國 立 交 通 大 學

統計學研究所

碩 士 論 文

Distributional and Inferential Properties of Capability Index C_{pk}^{T} for Processes with Multiple Characteristics

多品質特性製程的製程能力指標C_{pk}^{T}的分布和推論

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中 華 民 國 九 十 八 年 六 月

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碩 士 論 文

A Thesis

Submitted to Institute of Statistics

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Statistics

June 2009

Hsinchu, Taiwan, Republic of China

中華民國九十八年六月

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摘要

與良率有關的製程能力指標 C_{pk} 在製造業中已經廣泛的被使用來評量製程之表現。此方面多數的研究都著重在單一品質特性的製程，然而在實際的應用上，一個製程常常是具有多個品質特性，而每個特性都有不同的規格。在這一篇論文中，我們研究將 C_{pk} 推廣為多個獨立的品質特性製程的 C_{pk}，因此新定義一個指標 C_{pk}^{T}。

我們證明了 C_{pk}^{T} 像 C_{pk} 一樣與製程良率的上限跟下限有個一對一對應的關係。我們也以理論證明的方式找出估計量 $\widehat{C}_{pk}^{T}$ 的常態近似分配，透過常態近似分配，統計假設檢定、信賴區間、信賴下限都可以用來檢驗製程是否是有達到特定的標準。其中信賴下限在實務上特別重要，因為信賴下限可以用來估計製程的最小能力，與品質保證有很大的相關性；而精確性 (R) 則是估計量與信賴下限的比值，定義這個值是為了方便工程師每天量測製程的最小能力。接著透過資料模擬的方式，檢驗常態近似分配逼近估計量真正分配的準確性。

最後，我們用一個實際的製程－雙蕊光纖(a dual-fiber tip process) 作為例子，說明如何將新提出的指標及本文所提出之統計推論方法應用到實際的製程上。

Distributional and Inferential Properties of Capability Index C_{pk}^T for Processes with Multiple Characteristics

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Abstract

Process capability index C_{pk} has been popularly used in the manufacturing industry for measuring process performance based on yield (proportion of conformities). Most researches on C_{pk} focus on processes with single quality characteristic; but in many real applications, a process often has multiple quality characteristics. In this study, we extend C_{pk} to a new index C_{pk}^T for processes with multiple characteristics. We prove that the inequalities that link C_{pk} to the yield also hold for the new index. A natural estimator of $\hat{C}_{pk}^T$ is provided and a normal approximation to its distribution is derived. With this normal approximation, standard processes for statistical inferences such as hypothesis testing and confidence interval are developed for testing whether the process is capable and providing an interval estimate on C_{pk}^T, respectively. More importantly, we can obtain a confidence lower bound for C_{pk}^T, which measures the minimum process capability and is directly linked to quality assurance of products. The accuracy of the normal approximation is studied by simulation. Finally, we demonstrate how the new index C_{pk}^T as well as the inferential procedures developed in this

study can be used with a real example of a dual-fiber tip process.

致　謝

兩年的碩士生涯即將邁向終點，很快的即將踏入人生的另一個旅程。回首當初在交大統計所的榜單上查到自己是正取時，那種既興奮又期待的心情，以及現在豐收的喜悅，「不虛此行」的感觸油然而生。謝謝這兩年來，對於我的懵懂給予提攜以及在我無助時，對我伸出援手的所有人。

首先要感謝我的指導教授－彭文理老師。雖然，跟著老師做論文，挑戰性很大，遇到問題時要自己想辦法、找資源；出現瓶頸時，要靠自己去突破。老師只在每個階段的里程碑等著我，告訴我下個階段要達成的目標是什麼。這樣的過程，雖然很艱辛，但卻也讓我真的學會做論文。在寫論文時，老師也秉持著高標準來要求我，但卻也是這樣的要求激發了我嘗試想突破障礙、努力達到老師要求的潛能。因此，跟著老師做研究的這一年，讓我覺得生活過得既充實又有成就感。

再來，要感謝我的另一位指導老師－洪志真老師。感謝老師在我做論文遇到障礙時，給予我方向；碩士論文高標準的寫作要求，讓我學習到一篇論文要完成，並不是隨隨便便寫一寫就可以了，必須一而再，再而三的修改，讓我對論文編輯有了更上一層的進步。

除此之外，特別要感謝洪慧念老師，在我升碩二的暑假適逢洪志真老師出國的這段期間，每當我論文遇到瓶頸或是結果需要理論證明支持時，給予我很大的協助跟指導。

謝謝我的口試委員教授，在百忙之中為我口試，在口試時對於我的指導，更讓我發現本論文主題可以更深入探討的地方，這在我的工作領域上有很大的幫助，能夠真正達到學以致用的目標。

除此要特別感謝我們所上的助理郭碧芬小姐，她總是不辭勞苦幫我們處理很多行政事務上的問題，以及很好心的擔任當彭老師找我時的接線生，真的很感謝郭姐。

最後要感謝我的家人給我的支持，與家人相處的時間很少，因為爸媽的體諒，才能專心完成學業。

另外還有很多曾經幫助過我的朋友，因為有大家的幫助，才能有今天的成果。

2009 年 6 月　李美諭

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1. Introduction

Process capability indices (PCIs) are often used as a quality measure to evaluate the performance of a process. Because PCIs establish the relationship between the actual process and the manufacturing specifications, they have been widely used in the manufacturing industries in recent years. Basic capability indices, C_p, C_{PU}, C_{PL}, C_{pk}, C_{pm}, C_{pmk}, are developed for measuring whether a process has the reproduction capability (Kane (1986), Chan *et al*. (1988), Pearn *et al*. (1992), Kotz and Lovelace (1998), and Kotz and Johnson (2002)). Boyles (1991, 1994) proposed another index called S_{pk}. These indices are defined as:

$$C_p = \frac{USL-LSL}{6\sigma},\ C_{PU} = \frac{USL-\mu}{3\sigma},\ C_{PL} = \frac{\mu-LSL}{3\sigma},\ C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}, C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\},$$

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(\frac{USL-\mu}{\sigma}) + \frac{1}{2}\Phi(\frac{\mu-LSL}{\sigma})\right\},$$

where USL and LSL are the upper and the lower specification limits, respectively, μ is the process mean, σ is the process standard deviation, and T is the target value. While the indices C_p, C_{pk}, C_{pm}, C_{pmk}, and S_{pk} are appropriate for statistically in-control normal processes with two-sided specification limits, the indices C_{PU} and C_{PL} are designed specifically for processes with one-sided specification limit.

The first known index C_p measures only the distribution spread, which only reflects product quality consistency (in terms of process precision) but does not account for the location of process mean μ. The index C_{pk} not only takes into account the process variation and the extent of process centering, but also measures actual process performance based on yield (i.e., proportion of conformities). If the value of C_{pk} is given for a process, Boyles (1991) gave an upper bound and a lower bound for the process yield (denoted by *%Yield*) as

$$2\Phi(3C_{pk}) - 1 \le \%\ Yield \le \Phi(3C_{pk}), \tag{1.1}$$

where $\Phi(\bullet)$ is the cumulative distribution function (c.d.f.) of the standard normal distribution. For instance, if $C_{pk} = 1.33$, then the lower bound guarantees that the yield will be no less than 99.9934%, or equivalently, the process has no more than 66 parts per million (PPM) of non-conformities. The inequalities in (1.1) link the index C_{pk} with the process yield. The last index S_{pk} provides an exact measure of the process yield, because the index is defined as a monotonically increasing function of the yield. For example, if S_{pk} = 1.33, then the exact yield is 99.9933927%, or equivalently, 66.073 PPM of non-conformities.

The capability measuring for processes with single characteristic has been investigated extensively, but is comparatively neglected for processes involving multiple characteristics. However, it is quite common that industrial processes nowadays have more than one quality characteristic. Thus, the performance evaluation of multivariate processes has become more and more important.

Each of the multiple characteristics must meet certain specifications. However, the assessed quality of a product depends on the combined effects of the multiple characteristics, rather than on their individual values. For instance, a process manufacturing dual-fiber tips, a component is used to make fiber optic cables, has six quality characteristics, namely, the capillary diameter, length, wedge, core diameter, return loss, and polishing direction. These characteristics are related through the composition of the fiber tips. Therefore, it is natural to consider a multivariate characterization of this process.

The purpose of this study is to define a multivariate PCI, called C_{pk}^T, for multivariate processes, which is a natural extension of the index C_{pk} for univariate processes; and most importantly, the new index still retain the link to the yield as given in the expression (1.1). A natural estimator $\hat{C}_{pk}^T$ for C_{pk}^T is provided. Since the distribution of $\hat{C}_{pk}^T$ is mathematically intractable, we derive its asymptotic distribution and obtain a normal approximation accordingly. For quality assurance, a lower bound of the yield is a valuable quality measure. We use this normal approximation to obtain a confidence lower bound of C_{pk}^T from process data.

The contents of this thesis is divided into seven sections. In Section 2, we emphasize the importance of studying multivariate process capability indices and review recent studies on the performance evaluation of multivariate processes. In Section 3, we propose a new yield index C_{pk}^T for the overall process and relate it to the corresponding non-conformities in parts per million (NCPPM). In Section 4, we derive the asymptotic distribution of a natural estimator $\hat{C}_{pk}^{T}$ of C_{pk}^{T} and use this result to provide a confidence interval and a lower confidence bound of the new index for processes with multiple independent characteristics. To investigate how well the approximation is to the actual distribution, we compare the normal approximation with the actual distribution of $\hat{C}_{pk}^{T}$ by simulation. Moreover, the coverage rate and the confidence interval length are computed and the behavior of the lower confidence bound is investigated. In Section 5, we compute the most conservative lower confidence bound and the precision of the natural estimator for specified sample sizes, and investigate the accuracy of the normal approximation by simulation. A simulation study is conducted to investigate the bias of the natural estimator. In Section 6, as an illustrative example, we apply the methodology to a set of real data presented in Pearn and Wu (2005b). In Section 7, we conclude the thesis with a brief summary.

2. Capability Measures for Multiple Characteristics

In recent years, more and more researchers have been devoted to studying multivariate capability indices. For example, Chen (1994), Boyles (1996) and others presented multivariate capability indices for assessing capability. Wang and Chen (1998-1999) and Wang and Du (2000) proposed multivariate equivalents for C_p, C_{pk}, C_{pm}, and C_{pmk} based on the principal component analysis, which transforms the original correlated variables into a set of uncorrected variables that are linear combinations of the original variables. Moreover, a comparison of three recently proposed multivariate methodologies for assessing capability are illustrated and discussed in Wang et al. (2000). On the other hand, some researchers modified the univariate index for processes with multiple characteristics. For example, Chen and Pearn (2003) modified the process capability index S_{pk} to

$$S_{pk}^{T} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi(3S_{pk_i}) - 1\right) + 1\right]/2\right\},$$

where S_{pki} is the S_{pk} of the ith characteristic. Later, Pearn and Wu (2005a) proposed the following modified one-sided index, which is a generalization of the one-sided index C_{PU},

$$C_{PU}^{T} = \frac{1}{3}\Phi^{-1}\left\{\prod_{i=1}^{m}\Phi(3C_{PU_i})\right\},$$

where C_{PUi} is the C_{PU} of the ith characteristic.

Processes with multiple independent characteristics

When a processes has m (>1) independent characteristics, Bothe (1992) considered m yield measures $P_1,\ldots,P_m$ and suggested the overall process yield P to be measured by $P = min\{P_1,\ldots,P_m\}$. We can see that this approach does not reflect the real situation accurately. Assuming the process has five characteristics (m = 5) with equal yield measures $P_1 = P_2 = P_3 = P_4 = P_5$ = 99.9934%. Using Bothe's approach, the overall process yield is evaluated as $P = min\{P_1,\ldots,P_m\}$=99.9934% (or 66 PPM of non-conformities). Supposing that the five characteristics are mutually independent, then the overall process yield should be calculated as $P_1 \times P_2 \times P_3 \times P_4 \times P_5$ = 99.967% (or 330 PPM of non-conformities), which is significantly less than that suggested by Bothe (1992). (See Pearn and Wu (2005b)).

In the manufacturing industry, C_{pk} has been popularly used for measuring process performance because it can link to the process yield. In this paper, we define a new yield-related process capability index for processes of multiple independent characteristics. We consider a normal approximation to the distribution of the natural estimator to find the lower confidence bound, which gives us not only a clue on minimum actual performance related to the fraction of non-conforming units, but also is useful in decision making on the capability test.

3. A New Process Yield index for Multiple Independent Characteristics

3.1. The yield-related index C_{pk}^T

For a process with m quality characteristics, we assume the m characteristics follow mutually independent normal distribution, $N(\mu_i, \sigma_i^2),\ i = 1,\ldots,m.$ Denote the two-sided specification limits of the ith characteristic by USL_i and LSL_i, $i=1,\ldots,m$.

Given a value of C_{pki}, by (1.1), the individual yield of the ith characteristic has the following bounds

$$2\Phi(3C_{pki}) - 1 \le \%\ Yield_i \le \Phi(3C_{pki}),\ i = 1,\ldots,m. \tag{3.1}$$

If we wish to extend the notion of C_{pk} to a multivariate yield capability index C_{pk}^T, it is natural to require C_{pk}^T satisfying (3.1), that is,

$$2\Phi(3C_{pk}^{T})-1 \le \%Yield \le \Phi(3C_{pk}^{T}), \qquad (3.2)$$

where *%Yield* is the overall process yield of the multivariate process. Since the characteristics are mutually independent, by (3.1), $\prod_{i=1}^{m}\left(2\Phi(3C_{pki})-1\right)$ is a lower bound of the overall process yield. Thus, if we set

$$\prod_{i=1}^{m}\left(2\Phi(3C_{pki})-1\right) = 2\Phi(3C_{pk}^{T})-1, \qquad (3.3)$$

then the first inequality in (3.2) automatically holds. Therefore, by equation (3.3), it is natural to propose a new index as defined by

$$C_{pk}^{T} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi(3C_{pk_i})-1\right)+1\right]/2\right\}. \qquad (3.4)$$

It can be shown that the inequality $\%\,Yield \le \Phi(3C_{pk}^{T})$ holds as well. Derivation is given in Appendix A. Therefore, the new index defined as (3.4) satisfies (3.2). The new index C_{pk}^{T} may be viewed as a generalization of the single characteristic yield index C_{pk} and it provides a lower bound of the overall process yield. Therefore, the corresponding upper bound (UB) of non-conformities in parts per million, NCPPM, for a well-controlled process with multiple independent normal characteristics can be calculated as

$$\text{NCPPM} \le 10^{6} \times 2[1-\Phi(3C_{pk}^{T})]. \qquad (3.5)$$

Table 1 and Figure 1 present the corresponding upper bounds of NCPPM for C_{pk}^{T} = 1.00, 1.25, 1.33, 1.45, 1.50, 1.60, 1.67, and 2.00.

