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多變量製程能力指標之研究

A Study on Multivariate Process Capability Indices

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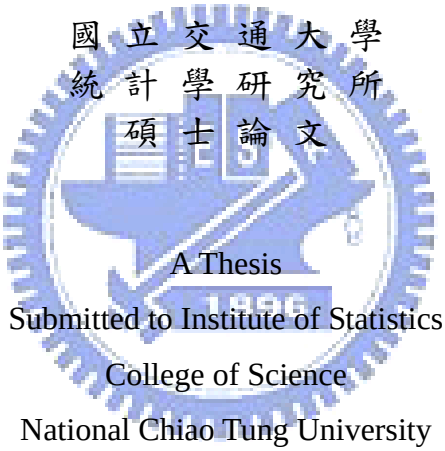

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摘要

製程能力指標近年來在業界廣泛的被使用於衡量製程製造產品的能力。衡量一維製程能力的指標已經被研究得相當完備，然而有關衡量二維製程能力的指標的研究雖然相當多，但指標和產品良率間的關係卻很少被強調，有鑑於此 Castagliola 和 Castellanos [4] 提出了兩個指標 BC_{pk} 以及 BC_p。這兩個指標是以超出凸多邊形規格區內不良品所占比例為基礎的指標。我們在本文章中將此指標推廣至多個產品特性的情形，並且提出估計這個指標在二維甚至多維製程的演算法，不像 Castagliola 和 Castellanos [4] 所使用的方法僅能運用在二維製程上。另外，我們利用四種拔靴法分別估計出 BC_{pk} 的信賴下限。至於 BC_p，由於原來的定義在不同的製程特性上給予不同的比例縮放時無法保持不變性，所以我們提出了一個預先修正的方法，解決原本指標沒有不變性的問題，另外，我們也推導出 $\widehat{BC_p}$ 的近似分佈，進而推導出信賴區間、信賴下限、以及假設檢定。最後我們用一個實際的例子來說明我們在文中所提出的估計方法。

關鍵詞：製程能力指標、二維常態分佈、良率、拔靴法

A Study on Multivariate Process Capability Indices

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Abstract

Process capability indices (PCIs) have been widely used in the industries for assessing the capability of manufacturing processes. The research of PCIs for univariate processes has been well developed. However, in the bivariate case, the PCI research may be plenty, but links between the index and the product yield are seldom emphasized. For this, by assuming a bivariate normal distribution and a rectangular specification region, Castagliola and Castellanos [4] proposed two indices BC_{pk} and BC_p. These two indices are defined based on the proportions of non-conforming products over convex polygons. We extend these indices to multivariate processes of more than two quality characteristics. We develop an algorithm for computing estimates of these indices, which is suitable for general multivariate processes, not like the algorithm in Castagliola and Castellanos [4] can only be used for bivariate processes. In addition, we estimate the lower confidence bound by bootstrap methods. As for BC_p, we find the original definition is not scale invariant, meaning that the BC_p value will vary with different scales on quality characteristics. We propose a pre-processing step to solve this problem. Moreover, we find an approximate distribution of $\widehat{BC}_p$, which enables us to develop statistical procedures for making inferences on process capability based on data, including hypothesis testing, confidence interval, and lower confidence bound. The latter is directly linked to the quality assurance. Finally, a real data set is used as an application example.

keywords: process capability indices, bivariate normal distribution, yield, bootstrap

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1 Introduction

Process Capability Indices (PCIs) are some indices developed for engineering convenience to quentify the performance of a process and are widely used in industries. Facing stronger competition than ever, nowadays, companies need to work very hard to improve the quality of their products. The measurement and evaluation of the process performance via PCIs becomes more and more important.

PCIs have been applied to manufacturing measurements since 1980s and some indices such as C_p, C_{pk}, C_{pm}, C_{pmk} are widely used. These indices have good performances for evaluating processes or product with one single quality characteristic. However, now with high-technology industries replacing traditional industries and becoming the main stream, many processes are so complex that many characteristics are affecting the quality simultaneously. Since these characteristics are measured on the same product/process items, they should be considered jointly, that is, the analysis should be based on multivariate statistical techniques. Therefore, PCIs that can assess a process or product of more than one quality characteristic are desirable.

Among univariate PCIs, C_{pk} could be the most popular one, not only because it accounts for both process mean and variance in the process assessment, but perhaps also because it links directly to the process yield by the following inequality [3]: $2\Phi(3C_{pk})-1 \leq$ % yield $\leq \Phi(3C_{pk})$, where %yield stands for the process yield. Therefore, C_{pk} sometimes is referred to as a yield assurance index. However, in the bivariate case, the PCI research may be plenty, but links between the index and the product yield are seldom emphasized. For this, by assuming a bivariate normal distribution and a rectangular specification region, Castagliola and Castellanos [4] proposed two indices, BC_{pk} and BC_p. These two indices are defined based on the proportions of non-conforming products over convex polygons. In this paper, we extend the notion of BC_{pk} to the multivariate case of more than two quality characteristics. The computation method of Castagliola and Castellanos [4] is only for bivariate processes with rectangular specification regions. We develop a computation method that can be implemented for more general multivariate processes with more flexible specification regions in higher dimensions. As for BC_p, we propose a pre-processing step to solve the problem that the original definition in [4] is

not scale invariant, meaning that the BC_p value will vary with different scales on quality characteristics. Moreover, we find an approximate distribution of $\widehat{BC}_p$, which enables us to develop statistical procedures for making inferences on process capability based on data, including hypothesis testing, confidence interval, and lower confidence bound. The latter is directly linked to the quality assurance.

This paper is organized as follows. We first review univariate and bivariate PCIs in the literature briefly in Section 2. In Section 3, we introduce the index BC_{pk} proposed by Castagliola and Castellanos [4] for bivariate processes. Then we establish connections between this index and the product yield. We further extend this index to multivariate processes of more than two characteristics. After that, we give the estimation algorithm for BC_{pk} and propose obtaining lower confidence bounds by bootstrap methods. We also study the distribution of a natural estimator of BC_{pk} by simulation. In Section 4, for BC_p, we propose a pre-processing algorithm to avoid the non-scale-invariance problem of the BC_p proposed by Castagliola and Castellanos [4]. We derive an approximate sampling distribution for a natural estimator of BC_p and develop statistical procedures for making inferences on process capability. In Section 5, we apply our methods to a set of real data. Finally, we conclude the paper with a brief summary in Section 6.

2 Literature Review

2.1 Indices for Univariate Process

In this subsection, consider an univariate process with a quality characteristic with the specification interval [LSL, USL]. Assume that process data follow a normal distribution with mean μ and variance σ^2.

PCIs for measuring processes with one quality characteristic have been investigated extensively for decades. Kane [14] proposed

$$C_p = \frac{USL - LSL}{6\sigma}$$

for measuring the variation of product characteristic relative to the specification. Since C_p cannot be used to measure yield, he proposed another index linked to the product yield as

$$C_{pk} = \min\left(\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right),$$

which accounts for not only the spread of the process but also the location of the process mean relative to the specification limits. As for the endurance of products, which is related how far a product deviates from the target T, Chan *et al.* [5] and Hsiang and Taguchi [12] proposed the following index:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}.$$

The loss function in the denominator is contributed by the deviation of the process mean μ from the target value T and process variance σ^2. Pearn *et al.* [16] proposed C_{pmk} by combining C_p and C_{pk} as

$$C_{pmk} = \min\left(\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right),$$

which is more sensitive to the deviation of process mean from the target value than C_p and C_{pk}.

2.2 Indices for Multivariate Processes

In this subsection, consider a multivariate process with k quality characteristics. Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are n_i i.i.d. $k \times 1$ random vectors of observations. $\bar{\boldsymbol{X}}$ is a $k \times 1$ vector representing the sample mean of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$.

Assuming the process data $\boldsymbol{X}$ follows a multivariate normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, denoted by $\boldsymbol{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Chan *et al.* [6] proposed an index for measuring how far the process mean $\boldsymbol{\mu}$ is from the target value $\boldsymbol{T}$ as

$$\widetilde{C}_{pm} = \left(\frac{k}{E[(\boldsymbol{X} - \boldsymbol{T})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{T})]} \right)^{1/2}.$$

Pearn *et al.* [16] introduced two multivariate PCIs, which are viewed as more natural generalizations of C_{pm} than the one proposed by Chan *et al.* [6]. They defined a multivariate C_p index as

$${}_kC_p^2 = \frac{c^2}{\chi^2_{k,0.0027}},$$

where $\chi^2_{k,\alpha}$ is the upper α quantile of a chi-square distribution with degrees of freedom k, and c is a constant satisfying $P\{(\boldsymbol{X} - \boldsymbol{T})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{T}) \leq c^2\} = 0.9973$. Analogously, they defined a multivariate C_{pm} index by

$${}_kC_{pm}^2 = \frac{{}_kC_p^2}{1 + (\boldsymbol{\mu} - \boldsymbol{T})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \boldsymbol{T})/k}. \tag{1}$$

Hubele *et al.* [13] proposed an index vector $(C_{PM}, PV, LI)^T$ for bivariate normal processes. The first component

$$C_{PM} = \frac{\text{area of specification}}{\text{area of modified process region}} = \left(\frac{\prod_{i=1}^{2} (USL_i - LSL_i)}{\prod_{i=1}^{2} (UPL_i - LPL_i)} \right)^{\frac{1}{2}}.$$

.

The modified process region is the smallest rectangle that can circumscribe $100(1-\alpha)\%$ of the process distribution (see Fig 1). The edges of the modified process region are defined as the lower and upper process limits, LPL_i and UPL_i, $i = 1, 2$. These four values can be obtained by solving the system of equations of first derivatives with respect to each x_i of

$$(\boldsymbol{X} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu}) = \chi^2_{2,\alpha}, \text{ where } \boldsymbol{X} = (X_1, X_2)^T \text{ and } \boldsymbol{\mu} = (\mu_1, \mu_2)^T.$$

The solutions are

$$UPL_1 = \mu_1 + \sqrt{\frac{\chi^2_{2,\alpha} det(\boldsymbol{\Sigma}_1^{-1})}{det(\boldsymbol{\Sigma}^{-1})}} \quad , \quad LPL_1 = \mu_1 - \sqrt{\frac{\chi^2_{2,\alpha} det(\boldsymbol{\Sigma}_1^{-1})}{det(\boldsymbol{\Sigma}^{-1})}},$$

$$UPL_2 = \mu_2 + \sqrt{\frac{\chi^2_{2,\alpha} det(\boldsymbol{\Sigma}_2^{-1})}{det(\boldsymbol{\Sigma}^{-1})}} \quad , \quad LPL_2 = \mu_2 - \sqrt{\frac{\chi^2_{2,\alpha} det(\boldsymbol{\Sigma}_2^{-1})}{det(\boldsymbol{\Sigma}^{-1})}},$$

where $\boldsymbol{\Sigma}_i^{-1}$, $i = 1, 2$, is the matrix obtained from $\boldsymbol{\Sigma}^{-1}$ by deleting the ith row and column. The meaning of this component is analogous to that of C_p, measuring the variation of product characteristics relative to the specifications.

