# 國立交通大學

## 應用數學系

## 碩 士 論 文

# 在線性累積前景理論下最佳投資策略的選擇

# Optimal Portfolio Selection under Linear Cumulative Prospect Theory


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## 摘　　　要

本論文我們關心的是『如何投資在股票市場將使我們獲利最大』，此問題針對於某些投資者在面對不確定的決策行為符合 Linear Cumulative Prospect Theory (線性累積前景理論, LCPT)。LCPT 為 Cumulative Prospect Theory 的一特例。本論文採用連續型 Black-Scholes 金融市場模型含有一股票和一銀行帳戶。我們推導出其最大獲利的總資產是由投資者的 probability weighting function (決策權數函數) 和 discounted Radon-Nikodym derivative 共同決定。在本論文的最後，我們給一例子算出其最大獲利，而且觀察當我們改變其參數時其最大獲利的變化。

# Optimal Portfolio Selection under Linear Cumulative Prospect Theory

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ABSTRACT

In this thesis we are concerned with the optimal portfolio selection for an investor who makes decision according to the Linear Cumulative Prospect Theory (LCPT). LCPT is a special case of Cumulative Prospect Theory. We investigate the case of a continuous-time economy model with one risk-free asset and one risky asset. The maximum value of terminal wealth is a supremum relative to the probability weighting function and the discounted Radon-Nikodym derivative. We derive some numerical results and illustrate how these parameters affects the maximum value.

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CHAPTER 1

# Introduction

The object of this thesis is to examine a very natural question: How can an investor optimize the portfolio investment in a continuous-time economy model with one risk-free asset and one risky asset under Linear Cumulative Prospect Theory (see, e.g., Schmidt and Zank [**18**]). Linear Cumulative Prospect Theory is a special case of Cumulative Prospect Theory(CPT). In this setting the utility function is linear. This question has already been extensively studied under Expected Utility Theory, see, e.g., Merton [**13**].

Expected Utility Theory (EUT), developed by von Neumann and Morgenstern [**22**] based on an axiomatic system, has an underlying assumption that decision makers are rational and risk averse when they face uncertainties. However, empirical research has shown that EUT fails to provide a good description of individual behavior under risk and uncertainty. Examples are the famous paradoxes of Allais [**1**] and Ellsberg [**7**]. This evidence has motivated the development of alternative theories, which are compatible with observed choice behavior. The following anomalies for daily life in EUT:

- People evaluate assets on gains and losses relative to a reference point, not on final wealth positions;
- People are not uniformly risk averse: they are risk averse on gains and risk taking on loses, and more sensitive to losses than to gains;
- People overweight small probabilities and underweight large probabilities.

In 1970s, the Prospect Theory (PT) is proposed by Kahneman and Tversky [**20**] for decision making under uncertainty as a psychologically realistic alternative to EUT. Starting from empirical evidence, the theory describes people decide which outcomes they see as basically identical and they set a reference point and consider lower outcomes as losses and larger as gains. And people behave as if they would compute their payoff utility, based on the potential outcomes and their respective probabilities. In contrast to EUT, it measures losses and gains, but not absolute wealth. Though prospect theory explained the major violations of EUT in decision making under risk, there exist a problems.

- Prospect Theory violated the first-order stochastic dominance.

Cumulative Prospect Theory (CPT) is introduced by Kahneman and Tversky in 1992 [**21**]. This theory is a further development and variance of PT. The difference from the original version of PT is that weighting is applied to the cumulative probability distribution function, as in rank-dependent expected utility theory, rather than to the probabilities of individual outcomes. In 2002, Daniel Kahneman was awarded the Nobel Memorial Prize in Economics for his work in Prospect theory.

The central model of this paper is Linear Cumulative Prospect Theory (LCPT) which is a special case of CPT. The main difference between CPT and LCPT is the utility function in LCPT is linear. Linear utility has a long tradition in theoretical and empirical research, in part due to its *tractability*. An axiomatic foundation of subjective expected utility with linear utility was provided by de Finetti [**8**]. Preston and Baratta [**15**] who used a linear utility model in order to estimate probability distortions. Edwards [**6**] collected amount of data from a series of experiments which support our model. He found evidence for sign-dependent probability distortions and also for linear utility. Many other studies observed linear utility for losses, in particular for small stakes. Handa [**9**] axiomatized a model of subjective expected value, which was implicitly used by Preston and Baratta and already discussed in Edwards. A model for decision under risk that combines linear utility and distorted probabilities is the dual theory of Yaari [**24**].

Some research has already been done on optimal investment under CPT. Most of the previous work takes place when no probability distortion exists or the form of the utility function cannot be linear. The optimal portfolio choice problem for a loss-averse investor is studied by Berkelaar, Kouweenberg and Post [**2**] in a complete market but where no distortion is applied to the probabilities. This problem has been recently solved by Jin and Zhou [**11**] in a continuous-time setting, within the complete market framework of Black and Scholes. Their result is thus only valid for non-linear utility function.

The rest of this thesis is organized as follows: In Chapter 2 we examine the main components of Linear Cumulative Prospect Theory and compares LCPT with CPT; In Chapter 3 we introduce the model and the portfolio selection problem; In Chapter 4 we derive the main method to solve the portfolio selection problem, and then find the maximum value and the optimal terminal wealth; In Chapter 5 a numerical example is considered and we illustrate how these parameters affects the maximum value.

CHAPTER 2

# The Comparison of LCPT and CPT

## 2.1. The form

Denote states space by $\Omega$ and subsets of $\Omega$ is denoted by $A, B, \ldots$. The state space $\Omega$ is endowed with a *$\sigma$-algebra* $\mathcal{F}$ of subsets of $\Omega$. Subsets of $\Omega$ which are contained in $\mathcal{F}$ are called *events*. A *partition* $\{A_1, A_2, \ldots, A_n\}$ of $\Omega$ is a collection of disjoint events and the union of which equals $\Omega$. The set of *outcomes* is $\mathbb{R}$ which indicates money. the element of the outcome is denoted by $x, y, z, \ldots$. People tend to think of possible outcomes usually relative to a *reference point* (also called the *status quo*) rather than to the final status. Outcomes above the reference point are called *gains* and outcomes below the reference point are called *losses*. Without loss of generality, we assume that reference point is given by zero. Therefore, we refer to positive outcomes as *gains* and to negative outcomes as *losses*.