4. Estimation of C_{pk}^{T}

4.1. The approximate distribution for a natural estimator of C_{pk}^{T}

Let $X_1,\ldots,X_n$ be independent and identically distributional (i.i.d.) as a multivariate normal process $N_m(\mu,\Sigma)$, where $X_j = (X_{1j},\ldots,X_{mj})'$, $\mu = (\mu_1,\ldots,\mu_m)'$, and Σ is the $m\times m$ diagonal matrix with diagonal elements $\sigma_1^2,\ldots,\sigma_m^2$.

Since the individually C_{pk} can be expressed as

$$C_{pki} = \frac{d_i - |\mu_i - m_i|}{3\sigma_i},\; i = 1,\ldots,m,$$

where μ_i is the mean of the ith characteristic, $m_i = (USL_i + LSL_i)/2$ is the mid-point of the specification interval, and $d_i = (USL_i - LSL_i)/2$ is the half width of the specification interval.

It is common to estimate C_{pki} by

$$\hat{C}_{pki} = \frac{d_i - |\overline{X}_i - m_i|}{3S_i}, \tag{4.1}$$

where $\overline{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ and $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_{ij} - \overline{X}_i)^2,\ i = 1, \ldots, m.$ (4.2)

To estimate the index C_{pk}^T, we consider the following natural estimator

$$\hat{C}_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi(3\hat{C}_{pki}) - 1\right) + 1\right] / 2\right\}, \tag{4.3}$$

where C_{pki} is the C_{pk} of the ith characteristic.

.

Theorem 1.

The exact distribution of $\hat{C}_{pk}^T$ is analytically extractable; however, it can be shown that $\hat{C}_{pk}^T$ has an asymptotic normal distribution as stated in the following theorem. The asymptotic distribution of $\hat{C}_{pk}^T$ is

$$\sqrt{n}\left(\hat{C}_{pk}^T - C_{pk}^T\right) \xrightarrow{d} N\left(0\,,\ \frac{1}{9\left[\phi(3C_{pk}^T)\right]^2}\sum_{i=1}^{m}(a_i^2 + b_i^2)\right) \text{ as } n \to \infty, \tag{4.4}$$

where $a_i = \left[\prod_{j=1, j\neq i}^{m}\left(2\Phi(3C_{pkj}) - 1\right)\right]\phi(3C_{pki})$ and $b_i = \frac{3}{\sqrt{2}}a_i C_{pki},\ i = 1, \ldots, m.$

We give two different proofs in Appendix B and Appendix C, respectively.

Note that, by equation (4.4), $\hat{C}_{pk}^T$ is asymptotically unbiased. To see how well the normal approximation is, we conduct a simulation study using the free statistical package R as follows. Four scenarios are considered in the simulation study, the combinations of two C_{pk}^T values (1 or 1.33) and two cases of process mean ($\mu_i \neq m_i$ or $\mu_i = m_i,\ i = 1, 2$). For each scenario, simulate 1,000,000 random samples of size n = 60, 200, 500, 1000 from $N_2(\mu_1, \mu_2, \sigma_1{}^2, \sigma_2{}^2, 0)$, a normal process with two independent characteristics. For each scenario, we compute 1,000,000 $\hat{C}_{pk}^T$ by (4.3). Figures 2-9 compare the simulated distribution obtained by 1,000,000 $\hat{C}_{pk}^{T}$'s to the normal approximation with their probability density functions (p.d.f.) and the cumulative distribution functions (c.d.f.). It is clear that as sample size n reaches 1000, the approximate and simulated distributions are very close. In fact, even with n = 60, the approximation is already quite reasonable for practical purposes.

4.2 Statistical inferences based on the normal approximation

With the asymptotic distribution given in equation (4.4), we now can make statistical inferences on C_{pk}^{T} based on a set of random samples, including the hypothesis testing, confidence interval, and lower confidence bound.

To test whether a given process is capable, we consider the following statistical hypothesis testing:

$$\begin{aligned} &\mathrm{H}_0:\ C_{pk}^{T} \le c \ \text{(the process is not capable)} \\ &\mathrm{H}_1:\ C_{pk}^{T} > c \ \text{(the process is capable)} \end{aligned} \tag{4.5}$$

where c is the minimal standard criterion on C_{pk}^{T}.

The test can be executed by considering the testing statistic

$$T = \frac{3\sqrt{n}\left(\hat{C}_{pk}^{T} - c\right)\phi(3\hat{C}_{pk}^{T})}{\sqrt{\sum_{i=1}^{m}\left(\hat{a}_i^2 + \hat{b}_i^2\right)}}, \tag{4.6}$$

where $\hat{a}_i = \left[\prod_{j=1, j\neq i}^{m}\left(2\Phi(3\hat{C}_{pkj}) - 1\right)\right]\phi(3\hat{C}_{pki})$ and $\hat{b}_i = \frac{3}{\sqrt{2}}\hat{a}_i\hat{C}_{pki}$, $i = 1,\ldots,m.$

Because we do not know the values of a_1, a_2, b_1, and b_2, we estimate them from data. The null hypothesis H_o is rejected at α level if $T > Z_\alpha$, where Z_α is the upper 100α% percentile of the standard normal distribution.

An approximate 100(1- α)% confidence interval for C_{pk}^{T} can be easily obtained as

$$\left[\hat{C}_{pk}^{T} - Z_\alpha\left\{\frac{1}{9n[\phi(3\hat{C}_{pk}^{T})]^2}\sum_{i=1}^{m}(\hat{a}_i^2 + \hat{b}_i^2)\right\}^{1/2},\ \hat{C}_{pk}^{T} + Z_\alpha\left\{\frac{1}{9n[\phi(3\hat{C}_{pk}^{T})]^2}\sum_{i=1}^{m}(\hat{a}_i^2 + \hat{b}_i^2)\right\}^{1/2}\right] \tag{4.7}$$

and an approximate 100(1- α)% lower confidence bound for C_{pk}^{T} can be expressed as

$$C_{pk}^{T\ LB} \approx \hat{C}_{pk}^{T} - Z_{\alpha}\left[\frac{1}{9n[\phi(3\hat{C}_{pk}^{T})]^2}\sum_{i=1}^{m}(\hat{a}_i^{2}+\hat{b}_i^{2})\right]^{1/2}, \tag{4.8}$$

where $\hat{a}_i$ and $\hat{b}_i$ are as before.

To evaluate the proposed confidence interval and lower bound of C_{pk}^{T}, we conduct a simulation study. We use equation (4.7) to obtain confidence interval and confidence interval length. Consider the case of $C_{pk}^{T}=1.33$ with 90% confidence level under a process of two independent characteristics. Note that there are infinite number of the combinations of the process distribution and the manufacturing specifications that would correspond to the same value of $C_{pk}^{T}=1.33$. We consider six scenarios as given in Table 3 in the study.

For each scenario, generate N=1,000,000 random samples of size n =30, 50, 100, 500, 1000 from $N_2(u_1, u_2, \sigma_1^2, \sigma_2^2, 0)$. For each case, we compute 1,000,000 $\hat{C}_{pk}^{T}$, the corresponding 1,000,000 confidence intervals and $C_{pk}^{T\ LB}$'s. Check if the true index C_{pk}^{T} is contained in the interval and if it is greater than $C_{pk}^{T\ LB}$.

Tables 2-5 present the coverage rate and the average length of 1,000,000 confidence intervals. We can find that the coverage rate approaches 0.9 (under α = 0.1) and the confidence interval length is decreasing to zero as the sample size n increases.

For the univariate case, Pearn and Shu (2003) examined the behavior of the lower confidence bound of C_{pk} against ξ, where $\xi = (\mu - m)/\sigma$. Since for a process with two characteristics, $C_{pk}^{T\ LB}$ involves ξ_1 and ξ_2. However, there are too many C_{pki}'s corresponding to one ξ_i. Therefore, instead, we explore the relationship between $C_{pk}^{T\ LB}$ and (C_{pk1}, C_{pk2}). To do this, instead of performing simulation experiments that require extensive calculations, we calculate and plot

$$C_{pk}^{T\ LB} \approx C_{pk}^{T} - Z_{\alpha}\left[\frac{1}{9n[\phi(3C_{pk}^{T})]^2}\sum_{i=1}^{m}(a_i^{2}+b_i^{2})\right]^{1/2} \tag{4.9}$$

versus C_{pk1} and C_{pk2} to examine the relationship between $C_{pk}^{T\ LB}$ and (C_{pk1}, C_{pk2}).

Based on $C_{pk}^{T\ LB}$ expressed in equation (4.9), given sample size n = 10, 30, 50, 70, 90, Figure 10 displays the curves of $C_{pk}^{T\ LB}$ versus various combinations of C_{pk1} and C_{pk2} with C_{pk}^{T} = 1.0, 1.33, 1.5, 1.67. Figure 11 plots curves of $C_{pk}^{T\ LB}$ versus C_{pk1} given sample size n =

10, 30, 50, 70, 90 for C_{pk}^{T} = 1.0, 1.33, 1.5, 1.67.

We examine the results of calculation and find that

- $C_{pk}^{T\ LB}$ reaches its absolute maximum when $C_{pk1} = C_{pk2}$.
- The minimum of $C_{pk}^{T\ LB}$ occurs when one of $C_{pk}'s$ approaches infinity, that is, when C_{pk}^{T} equals one of $C_{pk}'s$. This minimum is the most conservative lower confidence bound for a given C_{pk}^{T}.

From Figure 10 and Figure 11, we can also observe the above properties.

5. Accuracy of the Normal Approximation

For a given C_{pk}^{T}, by setting one C_{pk} at C_{pk}^{T} and the other C_{pk} at ∞, we can use equation (4.9) to compute the most conservative $C_{pk}^{T\ LB}$, which represents a measure of the minimum manufacturing capability of the process for the case when the process has two independent characteristics. For engineer convenience, if a process with two independent characteristics has $\hat{C}_{pk}^{T} = 1.33$ and sample size n =100, then we have 95% confidence to say that the true C_{pk}^{T} of this process is no less than 1.1392. Similarly, we can compute the largest possible $C_{pk}^{T\ LB}$ by setting $C_{pk1} = C_{pk2}$. The largest possible value of $C_{pk}^{T\ LB}$ may not have much of the practical value, but is of interest mathematically.

Tables 6 and 7 tabulate both the most conservative and largest possible $C_{pk}^{T\ LB}$ value for $\hat{C}_{pk}^{T} = 1(0.1)2$, n=10(10)200 and confidence level $\gamma = 95\%$.

5.1. Accuracy analysis of $C_{pk}^{T\ LB}$

Sample size determination is important, as it directly relates to the cost of the data collection. By equation (4.8), we have

$$n \approx (Z_{\alpha} / \hat{C}_{pk}^{T})^{2} \left[\frac{\sum_{i=1}^{m} (\hat{a}_i^2 + \hat{b}_i^2)}{9[\phi(3\hat{C}_{pk}^{T})]^2} \right] \left(1 - C_{pk}^{T\ LB} / \hat{C}_{pk}^{T} \right)^{-2}. \qquad (4.10)$$

where $\hat{a}_i = \left[\prod_{j=1, j\neq i}^{m} \left(2\Phi(3\hat{C}_{pkj}) - 1 \right) \right] \phi(3\hat{C}_{pki}),\ \hat{b}_i = \frac{3}{\sqrt{2}} \hat{a}_i \hat{C}_{pki},\ i = 1,\ldots,m.$

To evaluate the precision of the lower confidence bound C_{pm}^{L} for the index C_{pm} given earlier, Pearn and Shu (2003) defined a precision measure $R = C_{pm}^{L} / \hat{C}_{pm}$, where $\hat{C}_{pm}$ is a natural estimator of C_{pm}. Similarly, for $C_{pk}^{T\ LB}$ we can define $R = C_{pk}^{T\ LB} / \hat{C}_{pk}^{T}$. In Table 8, we tabulate the value R of the most conservative $C_{pk}^{T\ LB}$ for processes with two independent characteristics. These values can be useful for engineers and practitioners, because it would be convenient to assess the minimum capability of the process for engineers as a everyday work.

For example, if one requires a 95% lower confidence bound for C_{pk}^{T} to be of 85% precision of $\hat{C}_{pk}^{T}$ (i.e., $C_{pk}^{T\ LB} / \hat{C}_{pk}^{T} = 0.85$) for $\hat{C}_{pk}^{T} = 1.5$, then the most conservative sample size required for achieving this goal is 66, which can be computed by

$$n \approx (Z_{\alpha} / \hat{C}_{pk}^{T})^{2} \left[\frac{\sum_{i=1}^{2} (\hat{a}_i^2 + \hat{b}_i^2)}{9[\phi(3\hat{C}_{pk}^{T})]^{2}} \right] \left(1 - C_{pk}^{T\ LB} / \hat{C}_{pk}^{T} \right)^{-2}.$$

where $\hat{a}_1 = \phi(3\hat{C}_{pk}^{T})$, $\hat{a}_2 = 0$, $\hat{b}_1 = \frac{3}{\sqrt{2}} \phi(3\hat{C}_{pk}^{T})\hat{C}_{pk}^{T}$, and $\hat{b}_2 = 0$.

On the other hand, if one obtains a $\hat{C}_{pk}^{T} = 1.5$ from a set of data of size 66, then the most conservative 95% lower bound can be conveniently obtained by multiplying $\hat{C}_{pk}^{T}$ by the corresponding R (=0.85), i.e., the most conservative lower bound is $1.5 \times 0.85 = 1.275$. One then can conclude that the true value of the process capability C_{pk}^{T} is no less than 1.275 with 95% confidence.