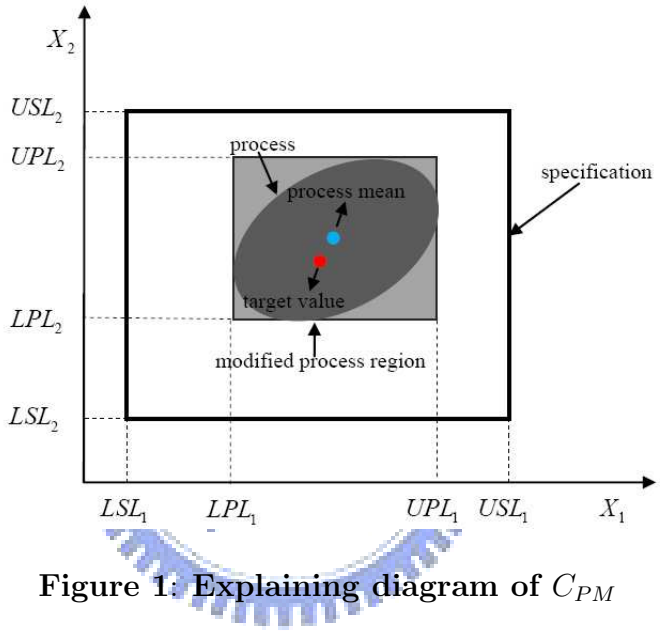

Figure 1: Explaining diagram of C_{PM}

The second component is the p-value of testing the difference between the center of specification (target value $\boldsymbol{T}$) and the process mean. Let the null hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{T}$, the Hotelling T^2 statistic [11] is

$$T^2 = n(\bar{\boldsymbol{X}} - \boldsymbol{T})^T \hat{\boldsymbol{\Sigma}}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{T}),$$

where $\bar{\boldsymbol{X}}$ is the sample mean and $\hat{\boldsymbol{\Sigma}}$ is the usual sample variance-covariance matrix of process data. Since $\frac{n-2}{2(n-1)} T^2$ follows $F_{2,n-2}$ distribution under null hypothesis, the p-value-based component PV is defined as

$$PV = P\left(T^2 \geq \frac{2(n-1)}{n-2} F_{2,n-2,\alpha}\right),$$

where $F_{2,n-2,\alpha}$ stands for the $100(1-\alpha)\%$ percentile of F distribution with degrees of freedom 2 and $n-2$. This component measures the distance of the process mean and the

target value. If the process mean is close to the target value, PV will be close to 1.

The third component LI provides the information about the location of the modified process region relative to the specification, defined as

$$LI = \max\left(1, \frac{|UPL_1 - USL_1|}{USL_1 - LSL_1}, \frac{|LPL_1 - LSL_1|}{USL_1 - LSL_1}, \frac{|UPL_2 - USL_2|}{USL_2 - LSL_2}, \frac{|LPL_2 - LSL_2|}{USL_2 - LSL_2}\right).$$

If this component is equal to 1, then the entire modified process region falls within or on the specification. If the component is greater than 1, then some or all modified process region falls out of the specification.

This index vector contains three components summarizing the size and location of process contour related to the specification.

Taam *et al.* [18] proposed an index as the ratio of two areas

$$\widetilde{C_p} = \frac{Area(R_1)}{Area(R_2)} = \frac{Area(\text{modified specification})}{Area(99.73\% \text{ process region})}, \tag{2}$$

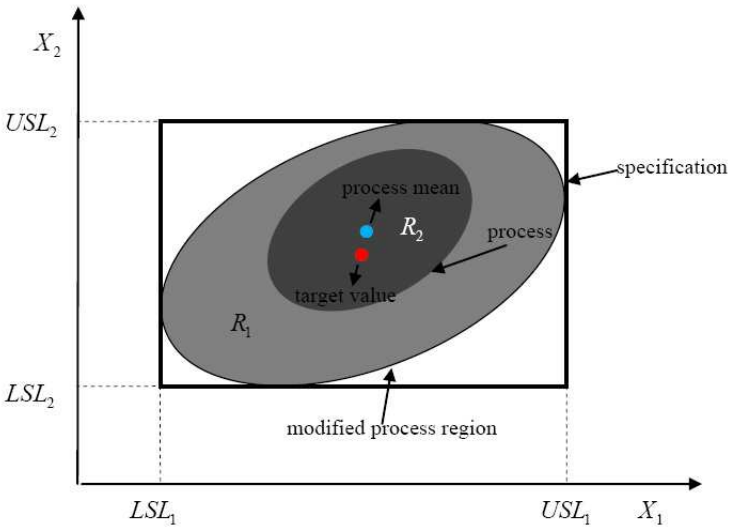

Figure 2: Explaining diagram of $\widetilde{C_p}$

where R_1 is a modified specification, which is the largest ellipsoid that is centered at the target value and completely within the original specification, R_2 is an elliptical region containing 99.73% of the bivariate normal distribution. This index is an extension of the univariate C_p for bivariate processes. Considering the shift of process mean from the

target value $\boldsymbol{T}$, Taam *et al.* [18] further modified this index by taking an adjustment factor D into account and defined a C_{pm} index for two quality characteristics as follows:

$$MC_{pm} = \frac{\widetilde{C_p}}{D}, \text{ where } D = \left(1 + (\boldsymbol{\mu} - \boldsymbol{T})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \boldsymbol{T})\right)^{\frac{1}{2}}.$$

$0 < D^{-1} < 1$ measures the closeness between the process mean and the target value. A larger value of $0 < D^{-1} < 1$ implies that the process mean is closer to the target value.

Chen [7] proposed an index using the concept of a specification zone expressed as

$$V(r_0) = \{\boldsymbol{x} \in R^k : h(\boldsymbol{x} - \boldsymbol{\mu}_0) \leq r_0\}, \tag{3}$$

where $h(\cdot)$ is a nonnegative homogeneous scalar function satisfying the condition $h(tx) = th(x)$ for all $t > 0$ and r_0 is a positive number. A process is considered capable if $P(\boldsymbol{X} \in V(r_0)) \geq 1 - \alpha$, where α is the allowable expected proportion of non-conforming products. Let $r = \min\{c : P(\boldsymbol{X} \in V(c)) \geq 1 - \alpha\}$. Then a process is considered to be capable if and only if $r \leq r_0$. This leads one to express an index for multivariate process in the form

$$MC_p = \frac{r_0}{r}.$$

According to Chen [7], this definition provides the following advantages: (i) allowing flexible specifications as general as given by $V(r_0)$ in (3), (ii) assuming no conditions on the underlying distribution, and (iii) permiting flexibility in setting a criterion for the capability of a process. For example, consider a rectangular specification zone

$$W = \{\boldsymbol{x} \in R^k : |\boldsymbol{x}_i - \boldsymbol{\mu}_i| \leq r_i,\ i = 1, \ldots, k\},$$

where $\boldsymbol{\mu}$ is the process mean and r_i's are positive constants. One can derive an alternative definition of MC_p as

$$MC_p = \frac{1}{r^*},$$

where r^* is a constant satisfying $P\left(\max\{|\boldsymbol{X}_i - \boldsymbol{\mu}_i|/r_i, i = 1, \ldots, k\} \leq r^*\right) = 1 - \alpha$. If $MC_p \geq 1$, the process is capable at $100(1 - \alpha)\%$ confidence level.

Pal [15] proposed an index defined as follows:

$$C_{PB} = \frac{S_R}{A_p} = \frac{(USL_1 - LSL_1)(USL_2 - LSL_2)}{\pi \chi^2_{2,0.0027} \sqrt{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2}},$$

where S_R represents the area of the specification rectangle and A_p represents the 99.73% area of the process region. This index is, in fact, an extension of the index (2) proposed by Taam *et al.* [18]. It is an area ratio of a rectangular region over an elliptical region while Taam *et al.* [18] used an elliptical region over another elliptical region as the area ratio.

Bothe [2] proposed a multivariate C_{pk} index defined as

$$MC_{pk} = \frac{Z_{P_T}}{3},$$

where Z_{P_T} is the P_Tth percentile of the standard normal distribution, and P_T is defined as

$$P_T = 1 - ((1 - P_{QC_1})(1 - P_{QC_2}) \cdots (1 - P_{QC_k}))^{\frac{1}{k}}$$

with P_{QC_i}, $i = 1, \ldots, k$, being the non-conforming rate of the ith quality characteristic. However, this index is designed only for independent process characteristics.

Wang and Du [19] proposed a method using principal component (PC) analysis to describe the performance of a process of multiple characteristics. In that paper, the procedures of obtaining the indices for normal data as well as non-normal data are described in the following:

$\widehat{MC}_p = \left(\prod_{i=1}^{k} \hat{C}_{p;PC_i}\right)^{1/k}$ for normal data, where $\hat{C}_{p;PC_i} = \dfrac{USL_{PC_i} - LSL_{PC_i}}{6\sqrt{S^2_{PC_i}}}$,

$\widehat{MC}_{pk} = \left(\prod_{i=1}^{k} \hat{C}_{pk;PC_i}\right)^{1/k}$ for normal data, where

$$\hat{C}_{pk;PC_i} = \frac{\min(USL_{PC_i} - \bar{X}_{PC_i},\ \bar{X}_{PC_i} - LSL_{PC_i})}{3\sqrt{S^2_{PC_i}}},$$

$\widehat{MC}_{pk} = \left(\prod_{i=1}^{k} \hat{C}_{pc;PC_i}\right)^{1/k}$ for non-normal data, where $\hat{C}_{pc;PC_i} = \dfrac{USL_{PC_i} - LSL_{PC_i}}{6\sqrt{\dfrac{\pi}{2\bar{d}}}}$,

and the elements in the above expressions are given in the following.

Suppose $\boldsymbol{S}$ is a non-singular $k \times k$ sample variance-covariance matrix. $\boldsymbol{LSL}$ and $\boldsymbol{USL}$ are $k \times 1$ vectors of lower and upper specification limits, respectively. Using spectral decomposition, we can obtain a matrix $\boldsymbol{D} = \boldsymbol{U}^T\boldsymbol{S}\boldsymbol{U}$, where $\boldsymbol{D} = diag(\lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2)$ with $\lambda_1^2 \geq$

$\lambda_2^2 \geq \cdots \geq \lambda_k^2$ being the eigenvalues of $\boldsymbol{S}$, and the columns of $\boldsymbol{U}$, $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, are the associated eigenvectors. As a result,

$$S_{PC_i} = \lambda_i,\ \bar{X}_{PC_i} = \boldsymbol{u}_i^T \bar{\boldsymbol{X}},$$
$$USL_{PC_i} = \boldsymbol{u}_i^T \boldsymbol{USL},\ LSL_{PC_i} = \boldsymbol{u}_i^T \boldsymbol{LSL},\ i = 1, \ldots, k,$$
$$\bar{d} = \frac{1}{k}\frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n}\left|\boldsymbol{u}_i{}^T \boldsymbol{X}_j - \frac{USL_{PC_i} + LSL_{PC_i}}{2}\right|.$$

Here we remark that the numerators of $\widehat{MC}_p$ and $\widehat{MC}_{pk}$ seem somewhat unreasonable, since the vectors $\boldsymbol{USL}$ and $\boldsymbol{LSL}$ no longer represent upper or lower bounds of the specification region in the directions of principal components. As a result, $USL_{PC_i} - LSL_{PC_i}$ sometimes may even become negative.