Consider a *prospect* (lottery, random variable) $f : \Omega \rightarrow \mathbb{R}$ which assigns to each state an outcome. The set of all prospects is denoted by $\mathcal{L}$. we assume that prospects are bounded (i.e., for any prospect $f$ there exists $c \in \mathbb{R}$ such that $|f(\omega)| \leq c$ for all states $\omega \in \Omega$) and $\mathcal{F}$-measurable (i.e., the inverse image of each interval of $\mathbb{R}$ is an event).

It is assumed that a decision maker has a preference relation over lotteries denoted by $\succeq$. As usual, $\succ$ denotes strict preference, $\sim$ denotes indifference. Sometimes we write $f \preceq g$ ($f \prec g$) instead of $g \succeq f$ ($g \succ f$). A functional $V : \mathcal{L} \rightarrow \mathbb{R}$ is a numerical representation that the preference relation $\succeq$, if for all $f, g \in \mathcal{L}$,

$$f \succeq g \qquad \Longleftrightarrow \qquad V(f) \geq V(g).$$

Before starting the Cumulative Prospect Theory, we introduce two important terminologies: utility function and probability weighting function.

**Definition** 2.1. (1) The *utility function* $u(\cdot)$ is defined as:

$$u(x) = \begin{cases} u^+(x) & \text{if } x \geq 0, \\ -u^-(-x) & \text{if } x \leq 0, \end{cases}$$

where $u^+ : \mathbb{R}^+ \to \mathbb{R}^+$ and $u^- : \mathbb{R}^+ \to \mathbb{R}^+$ are strictly increasing, concave with $u(0) = u^+(0) = u^-(0) = 0$.

(2) The *probability weighting functions* $w^+ : [0,1] \to [0,1]$ and $w^- : [0,1] \to [0,1]$ are differentiable and strictly increasing with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$.

**Cumulative Prospect Theory** (CPT) holds if the preference relation can be represented by the functional:

$$V^{CPT}(X) = \int_0^\infty w^+(\mathbb{P}\{u(X) > y\})\, dy - \int_{-\infty}^0 w^-(\mathbb{P}\{u(X) < y\})\, dy, \qquad (2.1)$$

where $u : \mathbb{R} \to \mathbb{R}$ is the utility function and $w^+, w^- : [0,1] \to [0,1]$ are two probability weighting functions defined as in **Definition** 2.1.

**Remark** 2.2. A *capacity* $\nu$ is a non-additive measure satisfying $\nu(\Omega) = 1$, $\nu(\phi) = 0$ and $\nu(A) \geq \nu(B)$ if $A \supseteq B$, e.g., the functions $\nu^+ = w^+ \circ \mathbb{P}$, $\nu^- = w^- \circ \mathbb{P}$ are two *capacities*. The integrals of equation (2.1) are called *Choquet integrals* with respect to $w^+ \circ \mathbb{P}$ and $w^- \circ \mathbb{P}$. (see, e.g., Choquet [**5**])

As constructed by Tversky and Kahneman (1992), CPT treats gains and losses separately.

**Linear Cumulative Prospect Theory** (LCPT) is a special case of CPT, the functions $u^+$ and $u^-$ are linear. More precisely, the utility function is of the form

$$u(x) = \begin{cases} x & \text{if } x \geq 0, \\ \lambda x & \text{if } x \leq 0, \end{cases}$$

with the loss aversion parameter $\lambda \geq 1$. In other words, LCPT holds if the preference relation can be represented by the functional:

$$V^{LCPT}(X) = \int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy - \int_{-\infty}^0 w^-(\mathbb{P}\{\lambda X < y\})\,dy, \tag{2.2}$$

where $w^+$, $w^-$ are two probability weighting functions. Denoted that

$$V_+(X) := \int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy, \tag{2.3}$$

$$V_-(X) := \int_{-\infty}^0 w^-(\mathbb{P}\{\lambda X < y\})\,dy. \tag{2.4}$$

## 2.2. The axiomatization

An important subset of prospects is the set of *rank-ordered simple prospects*. Rank-ordered simple prospects take only finitely many outcomes and arrange the outcomes in increasing order, such as

$$f = x_1 I_{A_1} + x_2 I_{A_2} + \cdots + x_n I_{A_n}, \qquad x_1 \leq x_2 \leq \cdots \leq x_n,$$

where $\{A_1, A_2, ..., A_n\}$ is a partition of $\Omega$ and $I_{A_i}$ is the indicator function of event $A_i$. It is understood that the rank-ordered simple prospect $f$ assigns outcome $x_i$ for states $\omega \in A_i$, $i = 1, 2, ..., n$.

**Notation** 2.3. We use the notation $x_A g$ for a prospect that giving outcome $x$ on event $A$ and with prospect $g$ on the complement $A^c$.

Now, we introduce several definitions of the preference relation $\succeq$:

**A1.** ***Weak order***: $\succeq$ is a weak order if $\succeq$ is complete ($f \succeq g$ or $g \succeq f$ for any two prospects $f, g$) and transitive ($f \succeq g$ and $g \succeq h$ implies $f \succeq h$).

**A2.** ***Continuity***: $\succeq$ is continuous if for any prospect $f$ the sets $\{g \in \mathcal{L} : g \succeq f\}$ and $\{g \in \mathcal{L} : g \preceq f\}$ are closed subsets under the supnorm $||f - g||_\infty = \sup_{\omega \in \Omega} |f(\omega) - g(\omega)|$.

**A3.** ***Stochastic dominance***: We say that $(y_1 I_{A_1} + \cdots + y_n I_{A_n})$ is stochastically dominant by $(x_1 I_{A_1} + \cdots + x_n I_{A_n})$ if

$$(x_1 I_{A_1} + \cdots + x_n I_{A_n}) \succ (y_1 I_{A_1} + \cdots + y_n I_{A_n})$$

whenever $x_i \geq y_i$ for all $i$ and $x_i > y_i$ for at least one $i$ with $\mathbb{P}(A_i) > 0$.

**Definition** 2.4. The preference relation $\succeq$ satisfies *sign-comonotonic tradeoff consistency* if there is *no* outcome $x, x', y, y'$ such that both of the following two statements hold at the same time.

