right) + \left(\int_{-3.86 \xrightarrow{+} u_1} h(u)\,du + \int_{-3.86 \xleftarrow{-} u_1} h(u)\,du\right)\\ &= 2\int_{u_2 \xrightarrow{+} -3.86} h(u)\,du + 2\int_{-3.86 \xrightarrow{+} u_1} h(u)\,du\\ &= 2\cdot(-1)\cdot \text{MATH}\left(\int_{1.63}^{0} h(-3.86+ri)\,i\,dr\right) + 2\cdot\text{MATH}\left(\int_{0}^{-1.63} h(-3.86+ri)\,i\,dr\right)\\ &= 0.35043\,.\end{aligned}$$

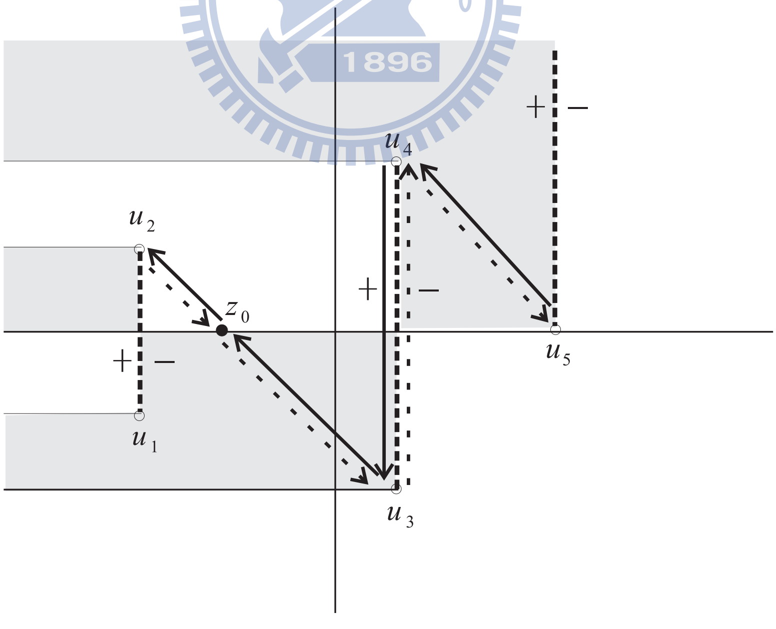


Figure 116

3. Evaluate $\int_{b_1} h(u)\,du$

Along $u_5 \longrightarrow u_4$, let $d = |u_4 - u_5|$.

$$u = 4.54 + re^{i(\pi - \tan^{-1}\frac{2.24}{4.54-1.59})},\quad r: 0 \longrightarrow d$$

$$
\begin{aligned}
\int_{b_1} h(z)\, dz &= \int_{u_5 \longrightarrow u_4} h(u)\, du + \int_{u_5 \dashleftarrow u_4} h(u)\, du \\
&= 2 \int_{u_5 \longrightarrow u_4} h(u)\, du \\
&= 2 \cdot (-1) \cdot \text{MATH} \left(\int_0^d h(4.54 + re^{i(\pi - \tan^{-1} \frac{2.24}{4.54-1.59})})\, e^{i(\pi - \tan^{-1} \frac{2.24}{4.54-1.59})}\, dr \right) \\
&= -0.293888 - 0.309729i\,.
\end{aligned}
$$

4. Evaluate $\int_{b_2} h(u)\, du$

$$
\begin{aligned}
\int_{b_2} h(u)\, du &= \int_{b_1} h(u)\, du + \left(\int_{u_4 \longrightarrow u_3} h(u)\, du + \int_{u_4 \dashleftarrow u_3} h(u)\, du \right) \\
&\quad + \left(\int_{u_3 \longrightarrow u_2} h(u)\, du + \int_{u_3 \dashleftarrow u_2} h(u)\, du \right) \\
&= \int_{b_1} h(u)\, du + 0 + \left(\int_{u_3 \longrightarrow u_2} h(u)\, du + \int_{u_3 \dashleftarrow u_2} h(u)\, du \right) \\
&= \int_{b_1} h(u)\, du + 2 \int_{u_3 \longrightarrow u_2} h(u)\, du \\
&= \int_{b_1} h(u)\, du + 2 \int_{u_3 \longrightarrow z_0} h(u)\, du + 2 \int_{z_0 \longrightarrow u_2} h(u)\, du
\end{aligned}
$$