Wang *et al.* [20] compared three process capability indices: $(C_{PM},\ PV,\ LI)^T$ proposed by Hubele *et al.* [13], MC_{pm} proposed by Taam *et al.* [18], and MC_p proposed by Chen [7]. They summarized that, in general, the multivariate indices could be obtained from (i) the area ratio of a specification region to a process region, (ii) the probability of a non-conforming product, and (iii) other approaches using loss functions or vector representation. The purpose of Wang *et al.* [20] is to illustrate the distinctions among the various meanings of capability in the multivariate case.

The purpose of this paper is to study yield related PCIs for multivariate processes. As mentioned in Section 1, BC_{pk} and BC_p proposed by Castagliola and Castellanos [4] are such indices. We shall give a more detailed review on these indices and then present how we would extend BC_{pk} to higher dimensions and how to modify BC_p to become scale-invariant in the later sections. And the last but not the least, we will provide methodologies on how to compute these indices.

3 Multivariate C_{pk} Index

3.1 Yield Measuring Index for Processes with Multiple Characteristics

In this subsection, we first introduce the bivariate C_{pk} index, BC_{pk}, proposed by Castagliola and Castellanos [4]. Then we provide the link between BC_{pk} and yield. Moreover, we extend this index to higher dimensions.

3.1.1 Alternative Definition of C_{pk}

Assume that the quality characteristic X of a product item is a $N(\mu, \sigma^2)$ random variable. Let [LSL, USL] be the corresponding lower and upper specification limits. Equivalent to the definition of Kane [14], an alternative definition for C_{pk} was proposed by Castagliola and Castellanos [4]. This definition is based on the lower and upper proportions of non-conforming products $p_L = P(X \leq LSL)$ and $p_U = P(X \geq USL)$. Since $X \sim N(\mu, \sigma^2)$, $p_L = \Phi(\frac{LSL-\mu}{\sigma})$ and $p_U = \Phi(\frac{-USL+\mu}{\sigma})$, where Φ is the cumulative distribution function (c.d.f.) of the standard normal distribution. Moreover, since the cumulative distribution function Φ is a strictly increasing function of the random variable, C_{pk} is equivalent to

$$\frac{1}{3}\min\{-\Phi^{-1}(p_U), -\Phi^{-1}(p_L)\}. \tag{4}$$

Similarly, the C_p in Kane [14] is equivalent to

$$\frac{1}{6}\left(-\Phi^{-1}(p_U) - \Phi^{-1}(p_L)\right).$$

3.1.2 Definition of BC_{pk}

Let X_1 and X_2 be the quality characteristics of interests with the specification limits $[LSL_1, USL_1]$ for X_1 and $[LSL_2, USL_2]$ for X_2. These limits define a rectangular specification area A. Assume that $\boldsymbol{X} = (X_1, X_2)^T$ follows a bivariate normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Applying eigenvalue-eigenvector decomposition to $\boldsymbol{\Sigma}$, we obtain two eigenvalues $\lambda_1^2 \geq \lambda_2^2 > 0$ and the associated eigenvectors, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Let $\boldsymbol{R} = [\boldsymbol{v_1}, \boldsymbol{v_2}]$, then $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$ and $\boldsymbol{\Sigma}$ can be expressed as $\boldsymbol{\Sigma} = \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^T$,

where $\boldsymbol{V}$ is the diagonal matrix with diagonal elements λ_1^2 and λ_2^2. In fact, the matrix $\boldsymbol{R}$ represents the rotation matrix that rotates the original axes to the main axes of the bivariate normal distribution (see Figure 3), $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ correspond to the main axes, and λ_1^2 and λ_2^2 are the variances on these main axes. More specifically, if we let $S_i = \boldsymbol{v}_i^T \boldsymbol{X}$, then $S_i \sim N(\boldsymbol{v}_i^T \boldsymbol{\mu}, \lambda_i^2)$, $i = 1, 2$, and S_1 and S_2 are independent. Suppose we move the origin to the process mean $\boldsymbol{\mu}$ and have the two new axes being in the directions of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then the two main axes divide the plane into four regions, A_1, A_2, A_3, and A_4. Obviously, $P(\boldsymbol{X} \in A_i) = 1/4$, $i = 1, \ldots, 4$. Denoting the specification region by A and $Q_i = A_i \cap A$, $i = 1, \ldots, 4$. Let $q_i = P(\boldsymbol{X} \in Q_i)$, $i = 1, \ldots, 4$. Then the probability that $\boldsymbol{X}$ is in A_i but not in the specification region is $p_i = 1/4 - q_i$ (see Figure 3).

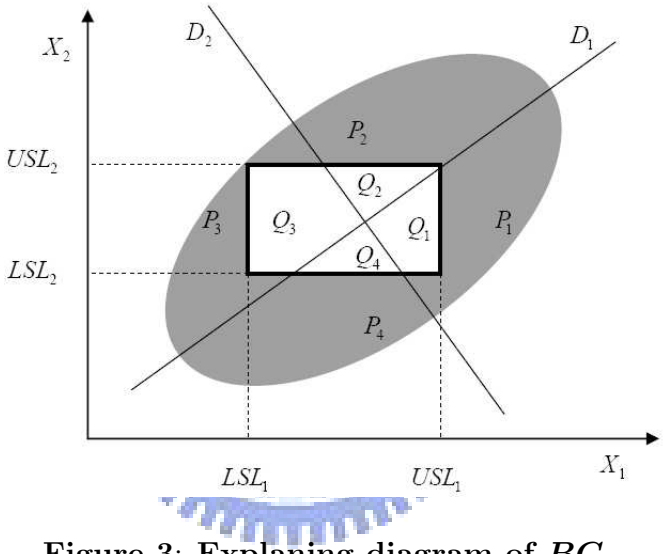

Figure 3: **Explaning diagram of BC_{pk}**

By analogy to the alternative definition of C_{pk} given in (4), Castagliola and Castellanos [4] defined a bivariate C_{pk} as

$$BC_{pk} = \frac{1}{3} min(-\Phi^{-1}(2p_1),\ -\Phi^{-1}(2p_2),\ -\Phi^{-1}(2p_3),\ -\Phi^{-1}(2p_4)).$$

This definition is similar to the alternative definition of C_{pk}, except that $0 \leq p_i \leq 1/4$, $i = 1, \ldots, 4$, in the bivariate case, while $0 \leq p_u\,(or\, p_L) \leq 1/2$ in the univarite case. We extend this definition to higher dimensions later.

3.1.3 Non-conforming Rate Based on BC_{pk}

According to the definition of BC_{pk} in the last subsection, we can establish a connection between the non-conforming rate ($\%NC$) and BC_{pk}. First note that

$$\begin{aligned} BC_{pk} &= \frac{1}{3}min(-\Phi^{-1}(2p_1),\ -\Phi^{-1}(2p_2),\ -\Phi^{-1}(2p_3),\ -\Phi^{-1}(2p_4)) \\ &= -\frac{1}{3}max(\Phi^{-1}(2p_1),\ \Phi^{-1}(2p_2),\ \Phi^{-1}(2p_3),\ \Phi^{-1}(2p_4)). \end{aligned}$$

Since $\Phi^{-1}(\cdot)$ is a strictly increasing function,

$$BC_{pk} = -\frac{1}{3}\Phi^{-1}(2p_{max}),$$

where $p_{max} = max(p_1,\ p_2,\ p_3,\ p_4)$. $\Phi^{-1}(\cdot)$ is a one-to-one function, so

$$p_{max} = \frac{1}{2}\Phi(-3BC_{pk}). \tag{5}$$

Note that $p_{max} \leq \%NC \leq 4p_{max}$. Plugging (5) into this inequality, we obtain

$$\frac{1}{2}\Phi(-3BC_{pk}) \leq \%NC \leq 2\Phi(-3BC_{pk}). \tag{6}$$

Although the lower bound of (6) is quite conservative, it is a convenient bound, meaning once the engineer gets a BC_{pk} value, he/she will know the bounds of non-conforming rate. The upper bound is very useful and is not a loose bound, meaning that it is reachable. Usually producers can take the upper bound of the non-conforming rate as an quality assurance to customers. For example, if the process is with BC_{pk}=1.00, one can guarantee that there will be 2700 non-conformities in 1,000,000 product items at most.

Table 1 gives the upper and lower bounds of the non-conforming rate $\%NC$ for various values of BC_{pk}. Figure 4 plots the bounds. We can see the bounds drop sharply as BC_{pk} increases and soon levels off when $BC_{pk} \geq 1.33$.

The second inequality of (6) is equivalent to

$$2\Phi(3BC_{pk}) - 1 \leq \%\text{ yield},$$

providing a same lower bound for the yield as in the univariate case [3]. The lower bound gives the worst level of the yield for a given BC_{pk}.