(1) There exist rank-ordered simple prospects $f, g$ which can be represented by the same partition $\{A_1, ..., A_n\}$, and a event $A_i$ such that

$$x_{A_i} f \succeq y_{A_i} g \quad \text{and} \quad x'_{A_i} f \preceq y'_{A_i} g.$$

(2) There exist rank-ordered simple prospects $h$, $k$ which can be represented using the same partition $\{B_1, ..., B_m\}$, and a event $B_i$ such that

$$x_{B_i}h \preceq y_{B_i}k \quad \text{and} \quad x'_{B_i}h \succ y'_{B_i}k.$$

whenever $x$, $y$, $x'$, $y'$ are of the *same sign* (i.e. either they are all gains or they are all losses) and all involved prospects are *comonotonic* (i.e. the rank-order of outcomes should remain the same).

**Proposition** 2.5 (Wakker and Tversky [**23**], Theorem 6.3). *Suppose that $\succeq$ is the preference relation on the set of prospects. The following two statements are equivalent:*

(1) *Cumulative Prospect Theory (CPT) holds with a continuous utility function;*
(2) *The preference relation $\succeq$ satisfies the following conditions:*
 (a) *weak ordering;*
 (b) *continuity;*
 (c) *stochastic dominance;*
 (d) *sign-comonotonic tradeoff consistency.*

*Further, both capacities are uniquely determined, and the utility function is a ratio scale.*

**Definition** 2.6. The preference relation $\succeq$ satisfies *independence of common increments* if for any two rank-ordered simple prospects $f, g$ with the same partition $\{A_1, ..., A_n\}$, we have:

$$x_{A_i}f \succeq y_{A_i}g \qquad \Rightarrow \qquad (x+a)_{A_i}f \succeq (y+a)_{A_i}g,$$

for any outcome $a \in \mathbb{R}$ whenever $x$, $y$, $x+a$, $y+a$ are of the same sign and all involved prospects are comonotonic.

Independence of common increments means that a common absolute change of an outcome of the same rank does not reverse the preference between two prospects as long as this change is not too large to affect the rank or the sign of the outcomes.

**Proposition** 2.7 (Schmidt and Zank [**19**], Theorem 4). *Suppose that $\succeq$ is the preference relation on the set of prospects. The following two statements are equivalent:*

(1) *Linear Cumulative Prospect Theory (LCPT) holds;*

(2) *The preference relation $\succeq$ satisfies the following conditions:*

 (a) *weak ordering;*

 (b) *continuity;*

 (c) *stochastic dominance;*

 (d) *independence of common increments;*

*Further, both capacities are uniquely determined, and the linear utility function is a ratio scale.*

CHAPTER 3

# The Model

## 3.1. The Black-Scholes model of the market

Let $T$ be a positive constant, called the *terminal time*. Consider $(\Omega, \mathcal{F}, \mathbb{P})$, a complete probability space and a standard Brownian motion $(W_t)_{0\le t\le T}$ with $W_0 = 0$. Define

$$\mathcal{F}_t^W := \sigma\big\{\, W_s : 0 \le s \le t \,\big\}, \quad t \in [\,0, T\,] \tag{3.1}$$

to be the natural filtration generated by $(W_t)$ and let $\mathcal{N}$ denote the collection of all $\mathbb{P}$-null subsets of $\mathcal{F}$. We shall use the augment filtration

$$\mathcal{F}_t := \sigma\big\{\, \mathcal{F}_t^W \cup \mathcal{N} \,\big\}, \quad t \in [\,0, T\,]. \tag{3.2}$$

Suppose that there is a market in which two assets are traded continuously. One is the bond with price process $B_t$ which is subject to the following (stochastic) ordinary differential equation:

$$dB_t = rB_t\, dt, \quad t \in [\,0, T\,]; \quad B_0 = 1, \tag{3.3}$$

where $r > 0$ is the constant annualized risk-free *interest rate*, continuously compounded. The other is a stock with price process $S_t$ satisfying the following stochastic differential equation (SDE):

$$dS_t = S_t\big[\alpha\, dt + \beta\, dW_t\big], \quad t \in [\,0, T\,]; \quad S_0 = s_0 > 0, \tag{3.4}$$

where $\alpha$ is the constant *drift rate* and $\beta$ is the constant *volatility*.

Consider an investor with initial endowment $x_0 \geq 0$. Assume that the trading takes place continuously in *self-financing* fashion, i.e. there is no consumption or income, and no transaction costs. If the investor invested the amount of money $\pi := (\pi_t)_{0\leq t\leq T}$ in the stock, then the corresponding wealth process $(X_t)_{0\leq t\leq T}$ depends on $x_0$ and $\pi$ is governed by the following equation (see, e.g., Karatzas and Shreve [**12**])

$$dX_t = \Big[rX_t + \pi_t\,(\alpha - r)\Big]\,dt + \pi_t\,\beta\,dW_t, \quad t \in [\,0, T\,]; \quad X_o = x_0. \tag{3.5}$$

Note that $\pi_t$ is the amount invested in stock at time $t$, not the number of shares held.

## 3.2. Problem 1

Before we formulate our portfolio selection problem, we specify the "allowable" investment policies with the following definition.

**Definition** 3.1. A portfolio process $\pi$ is said to be *admissible* if $\pi_t \in \mathcal{F}_t$ for all $0 \leq t \leq T$ and satisfies

$$E\left[\int_0^T {\pi_t}^2\,dt\right] < \infty. \tag{3.6}$$

Our portfolio selection problem is to find the optimal admissible portfolio $\pi^*$ in terms of maximizing the value of the terminal wealth $X_T$ under LCPT framework. The corresponding model can be formulated as follows:

$$\begin{aligned} &\text{Maximize} \quad V^{LCPT}(X_T) \\ &\text{subject to} \quad \begin{cases} (X_t,\, \pi_t) \text{ satisfies equation (3.5)}, \\ \pi \text{ is admissible.} \end{cases} \end{aligned} \qquad \textbf{(Problem 1)}$$

## 3.3. Problem 2

Observe that the wealth equation (3.5) admits an unique strong solution $X_t$ for any given portfolio $\pi_t$ at time $t$ by standard SDE theory. However, the wealth process $(X_t)_{0\le t\le T}$ in equation (3.5) might not be nonnegative process, i.e., the wealth process can take *negative* values. This is sometimes unacceptable for practical situation, since the most investors cannot buy assets when their wealth is negative. Therefore, an important restriction that we impose throughout this thesis is **the prohibition of bankruptcy** of the investor. That is, we limit our consideration to portfolio $\pi$ for which the corresponding wealth processes $(X_t)_{0\le t\le T}$ are such that $X_t \ge 0$, a.s., for all $t \in [\,0, T\,]$. Such bankruptcy-averting policy of investment does exist for it at least allows us to deposit all the money in the bank account.