5.2. Bias of the natural estimator of C_{pk}^{T}

In order to explore the bias of the natural estimator by simulation, we simulate a total of N=1,000,000 replications for each sample size of n = 30, 50, 100, 500, 1000. Take the average of N $\hat{C}_{pk}^{T}$'s to estimate $E(\hat{C}_{pk}^{T})$ and compare it with the true C_{pk}^{T}. The simulation results presented in Tables 9-12 indicate that the bias is negative for the cases under study. That is, we underestimate C_{pk}^{T} when the yields of the two independent characteristics are the same (i.e., C_{pk1} = C_{pk2}). On the other hand, when one %Yield almost reaches 100% (i.e., when $C_{pk}^{T} = C_{pk1}$ or $C_{pk}^{T} = C_{pk2}$), the problem is basically reduced to the univariate case, a situation previously studied by Kotz et al. (1993). They showed that the natural estimator $\hat{C}_{pk}$ of C_{pk}

is biased and the bias is positive when $\mu \neq m$ (μ is the process mean and m is the midpoint of the specifications). When $\mu = m$, the bias is positive for sample size ≤ 10, but is negative for larger values of sample size. See Kotz et al. (1993) for more details.

5.3. Sample size for required margin of error

From (4.8), the margin of sampling error is approximately note that

$$Z_{\alpha}\left[\frac{\sum_{i=1}^{m}(\hat{a}_i^2+\hat{b}_i^2)}{9n[\phi(3\hat{C}_{pk}^{T})]^2}\right]^{1/2}.$$

where $\hat{a}_i = \left[\prod_{j=1,\, j\neq i}^{m}\left(2\Phi(3\hat{C}_{pkj})-1\right)\right]\phi(3\hat{C}_{pki}),\ \hat{b}_i = \frac{3}{\sqrt{2}}\hat{a}_i\hat{C}_{pki},\ i = 1,\ldots,m.$

Table 13 gives the most conservative sample sizes required for the estimator of C_{pk}^{T} to be within a sample error less than 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 for various C_{pk}^{T} and significance level α. The most conservative case is used in calculation. (i.e., $C_{pk}^{T} = C_{pk1}$ or $C_{pk}^{T} = C_{pk2}$).

For example, for $C_{pk}^{T} = 1.33$ with $\alpha = 0.05$, a sample size ≥ 193 ensures that the sampling error of $\hat{C}_{pk}^{T}$ is no greater than 0.09.

6. An application example

For illustration, we consider a real example presented in Pearn and Wu (2005b), which is taken from an optical communication manufacturing factory located in Science-based Industrial Park in Taiwan. The example involves a process manufacturing dual-fiber tips, a component used in making fiber optic cables.

Figure 12 depicts a sample of the dual-fiber tips. Sixty dual-fiber tips were taken from a stable (i.e., in statistical control) process in the factory, and two product quality characteristics were measured, (i) Capillary length and (ii) Wedge. For a particular model of dual-fiber tips, the specifications of characteristics are listed in Table 14. According to Pearn and Wu (2005b), it is reasonable to assume that these 60 data were from a normal distribution with two independent quality characteristics. The sample mean, standard deviation, and specifications along with the individual $\hat{C}_{pk}$ of each characteristic are summarized in Table 14.

If the quality requirement was predefined as $C_{pk}^{T} \geq 1.33$, then we can make some statistical inferences on C_{pk}^{T} by using hypothesis testing and interval estimation. For testing the null hypothesis H_o as given in (4.5) with c =1.33, the testing statistic T given in (4.6) is 2.321008 > $Z_{0.05}$ = 1.645. Thus, H_o is rejected at α = 0.05. We conclude that the process meets the capability requirement of $C_{pk}^{T} > 1.33$ with 95% confidence.

Moreover, $\hat{C}_{pk}^{T}$ = 1.702917 and $C_{pk}^{T\ LB} = 1.438560$ by (4.3) and (4.8), respectively. Thus, we have 95% confidence to say C_{pk}^{T} is no less than 1.43856, or equivalently, there are no more than 16 PPM of non-conformities as given in (3.5).

7. Conclusions

Process yield is the most common criterion used in the manufacturing industry for measuring process performance. The widely used capability index C_{pk} is a yield-related index, in the sense that it can provide a lower bound for the yield of a process with single characteristic. But in many real applications, the process has multiple characteristics.

In this paper, we extend C_{pk} to an index C_{pk}^{T} to assess the yield of processes with multiple characteristics. It is shown that $2\Phi(3C_{pk}^{T}) - 1 \leq \%\ Yield \leq \Phi(3C_{pk}^{T})$, a property holds for the univariate C_{pk}. Based on the new index C_{pk}^{T}, the practitioners can make reliable decisions for capability testing and monitoring the overall performance of all process characteristics.

Unfortunately, the distributional properties of the natural estimator $\hat{C}_{pk}^{T}$ are mathematically intractable. We derive a normal approximation to the distribution of the $\hat{C}_{pk}^{T}$ by the first-order *Taylor* expansion and investigate the accuracy and precision of $\hat{C}_{pk}^{T}$ by simulation.

Applying the asymptotic distribution of $\hat{C}_{pk}^{T}$, hypothesis testing, confidence interval, and a confidence lower bound $C_{pk}^{T\ LB}$ are constructed. We investigate the behavior of $C_{pk}^{T\ LB}$ versus C_{pk1} and C_{pk2} for given $C_{pk}^{T}{}'s$ and find that the most conservative lower bound can be obtained by setting one of $C_{pk}{}'s$ at the given C_{pk}^{T} and the other at infinity. We also provide tables for engineers or practitioners to use in assessing their processes. On the other hand, it is also found that $C_{pk}^{T\ LB}$ reaches its absolute maximum when $C_{pk1} = C_{pk2}$.

As an illustrative example, an application example on dual-fiber tips taken from Pearn and Wu (2005b) is employed. The practical implementation of the statistical theory for manufacturing capability assessment bridges the gap between the theoretical development and the in-plant applications.

For the future research, we could consider the following topics:

- Use the second-order expansion of *Taylor series* to approximate the distribution of $\hat{C}_{pk}^{T}$ to get a more accurate approximation.
- Generalize C_{pk}^{T} for processes with asymmetric tolerances.
- Explore the similar research to C_{pk}^{T} for C_p, C_{PU}, C_{PL}, C_{pk}, C_{pm}, C_{pmk}.
- Develop appropriate process capability measurement based on C_{pk}^{T} when gauge measurement errors exist.

Followings are some other potential research topics:

- develop a powerful test for on-sided or two-sided supplier selection problem.
- develop a decision making method for product acceptance.
- develop tool replacement strategies for production with a low fraction of defectives.

References

1. Bickel, P. J. and Doksum, K. A. (2007). *Mathematical Statistics Basic Ideas and Selected Topics*, Volume I, 322－324. Pearson Prentice Hall.
2. Bothe, D. R. (1992). A capability study for an entire product. *ASQC Quality Congress Transactions*, 172-178.
3. Boyles, R. A. (1991). The Taguchi capability index. *Journal of Quality Technology*, 23, 17-26.
4. Boyles, R. A. (1994). Process capability with asymmetric tolerances. *Communications in Statistics: Simulations and Computation*, 23 (3), 615-643.
5. Boyles, R.A. (1996). Multivariate process analysis with lattice data. *Technometrics*, 38 (1), 37-49.
6. Chan, L. K., Cheng, S.W., and Spiring, F. A. (1988). A new measure of process capability: C_{pm}. *Journal of Quality Technology*, 20(3), 162-175.
7. Chen, H. (1994). A multivariate process capability index over a rectangular solid tolerance zone. *Statistica Sinica*, 4, 749-758.
8. Chen, K. S., Pearn, W. L., and Lin, P. C. (2003). Capability measures for processes with multiple characteristics. *Quality and Reliability Engineering International*, 19, 101-110.
9. Kane, V. E. (1986). Process capability indices. *Journal of Quality Technology*, 18, 41-52.
10. Kotz, S. and Johnson, N. L. (1993). *Process Capability Indices*. Chapman and Hall, London, U.K.
11. Kotz, S. and Johnson, N. L. (2002). Process capability indices－review, 1992－2000. *Journal of Quality Technology*, 34 (1), 1-19.
12. Kotz, S. and Lovelace, C. (1998). *Process Capability Indices in Theory and Practice*. Arnold, London, U.K.
13. Pearn, W. L. and Kotz, S. (2006). *Encyclopedia and Handbook of Process Capability Indices*. World Scientific.
14. Pearn, W. L., Kotz, S., and Johnson, N. L. (1992). Distributional and inferential properties of process capability indices. *Journal of Quality Technology*, 24(4), 216-231.
15. Pearn, W. L. and Shu, M. H. (2003). Lower confidence bounds with sample size information for C_{pm} applied to production yield assurance. *International Journal of Production Research*, 41(15), 3581-3599.

16. Pearn, W. L. and Wu, C. W. (2005a). Measuring manufacturing capability for couplers and wavelength division multiplexers. *International Journal of Advanced Manufacturing Technology*, 25, 533-541.

17. Pearn, W. L. and Wu, C. W. (2005b). Production quality and yield assurance for processes with multiple independent characteristics. *European Journal of Operational Research*, 173, 637-647.

18. Wang, F. K. and Chen, J. (1998-1999). Capability index using principal component analysis. *Quality Engineering*, 11, 21-27.

19. Wang, F. K. and Du, T. C. T. (2000). Using principal component analysis in process performance for multivariate data. *Omega—The international Journal of Management Science*, 28, 185-194.

20. Wang, F. K., Hubele, N. F., Lawrence, P., Miskulin, J. D., and Shahriari, H. (2000). Comparison of three multivariate process capability indices. *Journal of Quality Technology*, 32(3), 263-275.

Appendix A

Proposition $2\Phi(3C_{pk}^T)-1 \le \%\ Yield \le \Phi(3C_{pk}^T)$ (A1)

Lemma

$$\text{If } 0 \le P_i \le 1, \text{ then } 2\prod_{i=1}^{m} P_i - 1 \le \prod_{i=1}^{m}\left(2P_i - 1\right) \tag{A2}$$

Proof: The proof is by induction. We start with *m* = 2. To show $2P_1P_2 - 1 \le \left(2P_1 - 1\right)\left(2P_2 - 1\right)$, (A3)

it suffices to show $P_1P_2 - P_1 - P_2 + 1 \ge 0$, which holds since $0 \le P_1 \le 1$ and $0 \le P_2 \le 1$ Thus (A2) holds for *m*=2.

Assume (A2) holds for *m* = *k*, i.e.,

$$2\prod_{i=1}^{k} P_i - 1 \le \prod_{i=1}^{k}\left(2P_i - 1\right). \tag{A4}$$

For *m* = *k*+1, (A2) also holds because

$$\prod_{i=1}^{k+1}\left(2P_i - 1\right) = \left(\prod_{i=1}^{k}\left(2P_i - 1\right)\right)\left(2P_{k+1} - 1\right) \ge \left(2\prod_{i=1}^{k} P_i - 1\right)\left(2P_{k+1} - 1\right) \ge 2\prod_{i=1}^{k} P_i \cdot P_{k+1} - 1 = 2\prod_{i=1}^{k+1} P_i - 1,$$

where the first inequality holds by (A4) and the second inequality holds by (A3).

This completes the induction.

To prove (A1), it is easy to obtain $2\Phi(3C_{pk}^T) - 1 \le \%Yield$ by the definition of C_{pk}^T.

Since $2\Phi(3C_{pki}) - 1 \le \%\ Yield_i \le \Phi(3C_{pki}),\ i = 1, \ldots, m,$ the overall yield has an upper bound

$$\%\ Yield = \left(\prod_{i=1}^{m} \%\ Yield_i\right) \le \prod_{i=1}^{m} \Phi(3C_{pki}).$$

Then it suffices to show that $\prod_{i=1}^{m} \Phi(3C_{pki}) \le \Phi(3C_{pk}^T).$

By equation (3.3) and Lemma, we have

$$2\Phi(3C_{pk}^T) - 1 = \prod_{i=1}^{m}\left(2\Phi(3C_{pki}) - 1\right) \ge 2\prod_{i=1}^{m} \Phi(3C_{pki}) - 1,$$

which implies $\Phi(3C_{pk}^T) \ge \prod_{i=1}^{m} \Phi(3C_{pki})$. QED.

Appendix B

A proof of Theorem 1.