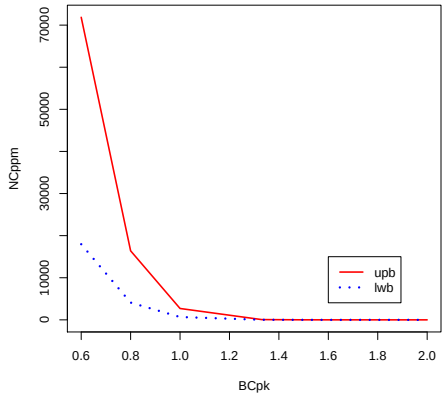

Figure 4: Bounds of non-conformity based on BC_{pk}

Table 1: Bounds of non-conformity based on BC_{pk}

BC_{pk}	Non-conformities in ppm	
	lwb	**upb**
0.60	17965.15956	71860.63823
0.80	4098.76796	16395.07185
1.00	674.94902	2699.79606
1.33	16.51832	66.07330
1.50	1.69884	6.79535
1.60	0.39666	1.58666
1.67	0.13608	0.54430
2.00	0.00049	0.00197

3.1.4 Extending C_{pk} to Higher Dimensions

Now we generalize the alternative definitions of C_{pk} and BC_{pk} to multivariate processes of $k > 2$ characteristics. By the same notion for the bivariate case, dividing the space R^k into 2^k subregions by the main axes of the k-variate distribution, we can define a multivariate C_{pk} index as

$$\begin{aligned} MC_{pk} &= \frac{1}{3} min(-\Phi^{-1}(2^{k-1}p_1),\ -\Phi^{-1}(2^{k-1}p_2), \ldots,\ -\Phi^{-1}(2^{k-1}p_{2^k})) \\ &= -\frac{1}{3} max(\Phi^{-1}(2^{k-1}p_1),\ \Phi^{-1}(2^{k-1}p_2), \ldots,\ \Phi^{-1}(2^{k-1}p_{2^k})) \\ &= -\frac{1}{3}\Phi^{-1}(2^{k-1}p_{max}), \end{aligned}$$

where p_i is the probability of a randomly selected sample being in the ith subregion, but not meeting the specification and $p_{max} = max(p_1,\ p_2, \ldots,\ p_{2^k})$. Equivalently,

$$p_{max} = \frac{1}{2^{k-1}}\Phi(-3MC_{pk}).$$

Since $p_{max} \leq \%NC \leq 2^k p_{max}$, we can also get an inequality of non-conforming rate in the general multivariate case as

$$\frac{1}{2^{k-1}}\Phi(-3MC_{pk}) \leq \%NC \leq 2\Phi(-3MC_{pk}),$$

which is equivalent to

$$1 - \frac{1}{2^{k-1}}\Phi(-3MC_{pk}) \leq \%\text{yield} \leq 1 - 2\Phi(-3MC_{pk}), \tag{7}$$

3.2 Estimation of BC_{pk}

In this subsection, we propose an algorithm to calculate $\widehat{MC}_{pk}$ for k-dimensional quality characteristic vector. We apply this algorithm to a bivariate case for calculating $\widehat{BC}_{pk}$. Then we use bootstrap approaches to make inferences on BC_{pk}.

3.2.1 A Natural Estimator of BC_{pk}

Algorithm for calculating $\hat{MC}_{pk}$:

1. Estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X_i} \text{ and } \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X_i} - \hat{\boldsymbol{\mu}})^T(\boldsymbol{X_i} - \hat{\boldsymbol{\mu}}).$$

2. Compute eigenvalues $\hat{\lambda}_1^2,\ \hat{\lambda}_2^2, \ldots, \hat{\lambda}_k^2$ and eigenvectors $\hat{\boldsymbol{v}}_1^2,\ \hat{\boldsymbol{v}}_2^2, \ldots, \hat{\boldsymbol{v}}_k^2$ of $\hat{\boldsymbol{\Sigma}}$.

3. Compute an estimate $\hat{q}_i$ of q_i and an estimate $\hat{p}_i$ of p_i by Monte Carlo integration as follows. Generate a very large number of data from $N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then compute

$$\hat{p}_i = \frac{\text{number of simulated data in } P_i}{\text{number of simulated data}} \text{ and } \hat{q}_i = \frac{\text{number of simulated data in } Q_i}{\text{number of simulated data}}. \tag{8}$$

4. Compute the estimate for MC_{pk} by

$$\widehat{MC}_{pk} = -\frac{1}{3}\Phi^{-1}(2^{k-1}\hat{p}_{max}),$$

where $\hat{p}_{max} = max(\hat{p}_1,\ \hat{p}_2, \ldots,\ \hat{p}_{2^k})$.

The method to calculate $\hat{p}_i$ and $\hat{q}_i$ proposed by Castagliola and Castellanos [4] is an integration method based on Green's formula. This method connot be directly extend to higher dimensions. Our Monte Carlo integration method, although requires intensive computation for higher dimensions, works for all dimensions.

However, when studying the distribution of $\widehat{BC}_{pk}$, we need to repeat the estimation procedure for each replication of generating $\widehat{BC}_{pk}$. Then the algorithm described above becomes computationally infeasible. More specifically, to generate one $\widehat{BC}_{pk}$ when $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we simulate n normal data from $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Normally, we could perform the algorithm described earlier to obtain a $\widehat{BC}_{pk}$, which requires generating N=1,000,000 (say) data from $N(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$. But if we need a large number of replications, say, thousands or hundreds of thousands, then the procedure becomes infeasible. To overcome this difficulty, we develop a method that requires generating the large amount of the standard multivariate normal data only once.

It is well known that the affine transformation of $\boldsymbol{Z} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})$ will transform a random vector $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to a standard multivariate normal $\boldsymbol{Z} \sim N(\boldsymbol{O}, \boldsymbol{I})$. So instead of generating N data from $N(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$, which is different for each replication, to perform Monte Carlo integration, we generate N data from $N(\boldsymbol{O}, \boldsymbol{I})$, which can be reused for all replications. To compute p_i's, we need to transform the specification region accordingly. When the specification region is a rectangle (or cube), we only need to transform the vertices and then construct the specification region in the transformed space. After that, we only need to compute q_i's and p_i's by (8).

One of the most important advantages of our method in calculating $\hat{p}_i$'s and $\hat{q}_i$'s is that it allows the computation of $\widehat{BC}_{pk}$ not only for rectangular specifications but also for specification regions of any shape as long as we have a way to describe them. However, for convenience, we still illustrate our method by examples of rectangular specifications, since this shape is widely used in factories.

Four different examples are considered in the study. Table 2 lists the distribution parameters and specifications for each case.

Table 2: Parameters and specifications of BC_{pk} examples

	Distribution parameters				
	μ_1	σ_1^2	μ_2	σ_2^2	ρ
Case1	6.0	0.8	7.0	1.0	0.0
Case2	5.0	0.5	6.0	0.45	0.5
Case3	3.0	1.0	6.0	1.0	0.2
Case4	1.0	1.0	1.0	1.0	0.2

	X_1 spec		X_2 spec	
	LSL_1	USL_1	LSL_2	USL_2
Case1	2.0	10.0	3.0	10.0
Case2	2.0	9.0	3.0	10.0
Case3	0.5	6.5	1.0	7.0
Case4	3.0	5.0	1.0	3.0

Note that

- Case 1 is supposed to be a capable process.
- Case 2 is supposed to be a satisfactory process.
- Case 3 is supposed to be an inadequate process because of large variability and shift.
- Case 4 is supposed to be a poor process because the variability of the process is large and the process mean is out of specification.

Figure 5 plots, for each case, a set of sample data, the specification region, and two orthogonal lines passing through sample mean and having eigenvectors of $\hat{\mathbf{\Sigma}}$ as their directions. To investigate the distribution of $\widehat{BC}_{pk}$, we generate 1000 sets of data for each case. Tables 3-6 present Cases 1-4, respectively, with the true values, sample mean and sample standard deviation of 1000 values of $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$, $\hat{q}_4$, and $\widehat{BC}_{pk}$. The bias defined as the difference of the sample mean and the true value is also included. Results indicate that $\widehat{BC}_{pk}$ is a reasonable estimator.

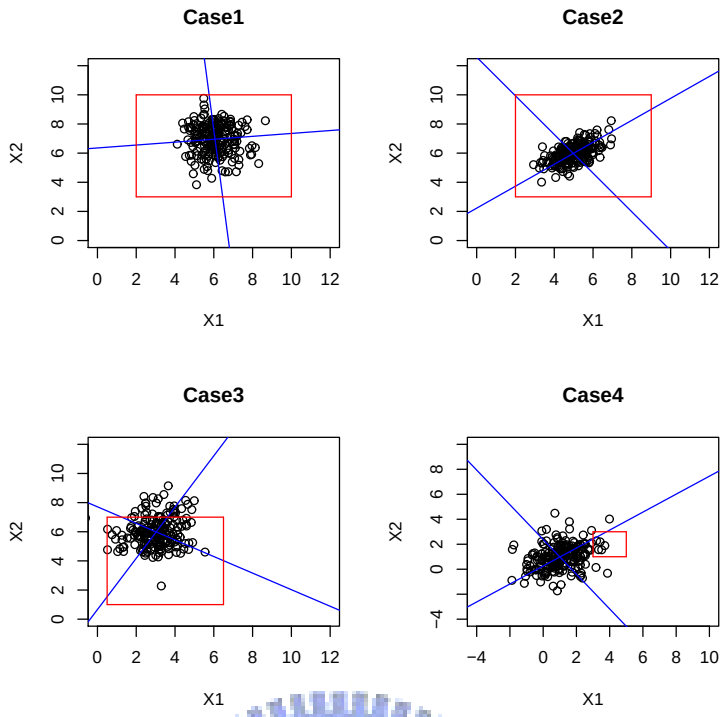

Figure 5: Plots of 4 cases under study

Table 3: Estimation results of Case 1

	True value	Sample mean	Sample sd.	Bias
q_1	0.247914	0.247572	0.000983	0.000342
q_2	0.247901	0.247554	0.001018	0.000347
q_3	0.249926	0.249791	0.000061	0.000135
q_4	0.249921	0.249801	0.000063	0.000012
$\boldsymbol{BC_{pk}}$	**0.878933**	**0.862306**	**0.1085323**	**0.016627**

Table 4: Estimation results of Case 2

	True value	Sample mean	Sample sd.	Bias
q_1	0.249998	0.249992	0.000024	0.000006
q_2	0.250000	0.249999	0.000003	0.000001
q_3	0.249947	0.249905	0.000130	0.000042
q_4	0.249942	0.249098	0.000131	0.000844
$\boldsymbol{BC_{pk}}$	**1.227117**	**1.238882**	**0.111481**	**0.011765**

Table 5: Estimation results of Case 3

	True value	Sample mean	Sample sd.	Bias
q_1	0.121880	0.121354	0.035231	0.000526
q_2	0.121316	0.120687	0.034688	0.000629
q_3	0.247951	0.247775	0.001437	0.000176
q_4	0.249857	0.249806	0.000413	0.000051
$\boldsymbol{BC_{pk}}$	**0.217160**	**0.221018**	**0.073617**	**0.003858**

Table 6: Estimation results of Case 4

	True value	Estimate mean	Estimate sd.	Bias
q_1	0.015005	0.015057	0.006267	0.000221
q_2	0.002751	0.246892	0.002791	0.003108
q_3	0.000000	0.000000	0.000000	0.000000
q_4	0.000000	0.000000	0.000000	0.000000
$\boldsymbol{BC_{pk}}$	**0.000000**	**0.000000**	**0.000000**	**0.000000**

3.2.2 Estimating Lower Confidence Bound by Bootstrap Approach

Note that we can only obtain a $\widehat{BC}_{pk}$ with a set of data. To infer anything on BC_{pk} directly, say, by obtaining a lower confidence bound, usually we would need many $\widehat{BC}_{pk}$'s. However, in many applications, repeating experiments to have a number of estimates is not possible or economical. So Efron [9] introduced a computationally intensive but effective estimation method called "bootstrap", which is a data-driven technique for statistical inferences. One can repeat the small-data-size resampling procedure many times to infer parameters in population without model assumptions.

In this subsection, we emphasize the lower confidence bound rather than the confidence interval. Since BC_{pk} is an index of yield assurance, lower confidence bound represents the worst yield at a certain confidence level, that is, the true yield will not be worse than the yield corresponding to the lower confidence bound.