Our first result makes the simplifying observation that the wealth process for all $0 \le t \le T$, $X_t$ is nonnegative if and only if the terminal wealth $X_T$ is nonnegative.

**Proposition** 3.2 (Bielecki, Jin, Pliska and Zhou [**3**], Proposition 2.1). *Let $(X_t)_{0\le t\le T}$ be a wealth process with respect to an admissible portfolio $\pi$ and let $T$ be the terminal time. Then*

$$X_T \ge 0,\ a.s. \qquad \Longleftrightarrow \qquad X_t \ge 0,\ a.s.,\ \forall\, t \in [0,T]. \tag{3.7}$$

The importance of **Proposition** 3.2 is that it enables us to replace the constraint $X_t \ge 0$, for all $t \in [\,0, T\,]$, by the terminal constraint $X_T \ge 0$.

**Assumption** 3.3. The terminal wealth $X_T \ge 0$, a.s.

Therefore, **Assumption** 3.3 greatly simplifies our problem, which is formulated as follows:

$$\text{Maximize} \quad V_+(X_T) = \int_0^\infty w^+(\mathbb{P}\{X_T > y\})\, dy$$
$$\text{subject to} \quad \begin{cases} (X_t, \pi_t) \text{ satisies equation (3.5)}, \\ \pi \text{ is admissible}, \\ X_T \geq 0, \quad a.s. \end{cases} \qquad \textbf{(Problem 2)}$$

### 3.4. Problem 3

Define

$$\theta := \frac{\alpha - r}{\beta} \tag{3.8}$$

as the usual *market price of risk*. Applying Girsanov's Theorem, consider a risk-neutral probability measure $\mathbb{Q}$ defined by

$$\mathbb{Q}(A) = \int_A Z_t\, d\mathbb{P} \quad \text{for all } A \in \mathcal{F}_t, \tag{3.9}$$

where

$$Z_t := \exp\Big\{ -\frac{1}{2}\theta^2 t - \theta W_t \Big\} \tag{3.10}$$

is the *Radon-Nikodym derivative*. Note that $E[Z_T] = 1$. In particular, under this risk-neutral probability measure $\mathbb{Q}$, the discounted portfolio value process $(e^{-rt}X_t)_{0\leq t\leq T}$ is a *martingale*. This implies that

$$X_t = e^{rt} E_{\mathbb{Q}}\Big[ e^{-rT} X_T \Big| \mathcal{F}_t \Big] = \frac{e^{-r(T-t)}}{Z_t} E\Big[ Z_T X_T \Big| \mathcal{F}_t \Big], \quad a.s., \quad t \in [0, T], \tag{3.11}$$

where $E_{\mathbb{Q}}$ means the expectation with respect to the probability $\mathbb{Q}$. Equation (3.11) tells us that the process $(X_t)_{0\leq t\leq T}$ also is uniquely determined when the random variable $X_T$ is given. This leads to the following proposition.

**Proposition** 3.4 (Karatzas and Shreve, [**12**], Definition 6.1 and Theorem 6.6). *Let $X$ be an $\mathcal{F}_T$-measurable random variable such that $X \geq 0$, a.s., and*

$$E_{\mathbb{Q}}\Big[e^{-rT}X\Big] = E\Big[e^{-rT}Z_T X\Big] = x_o, \tag{3.12}$$

*then there exists an admissible portfolio $\pi$ such that the corresponding wealth process $(X_t)_{0\leq t\leq T}$ satisfies $X_T = X$, a.s., and $X_0 = x_0$.*

**Notation** 3.5. Denoted the discounted Radon-Nikodym derivative by

$$\rho_t := e^{-rt}Z_t = \exp\Big\{ - \big(r + \frac{1}{2}\,\theta^2\big)\,t - \theta\,W_t\Big\} \tag{3.13}$$

and

$$\rho := \rho_T = \textit{state price density random variable}. \tag{3.14}$$

Due to (3.13) we see that $\rho$ is a *log-normal* random variable. Let $N(\cdot)$ and $\psi(\cdot)$ be the distribution function and probability density function of a *standard normal random variable*, respectively. Therefore, the distribution function $F_\rho$ and probability density function $f_\rho$ of $\rho$ are given by

$$F_\rho(x) := N\Big(\frac{\ln x - \mu}{\sigma}\Big) \;\text{ and }\; f_\rho(x) := \frac{1}{x\sigma}\,\psi\Big(\frac{\ln x - \mu}{\sigma}\Big) \tag{3.15}$$

where $\mu$ and $\sigma$ are the mean and standard deviation of the random variable $\ln\rho$. Precisely,

$$\mu = -(\,r + \frac{1}{2}\,\theta^2\,)T \qquad \text{and} \qquad \sigma^2 = \theta^2\,T \tag{3.16}$$

In view of **Proposition** 3.4 and using equation (3.14), in order to solve **Problem 2** we only need first to solve the following maximization problem in the terminal wealth, $X$:

$$
\begin{aligned}
&\text{Maximize} \quad V_+(X) = \int_0^\infty w^+(\mathbb{P}\{X > y\})\, dy \\
&\text{subject to} \quad \begin{cases} E[\,\rho X\,] = x_0, \\ X \text{ is } \mathcal{F}_T\text{-measurable}, \\ X \geq 0, \quad a.s. \end{cases}
\end{aligned}
\qquad \textbf{(Problem 3)}
$$

**Remark** 3.6. Once **Problem 3** is solved with an optimal solution $X^*$, the optimal wealth process $(X_t^*)_{0\leq t\leq T}$ can be got by equation (3.11). Therefore, in the rest of the thesis we will focus on **Problem 3**.

CHAPTER 4

# Main Results

## 4.1. Problem 4

Now we consider the general maximization problem :

$$
\begin{aligned}
&\text{Maximize} \quad V_+(X) = \int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy \\
&\text{subject to} \quad \begin{cases} E[\,\rho X\,] = x_0, \\ X \geq 0, \quad a.s. \end{cases}
\end{aligned}
\qquad \textbf{(Problem 4)}
$$

The objective of **Problem 4** is to find the optimal random variable $X^*$. Here we turn finding the optimal random variable $X^*$ into seeking the distribution function of $X^*$ by the following steps.