Along $u_3 \longrightarrow z_0$, let $d = |u_3 - z_0|$.

$$
u = 1.59 - 2.24i + re^{i(\pi - \tan^{-1} \frac{1.63-(-1.63)}{1.59-(-3.86)})}, \quad r : 0 \longrightarrow \frac{1.63}{1.63 + 2.24} d
$$

$$
\begin{aligned}
&\int_{u_3 \longrightarrow z_0} h(u)\, du \\
&= 2 \cdot (-1) \cdot \text{MATH} \left(\int_0^{\frac{1.63}{1.63+2.24} d} h(1.59 - 2.24i + re^{i(\pi - \tan^{-1} \frac{1.63-(-1.63)}{1.59-(-3.86)})})\, e^{i(\pi - \tan^{-1} \frac{1.63-(-1.63)}{1.59-(-3.86)})}\, dr \right) \\
&= -0.106926 - 0.209149i\,.
\end{aligned}
$$

$$
\begin{aligned}
&\int_{z_0 \longrightarrow u_2} h(u)\, du \\
&= 2 \cdot \text{MATH} \left(\int_{\frac{1.63}{1.63+2.24} d}^{d} h(1.59 - 2.24i + re^{i(\pi - \tan^{-1} \frac{1.63-(-1.63)}{1.59-(-3.86)})})\, e^{i(\pi - \tan^{-1} \frac{1.63-(-1.63)}{1.59-(-3.86)})}\, dr \right) \\
&= -0.0424914 + 0.0339204i\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{b_2} h(u)\, du &= \int_{b_1} h(u)\, du + 2 \int_{u_3 \longrightarrow z_0} h(u)\, du + 2 \int_{z_0 \longrightarrow u_2} h(u)\, du \\
&= (-0.293888 - 0.309729i) + (-0.106926 - 0.209149i) + (-0.0424914 + 0.0339204i) \\
&= -0.149418 - 0.332545i\,.
\end{aligned}
$$

Example 28. *Given* $E = 10$ *and given six points,* $u_1 = 2i, u_2 = i, u_3 = 1, u_4 = 2, u_5 = 2 - i, u_6 = 4 + i$. *Then*

$$\begin{aligned}
&\prod_{j=1}^{6}(u - u_j) \\
&= [u - i][u - 2i][u - 1][u - 2][u - (2 - i)][u - (4 + i)] \\
&= (-36 + 8i) + (66 - 66i)u - (22 - 117i)u^2 - (27 + 81i)u^3 + (27 + 25i)u^4 - (9 + 3i)u^5 + u^6 \\
&= 2[10 - F(u)].
\end{aligned}$$

Let

$$\begin{aligned}
h(u) &= \frac{1}{\sqrt{2[10 - F(u)]}} \\
&= \frac{1}{\sqrt{[u - i][u - 2i][u - 1][u - 2][u - (2 - i)][u - (4 + i)]}}.
\end{aligned}$$