Suppose that we have a sample $\{X_1, X_2, \ldots, X_n\}$ of size n from a population F with parameter $\theta = \theta(F)$. Resampling a sample of size n with replacement from $\{X_1, X_2, \ldots, X_n\}$,

we can get

$$X_1{}^*,\ X_2{}^*,\ldots,\ X_n{}^* \overset{iid}{\sim} F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1(X_j \le x).$$

Repeating this for B times and calculate $\hat{\theta}_i^* = \theta(F_n{}^*)$, $i = 1,\ldots,B$, where $F_n{}^*(x) = \frac{1}{B}\sum_{i=1}^{B} 1(X_j{}^* \le x)$. Then we can make some inferences on θ based on the bootstrap estimates $\hat{\theta}_1^*$, $\hat{\theta}_2^*$,..., $\hat{\theta}_B^*$.

In this section, we try four types of bootstrap methods to estimate the confidence lower bound [8] [10] of BC_{pk}, including Basic Bootstrap Method, Percentile Bootstrap Method, Standard Bootstrap Method, and Bias Corrected accelerated Percentile Bootstrap (BCPB) Method.

Basic Bootstrap Method

Following [8], we can get a $100(1-\alpha)\%$ confidence interval by the basic bootstrap method as

$$\left[2\hat{\theta} - \hat{\theta}^*_{([B(1-\frac{\alpha}{2})])},\ 2\hat{\theta} - \hat{\theta}^*_{([B(\frac{\alpha}{2})])}\right],$$

where $\hat{\theta}^*_{(i)}$ is the ith ordered estimate from the bootstrap procedure and $\hat{\theta}$ is the estimate from the original sample. To provide a lower bound on the process yield, we are interested in getting a 100(1- α)% lower confidence bound. The basic bootstrap method gives such a lower confidence bound as $2\hat{\theta} - \hat{\theta}^*_{([B(1-\alpha)])}$.

Standard Bootstrap Method

The average and standard deviation of B bootstrap estimates $\hat{\theta}_1^*$, $\hat{\theta}_2^*,\ldots,$ $\hat{\theta}_B^*$ are, respectively,

$$\bar{\theta}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{\theta}_i^* \text{ and } \hat{S}^* = \sqrt{\frac{1}{B-1}\sum_{i=1}^{n}(\hat{\theta}_i^* - \bar{\theta}^*)^2}.$$

We can use normal approximation to obtain the 100(1-α)% confidence interval of θ based on the standard method as

$$\left[\bar{\theta}^* - Z_{\frac{\alpha}{2}}\hat{S}^*,\ \bar{\theta}^* + Z_{\frac{\alpha}{2}}\hat{S}^*\right].$$

The 100(1- α)% lower confidence bound of the standard method is $\bar{\theta}^* - Z_\alpha \hat{S}^*$.

Percentile Bootstrap Method

We take the α and the $1-\alpha$ percentage points to construct the 100(1- α)% confidence interval of the percentile method as

$$\left[\hat{\theta}^*_{([B(\frac{\alpha}{2})])},\ \hat{\theta}^*_{([B(1-\frac{\alpha}{2})])}\right]$$

and the 100(1- α)% lower confidence bound is $\hat{\theta}^*_{([B(\alpha)])}$.

Bias-Corrected Percentile Bootstrap Method

It is possible that bootstrap distributions obtained using only a sample of the complete bootstrap distribution may be shifted higher or lower than would be expected, thus Bias-Corrected Percentile Bootstrap Method (BCPB) was suggested by [10] to correct this bias.

First, using the distribution of $\hat{\theta}^*$, calculate the probability

$$p_0 = P(\hat{\theta}^* \leq \hat{\theta})$$

by the proportion of $\hat{\theta}^*_i$'s satisfying $\hat{\theta}^*_i \leq \hat{\theta}$.
Second, calculate

$$\begin{aligned} z_0 &= \Phi^{-1}(p_0), \\ P_{L,\alpha/2} &= \Phi(2z_0 - z_{\alpha/2}), \\ P_{U,\alpha/2} &= \Phi(2z_0 + z_{\alpha/2}). \end{aligned}$$

Finally, the 100(1- α)% confidence interval obtained from BCPB method is

$$\left[\hat{\theta}^*_{([BP_{L,\alpha/2}])},\ \hat{\theta}^*_{([BP_{U,\alpha/2}])}\right].$$

The 100(1- α)% lower confidence bound is $\hat{\theta}^*_{([BP_{L,\alpha}])}$.

As an illustrative example, we use the above four bootstrap methods to estimate 90% lower confidence bound for the case of $BC_{pk} = 1.00$. Consider various sample size

n=30(10)100, 125(25)200, 250, 300. For each sample, we use the algorithm described before to obtain a $\widehat{BC}_{pk}$. Here for the Monte Carlo integration, we generate $N = 1,000,000$ data from $N(\boldsymbol{O}, \boldsymbol{I})$. We then perform the bootstrap resampling $B = 3,000$ times to obtain 3,000 bootstrap estimates of BC_{pk}. With these 3,000 estimates, we obtain a lower confidence bound (LCB) for each of the four bootstrap methods. Repeating the above steps for 100 times, we then have 1000 LCBs.

The bootstrap results are shown in Table 7. Table 7 lists the estimated LCBs (with 90% confidence level) and their tolerance errors. The estimated LCB is the mean of 100 LCB estimates. The tolerance error is three times the standard error of the estimated lower confidence bounds, which is obtained by dividing the sample standard deviation of the 100 LCBs estimates by 10. The true lower bound of BC_{pk} will likely be between the estimated LCBs $\pm$ tolerance error. In general, the LCB will get closer to the true value 1.00 and the tolerance error becomes smaller as data size n gets larger. Comparing the performances of four methods from this table, we find that, the basic method has the closest lower bound to 1.00, while the standard method has the poorest. As for the tolerance error, BCPB has the smallest while basic method has the largest. We can conclude that, the basic method has the closest lower bound to 1.00 but with the largest variance, the percentile and BCPB methods both have high accuracy in estimating the lower bound but the lower bound is not as close to 1.00 as the basic method, and the standard method has the poorest lower bound but the tolerance error is acceptable. When data size is small, the standard method fails, the basic method performs poorly, the percentile method and the BCPB perform satisfactorily with the latter slightly better.

Table 7: Bootstrap results of LCB when $BC_{pk} = 1$

Basic			Percentile		
Data size	LCB	Tolerance error	Data size	LCB	Tolerance error
300	0.9196162	0.0156318	300	0.9079113	0.0122382
250	0.9150506	0.0177561	250	0.9040253	0.0142644
200	0.9094465	0.0184926	200	0.8980887	0.0150957
175	0.8972231	0.0205290	175	0.8901748	0.0253932
150	0.8872071	0.0218907	150	0.8796073	0.0185502
125	0.8810805	0.0240291	125	0.8717951	0.0205026
100	0.8734203	0.0252222	100	0.8663372	0.0223302
90	0.8636612	0.0277125	90	0.8613130	0.0238812
80	0.8557330	0.0286449	80	0.8547115	0.0242241
70	0.8584181	0.0340782	70	0.8511570	0.0281736
60	0.8400805	0.0357033	60	0.8412904	0.0201740
50	0.8309082	0.0415170	50	0.8408328	0.0327690
40	0.8095727	0.0490749	40	0.8249969	0.0376905
30	0.5616968	0.5741250	30	0.7966476	0.0402990
Standard			**Bias-Corrected Percentile**		
Data size	LCB	Tolerance error	Data size	LCB	Tolerance error
300	0.8811285	0.0139002	300	0.9078527	0.0120291
250	0.8740663	0.0159999	250	0.9045532	0.0140316
200	0.8640700	0.0167988	200	0.8984700	0.0148947
175	0.8513862	0.0185649	175	0.8916538	0.0159915
150	0.8384348	0.0200160	150	0.8818719	0.0184107
125	0.8272374	0.0219966	125	0.8745090	0.0205578
100	0.8150309	0.0235320	100	0.8707972	0.0222366
90	0.8046600	0.0257322	90	0.8657128	0.0233943
80	0.7932829	0.0262065	80	0.8615557	0.0244539
70	0.7782295	0.0316164	70	0.8570728	0.0280749
60	0.5724551	0.0306450	60	0.8485384	0.0298275
50	0.6579983	0.0424130	50	0.8569380	0.0375157
40	0.4642365	0.1272390	40	0.8365426	0.0371589
30	0.0183462	0.5531538	30	0.8455312	0.0541467

3.2.3 Empirical Cumulative Distribution Function of BC_{pk}

In this subsection, we describe how to simulate empirical cumulative distribution function (ECDF) of $\widehat{BC}_{pk}$. First, we generate 100,000 $\widehat{BC}_{pk}$ by the algorithm given in subsection 3.2.1 with sample data of size 500 and 10,000,000 simulated $N_2(\boldsymbol{O}, \boldsymbol{I})$ data for Monte Carlo integration. Then we calculate the ECDF and compare it with a normal distribution.

We perform the simulation with the procedures mentioned above for two special cases: BC_{pk}=1.00 and BC_{pk}=1.33, each has 100,000 replications to find the sampling distribution of $\widehat{BC}_{pk}$. To see if $\widehat{BC}_{pk}$ behaves like a normal distribution. Figures 6-8 plot ECDF, Q-Q plots, and the histograms of the 100,000 simulated $\widehat{BC}_{pk}$. From these plots, we can conclude that the sampling distribution of $\widehat{BC}_{pk}$ is fairly close to a normal distribution. We also find that the sampling distribution for BC_{pk}=1.00 is closer to normal than that for BC_{pk}=1.33. We note that the ECDF curve of BC_{pk}=1.33 in Figure 6(b) moves off a little from the normal curve, but we have no explanation for it.