Theorem 1. The asymptotic distribution of $\hat{C}_{pk}^{T}$ is

$$\sqrt{n}\left(\hat{C}_{pk}^{T}-C_{pk}^{T}\right)\xrightarrow{d}N\left(0\ ,\ \frac{1}{9\left[\phi(3C_{pk}^{T})\right]^{2}}\sum_{i=1}^{m}(a_i^2+b_i^2)\right)\ \text{as } n\to\infty \qquad (4.4)$$

where $a_i = \left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})$ and $b_i = \frac{3}{\sqrt{2}}a_iC_{pki},\ i=1,\dots,m.$

Proof. By definition, we have

$$C_{pki}=\frac{d_i-|\mu_i-m_i|}{3\sigma_i},\ i=1,..,m,$$

where μ_i is the mean of the ith characteristic, m_i = $(USL_i+LSL_i)/2$ is the mid-point of the specification interval, and d_i = $(USL_i-LSL_i)/2$ is the half width of the specification interval, for $i=1,\dots,m$. By definition,

$$C_{pk}^{T}=\frac{1}{3}\Phi^{-1}\left\{\left[\prod_{i=1}^{m}\left(2\Phi(3C_{pk_i})-1\right)+1\right]/2\right\}. \qquad \text{(B1)}$$

Since C_{pki} is a function of μ_i and σ_i^2, by (B1), C_{pk}^{T} is a function of $\mu_1,\dots,\mu_m,\sigma_1^2,\dots,\sigma_m^2$. Denote this function by f. Then $\hat{C}_{pk}^{T}$ = $f(\hat{\mu}_1,\dots,\hat{\mu}_m;\hat{\sigma}_1^2,\dots,\hat{\sigma}_m^2)$, where $\hat{\mu}_i=\bar{\mathrm{X}}_i$ and $\hat{\sigma}_i^2=S_i^2=\sum_{k=1}^{n}(\mathrm{X}_{ik}-\bar{\mathrm{X}}_i)^2/n-1$, i = $1,\dots,m.$

Employing the first-order expansion of m-variates *Taylor series*, we can obtain

$$\begin{aligned}\hat{C}_{pk}^{T} \approx\ & f(\mu_1,\dots,\mu_m;\sigma_1^2,\dots,\sigma_m^2)+\frac{\partial f(\mu_1,\dots,\mu_m;\sigma_1^2,\dots,\sigma_m^2)}{\partial\hat{u}_1}(\hat{\mu}_1-\mu_1)+\dots\\ &+\frac{\partial f(\mu_1,\dots,\mu_m;\sigma_1^2,\dots,\sigma_m^2)}{\partial\hat{u}_m}(\hat{\mu}_m-\mu_m)+\frac{\partial f(\mu_1,\dots,\mu_m;\sigma_1^2,\dots,\sigma_m^2)}{\partial\hat{\sigma}_1^2}(\hat{\sigma}_1^2-\sigma_1^2)+\dots\\ &+\frac{\partial f(\mu_1,\dots,\mu_m;\sigma_1^2,\dots,\sigma_m^2)}{\partial\hat{\sigma}_m^2}(\hat{\sigma}_m^2-\sigma_m^2).\end{aligned}$$

Differentiating with respect to $\hat{\mu}_i$ and $\hat{\sigma}_i^2$, $i=1,\dots,m$ gives

$$\frac{\partial f(\mu_1,...,\mu_m,\sigma_1^2,...,\sigma_m^2)}{\partial \hat{u}_i} = \begin{cases} \dfrac{1}{-3\phi(3C_{pk}^T)}\dfrac{1}{\sigma_i}\left\{\left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\}, & \text{if } \mu_i \geq m_i \\ \dfrac{1}{3\phi(3C_{pk}^T)}\dfrac{1}{\sigma_i}\left\{\left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\}, & \text{if } \mu_i < m_i \end{cases}$$

$$\frac{\partial f(\mu_1,...,\mu_m,\sigma_1^2,...,\sigma_m^2)}{\partial \hat{\sigma}_i^2} = \frac{-1}{3\phi(3C_{pk}^T)}\left\{\frac{3C_{pki}}{2\sigma_i^2}\left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\},$$

for $i = 1,..,m.$

Denote

$$W_i = \frac{1}{\sigma_i}\left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pki})-1\right)\right]\phi(3C_{pki})\left(\overline{\mathrm{X}}_i - \mu_i\right)$$

and

$$G_i = \frac{3C_{pki}}{2\sigma_i^2}\left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\left(S_i^2 - \sigma_i^2\right), \text{ for } i = 1,...,m.$$

Then

$$\hat{C}_{pk}^T \approx C_{pk}^T + \frac{-1}{3\phi(3C_{pk}^T)}\sum_{i=1}^{m}(W_i + G_i).$$

Let $Z_i = \sqrt{n}\left(\overline{\mathrm{X}}_i - \mu_i\right)$ and $Y_i = \sqrt{n}\left(S_i^2 - \sigma_i^2\right)$, i = 1,…,m. Then Z_i and Y_i are independent. Because the first two moments of $\overline{\mathrm{X}}_i$ and S_i^2 exist, by the Central Limit Theorem, Z_i and Y_i converge to $N\left(0\ ,\sigma_i^2\right)$ and $N\left(0\ ,2\sigma_i^4\right)$, respectively, $i = 1,...,m.$

So we obtain $E(\hat{C}_{pk}^T) = C_{pk}^T$ and

$$Var(\hat{C}_{pk}^T) \approx Var\left\{C_{pk}^T + \frac{-1}{3\phi(3C_{pk}^T)}\sum_{i=1}^{m}(W_i + G_i)\right\} \approx \frac{1}{9n\left[\phi(3C_{pk}^T)\right]^2}\sum_{i=1}^{m}\left(a_i^2 + b_i^2\right),$$

where $a_i = \left[\prod_{j=1,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})$ and $b_i = \dfrac{3C_{pki}}{\sqrt{2}}a_i,\ i = 1,...,m.$

Appendix C

Another proof of Theorem 1.

Since $C_{pk}^{T} = f(\mu_1,...,\mu_m;\sigma_1^2,...,\sigma_m^2)$ and $\hat{C}_{pk}^{T} = f(\hat{\mu}_1,...,\hat{\mu}_m;\hat{\sigma}_1^2,...,\hat{\sigma}_m^2)$, where $\hat{\mu}_i = \overline{\mathrm{X}}_i$ and $\hat{\sigma}_i^2 = S_i^2 = \dfrac{1}{n-1}\sum_{k=1}^{n}(\mathrm{X}_{ik}-\overline{\mathrm{X}}_i)^2$, $i = 1,...,m$. Because $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are MLE of μ_i and σ_i^2, respectively, $\hat{C}_{pk}^{T}$ is the MLE of $C_{pk}^{T} = f(\mu_1,...,\mu_m;\sigma_1^2,...,\sigma_m^2)$.

Let $\theta = (\mu_1,...,\mu_m,\sigma_1^2,...,\sigma_m^2)$. The Fisher information is evaluated as

$$I_{m\times m}(\theta) = \begin{bmatrix} \sigma_1^{-2} & 0... & & & 0 \\ & ... & & & \\ 0... & 0 & \sigma_m^{-2} & 0\ ... & 0 \\ 0\ ... & 0 & 2\sigma_1^{-4} & 0... & 0 \\ & & ... & & \\ 0\ ... & & & 0 & 2\sigma_m^{-4} \end{bmatrix}_{m\times m}.$$

$$\frac{\partial f(\mu_1,...,\mu_m,\sigma_1^2,...,\sigma_m^2)}{\partial \hat{u}_i} = \begin{cases} \dfrac{1}{-3\phi(3C_{pk}^{T})}\dfrac{1}{\sigma_i}\left\{\left[\prod_{j=1,\,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\}, & \text{if } \mu_i \geq m_i \\ \dfrac{1}{3\phi(3C_{pk}^{T})}\dfrac{1}{\sigma_i}\left\{\left[\prod_{j=1,\,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\}, & \text{if } \mu_i < m_i \end{cases}$$

$$\frac{\partial f(\mu_1,...,\mu_m,\sigma_1^2,...,\sigma_m^2)}{\partial \hat{\sigma}_i^2} = \frac{-1}{3\phi(3C_{pk}^{T})}\left\{\frac{3C_{pki}}{2\sigma_i^2}\left[\prod_{j=1,\,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})\right\},$$

for $i = 1,..,m$.

By Theorem 5.3.5. in Bickel, and Doksum (2007), we can obtain the desired result:

$$\sqrt{n}\left(\hat{C}_{pk}^{T} - C_{pk}^{T}\right)\xrightarrow{d} N\left(0,\frac{1}{9[\phi(3C_{pk}^{T})]^2}\sum_{i=1}^{m}\left(a_i^2+b_i^2\right)\right),\ \text{as } n\to\infty,$$

where $a_i = \left[\prod_{j=1,\,j\neq i}^{m}\left(2\Phi(3C_{pkj})-1\right)\right]\phi(3C_{pki})$ and $b_i = \dfrac{3C_{pki}}{\sqrt{2}}a_i$, $i = 1,...,m$. Q.E.D.

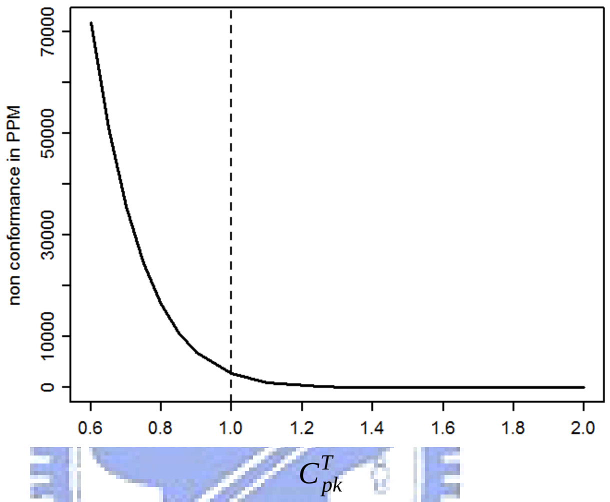


Figure 1. Upper bounds on NCPPM values of C_{pk}^{T}.

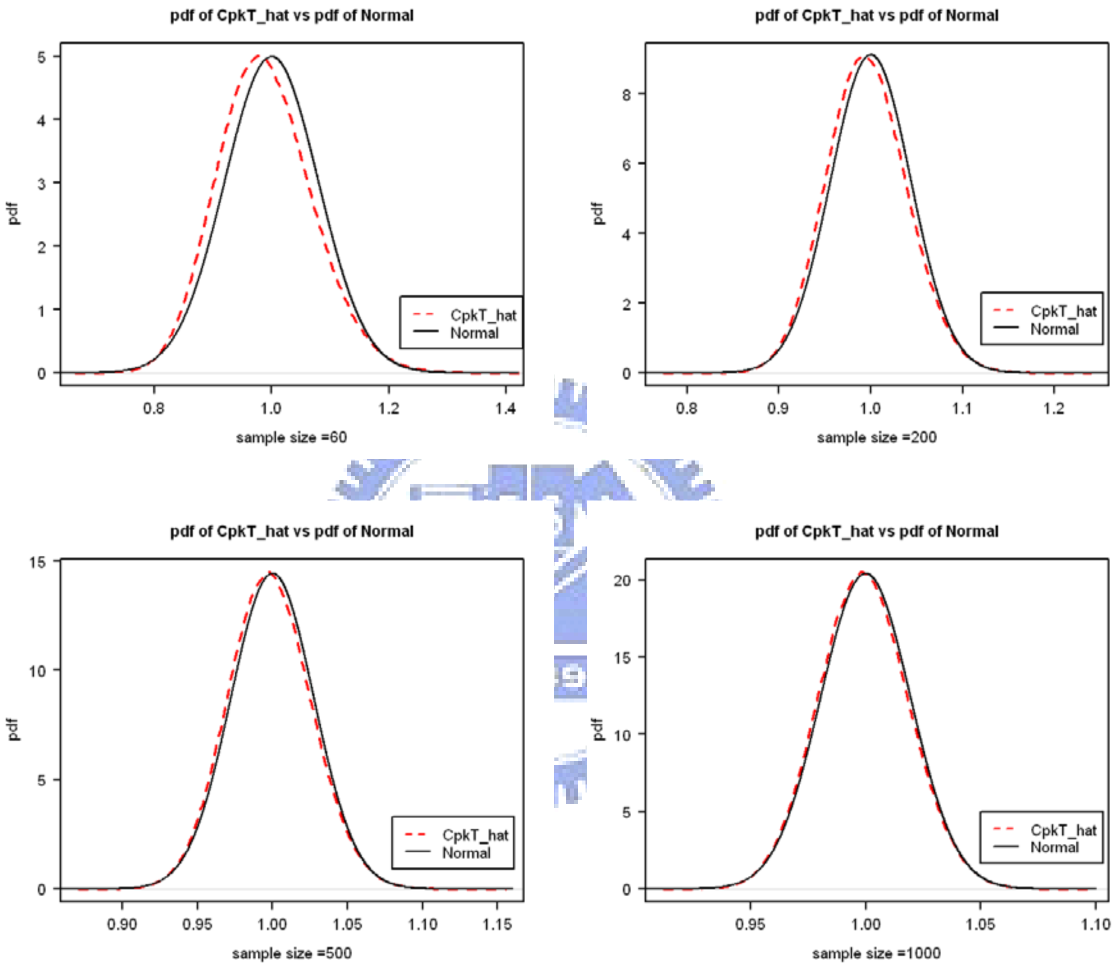


Figure 2. Comparison of the probability density function (dash line) of $\hat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =60, 200, 500, 1000 ($C_{pk}^{T}=1.0$ and $\mu_1 \neq m_1$ and $\mu_2 \neq m_2$).

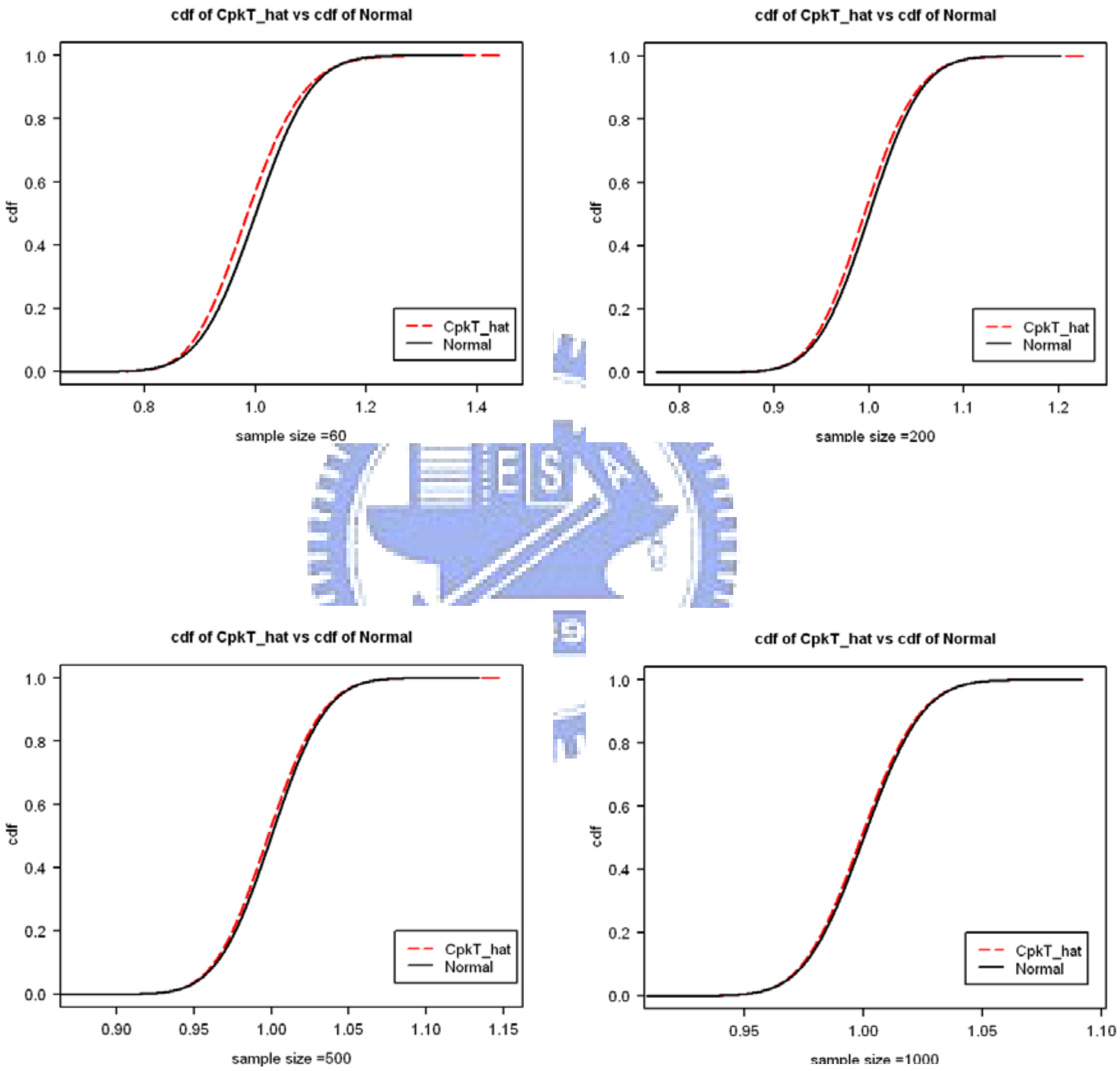


Figure 3. Comparison of the distribution function (dash line) of $\widehat{C}^{T}_{pk}$ obtained by simulation and its normal approximation (solid line) for n =50, 200, 500, 1000 ($C^{T}_{pk}=1.0$ and $\mu_1 \neq m_1$ and $\mu_2 \neq m_2$).