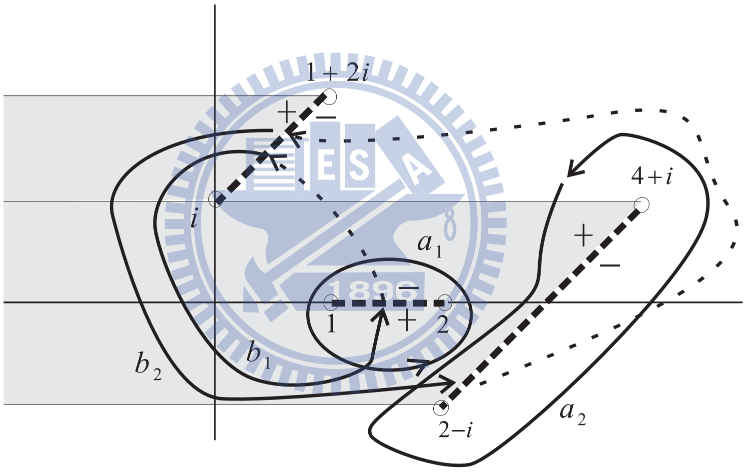


Figure 117

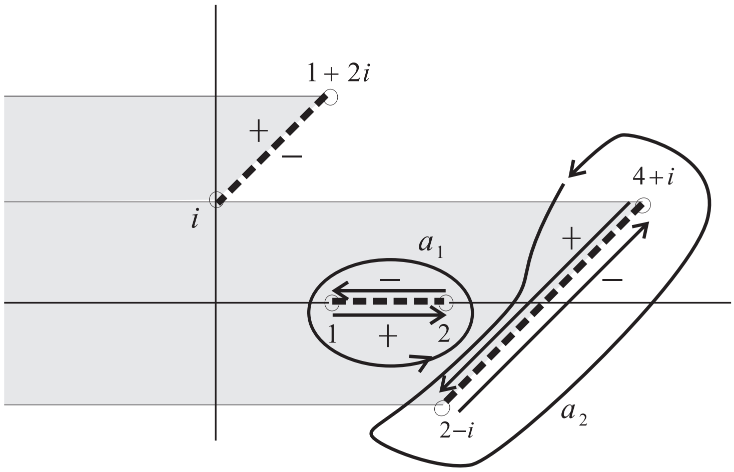


Figure 118

1. Evaluate $\int_{a_1} h(u)\, du$

$$\begin{aligned}\int_{a_1} h(u)\, du &= \int_{1 \xrightarrow{+} 2} h(u)\, du + \int_{1 \xleftarrow{-} 2} h(u)\, du \\ &= 2 \int_{1 \xrightarrow{+} 2} h(u)\, du \\ &= 2 \cdot \text{MATH} \left(\int_1^2 h(x)\, dx \right) \\ &= 1.80392 - 0.410359i\,.\end{aligned}$$

2. Evaluate $\int_{a_2} h(u)\, du$

$$\begin{aligned}\int_{a_2} h(u)\, du &= \int_{4+i \xrightarrow{+} 2-i} h(u)\, du + \int_{4+i \xleftarrow{-} 2-i} h(u)\, du \\ &= 2 \int_{4+i \xrightarrow{+} 2-i} h(u)\, du \\ &= 2 \cdot (-1) \cdot \text{MATH} \left(\int_{2\sqrt{2}}^{0} h(2 - i + re^{i(\frac{-7\pi}{4})})\, e^{i(\frac{-7\pi}{4})}\, dr \right) \\ &= -0.849439 + 0410231i\,.\end{aligned}$$

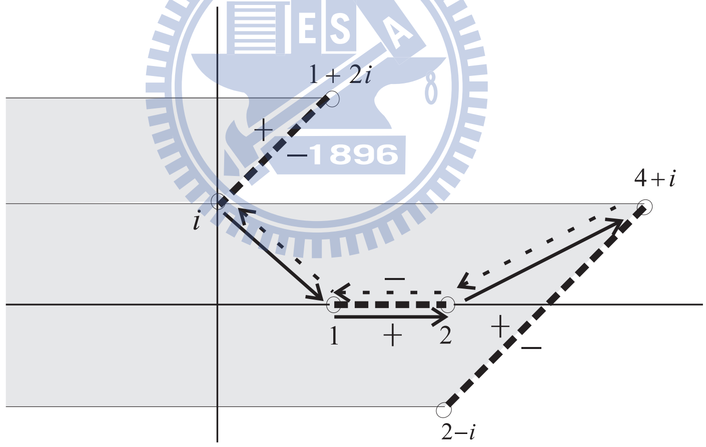


Figure 119

3. Evaluate $\int_{b_1} h(u)\, du$

$$\begin{aligned}\int_{b_1} h(z)\, dz &= \int_{i \longrightarrow 1} h(u)\, du + \int_{i \dashleftarrow 1} h(u)\, du \\ &= 2 \int_{i \longrightarrow 1} h(u)\, du \\ &= 2 \cdot (-1) \cdot \text{MATH} \left(\int_0^{\sqrt{2}} h(i + re^{i(\frac{-\pi}{4})})\, e^{i(\frac{-\pi}{4})}\, dr \right) \\ &= 0.314384 + 1.42251i\,.\end{aligned}$$