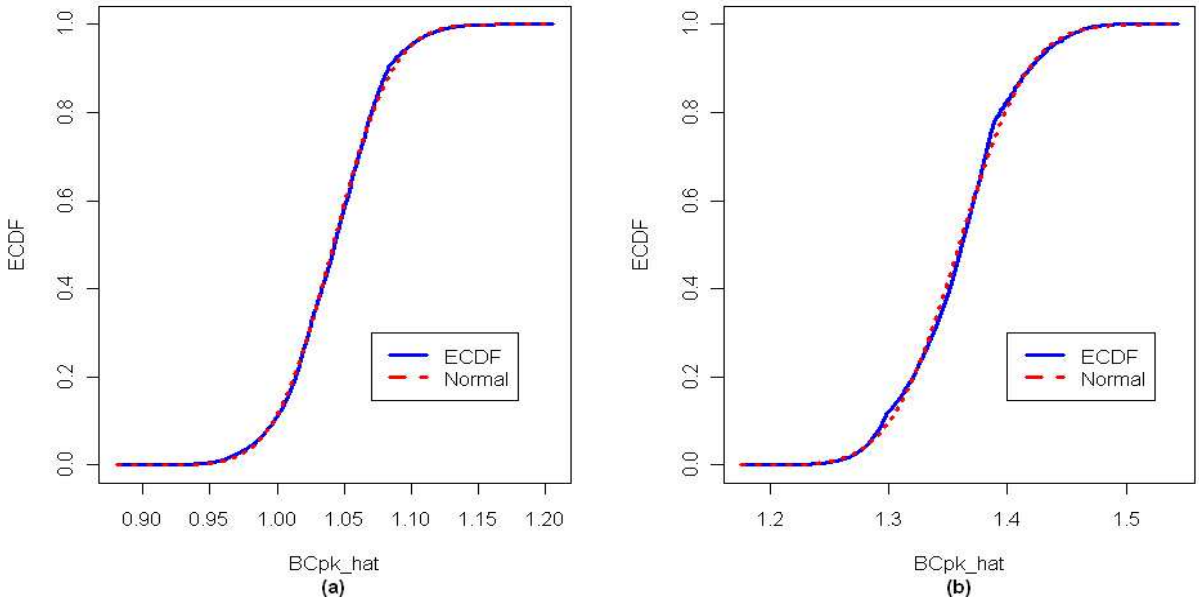

Figure 6: Comparing empirical cumulative distribution function and a normal distribution, when (a) $BC_{pk} = 1.00$ (b) $BC_{pk} = 1.33$

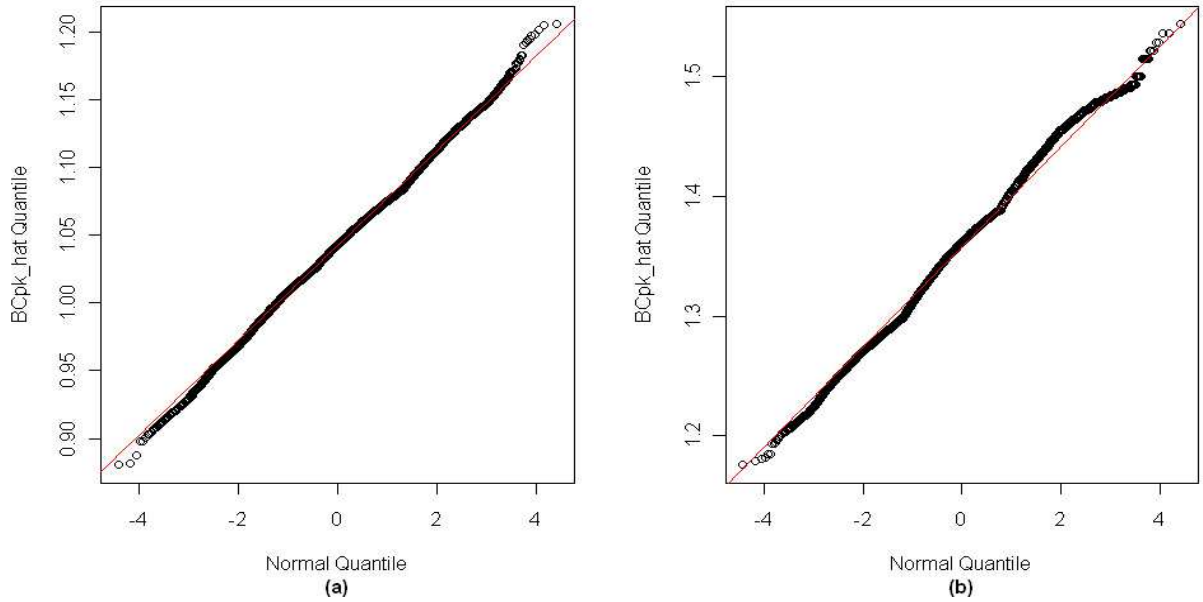

Figure 7: **Q-Q plot of $\widehat{BC}_{pk}$, when (a) $BC_{pk} = 1.00$ (b) $BC_{pk} = 1.33$**

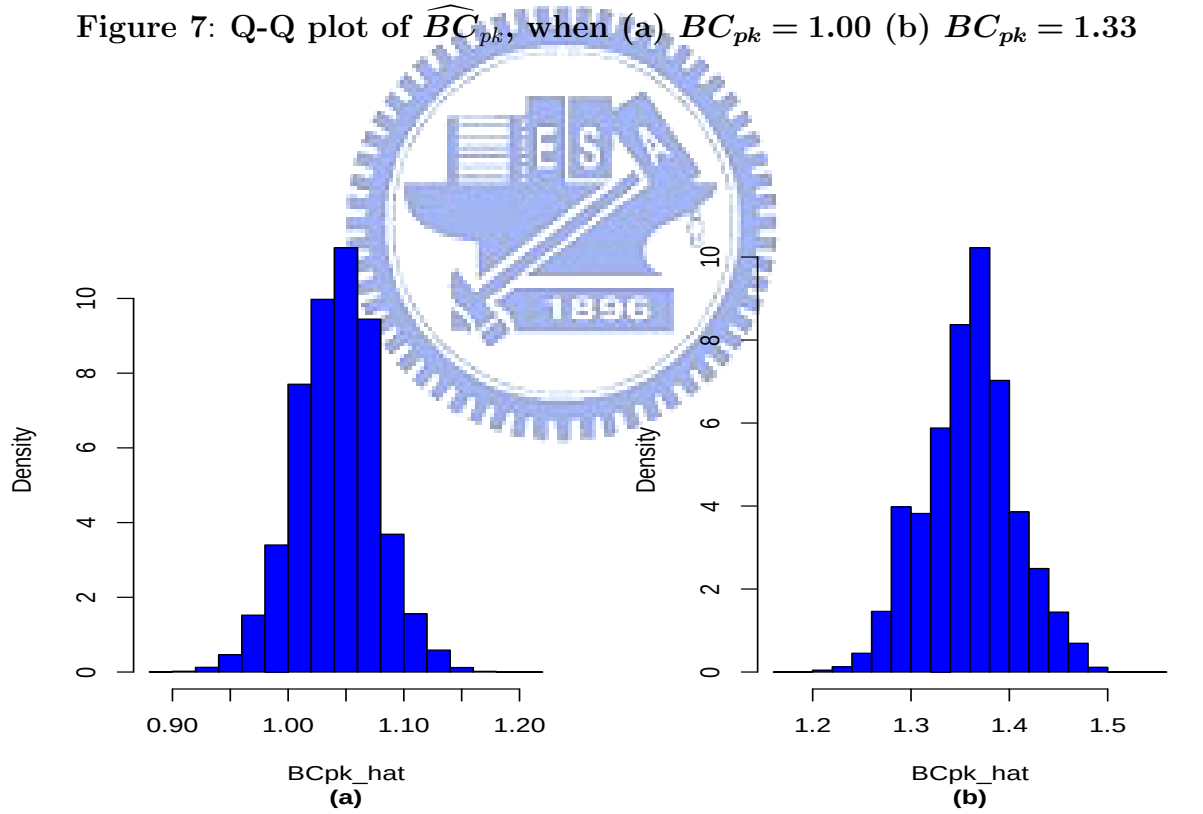

Figure 8: **Histogram of 100,000 $\widehat{BC}_{pk}$, when (a) $BC_{pk} = 1.00$ (b) $BC_{pk} = 1.33$**

4 Bivariate C_p Index: BC_p

4.1 Variation Measuring Index for Processes with Multiple Characteristics

4.1.1 Definition of BC_p

By analogy to the univariate case, Castagliola and Castellanos [4] defined a new bivariate C_p index, called BC_p, as the maximum value of BC_{pk}, i.e.,

$$BC_p \equiv \max_{\boldsymbol{\mu},\theta} BC_{pk}.$$

Because of BC_{pk} itself is defined as the minimum of the four values, $-\Phi^{-1}(2p_1)$, $-\Phi^{-1}(2p_2)$, $-\Phi^{-1}(2p_3)$, and $-\Phi^{-1}(2p_4)$, the maximum value of BC_{pk} is necessarily reached when $-\Phi^{-1}(2p_1) = -\Phi^{-1}(2p_2) = -\Phi^{-1}(2p_3) = -\Phi^{-1}(2p_4)$, i.e., when $p_1 = p_2 = p_3 = p_4 = p/4$ and $p = p_1 + p_2 + p_3 + p_4$ is minimum, or when $q_1 = q_2 = q_3 = q_4 = q/4$ and $q = q_1 + q_2 + q_3 + q_4$ is maximum. Therefore

$$BC_p = -\frac{1}{3}\Phi^{-1}(2 \times \frac{p}{4}) = -\frac{1}{3}\Phi^{-1}(\frac{p}{2}) = -\frac{1}{3}\Phi^{-1}\left(2\left(\frac{1}{4} - q_1\right)\right).$$

Now the question is how to get $p_1 = p_2 = p_3 = p_4 = p/4$ (or $q_1 = q_2 = q_3 = q_4 = q/4$). In the univariate case, the answer is that, when $\mu = (LSL + USL)/2$ (i.e., the process mean is on the center of specification), $C_p = C_{pk}$. In the bivariate case, Castagliola and Castellanos [4] let $\mu_1 = (LSL_1 + USL_1)/2$, $\mu_2 = (LSL_2 + USL_2)/2$, and found the optimal p by varying the rotation angle θ of the rotation matrix $\boldsymbol{R}$. However, we find that BC_p defined above is not scale-invariant, in the sense that BC_p will change its value if we scale the process with different scales in X_1 and X_2 coordinates. Luckily we find that BC_{pk} is scale-invariant. Take the four examples in Subsection 3.2.1 as examples to demonstrate this scaling problem. We scale each case by $X_1' = 2X_1$ and $X_2' = 3X_2$. The parameters and specifications of each case after scaling are listed in Table 8.

Table 9 presents the values of BC_p and BC_{pk} for all cases. We note that, θ, p, and BC_p may change their values after scaling. In practice, using different units for quality characteristics is fairly common and should not affect process assessment. Thus a well-defined capability index should be invariant of scaling. In the next subsection, we propose a simple solution to fix this problem.