**Lemma** 4.1 (Ross [**17**], Chapter 5 theoretical exercises 28.). *Let $X$ be a continuous random variable having the distribution function $F(\cdot)$. Then the random variable $1-F(X)$ follows uniform distribution over the interval $(0,1)$, that is,*

$$1 - F(X) \sim U(0,1). \tag{4.1}$$

Recall that for the distribution function $F(\cdot)$ of $X$ defined by $F(x) = \mathbb{P}\{X \leq x\}$ and $F(\cdot)$ satisfies

1. $F(\cdot)$ is nondecreasing and right continuous.
2. $\lim_{b\to\infty} F(b) = 1$ and $\lim_{b\to-\infty} F(b) = 0$.

If the distribution function $F(\cdot)$ is strictly increasing and continuous, then $F^{-1} : [0,1] \to \mathbb{R}$ exists. Unfortunately, the distribution function does not have an inverse in general. Therefore, we defined the following general inverse function of the distribution function.

**Definition** 4.2. Let $F(\cdot)$ be the distribution function of $X$. We defined the *inverse function* of $F(\cdot)$, for $y \in [0,1]$,

$$F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\} \qquad \text{with } \inf \phi = \infty. \tag{4.2}$$

Some properties of the inverse of the distribution function are :

1. $F^{-1}(\cdot)$ is nondecreasing and left continuous.
2. $F^{-1}(F(x)) \leq x$.
3. $F(F^{-1}(y)) \geq y$.
4. $F^{-1}(y) \leq x$ if and only if $y \leq F(x)$.
5. If $Y \sim U(0,1)$ then $F^{-1}(Y)$ has the same distribution as $X$.

The property 5 tells us that the inverse of the distribution function can translate results the uniform distribution to the other distributions.

**Proposition** 4.3 (Jin and Zhou [**11**], Lemma C.1). *If $X^*$ is the optimal solution for **Problem 4** and $G^*(\cdot)$ is the distribution function of $X^*$, then $(G^*)^{-1}\big(1 - F_\rho(\rho)\big)$ has the same distribution as $X^*$ and*

$$X^* = (G^*)^{-1}\big(1 - F_\rho(\rho)\big), \tag{4.3}$$

*where $(G^*)^{-1}(\cdot)$ is the inverse function of $G^*(\cdot)$.*

We denote

$$U := 1 - F_\rho(\rho) \tag{4.4}$$

where $F_\rho(\cdot)$ is the distribution function of $\rho$. Then $U \sim U(0,1)$, $\rho = F_\rho^{-1}(1-U)$ and $X = G^{-1}(U)$, a.s. Therefore,

$$E[\,\rho X\,] = E[\,F_\rho^{-1}(1-U)G^{-1}(U)\,] = \int_0^1 G^{-1}(s)\cdot F_\rho^{-1}(1-s)\,ds. \tag{4.5}$$

Now we turn to the objective functional of **Problem 4**, $\int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy$, set $X = G^{-1}(U)$, we have

$$\begin{aligned}
&\int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy \\
=&\int_0^\infty w^+(1-\mathbb{P}\{G^{-1}(U) \le y\})\,dy \\
=&\int_0^\infty w^+(1-\mathbb{P}\{U \le G(y)\})\,dy \\
=&\int_0^\infty w^+(1-G(y))\,dy \\
=&\Big[y\cdot w^+\big(1-G(y)\big)\Big]_0^\infty + \int_0^\infty y\cdot(w^+)'(1-G(y))G'(y)\,dy
\end{aligned}$$

Here we need an important assumption.

<u>**Assumption**</u> 4.4. We assume that the limit

$$\lim_{y\to\infty}[y\cdot w^+(1-G(y))] = 0 \tag{4.6}$$

This assumption is very rational because $\lim_{y\to\infty} w^+(1-G(y)) = 0$ and usually the utility function $u(\cdot)$ is bounded.

Let $s = G(y)$. Then we can get

$$\int_0^\infty w^+(\mathbb{P}\{X > y\})\,dy = \int_0^1 G^{-1}(s)\cdot(w^+)'(1-s)\,ds \tag{4.7}$$

## 4.2. Problem 5

**Proposition** 4.3 suggests that in order to solve **Problem 4** we only needs to seek among random variables of the form $G^{-1}(U)$, where $G(\cdot)$ is the distribution function of a nonnegative random variable. Applying (4.5) and (4.7), we turn **Problem 4** into the following problem.

$$\begin{array}{ll} \text{Maximize} & v(G) := \displaystyle\int_0^1 G^{-1}(s) \cdot (w^+)'(1-s)\, ds \\ \text{subject to} & \begin{cases} \displaystyle\int_0^1 G^{-1}(s) \cdot F_\rho^{-1}(1-s)\, ds = x_0, \\ G(\cdot) \text{ is the distribution function of a nonnegative r.v.} \end{cases} \end{array} \qquad (\textbf{Problem 5})$$

The following result, which is straightforward in view of Lemma 4.1 and Proposition 4.3, means that **Problem 4** is equivalent to **Problem 5**.

**Proposition** 4.5 (Jin and Zhou [**11**], Proposition C.1).
*If $G^*(\cdot)$ is optimal for Problem 5, then*

$$X^* := (G^*)^{-1}(U)$$

*is optimal for Problem 4. Conversely, if $X^*$ is optimal for Problem 4, then its distribution function $G^*(\cdot)$ is optimal for Problem 5 and $X^* = (G^*)^{-1}(U)$, a.s..*

## 4.3. Problem 6

Denoting

$$g(\cdot) := G^{-1}(\cdot). \tag{4.8}$$

Since $g(\cdot)$ is the inverse of the distribution function, so $g : [0,1] \mapsto [0,\infty]$ is nondecreasing and left continuous with $g(0) = 0$.