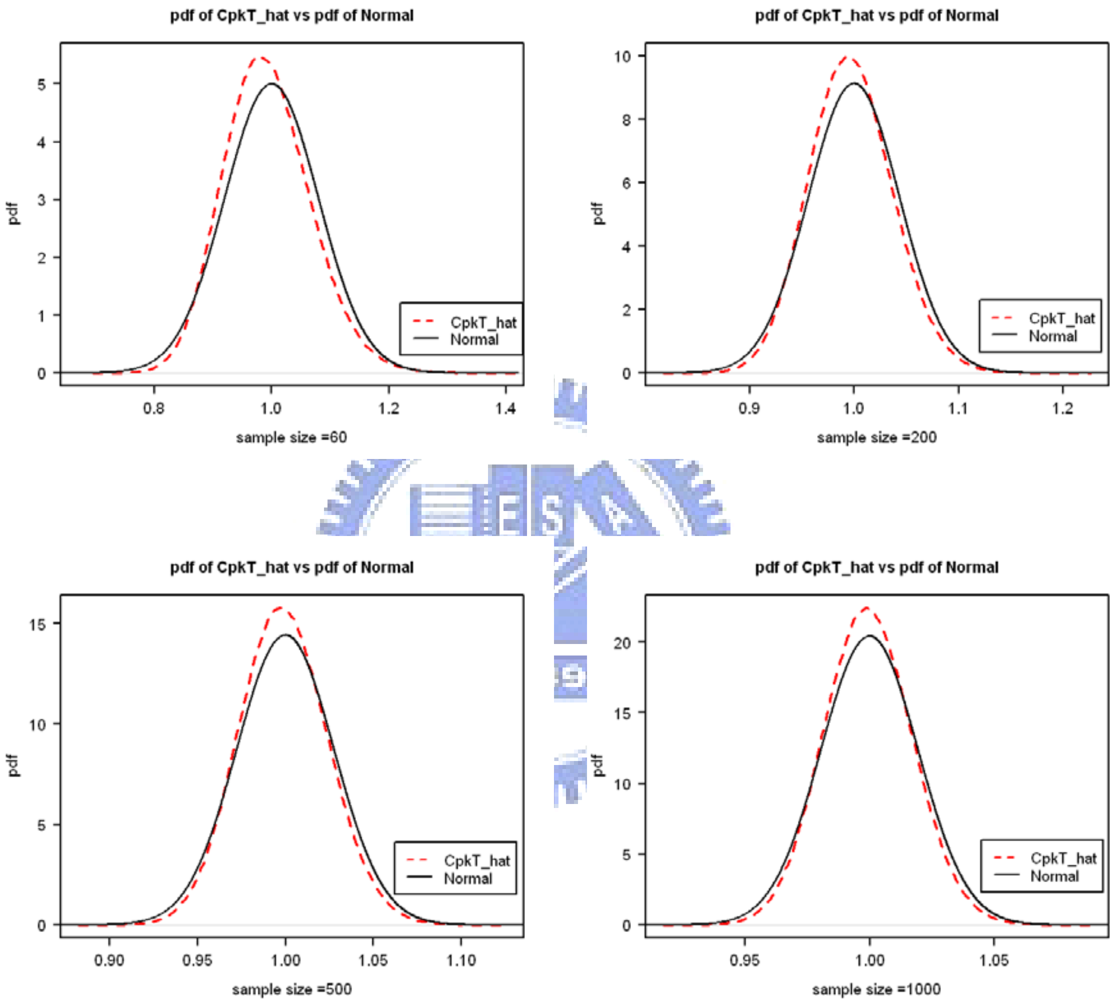


Figure 4. Comparison of the probability density function (dash line) of $\hat{C}^T_{pk}$ obtained by simulation and its normal approximation (solid line) for n =60, 200, 500, 1000 ($C^T_{pk}=1.0$ and $\mu_1 = m_1$ and $\mu_2 = m_2$).

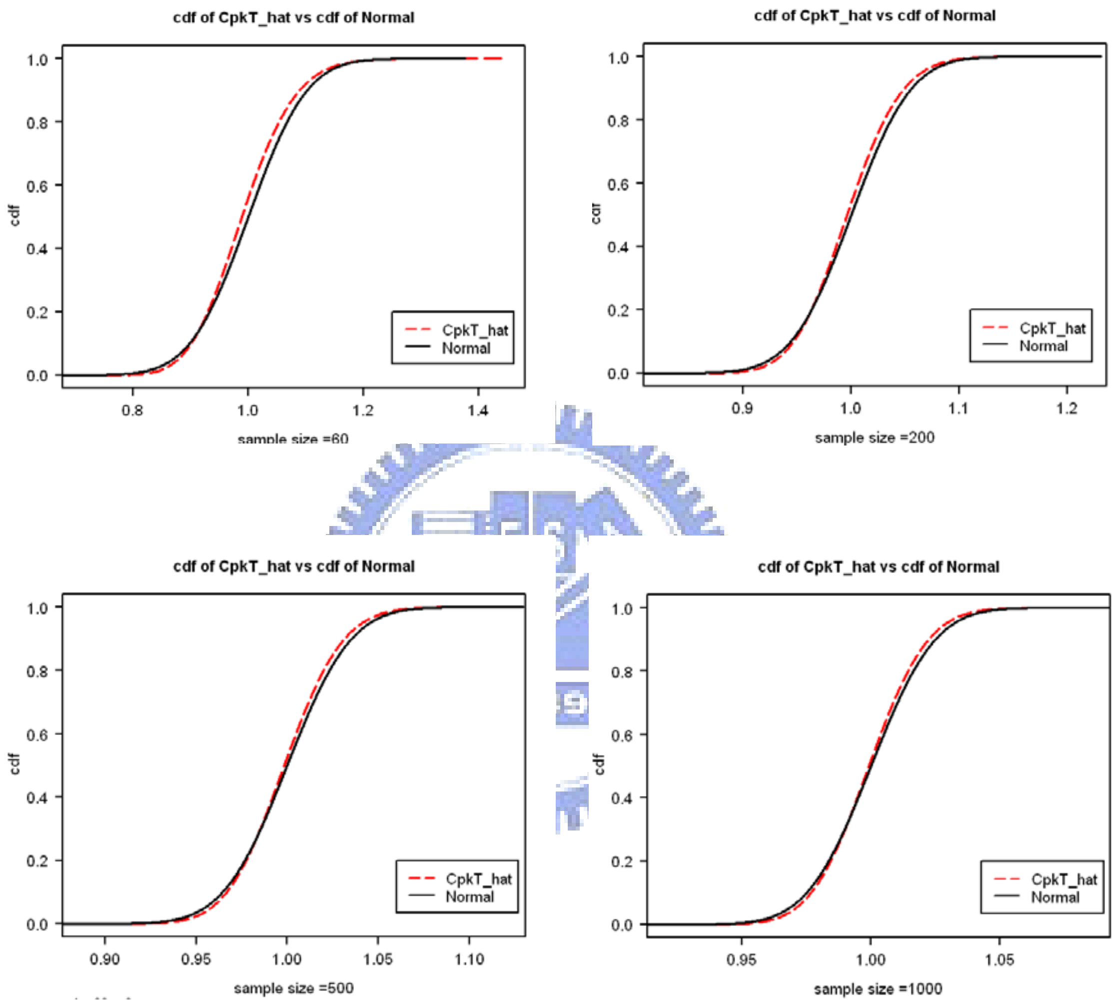


Figure 5. Comparison of the distribution function (dash line) of $\widehat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =50, 200, 500, 1000 ($C_{pk}^{T}=1.0$ and $\mu_1 = m_1$ and $\mu_2 = m_2$).

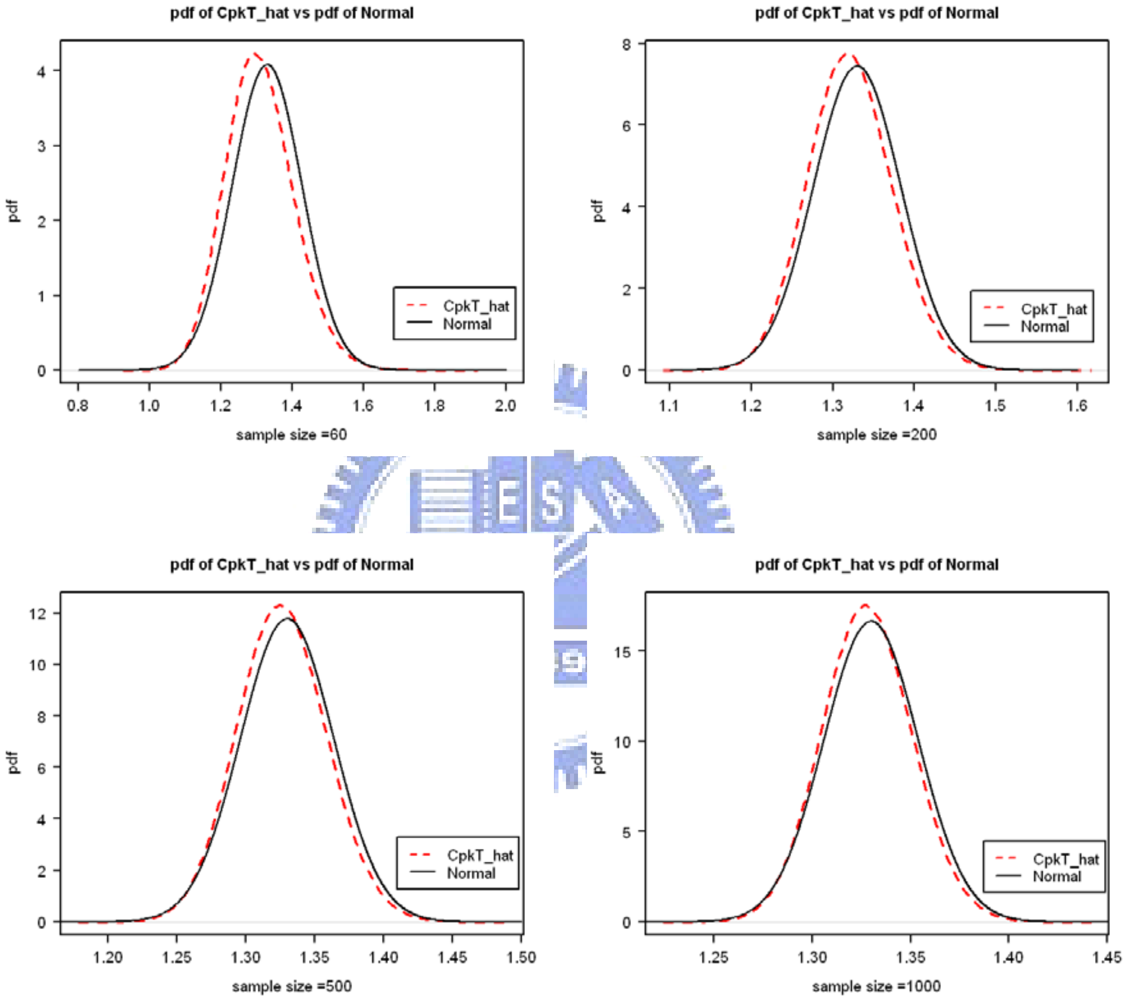


Figure 6. Comparison of the probability density function (dash line) of $\hat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =60, 200, 500, 1000 (C_{pk}^{T} = 1.33 and $\mu_1 \neq m_1$ and $\mu_2 \neq m_2$).