Table 8: Parameters and specifications of 4 examples before and after scaling

	Distribution parameters				
	μ_1	σ_1^2	μ_2	σ_2^2	ρ
Case1'	12.0	3.2	21.0	9.0	0.0
Case2'	10.0	2.0	18.0	4.05	0.5
Case3'	6.0	4.0	18.0	9.0	0.2
Case4'	2.0	4.0	3.0	9.0	0.2

	X_1 spec		X_2 spec	
	LSL_1	USL_1	LSL_2	USL_2
Case1'	4.0	20.0	9.0	30.0
Case2'	4.0	18.0	9.0	30.0
Case3'	1.0	13.0	3.0	21.0
Case4'	6.0	10.0	3.0	9.0

Table 9: BC_p and BC_{pk} values of 4 examples before and after scaling

	BC_{pk}	BC_p	
Case1	0.878933	1.169641	$\theta = 0.0000,\ p = 0.000473$
Case1'	0.878933	1.203389	$\theta = 22.3199,\ p = 0.000306$
Case2	1.227117	1.163055	$\theta = 45.0000,\ p = 0.000001$
Case2'	1.227117	1.421630	$\theta = 90.0000,\ p = 0.000020$
Case3	0.217160	0.873872	$\theta = 45.0000,\ p = 0.008814$
Case3'	0.217160	0.936945	$\theta = 90.0000,\ p = 0.004999$
Case4	0.000000	0.209254	$\theta = 45.0000,\ p = 0.0530151$
Case4'	0.000000	0.210699	$\theta = 90.0000,\ p = 0.527320$

4.1.2 Pre-processing Data

Our solution for BC_p to be invariant is fairly simple: standardize the data and specifications such that the specification rectangle becomes a square centered at origin (0,0). Let the quality characteristic vector $\boldsymbol{X} = (X_1, X_2)^T$ and the specification region be $[LSL_1, USL_1] \times [LSL_2, USL_2]$. We transform $\boldsymbol{X} = (X_1, X_2)^T$ into $\boldsymbol{X'} = (X_1', X_2')^T$ by

$$X_1' = \frac{1}{USL_1 - LSL_1}\left(X_1 - \frac{LSL_1 + USL_1}{2}\right),$$
$$X_2' = \frac{1}{USL_2 - LSL_2}\left(X_2 - \frac{LSL_2 + USL_2}{2}\right),$$

which transforms the specification region into an unit square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. In fact, the unit length is not necessary, a square centered at the origin is sufficient.

Suppose $\boldsymbol{X} = (X_1, X_2)^T$ is a bivariate normal random vector following $N(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$. Then $\boldsymbol{X'} = (X_1', X_2')^T$ follows $N(\mu_1',\ \sigma_1^{2'},\ \mu_2',\ \sigma_2^{2'},\ \rho' = \rho)$, where

$$
\begin{aligned}
\mu_1' &= \frac{1}{USL_1 - LSL_1}\left(\mu_1 - \tfrac{LSL_1+USL_1}{2}\right), \\
\sigma_1^{2'} &= \frac{\sigma_1^2}{(USL_1 - LSL_1)^2}, \\
\mu_2' &= \frac{1}{USL_2 - LSL_2}\left(\mu_2 - \tfrac{LSL_2+USL_2}{2}\right), \\
\sigma_2^{2'} &= \frac{\sigma_2^2}{(USL_2 - LSL_2)^2}, \\
\rho' &= \frac{E(X_1'X_2') - \mu_1'\mu_2'}{\sigma_1'\sigma_2'} \\
&= \frac{E(X_1X_2) - \frac{LSL_2+USL_2}{2}E(X_1) - \frac{LSL_1+USL_1}{2}E(X_2) + \frac{LSL_1+USL_1}{2}\frac{LSL_2+USL_2}{2}}{\sigma_1\sigma_2} \\
&\quad - \frac{(\mu_1 - \frac{LSL_1+USL_1}{2})(\mu_2 - \frac{LSL_2+USL_2}{2})}{\sigma_1\sigma_2} \\
&= \rho.
\end{aligned}
$$

Table 10 presents the BC_p values obtained with the pre-processing step for 4 examples before and after scaling. In fact, since the distribution of $\boldsymbol{X}$ and the specification region will be the same after the transformation for processes with different scales, BC_p will be the same.

Table 10: **The results of BC_p after pre-processing**

Case	BC_p	p
1	0.944225	0.0046920
1'	0.944225	0.0046920
2	1.243213	0.0001918
2'	1.243213	0.0001918
3	0.873872	0.0088140
3'	0.873872	0.0088140
4	0.209254	0.0530151
4'	0.209254	0.0530151

4.2 Estimation of BC_p

In this subsection, we first develop an algorithm to calculate $\widehat{BC}_p$, and then we derive an approximate normal distribution for $\widehat{BC}_p$ by Taylor expansion. Based on this normal approximation, we will develop procedures for making statistical inferences on BC_p, including testing whether process is capable or not by hypothesis testing and constructing a confidence interval of BC_p to obtain the precision of the estimate, and providing a lower confidence bound.

4.2.1 Estimation of BC_p

Algorithm for calculating $\widehat{BC}_p$:

1. Transform data and specification by
$X_1' = \frac{1}{USL_1 - LSL_1}\left(X_1 - \frac{LSL_1 + USL_1}{2}\right)$,
$X_2' = \frac{1}{USL_2 - LSL_2}\left(X_2 - \frac{LSL_2 + USL_2}{2}\right)$,
$[LSL_1', USL_1'] = [-\frac{1}{2}, \frac{1}{2}]$, and $[LSL_2', USL_2'] = [-\frac{1}{2}, \frac{1}{2}]$

2. Set $\hat{\mu}_1 = 0$, and $\hat{\mu}_2 = 0$. Compute the sample covariance matrix $\hat{\Sigma}$ from the transformed data.

3. Calculate the eigenvalues $\hat{\lambda}_1^2$ and $\hat{\lambda}_2^2$ of $\hat{\Sigma}$.

4. Looping θ over $0 \leq \theta \leq 360°$ to rotate the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ by an angle θ and compute $\hat{q}(\theta)$, the probability of a bivariate normal vector $\boldsymbol{X}$ falls in the rotated square by Monte Carlo integration (see Figure 9).

5. Find $\hat{q} = \max_{0 \leq \theta \leq 360°} \hat{q}(\theta)$ and $q_1 = q_2 = q_3 = q_4$.

6. Compute $\widehat{BC}_p = -\frac{1}{3}\Phi^{-1}(\frac{1-\hat{q}}{2})$.

We apply this estimating algorithm to various bivariate normal processes and find that

we always get $\theta = 45°$ for the optimal $\hat{q}$, which is intuitively reasonable. Therefore, we can simplify the algorithm by setting $\theta = 45°$ in Step 4 instead of looping over various θ to calculate the optimal $\hat{q}$, then compute $\widehat{BC}_p$. This would save computing time tremendously. We conjecture that $\theta = 45°$ will give the BC_p value, but we do not have a formal proof yet.

4.2.2 Normal Approximation

The exact distribution of $\widehat{BC}_p$ is mathematically intractable. However, we can obtain a normal approximation to the distribution of $\widehat{BC}_p$ by taking its first-order Taylor expansion.

Since BC_p is a function of q, we first show the formula of calculating q. According to the algorithm in the last section, if we just consider the relative position of the process and specification region, we can consider the process as following $N(0, \lambda_1^2, 0, \lambda_2^2, 0)$ and the square with vertices $(\sqrt{2}/2, 0)$, $(0, \sqrt{2}/2)$, $(-\sqrt{2}/2, 0)$, $(0, -\sqrt{2}/2)$ as the specification region. Because of specification is symmetric about X_1 and X_2 axes (see Figure 9), we can focus on the yield in the light gray area. Then

$$\begin{aligned}
q &= 4\int_0^{\sqrt{2}/2}\int_0^{-x_1+\sqrt{2}/2} \frac{1}{2\pi\lambda_1\lambda_2}\, e^{-\frac{x_1^2}{2\lambda_1^2}-\frac{x_2^2}{2\lambda_2^2}}\, dx_2 dx_1 \\
&= 4\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2\pi}\lambda_1}\, e^{-\frac{x_1^2}{2\lambda_1^2}} \int_0^{-x_1+\sqrt{2}/2} \frac{1}{\sqrt{2\pi}\lambda_2}\, e^{-\frac{x_2^2}{2\lambda_2^2}}\, dx_2 dx_1 \\
&= 4\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2\pi}\lambda_1}\, e^{-\frac{x_1^2}{2\lambda_1^2}} \left(\Phi\left(\frac{-x_1+\sqrt{2}/2}{\lambda_2}\right) - 1/2\right) dx_1 \\
&= 4\int_0^{\sqrt{2}/2} \phi\left(\frac{x_1}{\lambda_1}\right) \Phi\left(\frac{-x_1+\sqrt{2}/2}{\lambda_2}\right) dx_1 - 2\Phi\left(\frac{\sqrt{2}}{2\lambda_1}\right) + 1.
\end{aligned}$$

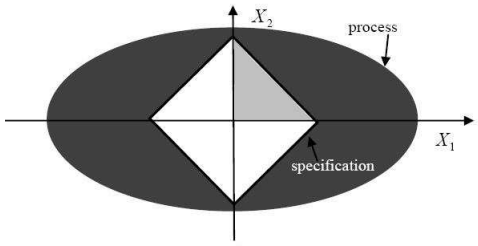

Figure 9: **The relative position of the specification square w.r.t. the bivariate distribution corresponding to BC_p**

Let $q(\lambda_1, \lambda_2) = q = 4\int_0^{\sqrt{2}/2} \phi\left(\frac{x_1}{\lambda_1}\right) \Phi\left(\frac{-x_1+\sqrt{2}/2}{\lambda_2}\right) dx_1 - 2\Phi\left(\frac{\sqrt{2}}{2\lambda_1}\right) + 1$. The partial derivatives of $q(\lambda_1, \lambda_2)$ with respective to λ_1 and λ_2 are, respectively,

$$\begin{aligned}
\frac{\partial q(\lambda_1,\lambda_2)}{\partial \lambda_1} &= 4\int_0^{\sqrt{2}/2} \phi\left(\frac{x_1}{\lambda_1}\right)\left(\frac{x_1^2}{\lambda_1^2} - \frac{1}{\lambda_1}\right) \Phi\left(\frac{-x_1+\sqrt{2}/2}{\lambda_2}\right) dx_1 + \phi\left(\frac{\sqrt{2}}{2\lambda_1}\right)\left(\frac{\sqrt{2}}{\lambda_1^2}\right), \\
\frac{\partial q(\lambda_1,\lambda_2)}{\partial \lambda_2} &= 4\int_0^{\sqrt{2}/2} \phi\left(\frac{x_1}{\lambda_1}\right) \phi\left(\frac{-x_1+\sqrt{2}/2}{\lambda_2}\right)\left(\frac{x_1-\sqrt{2}/2}{\lambda_2^2}\right) dx_1,
\end{aligned}$$

which can be evaluated numerically when given λ_1 and λ_2.

Denote $Q_1(\lambda_1, \lambda_2) \equiv \frac{\partial q(\lambda_1,\lambda_2)}{\partial \lambda_1}$ and $Q_2(\lambda_1, \lambda_2) \equiv \frac{\partial q(\lambda_1,\lambda_2)}{\partial \lambda_2}$. Then an approximate distribution of $\widehat{BC}_p$ can be obtained as

$$\widehat{BC}_p \approx N\left(BC_p, \frac{Q_1^2(\lambda_1,\lambda_2)\lambda_1^2 + Q_2^2(\lambda_1,\lambda_2)\lambda_2^2}{2 \times 36n[\phi(3BC_p)]^2}\right)$$

Detail of using Taylor expansion to derive the normal approximation is given below.