Then we can rewrite **Problem 5** into

$$\text{Maximize} \quad \int_0^1 g(s)\cdot (w^+)'(1-s)\,ds$$
$$\text{subject to} \quad \begin{cases} \displaystyle\int_0^1 g(s)\cdot F_\rho^{-1}(1-s)\,ds = x_0, \\ g:[0,1]\mapsto[0,\infty] \text{ is nondecreasing and left continuous with } g(0)=0. \end{cases}$$

(**Problem 6**)

## 4.4. Main idea and results

Our main idea is to find an inequality, with which we can solve **Problem 6**. The inequality have relation that the objective is less than or equal to the first constrain. Since the integrations of Problem 6 can be express *convolution* or *inner produce* type, we found some useful weighted inequalities for monotone functions, which play a key role in solving **Problem 6** :



**Proposition** 4.6 (Heinig and Maligranda [**10**], Theorem 2.1). *Let* $0<p\le q<\infty$, $u(s),v(s)\ge 0$ *and* $f(0)=0$. *The inequality*

$$\Big(\int_0^1 u(s)f(s)^q\,ds\Big)^{1/q} \le M\Big(\int_0^1 v(s)f(s)^p\,ds\Big)^{1/p} \tag{4.9}$$

*holds for all nondecreasing* $f:[0,1]\to[0,\infty]$ *if and only if*

$$\Big(\int_t^1 u(s)\,ds\Big)^{1/q} \le M\Big(\int_t^1 v(s)\,ds\Big)^{1/p} \quad \text{for all } 0\le t<1. \tag{4.10}$$

Taking $p=q=1$, we get the following corollary.

**Corollary 4.7.** *Let* $u(s),v(s)\ge 0$ *and* $f(0)=0$. *The inequality*

$$\Big(\int_0^1 u(s)f(s)\,ds\Big) \le M\Big(\int_0^1 v(s)f(s)\,ds\Big), \tag{4.11}$$

*holds for all nondecreasing* $f : [0,1] \to [0,\infty]$ *if*

$$M = \sup_{0\le t<1} \Big(\int_t^1 u(s)\,ds\Big)\Big(\int_t^1 v(s)ds\Big)^{-1}. \tag{4.12}$$

*Moreover,* $M$ *is the best constant satisfying (4.11). Equation (4.12) admits an optimal solution* $t^*$*, then "=" holds when* $f(x) = \lambda I_{(t^*,1]}(x)$ *where* $\lambda$ *is any nonnegative constant.*

Since the probability weighting function $w^+(\cdot)$ is strictly increasing and differentiable, we get the derivative $(w^+)'(s) \ge 0$ for all $0 \le s \le 1$. And $F_\rho^{-1}(\cdot)$ is the inverse function of the distribution function, so $F_\rho^{-1}(\cdot)$ is nondecreasing, that is, $F_\rho^{-1}(s) \ge 0$ for all $0 \le s \le 1$. We use the result of **Corollary 4.7** for **Problem 6**. If

$$M = \sup_{0\le t<1} \Big(\int_t^1 (w^+)'(1-s)\,ds\Big)\Big(\int_t^1 F_\rho^{-1}(1-s)ds\Big)^{-1}, \tag{4.13}$$

then the following inequality

$$\Big(\int_0^1 g(s)\cdot(w^+)'(1-s)\,ds\Big) \le M\Big(\int_0^1 g(s)\cdot F_\rho^{-1}(1-s)\,ds\Big) \tag{4.14}$$

holds for all nondecreasing $g : [0,1] \to [0,\infty]$ with $g(0) = 0$. That is, the optimal value of **Problem 6** is $M \cdot x_0$ if the equality of (4.14) holds. The constant $M$ given by (4.13) can be simplified by

$$M = \sup_{0\le t<1} \Big(w^+(1-t) - w^+(0)\Big)\Big(\int_0^{1-t} F_\rho^{-1}(s)\,ds\Big)^{-1} \tag{4.15}$$

$$= \sup_{0<c\le 1} \Big[w^+(c)\cdot\Big(\int_0^c F_\rho^{-1}(s)\,ds\Big)^{-1}\Big] \tag{4.16}$$

Equation (4.16) admits an optimal solution $c^*$, then the optimal function of **Problem 6** is of the form

$$g^*(x) = (G^*)^{-1}(x) = \lambda I_{(1-c^*,1]}(x),\ x \in [0,1] \tag{4.17}$$

where $\lambda > 0$ is the constant satisfying $\lambda \cdot \int_0^{c^*} F_\rho^{-1}(s)\,ds = x_0$. But when $M = \infty$, then $\int_0^{c^*} F_\rho^{-1}(s)\,ds = 0$ and $\lambda$ does not exist.

**Remark** 4.8. A maximization problem is called *well-posed* if the supremum of its objective is finite; otherwise it is called *ill-posed*.

**Theorem** 4.9. *The following statements are equivalent:*

(1) ***Problem 6*** *is well-posed for any* $x_0 \geq 0$.

(2) *The optimal ratio* $M = \sup_{0 \leq c \leq 1} \left[ w^+(c) \cdot \left( \int_0^c F_\rho^{-1}(s)\, ds \right)^{-1} \right] < \infty$.

(3) $\lim_{c \to 0} \dfrac{(w^+)'(c)}{F_\rho^{-1}(c)} < \infty$.

*Furthermore, when one of the above (1)-(3) holds, the optimal solution to* ***Problem 6*** *is of the form*

$$g^*(x) = (G^*)^{-1}(x) = x_0 \left( \int_0^{c^*} F_\rho^{-1}(s)\, ds \right)^{-1} \boldsymbol{I}_{(1-c^*,1]}(x),\ x \in [0,1]. \tag{4.18}$$

Proof. (1) $\Longleftrightarrow$ (2) is clear.

(2) $\Longleftrightarrow$ (3). Since $0 \leq w^+(c) \leq 1$, $0 \leq \int_0^c F_\rho^{-1}(s)\, ds \leq e^{rT}$ for $0 \leq c \leq 1$ and $w^+(c)$, $\int_0^c F_\rho^{-1}(s)\, ds$ are strictly increasing for $c$, then the $\left[ w^+(c) \cdot \left( \int_0^c F_\rho^{-1}(s)\, ds \right)^{-1} \right]$ may be infinity only when $c = 0$. Therfore,

$$(2) \Longleftrightarrow \lim_{c \to 0} \left[ w^+(c) \cdot \left( \int_0^c F_\rho^{-1}(s)\, ds \right)^{-1} \right] < \infty,$$

Applying l'Hôpital Rule to $w^+(c) / \int_0^c F_\rho^{-1}(s)\, ds$, we obtain (3). Equation (4.17) says $g^*(x) = \lambda I_{(1-c^*,1]}(x)$ where $\lambda > 0$ is a constant. $\lambda$ must satisfy $\int_0^1 g^*(s) \cdot F_\rho^{-1}(1-s)\, ds = x_0$, then we get $\lambda = x_0 \left( \int_0^{c^*} F_\rho^{-1}(s)\, ds \right)^{-1}$.

□

We now summarize the main result in the following theorem.