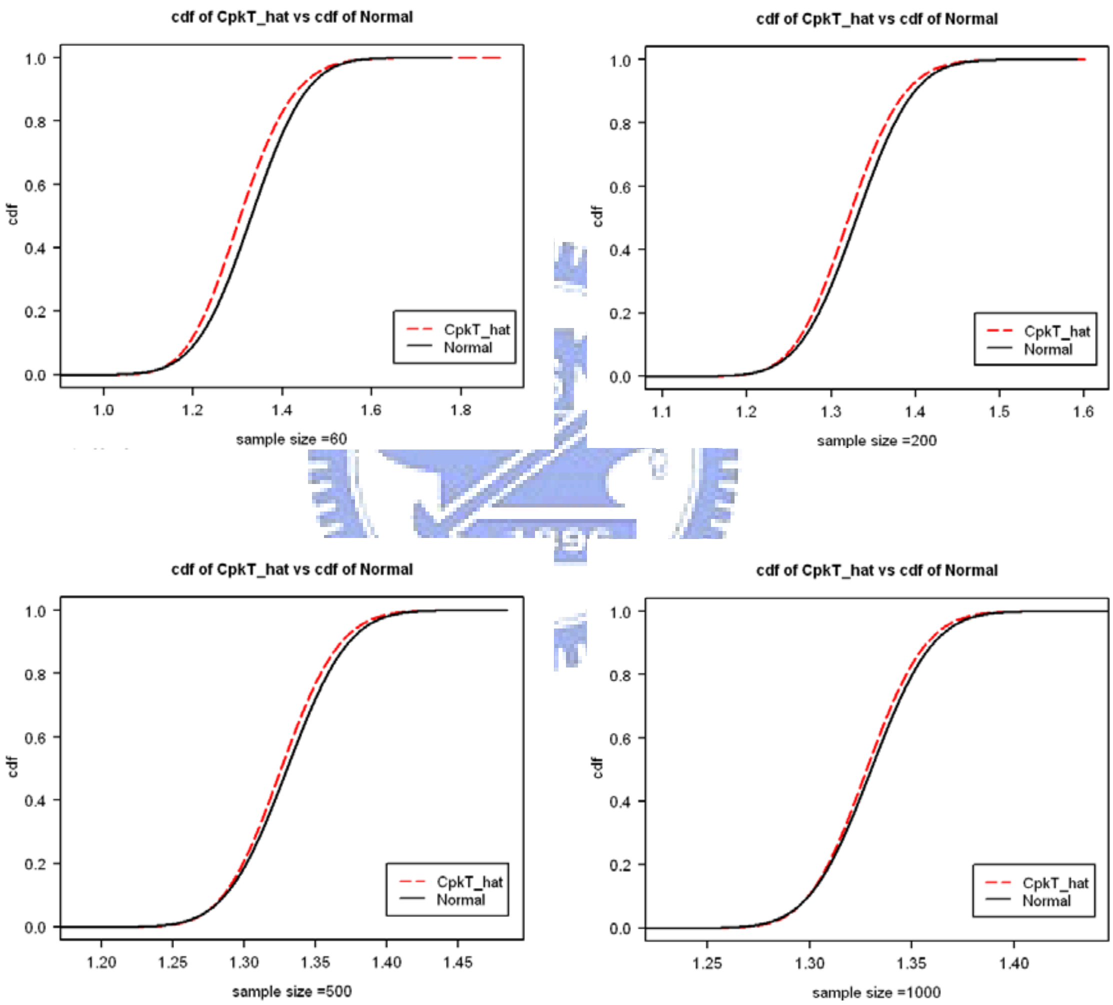


Figure 7. Comparison of the distribution function (dash line) of $\widehat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =50, 200, 500, 1000 ($C_{pk}^{T}=1.33$ and $\mu_1 \neq m_1$ and $\mu_2 \neq m_2$).

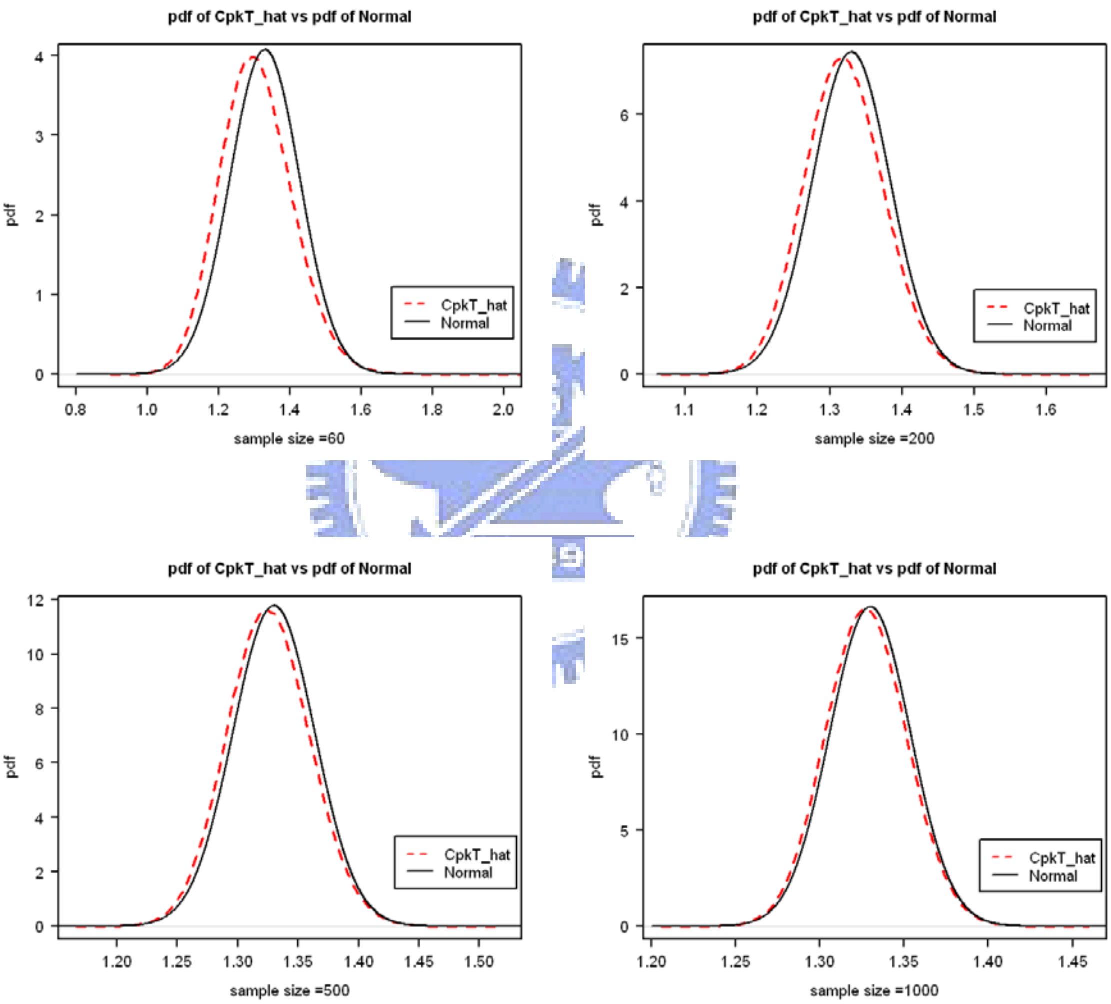


Figure 8. Comparison of the probability density function (dash line) of $\widehat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =60, 200, 500, 1000 ($C_{pk}^{T}=1.33$ and $\mu_1 = m_1$ and $\mu_2 = m_2$).

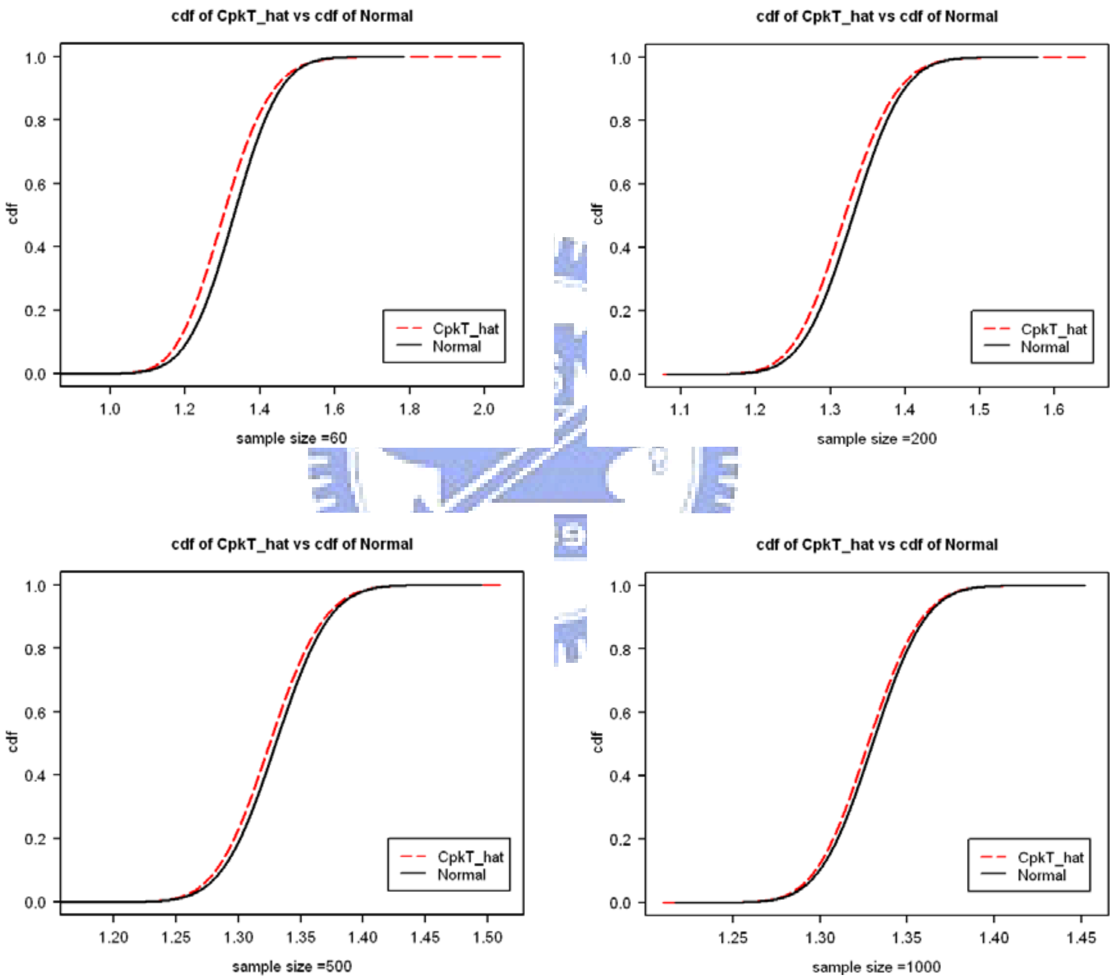
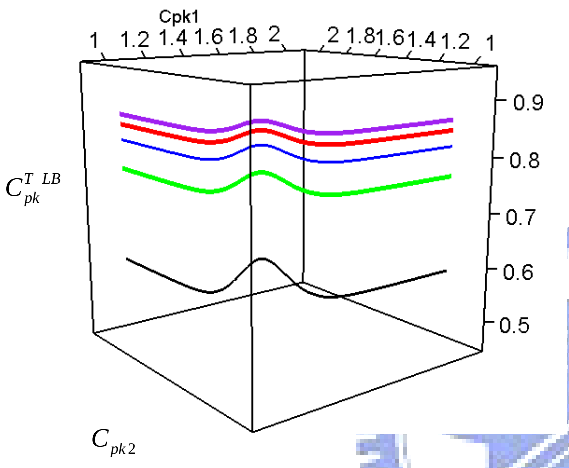


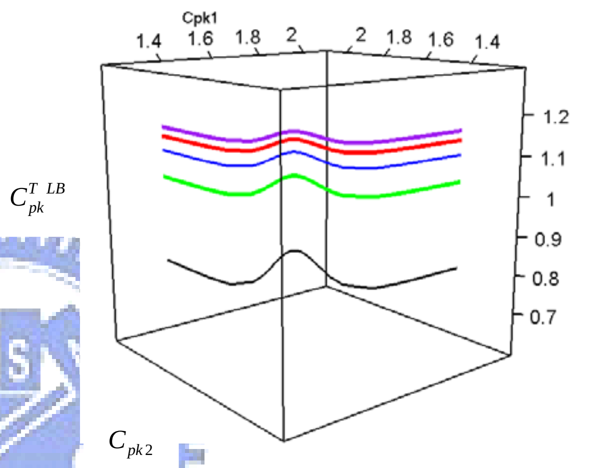
Figure 9. Comparison of the distribution function (dash line) of $\hat{C}_{pk}^{T}$ obtained by simulation and its normal approximation (solid line) for n =50, 200, 500, 1000 (C_{pk}^{T} = 1.33 and $\mu_1 = m_1$ and $\mu_2 = m_2$).

(a)



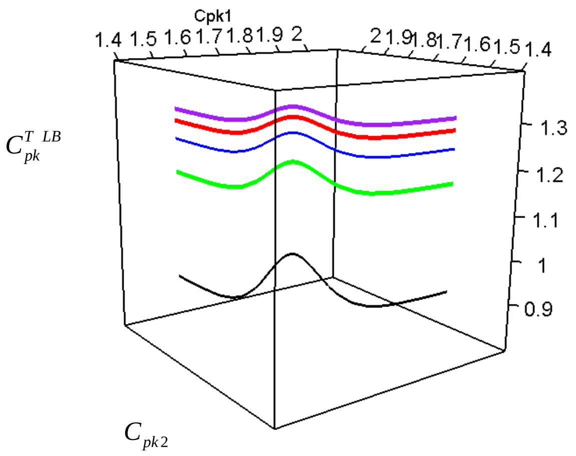
(a) $1.0 \le C_{pk1} \le 2.0$, $1.0 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1$

(b)



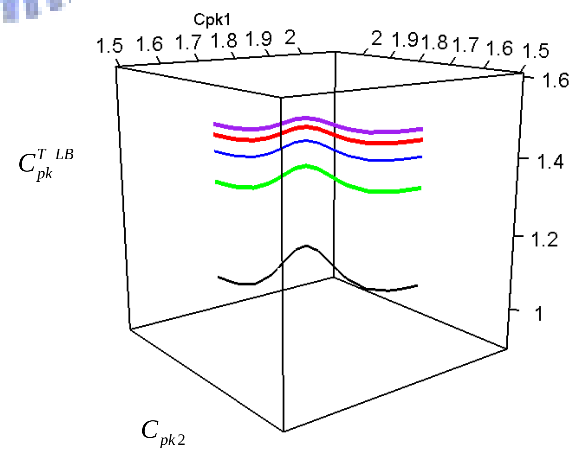
(b) $1.33 \le C_{pk1} \le 2.0$, $1.33 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.33$

(c)



(c) $1.5 \le C_{pk1} \le 2.0$, $1.5 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.5$

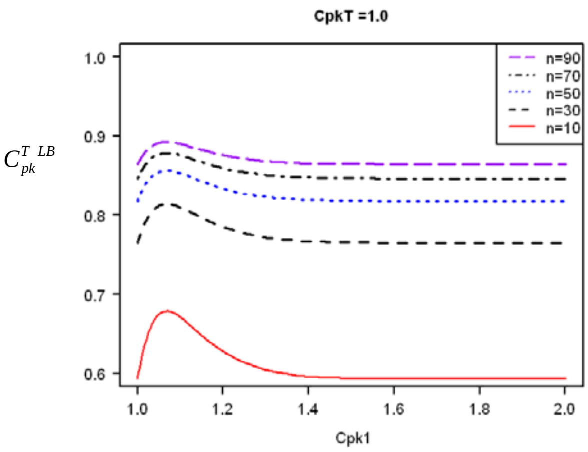
(d)



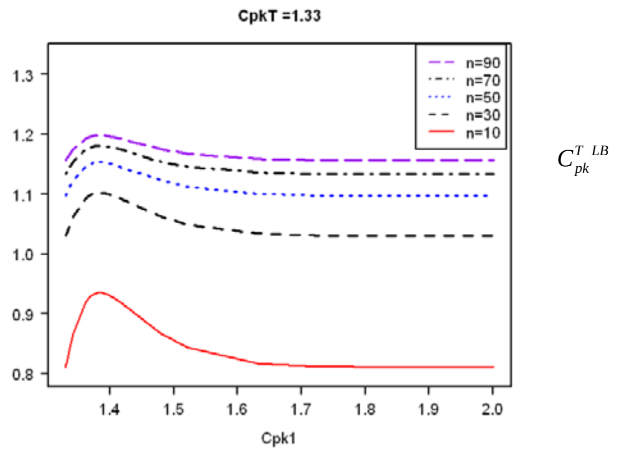
(d) $1.67 \le C_{pk1} \le 2.0$, $1.67 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.67$

Figure 10. Curves of $C_{pk}^{T\ LB}$ versus (C_{pk1}, C_{pk2}) with α = 0.05 and n =10(20)90 (from bottom to top in plot).