Let $f(q) = BC_p = -\frac{1}{3}\Phi^{-1}\left(\frac{1-q}{2}\right)$. Then

$$\begin{aligned}
\widehat{BC}_p &= f(\hat{q}) \\
&\approx f(q) + \frac{df(q)}{dq}(\hat{q} - q) \\
&= -\frac{1}{3}\Phi^{-1}\left(\frac{1-q}{2}\right) + \frac{1}{6\phi\left(\Phi^{-1}\left(\frac{1-q}{2}\right)\right)}(\hat{q} - q) \\
&= BC_p + \frac{1}{6\phi(3BC_p)}(\hat{q} - q).
\end{aligned}$$

By [1], we have

$$\sqrt{n}\left(\begin{bmatrix}\hat{\lambda}_1\\ \hat{\lambda}_2\end{bmatrix}-\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}\right)\xrightarrow{d} N\left(0,\begin{bmatrix}\frac{1}{2}\lambda_1^2 & 0\\ 0 & \frac{1}{2}\lambda_2^2\end{bmatrix}\right) \text{ as } n\to\infty.$$

One can infer that as $n\to\infty$,

$$\sqrt{n}(\hat{q}(\hat{\lambda}_1,\hat{\lambda}_2)-q(\lambda_1,\lambda_2))\xrightarrow{d} N\left(0,\ \frac{1}{2}\left(Q_1^2(\lambda_1,\lambda_2)\lambda_1^2+Q_2^2(\lambda_1,\lambda_2)\lambda_2^2\right)\right).$$

Now let $Z_q=\sqrt{n}(\hat{q}(\hat{\lambda_1},\hat{\lambda_2})-q(\lambda_1,\lambda_2))$, then

$$\widehat{BC}_p\approx BC_p+\frac{1}{6\sqrt{n}\phi(3BC_p)}Z_q.$$

As a result, $\widehat{BC}_p$ has an approximate normal distribution with mean and variance:

$$E(\widehat{BC}_p)\approx BC_p,\ \ Var(\widehat{BC}_p)\approx\frac{Q_1^2(\lambda_1,\lambda_2)\lambda_1^2+Q_2^2(\lambda_1,\lambda_2)\lambda_2^2}{2\times 36n[\phi(3BC_p)]^2}.$$

We can establish a 100(1-α)% approximate confidence interval and lower confidence bound from the approximate distribution of $\widehat{BC}_p$ as follows:

$$\text{Lower confidence bound: } \widehat{BC}_p-Z_\alpha\frac{\left(Q_1^2(\lambda_1,\lambda_2)\lambda_1^2+Q_2^2(\lambda_1,\lambda_2)\lambda_2^2\right)^{1/2}}{\sqrt{72n}\phi(3\widehat{BC}_p)}, \tag{9}$$

$$\text{Confidence Interval: } \widehat{BC}_p\pm Z_\alpha/2\frac{\left(Q_1^2(\lambda_1,\lambda_2)\lambda_1^2+Q_2^2(\lambda_1,\lambda_2)\lambda_2^2\right)^{1/2}}{\sqrt{72n}\phi(3\widehat{BC}_p)}. \tag{10}$$

4.2.3 Hypothesis Testing

Hypothesis testing with significance level α used for examining whether the process capability meets customers' demands can be stated as follows:

$$\begin{aligned} H_0 &: BC_p\le C \text{ (process is not capable)}\\ H_1 &: BC_p> C \text{ (process is capable)}\end{aligned}$$

for some $C>0$. The testing statistics is

$$Z^*=\frac{\widehat{BC}_p-C}{\left(Q_1^2(\lambda_1,\lambda_2)\lambda_1^2+Q_2^2(\lambda_1,\lambda_2)\lambda_2^2\right)^{1/2}/\sqrt{72n}\phi(3\widehat{BC}_p)}$$

The decision making rule is

$$\text{reject } H_0, \quad \text{if } Z^* > Z_\alpha;$$
$$\text{do not reject } H_0, \quad \text{if } Z^* \leq Z_\alpha,$$

where Z_α is the αth upper quantile of the standard normal distribution.

4.2.4 Illustrative Examples

We apply our pre-processing and estimating methods to four examples in Table 2. We generate 1000 sets of 100 bivariate normal data for each case. Table 11 presents for Cases 1-4, the true values, sample mean and sample standard deviation of 1000 $\widehat{BC}_p{}'s$. For each data set, we can obtain a lower confidence bound (LCB) and a confidence interval (CI) by (9) and (10), respectively. Table 12 shows a LCB and a CI using one data set for each of Cases 1-4.

Table 11: Estimation results of BC_p

Case	True value	Sample mean	Sample sd.	Bias
1	0.944225	0.941910	0.003219	0.002315
2	1.243213	1.238305	0.005577	0.004908
3	0.872872	0.877583	0.030057	0.004711
4	0.209254	0.214192	0.017330	0.004938

Table 12: Approximate 90% lower confidence bounds and confidence interval of BC_p

Case	LCB	CI
1	0.920824	[0.915502, 0.963738]
2	1.249896	[1.246603, 1.259673]
3	0.877702	[0.871465, 0.927997]
4	0.211038	[0.206967, 0.243863]

5 An Application Example

In this section we employ our estimating methods given in the last two sections to an industrial example described by Chen [7]. Chen [7] discussed a real example presented by Sultan [17] regarding an industrial process in which the brinell hardness (H) and the tensile strength (S) are the quality characteristics. The Chen-Sultan's data consists of twenty five samples taken from a process with the specifications for H and S being [112.7, 241.3] and [32.7, 73.3], respectively. Figure 10 depicts the data and the specification region. Table 13 gives the estimate $\widehat{BC}_{pk}$ along with the 90% bootstrap lower confidence bounds and their tolerance errors with $B = 3000$. Table 14 gives the estimate $\hat{C}_p$ and the 90% confidence interval and 90% lower confidence bound obtained by (9) and (10), respectively.

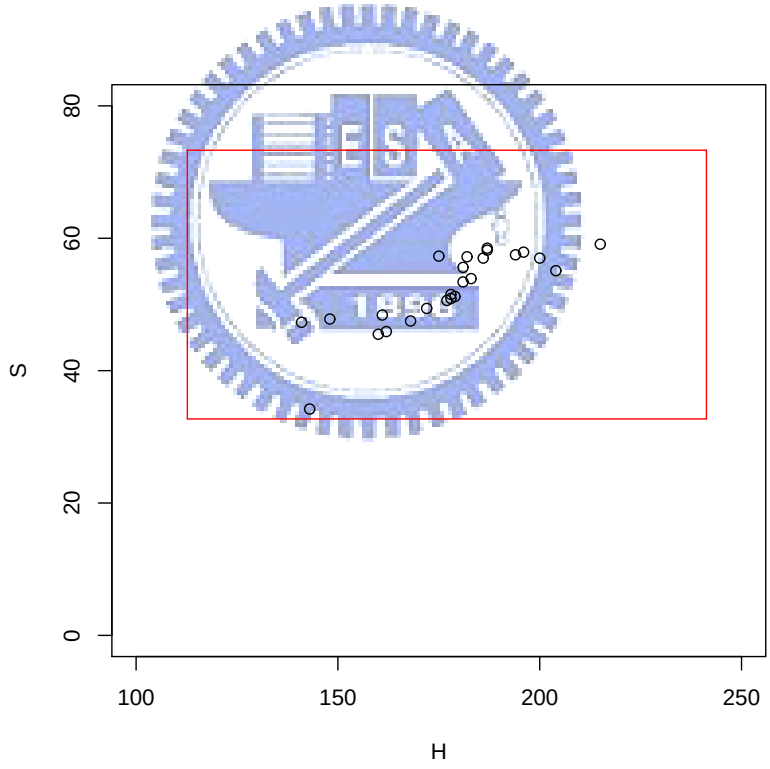

Figure 10: **Data and specification region of Chen-Sultan's example**

Table 13: $\boldsymbol{BC_{pk}}$ **estimation results of Chen-Sultan's data**

$\widehat{BC}_{pk}$=0.503846			
$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$	$\hat{p}_4$
0.013065	0.032663	0.019598	0.012852
bootstrap lower confidence bounds			
Basic	Percentile	Standard	Bias-corrected percentile
0.443303	0.419805	0.391496	0.402532
Tolerance error of Bootstrap lower confidence bounds			
Basic	Percentile	Standard	Bias-corrected percentile
0.0002362	0.0006795	0.0004225	0.0010551

Table 14: $\boldsymbol{BC_p}$ **estimation results of the Chen-Sultan example**

$\hat{p}$	$\widehat{BC}_p$	90% confidence interval	90% lower confidence bound
0.0047052	0.943900	[0.897111, 0.990689]	0.907436

With $\widehat{BC}_{pk} = 0.503846$, we have a corresponding upper bound of non-conforming rate 130651 ppm. Take the lowest LCB among the four bootstrap method, 0.391496, one may say that, with 90% confidence, the yield of the product is at least 82%. These indicate that this process is inadequate that either process variation needs to be reduced or process mean needs to be adjusted to get closer to the target value.

6 Conclusions

In this paper, we study the bivariate PCIs, BC_{pk} and BC_p, proposed by Castagliola and Castellanos [4]. We summarize our work below. For BC_{pk}, we

- establish a link between BC_{pk} and the process yield by showing that $2\Phi(3BC_{pk}) - 1 \leq$ %yield, an inequality links the index C_{pk} and the yield in the univariate case. This lower bound provides a measure for quality assurance.
- extend BC_{pk} to MC_{pk}, an index for processes of more than two characteristics, with the same notion. The lower bound inequality $2\Phi(3MC_{pk}) - 1 \leq$ %yield also holds.
- provide a new algorithm for computing the estimate $\hat{BC}_{pk}$ of BC_{pk}. The new algorithm can be used for processes with more general specification regions and/or higher dimensions.
- utilize bootstrap methods to obtain lower confidence bounds of BC_{pk}. Among the four bootstrap methods, we recommend Bias-Corrected Percentile Bootstrap (BCPB) method since it has the smallest tolerance error.

For BC_p, we

- find that the original definition is not scale-invariant. We propose a pre-processing step to fix the problem.
- develop an efficient algorithm for computing the natural estimate of BC_p, which is a lot faster than the method given in Castagliola and Castellanos [4].
- derive an approximate normal distribution for $\widehat{BC}_p$ by taking its first-order Taylor expansion. This enables us to develop statistical procedures for making inferences on process capability based on data, including hypothesis testing, confidence interval, and lower confidence bound.

Lastly, we illustrate the estimation methods with a set of real data.

As for the future research, first we need to prove the BC_p indeed can be obtained by rotating the coordinates by an angle of $\theta = 45°$. Second, our simulation indicates that the sampling distribution of $\widehat{BC}_{pk}$ is fairly close to normal distribution. If we could find a

normal approximation to it, it would be very helpful in developing procedures for making inferences on BC_{pk}.

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