**Theorem** 4.10. *Assume that* $\lim_{c\to 0}\big[(w^+)'(c)/F_\rho^{-1}(c)\big] < \infty$. *Then the maximal value at terminal time* $T$ *under LCPT is given by*

$$\sup_{0\le c\le 1}\left[w^+(c)\cdot\Big(\int_0^c F_\rho^{-1}(s)\,ds\Big)^{-1}\right]\cdot x_0 \tag{4.19}$$

*Equation (4.19) admits an optimal solution* $c^*$. *Then the corresponding optimal terminal wealth to* ***Problem 3*** *is*

$$X^* = x_0\Big(E[\,\rho\,\boldsymbol{I}_{\{\rho<F_\rho^{-1}(c^*)\}}\,]\Big)^{-1}\boldsymbol{I}_{\{\rho<F_\rho^{-1}(c^*)\}}\ \ a.s. \tag{4.20}$$

**Remark** 4.11. Since $E[\,\rho\,] = e^{-rT}$, so $\Big(E[\,\rho\,\mathbf{I}_{\{\rho<F_\rho^{-1}(c^*)\}}\,]\Big)^{-1} \ge e^{rT}$ and $e^{rT}$ is the ratio that all the money put in the bank account. Therefore, It means that the payoff $\Big(E[\,\rho\,\mathbf{I}_{\{\rho<F_\rho^{-1}(c^*)\}}\,]\Big)^{-1}\cdot x_0$ is better than the payoff if we invest all money in the bond.

The optimal terminal wealth $X^*$ mentioned in (4.20) tells us two different stories in economical view at terminal time. In the cases of $\{\rho < F_\rho^{-1}(c^*)\}$ the payoff we gain is more than that we get from the bond market, and this payoff is fixed due to the deterministic coefficient. Furthermore, in the rest of part $\{\rho \ge F_\rho^{-1}(c^*)\}$ all the assets turn out to be zero. Those cases of $\{\rho < F_\rho^{-1}(c^*)\}$ are profitable to the investors. It might be that the noise $W_t$ of the stock price does not fluctuate dramatically.

### 4.5. The optimal wealth process and the optimal strategies

In this section, we want to find a portfolio $\pi$ replicating the optimal terminal wealth $X^*$ of (4.20). Recall that $\rho = \rho_T$ with

$$\rho_t := \exp\Big\{-\big(r+\frac{1}{2}\,\theta^2\big)\,t-\theta\,W_t\Big\},\qquad 0\le t\le T.$$

Let $N(\cdot)$ and $\psi(\cdot)$ be the distribution function and probability density function of a standard normal random variable respectively. $\rho(t,T) := \rho_T/\rho_t$ conditional on $\mathcal{F}_t$ follows a log-normal distribution with parameter $(\mu_t, \sigma_t^2)$, where

$$\mu_t = -(\, r + \frac{1}{2}\,\theta^2\,)(T-t) \qquad \text{and} \qquad \sigma_t^2 = \theta^2(T-t) \tag{4.21}$$

and $\lambda := x_0 \Big( E[\,\rho\,\mathbf{I}_{\{\rho < F_\rho^{-1}(c^*)\}}\,] \Big)^{-1}$.

By (3.11), the replicating wealth process is given by

$$\begin{aligned} X_t &= E\big[\,\rho(t,T)X^* \big| \mathcal{F}_t\,\big] \\ &= \lambda\, E\big[\,\rho(t,T)\mathbf{I}_{\{\rho < F_\rho^{-1}(c^*)\}} \big| \mathcal{F}_t\,\big] \\ &= \lambda\, E\big[\,\rho(t,T)\mathbf{I}_{\{\rho(t,T) < F_\rho^{-1}(c^*)/\rho_t\}} \big| \mathcal{F}_t\,\big] \\ &= \frac{\lambda}{\sigma_t} \int_0^{F_\rho^{-1}(c^*)/\rho_t} \psi\Big(\frac{\ln y - \mu_t}{\sigma_t}\Big)\, dy. \end{aligned}$$

Define

$$f(t,x) := \frac{\lambda}{\sigma_t} \int_0^{F_\rho^{-1}(c^*)/x} \psi\Big(\frac{\ln y - \mu_t}{\sigma_t}\Big)\, dy.$$

It is well known that the replicating portfolio is

$$\pi_t = -\Big(\frac{\alpha - r}{\beta^2}\Big) f_x(t,\rho_t)\,\rho_t; \tag{4.22}$$

see, e.g., Bielecki [**3**], Equation(7.6). Now we calculate

$$f_x(t,\rho_t) = -\lambda\,\psi\Big(\frac{\ln F_\rho^{-1}(c^*) - \mu_t - \ln \rho_t}{\sigma_t}\Big)\Big(\frac{F_\rho^{-1}(c^*)}{{\rho_t}^2}\Big). \tag{4.23}$$

Plugging it in (4.22), we get the following result.

**Theorem** 4.12. *The wealth-portfolio pair replicating $X^*$ is given by*

$$X_t = \frac{\lambda}{\sigma_t} \int_0^{F_\rho^{-1}(c^*)/\rho_t} \psi\Big(\frac{\ln y - \mu_t}{\sigma_t}\Big)\, dy, \tag{4.24}$$

$$\pi_t = \lambda\Big(\frac{\alpha - r}{\beta^2}\Big)\psi\Big(\frac{\ln F_\rho^{-1}(c^*) - \mu_t - \ln \rho_t}{\sigma_t}\Big)\Big(\frac{F_\rho^{-1}(c^*)}{\sigma_t \rho_t}\Big), \tag{4.25}$$

*where $\lambda = x_0 \Big( E[\,\rho\,\boldsymbol{I}_{\{\rho < F_\rho^{-1}(c^*)\}}\,] \Big)^{-1}$.*

CHAPTER 5

# Numerical Result

In this chapter, we give some numerical results for the maximum value at the terminal time $T$

$$\sup_{0\le c\le 1}\left[w^+(c)\cdot\Big(\int_0^c F_\rho^{-1}(s)\,ds\Big)^{-1}\right]\cdot x_0 =: M\cdot x_0$$

of **Theorem 4.10**, and observe how these parameters affect the maximal value.

**Example** 5.1. consider the case where the terminal time $T = 5$ (years), the interest rate $r = 0.01$, the drift rate of the stock $\alpha = 0.05$ and the volatility of the stock $\beta = 0.2$. We assume the probability weighting function for gain $w^+(\cdot)$ is of the form

$$w^+(p) = e^{-(-\ln p)^{1.2}},$$

which is proposed by Prelec [**14**].