(a)



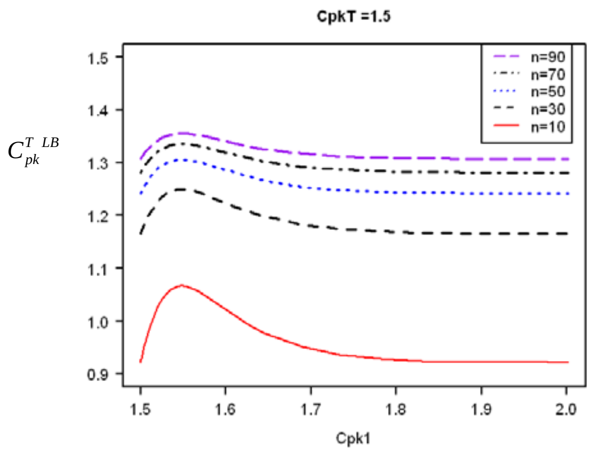
(b)



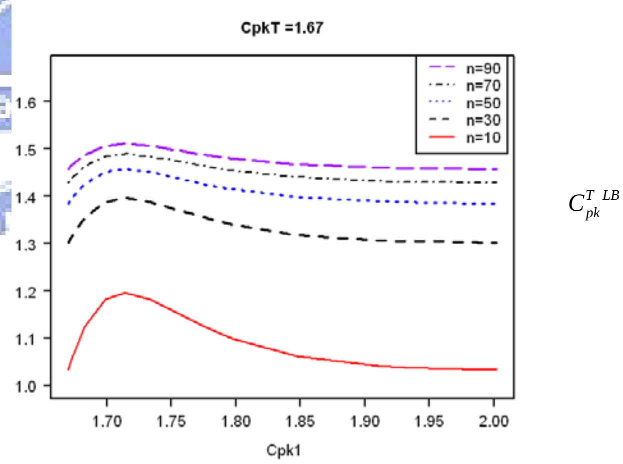
(a) $1.0 \le C_{pk1} \le 2.0$, $1.0 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1$

(b) $1.33 \le C_{pk1} \le 2.0$, $1.33 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.33$

(c)



(d)



(c) $1.5 \le C_{pk1} \le 2.0$, $1.5 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.5$

(d) $1.67 \le C_{pk1} \le 2.0$, $1.67 \le C_{pk2} \le 2.0$ and $C_{pk}^{T} = 1.67$.

Figure 11. Curves of $C_{pk}^{T\ LB}$ versus C_{pk1} for various C_{pk}^{T}, α = 0.05, and n=10(20)90 (from bottom to top in plot).

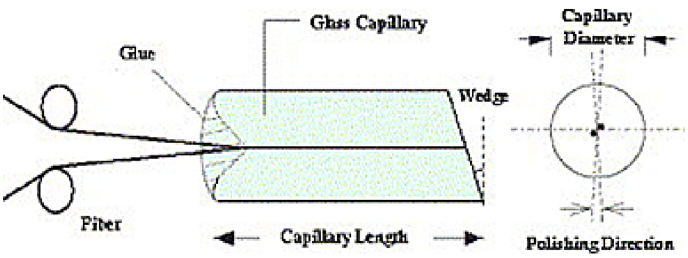


Figure 12. A sample of the dual-fiber tips. (from Pearn and Wu (2005b)).

Table 1. Corresponding upper bounds of NCPPM for some specific values of C_{pk}^{T}.

C_{pk}^{T}	Upper bound of NCPPM
1.00	2699.796
1.25	176.835
1.33	66.073
1.45	13.614
1.50	6.795
1.60	1.587
1.67	0.544
2.00	0.002

Table 2. Coverage rate and 90% confidence interval length (in parentheses) for various cases of C_{pk}^{T} =1.0 with n =30, 50, 100, 500, 1000.

C_{pk1}	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683
C_{pk2}	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683
$\frac{d_1}{\sigma_1}$	3.2050	3.5611	4.0062	2.1366	4.0062	4.0062
$\frac{d_2}{\sigma_2}$	3.2050	3.5611	4.0062	2.1366	5.3416	3.5611
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0	0.3561	0.8012	5.3416	0.8012	0.8012
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0	0.3561	0.8012	5.3416	2.1366	0.3561
sample size *n*	**coverage rate (confidence interval length)**					
30	0.9436 (0.4099)	0.9152 (0.4104)	0.9144 (0.4108)	0.9143 (0.4108)	0.9143 (0.4107)	0.9142 (0.4106)
50	0.9405 (0.3107)	0.9129 (0.3119)	0.9126 (0.3120)	0.9126 (0.3119)	0.9130 (0.3120)	0.9127 (0.3119)
100	0.9371 (0.2140)	0.9095 (0.2150)	0.9092 (0.2150)	0.9095 (0.2150)	0.9092 (0.2150)	0.9097 (0.2150)
500	0.9302 (0.0922)	0.9032 (0.0924)	0.9029 (0.0924)	0.9028 (0.0924)	0.9035 (0.0924)	0.9034 (0.0924)
1000	0.9296 (0.0647)	0.9020 (0.0648)	0.9021 (0.0648)	0.9018 (0.0648)	0.9019 (0.0648)	0.9021 (0.0648)

Table 3. Coverage rate and 90% confidence interval length (in parentheses) for various cases of C_{pk}^{T} =1.33 with n =30, 50, 100, 500, 1000.

C_{pk1}	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838
C_{pk2}	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838
$\frac{d_1}{\sigma_1}$	4.6452	5.1613	5.8065	6.6360	7.7420	6.6360
$\frac{d_2}{\sigma_2}$	4.6452	5.1613	5.8065	6.6360	7.7420	7.7420
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0	0.5161	1.1613	1.9908	3.0968	1.9908
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0	0.5161	1.1613	1.9908	3.0968	3.0968
sample size *n*	**coverage rate (confidence interval length)**					
30	0.9295 (0.5967)	0.9135 (0.5968)	0.9135 (0.5969)	0.9133 (0.5968)	0.9131 (0.5968)	0.9135 (0.5968)
50	0.9297 (0.4543)	0.9137 (0.4551)	0.9135 (0.4550)	0.9143 (0.4551)	0.9144 (0.4550)	0.9139 (0.4552)
100	0.9283 (0.3123)	0.9135 (0.3133)	0.9137 (0.3132)	0.9130 (0.3132)	0.9136 (0.3132)	0.9135 (0.3132)
500	0.9216 (0.1300)	0.9076 (0.1304)	0.9081 (0.1304)	0.9075 (0.1304)	0.9077 (0.1304)	0.9074 (0.1304)
1000	0.9194 (0.0898)	0.9047 (0.0900)	0.9048 (0.0900)	0.9054 (0.0900)	0.9047 (0.0900)	0.9055 (0.0900)

Table 4. Coverage rate and 90% confidence interval length (in parentheses) for various cases of C_{pk}^{T} =1.5 with n =30, 50, 100, 500, 1000.

C_{pk1}	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484
C_{pk2}	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484
$\frac{d_1}{\sigma_1}$	4.1515	4.7445	5.1893	5.5353	6.6423	8.3029
$\frac{d_2}{\sigma_2}$	4.1515	4.7445	5.1893	5.5353	6.6423	8.3029
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0	0.5931	1.0379	1.3838	2.4909	4.1515
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0	0.5931	1.0379	1.3838	2.4909	4.1515
sample size *n*	**coverage rate (confidence interval length)**					
30	0.9340 (0.5309)	0.9138 (0.5313)	0.9143 (0.5312)	0.9141 (0.5313)	0.9144 (0.5311)	0.9133 (0.5312)
50	0.9330 (0.4032)	0.9138 (0.4041)	0.9137 (0.4041)	0.9147 (0.4042)	0.9139 (0.4042)	0.9139 (0.4041)
100	0.9305 (0.2767)	0.9126 (0.2777)	0.9122 (0.2777)	0.9122 (0.2777)	0.9130 (0.2777)	0.9127 (0.2777)
500	0.9228 (0.1161)	0.9062 (0.1164)	0.9057 (0.1164)	0.9058 (0.1164)	0.9059 (0.1164)	0.9058 (0.1164)
1000	0.9215 (0.0807)	0.9040 (0.0809)	0.9037 (0.0809)	0.9036 (0.0809)	0.9041 (0.0809)	0.9035 (0.0809)

Table 5. Coverage rate and 90% confidence interval length (in parentheses) for various cases of C_{pk}^{T} =2.0 with n =30, 50, 100, 500, 1000.

C_{pk1}	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372
C_{pk2}	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372
$\frac{d_1}{\sigma_1}$	6.1116	6.9846	2.0372	3.6669	6.1116	6.1116
$\frac{d_2}{\sigma_2}$	6.1116	6.9846	2.0372	3.6669	6.9846	12.223
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0	0.8731	8.1488	9.7785	0	0
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0	0.8731	8.1488	9.7785	0.8731	6.1116
sample size *n*	**coverage rate (confidence interval length)**					
100	0.9227 (0.4237)	0.9137 (0.4243)	0.9135 (0.4242)	0.9139 (0.4242)	0.9181 (0.4239)	0.9185 (0.4240)
300	0.9220 (0.2333)	0.9130 (0.2339)	0.9130 (0.2338)	0.9126 (0.2339)	0.9173 (0.2336)	0.9178 (0.2336)
500	0.9204 (0.1760)	0.9111 (0.1765)	0.9113 (0.1765)	0.9112 (0.1765)	0.9159 (0.1763)	0.9158 (0.1763)
700	0.9188 (0.1462)	0.9107 (0.1466)	0.9107 (0.1466)	0.9104 (0.1466)	0.9145 (0.1464)	0.9143 (0.1464)
1000	0.9177 (0.1202)	0.9092 (0.1205)	0.9084 (0.1205)	0.9083 (0.1205)	0.9134 (0.1203)	0.9141 (0.1203)

Table 6. The most conservative 95% lower confidence bounds $C_{pk}^{T\,LB}$ of C_{pk}^{T} for $\widehat{C}_{pk}^{T}$ =1(0.1)2, n =10(10)200.

n	$\widehat{C}_{pk}^{T}$										
	1.0	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
10	0.5934	0.6598	0.7258	0.7914	0.8567	0.9217	0.9865	1.0511	1.1156	1.1800	1.2442
20	0.7125	0.7888	0.8647	0.9404	1.0158	1.0911	1.1662	1.2412	1.3161	1.3909	1.4656
30	0.7652	0.8459	0.9262	1.0064	1.0863	1.1661	1.2458	1.3254	1.4049	1.4843	1.5637
40	0.7967	0.8799	0.9629	1.0457	1.1283	1.2108	1.2933	1.3756	1.4578	1.5400	1.6221
50	0.8182	0.9032	0.9879	1.0725	1.1570	1.2414	1.3256	1.4098	1.4939	1.5780	1.6620
60	0.8340	0.9203	1.0064	1.0924	1.1782	1.2639	1.3495	1.4351	1.5206	1.6061	1.6915
70	0.8463	0.9336	1.0208	1.1078	1.1946	1.2814	1.3681	1.4548	1.5413	1.6279	1.7144
80	0.8562	0.9444	1.0323	1.1202	1.2079	1.2955	1.3831	1.4706	1.5580	1.6454	1.7328
90	0.8645	0.9533	1.0419	1.1305	1.2189	1.3072	1.3955	1.4837	1.5719	1.6600	1.7481
100	0.8714	0.9608	1.0500	1.1392	1.2282	1.3171	1.4060	1.4948	1.5836	1.6723	1.7610
110	0.8774	0.9673	1.0570	1.1466	1.2362	1.3256	1.4150	1.5044	1.5937	1.6829	1.7721
120	0.8826	0.9729	1.0631	1.1532	1.2432	1.3331	1.4229	1.5127	1.6024	1.6922	1.7818
130	0.8872	0.9779	1.0685	1.1589	1.2493	1.3396	1.4298	1.5200	1.6102	1.7003	1.7904
140	0.8913	0.9824	1.0733	1.1641	1.2548	1.3454	1.4360	1.5266	1.6171	1.7076	1.7980
150	0.8950	0.9863	1.0776	1.1687	1.2597	1.3507	1.4416	1.5325	1.6233	1.7141	1.8049
160	0.8983	0.9900	1.0815	1.1728	1.2642	1.3554	1.4466	1.5378	1.6289	1.7200	1.8111
170	0.9014	0.9932	1.0850	1.1766	1.2682	1.3597	1.4512	1.5426	1.6340	1.7254	1.8167
180	0.9042	0.9963	1.0882	1.1801	1.2719	1.3637	1.4554	1.5471	1.6387	1.7303	1.8219
190	0.9067	0.9990	1.0912	1.1833	1.2754	1.3673	1.4593	1.5511	1.6430	1.7348	1.8266
200	0.9091	1.0016	1.0940	1.1863	1.2785	1.3707	1.4628	1.5549	1.6470	1.7390	1.8310