4. Evaluate $\int_{b_2} h(u)\,du$

$$\begin{aligned}
\int_{b_2} h(u)\,du &= \int_{b_1} h(u)\,du + \left(\int_{1\xrightarrow{+}2} h(u)\,du + \int_{1\overset{-}{\dashleftarrow}2} h(u)\,du\right) \\
&\quad + \left(\int_{2\longrightarrow 4+i} h(u)\,du + \int_{2\dashleftarrow 4+i} h(u)\,du\right) \\
&= \int_{b_1} h(u)\,du + 0 + \left(\int_{2\longrightarrow 4+i} h(u)\,du + \int_{2\dashleftarrow 4+i} h(u)\,du\right) \\
&= \int_{b_1} h(u)\,du + 2\int_{2\longrightarrow 4+i} h(u)\,du \\
&= (0.314384 + 1.42251i) + 2\cdot(-1)\textsc{math}\left(\int_0^{\sqrt{5}} h(2 + re^{i\tan^{-1}\frac{1}{2}})\, e^{i\tan^{-1}\frac{1}{2}}\,dr\right) \\
&= 0.133229 + 0.00918344i\,.
\end{aligned}$$

7 Conclusion

In order to solve equations of the form $u'' + f(u) = 0$, we need to evaluate the integrals of the form

$$\int \frac{1}{\sqrt{\prod_{j=1}^{n}(u-u_j)}}\, du,$$

or

$$\int \sqrt{\prod_{j=1}^{n}(u-u_j)}\, du,$$

where the $u_k's$ play the roles of branch points. We build the Riemann surfaces of genus N for

$$g(z) = \sqrt{\prod_{j=1}^{2N+1}(z-z_j)}\, dz,$$

or

$$g(z) = \sqrt{\prod_{j=1}^{2N+2}(z-z_j)}\, dz.$$

Then we evaluate integrals along $a-cycles$ and $b-cycles$. We compute the values of those integrals using the software "*Mathematica*". If we evaluate integrals using Mathematica, the signs of values computed by Mathematica is different from the signs of values computed theoretically.

Suppose that sheet I and sheet II of cut plane for $f(z) = \sqrt{\prod_{j=1}^{2N+1}(z-z_j)}\, dz$ are

$$P_I = \{z \in \mathbb{C} | \alpha - 2\pi \leq \arg z < \alpha\}, \text{ and}$$
$$P_{II} = \{z \in \mathbb{C} | \alpha \leq \arg z < \alpha + 2\pi\}.$$

Let $I_c = [\alpha - 2\pi, -\pi]$ and let $z \in P_I$. In Mathematica,

$$\arg(z-z_j) \in I_c \Longrightarrow \sqrt{z-z_j} = (-1) \cdot \text{MATH}\left(\sqrt{z-z_j}\right)$$
$$\arg(z-z_j) \notin I_c \Longrightarrow \sqrt{z-z_j} = \text{MATH}\left(\sqrt{z-z_j}\right)$$

If we want to evaluate integrals in sheet II, we use the property $f(z)|_{II} = f(z)|_I$ to obtain the correct values. When the cut plane is more complicated, we can use sign-regions to help us to determine the signs of values computed by Mathematica.

The Riemann surfaces discussed in this thesis is two-sheeted. Of course, it is able to discuss the algebraic structure of the corresponding cut plane for functions of the form $f(z) = \sqrt[3]{\prod_{j=1}^{2N+1}(z-z_j)}\, dz$.

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[5] Jie-Ru Wu, "The Path-Integral Computations on Two-Sheeted Riemann Surfaces of Genus N", NCTU, 2003.