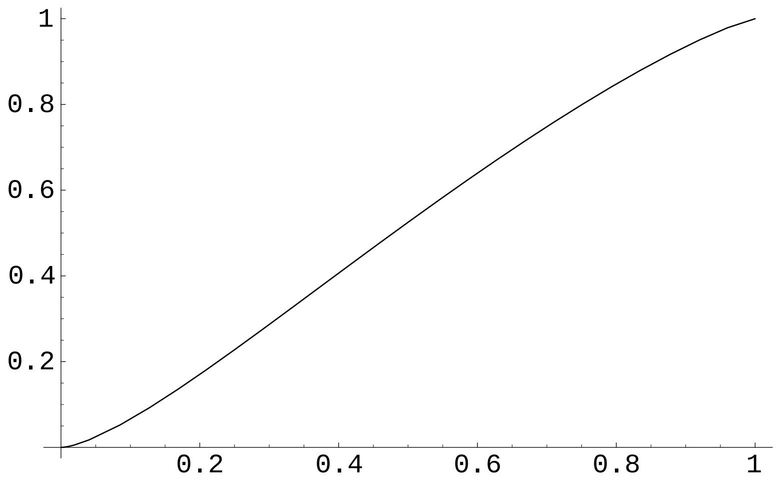


FIGURE 5.1. The probability weighting function $w^+(p) = e^{-(-\ln p)^{1.2}}$

Recall that the distribution function of $\rho$ is

$$F_\rho(x) = N\Big(\frac{\ln x - \mu}{\sigma}\Big) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\Big(\frac{\ln x - \mu}{\sigma}\Big), \tag{5.1}$$

where

$$\begin{aligned}
\mu &= -(r + \frac{1}{2}\theta^2)\,T \,:\, \text{the mean of } \ln\rho \\
\sigma^2 &= \theta^2\,T \,:\, \text{the variance of } \ln\rho \\
\theta &= \frac{\alpha - r}{\beta} \,:\, \text{the usual market price of risk} \\
\mathrm{erf}\,(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \,:\, \text{the error function}
\end{aligned}$$

Therefore, the inverse function of $F_\rho(\cdot)$ is $F_\rho^{-1}(y) = \exp\Big\{\mu + \sigma\sqrt{2}\,\mathrm{erf}^{-1}(2y-1)\Big\}$. In this example, the values for parameters are shown in Table 1.

| Parameters | $T$ | $r$ | $\alpha$ | $\beta$ | $\theta$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|---|---|---|
| Values | 5 | 0.01 | 0.05 | 0.2 | 0.2 | -0.15 | 0.447214 |

TABLE 1. Parameters in Example

We plot the graph of $w^+(c)\cdot\Big(\int_0^c F_\rho^{-1}(s)\,ds\Big)^{-1}$ for $0 < c \le 1$ as in the Figure 5.2.. From the graph , we observe that the maximum does exist and the approximate values of $c^*$ and $M$ are

$$\begin{aligned}
c^* &\approx 0.253411, \\
M &\approx 1.82713.
\end{aligned}$$

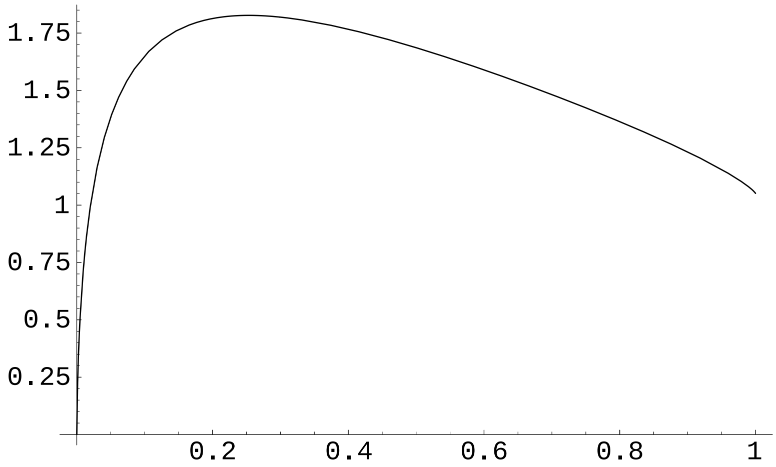


FIGURE 5.2. The graph of $w^{+}(c) \cdot \left( \int_0^c F_\rho^{-1}(s)\, ds \right)^{-1}$

Second, we only change the terminal time $T$ when the other parameters fix and observe the $c^*$ and $M$ in the Table 2.

| $T$ (years) | 0.5 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $M$ | 1.18922 | 1.26449 | 1.40291 | 1.5398 | 1.68045 | 1.82713 | 1.98128 |
| $e^{rT}$ | 1.00501 | 1.01005 | 1.0202 | 1.03045 | 1.04081 | 1.05127 | 1.06184 |

TABLE 2

Finally, we change the drift rate of the stock $\alpha$ and the volatility of the stock $\beta$ when the other parameters fix. Observe that $\alpha$ and $\beta$ how to affect the maximum ratio $M$ in the Table 3.

| | | The value of drift rate $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $M$ | $-0.03$ | $-0.02$ | $-0.01$ | 0 | 0.01 | 0.02 | 0.03 |
| | 0.2 | 1.82713 | 1.54742 | 1.34950 | 1.21287 | 1.12410 | 1.21287 | 1.34950 |
| The | 0.3 | 1.47355 | 1.34950 | 1.25255 | 1.17848 | 1.12410 | 1.17848 | 1.25255 |
| value | 0.4 | 1.34950 | 1.27450 | 1.21287 | 1.16314 | 1.12410 | 1.16314 | 1.21287 |
| of | 0.5 | 1.28838 | 1.23602 | 1.19163 | 1.15451 | 1.12410 | 1.15451 | 1.19163 |
| volatility | 0.6 | 1.25255 | 1.21287 | 1.17848 | 1.14899 | 1.12410 | 1.14899 | 1.17848 |
| $\beta$ | 0.7 | 1.22921 | 1.19751 | 1.16957 | 1.14516 | 1.12410 | 1.14516 | 1.16957 |
| | 0.8 | 1.21287 | 1.1866 | 1.16314 | 1.14235 | 1.12410 | 1.14235 | 1.16314 |

TABLE 3

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