Table 7. The largest possible 95% lower confidence bounds $C_{pk}^{T\ LB}$ of C_{pk}^{T} for $\widehat{C}_{pk}^{T} = 1(0.1)2$, n =10(10)200.

n	$\widehat{C}_{pk}^{T}$										
	1.0	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
10	0.6788	0.7573	0.8353	0.9127	0.9898	1.0666	1.1430	1.2192	1.2952	1.3709	1.4466
20	0.7729	0.8577	0.9421	1.0262	1.1100	1.1935	1.2768	1.3600	1.4430	1.5259	1.6087
30	0.8146	0.9021	0.9894	1.0764	1.1632	1.250	1.3361	1.4224	1.5085	1.5945	1.6805
40	0.8394	0.9287	1.0176	1.1064	1.1949	1.2833	1.3715	1.4596	1.5476	1.6355	1.7233
50	0.8564	0.9467	1.0369	1.1268	1.2166	1.3062	1.3956	1.4850	1.5742	1.6634	1.7525
60	0.8689	0.9601	1.0511	1.1419	1.2325	1.3230	1.4134	1.5037	1.5939	1.6840	1.7741
70	0.8786	0.9705	1.0621	1.1536	1.2450	1.3362	1.4273	1.5183	1.6092	1.7000	1.7908
80	0.8865	0.9788	1.0710	1.1631	1.2550	1.3468	1.4384	1.5300	1.6215	1.7129	1.8043
90	0.8929	0.9858	1.0784	1.1709	1.2633	1.3555	1.4477	1.5398	1.6317	1.7236	1.8155
100	0.8984	0.9916	1.0847	1.1775	1.2703	1.3629	1.4555	1.5480	1.6404	1.7327	1.8250
110	0.9032	0.9967	1.0900	1.1832	1.2763	1.3693	1.4622	1.5550	1.6478	1.7405	1.8331
120	0.9073	1.0011	1.0947	1.1882	1.2816	1.3749	1.4681	1.5612	1.6543	1.7473	1.8402
130	0.9109	1.0050	1.0988	1.1926	1.2862	1.3798	1.4732	1.5666	1.6600	1.7533	1.8465
140	0.9142	1.0084	1.1025	1.1965	1.2904	1.3842	1.4779	1.5715	1.6651	1.7586	1.8521
150	0.9171	1.0115	1.1058	1.2000	1.2941	1.3881	1.4820	1.5759	1.6696	1.7634	1.8571
160	0.9197	1.0143	1.1088	1.2032	1.2975	1.3916	1.4857	1.5798	1.6738	1.7677	1.8616
170	0.9221	1.0169	1.1115	1.2061	1.3005	1.3949	1.4892	1.5834	1.6776	1.7717	1.8658
180	0.9243	1.0192	1.1140	1.2087	1.3033	1.3978	1.4923	1.5867	1.6810	1.7753	1.8696
190	0.9263	1.0214	1.1163	1.2112	1.3059	1.4006	1.4951	1.5897	1.6842	1.7786	1.8730
200	0.9282	1.0234	1.1184	1.2134	1.3083	1.4031	1.4978	1.5925	1.6871	1.7817	1.8762

Table 8. The precision R for the most conservative 95% lower confidence bounds $C_{pk}^{T\ LB}$ of C_{pk}^{T} for $\hat{C}_{pk}^{T}$ =1(0.1)2, n =10(10)200.

n	R										
	1.0	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
10	0.5934	0.5998	0.6048	0.6088	0.6119	0.6145	0.6166	0.6183	0.6198	0.6211	0.6221
20	0.7125	0.7171	0.7206	0.7234	0.7256	0.7274	0.7289	0.7301	0.7312	0.7321	0.7328
30	0.7652	0.7690	0.7718	0.7742	0.7759	0.7774	0.7786	0.7796	0.7805	0.7812	0.7819
40	0.7967	0.7999	0.8024	0.8044	0.8059	0.8072	0.8083	0.8092	0.8099	0.8105	0.8111
50	0.8182	0.8211	0.8233	0.8250	0.8264	0.8276	0.8285	0.8293	0.8299	0.8305	0.8310
60	0.8340	0.8366	0.8387	0.8403	0.8416	0.8426	0.8434	0.8442	0.8448	0.8453	0.8458
70	0.8463	0.8487	0.8507	0.8522	0.8533	0.8543	0.8551	0.8558	0.8563	0.8568	0.8572
80	0.8562	0.8585	0.8603	0.8617	0.8628	0.8637	0.8644	0.8651	0.8656	0.8660	0.8664
90	0.8645	0.8666	0.8683	0.8696	0.8706	0.8715	0.8722	0.8728	0.8733	0.8737	0.8741
100	0.8714	0.8735	0.8750	0.8763	0.8773	0.8781	0.8788	0.8793	0.8798	0.8802	0.8805
110	0.8774	0.8794	0.8808	0.8820	0.8830	0.8837	0.8844	0.8849	0.8854	0.8857	0.8861
120	0.8826	0.8845	0.8859	0.8871	0.8880	0.8887	0.8893	0.8898	0.8902	0.8906	0.8909
130	0.8872	0.8890	0.8904	0.8915	0.8924	0.8931	0.8936	0.8941	0.8946	0.8949	0.8952
140	0.8913	0.8931	0.8944	0.8955	0.8963	0.8969	0.8975	0.8980	0.8984	0.8987	0.8990
150	0.8950	0.8966	0.8980	0.8990	0.8998	0.9005	0.9010	0.9015	0.9018	0.9022	0.9025
160	0.8983	0.9000	0.9013	0.9022	0.9030	0.9036	0.9041	0.9046	0.9049	0.9053	0.9056
170	0.9014	0.9029	0.9042	0.9051	0.9059	0.9065	0.9070	0.9074	0.9078	0.9081	0.9084
180	0.9042	0.9057	0.9068	0.9078	0.9085	0.9091	0.9096	0.9101	0.9104	0.9107	0.9110
190	0.9067	0.9082	0.9093	0.9102	0.9110	0.9115	0.9121	0.9124	0.9128	0.9131	0.9133
200	0.9091	0.9105	0.9117	0.9125	0.9132	0.9138	0.9143	0.9146	0.9150	0.9153	0.9155

Table 9. Estimates of $E(\widehat{C}_{pk}^{T})$ and their standard errors (in parentheses) for some cases of $C_{pk}^{T}=1.0$ and various *n*.

C_{pk1}	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683
C_{pk2}	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683	1.0683
$\frac{d_1}{\sigma_1}$	3.2050	4.0062	3.5611	5.3416	3.5611	3.5611	3.2050	3.2050	5.3416	3.2050
$\frac{d_2}{\sigma_2}$	3.2050	4.0062	3.5611	5.3416	5.3416	4.0062	3.5611	5.3416	4.0062	4.0062
$\frac{(\mu_1-m_1)}{\sigma_1}$	0	0.8012	0.3561	2.1366	0.3561	0.3561	0	0	2.1366	0
$\frac{(\mu_2-m_2)}{\sigma_2}$	0	0.8012	0.3561	2.1366	2.1366	0.8012	0.3561	2.1366	0.8012	0.8012
n	Estimate of $E(\widehat{C}_{pk}^{T})$ and its standard error (in parentheses)									
30	0.9881 (1.1×10^{-4})	0.9826 (1.2×10^{-4})	0.9815 (1.1×10^{-4})	0.9824 (1.1×10^{-4})	0.9819 (1.1×10^{-4})	0.9819 (1.1×10^{-4})	0.9848 (1.1×10^{-4})	0.9853 (1.1×10^{-4})	0.9823 (1.1×10^{-4})	0.9853 (1.1×10^{-4})
50	0.9925 (7.3×10^{-5})	0.9889 (8.1×10^{-5})	0.9890 (8.1×10^{-5})	0.9888 (8.1×10^{-5})	0.9872 (8.9×10^{-5})	0.9872 (8.9×10^{-5})	0.9893 (8.5×10^{-5})	0.9893 (8.5×10^{-5})	0.9874 (8.9×10^{-5})	0.9893 (8.5×10^{-5})
100	0.9948 (5.7×10^{-5})	0.9926 (6.3×10^{-5})	0.9926 (6.2×10^{-5})	0.9926 (6.2×10^{-5})	0.9926 (6.2×10^{-5})	0.9927 (6.2×10^{-5})	0.9938 (6.0×10^{-5})	0.9938 (6.0×10^{-5})	0.9927 (6.2×10^{-5})	0.9937 (6.0×10^{-5})
500	0.9988 (2.5×10^{-5})	0.9983 (2.8×10^{-5})	0.9983 (2.8×10^{-5})	0.9983 (2.8×10^{-5})	0.9983 (2.8×10^{-5})	0.9983 (2.8×10^{-5})	0.9986 (2.7×10^{-5})	0.9985 (2.7×10^{-5})	0.9983 (2.8×10^{-5})	0.9986 (2.7×10^{-5})

Table 10. Estimates of $E(\widehat{C}_{pk}^{T})$ and their standard errors (in parentheses) for some cases of $C_{pk}^{T} = 1.33$ and various n.

C_{pk1}	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838
C_{pk2}	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838	1.3838
$\frac{d_1}{\sigma_1}$	4.1515	4.7445	5.1893	5.5353	6.6423	8.3029	4.7445	4.7445	4.7445	4.7445
$\frac{d_2}{\sigma_2}$	4.1515	4.7445	5.1893	5.5353	6.6423	8.3029	5.1893	5.5353	6.6423	8.3029
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0	0.5931	1.0379	1.3838	2.4909	4.1515	0.5931	0.5931	0.5931	0.5931
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0	0.5931	1.0379	1.3838	2.4909	4.1515	1.0379	1.3838	2.4909	4.1515
n	Estimate of $E(\widehat{C}_{pk}^{T})$ and its standard error (in parentheses)									
30	1.2989 (1.4×10^{-4})	1.2945 (1.4×10^{-4})	1.2944 (1.4×10^{-4})	1.2943 (1.4×10^{-4})	1.2945 (1.4×10^{-4})	1.2944 (1.4×10^{-4})	1.2944 (1.4×10^{-4})	1.2945 (1.4×10^{-4})	1.2943 (1.4×10^{-4})	1.2945 (1.4×10^{-4})
50	1.3069 (1.1×10^{-4})	1.3034 (1.1×10^{-4})	1.3033 (1.1×10^{-4})	1.3035 (1.1×10^{-4})	1.3032 (1.1×10^{-4})	1.3034 (1.1×10^{-4})	1.3034 (1.1×10^{-4})	1.3032 (1.1×10^{-4})	1.3034 (1.1×10^{-4})	1.3034 (1.1×10^{-4})
100	1.3159 (7.3×10^{-5})	1.3134 (7.8×10^{-5})	1.3134 (7.8×10^{-5})	1.3134 (7.8×10^{-5})	1.3135 (7.8×10^{-5})	1.3133 (7.8×10^{-5})	1.3134 (7.8×10^{-5})	1.3134 (7.8×10^{-5})	1.3135 (7.8×10^{-5})	1.3133 (7.8×10^{-5})
500	1.3263 (3.3×10^{-5})	1.3256 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3256 (3.4×10^{-5})	1.3257 (3.4×10^{-5})	1.3256 (3.4×10^{-5})

Table 11. Estimates of $E(\hat{C}_{pk}^{T})$ and their standard errors (in parentheses) for some cases of $C_{pk}^{T}=1.5$ and various *n*.

C_{pk1}	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484
C_{pk2}	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484	1.5484
$\frac{d_1}{\sigma_1}$	4.6452	5.1613	5.8065	6.6360	7.7420	5.1613	5.8065	6.6360	7.7420	4.6452
$\frac{d_2}{\sigma_2}$	4.6452	5.1613	5.8065	6.6360	7.7420	5.8065	6.6360	7.7420	5.1613	7.7420
$\frac{(\mu_1-m_1)}{\sigma_1}$	0	0.5161	1.1613	1.9908	7.7420	0.5161	1.1613	1.9908	3.0968	0
$\frac{(\mu_2-m_2)}{\sigma_2}$	0	0.5161	1.1613	1.9908	7.7420	1.1613	1.9908	3.0968	0.5161	3.0968
n	Estimate of $E(\hat{C}_{pk}^{T})$ and its standard error (in parentheses)									
30	1.4103 (1.5×10^{-4})	1.4545 (1.6×10^{-4})	1.4542 (1.6×10^{-4})	1.4547 (1.6×10^{-4})	1.4547 (1.6×10^{-4})	1.4545 (1.6×10^{-4})	1.4543 (1.6×10^{-4})	1.4543 (1.6×10^{-4})	1.4543 (1.6×10^{-4})	1.4566 (1.6×10^{-4})
50	1.4305 (1.2×10^{-4})	1.4652 (1.2×10^{-4})	1.4654 (1.2×10^{-4})	1.4655 (1.2×10^{-4})	1.4652 (1.2×10^{-4})	1.4653 (1.2×10^{-4})	1.4654 (1.2×10^{-4})	1.4653 (1.2×10^{-4})	1.4655 (1.2×10^{-4})	1.4671 (1.2×10^{-4})
100	1.4526 (8.3×10^{-5})	1.4776 (8.7×10^{-5})	1.4776 (8.7×10^{-5})	1.4778 (8.7×10^{-5})	1.4778 (8.7×10^{-5})	1.4777 (8.7×10^{-5})	1.4778 (8.7×10^{-5})	1.4778 (8.7×10^{-5})	1.4776 (8.7×10^{-5})	1.4789 (8.7×10^{-5})
500	1.4822 (3.7×10^{-5})	1.4938 (3.8×10^{-5})	1.4938 (3.8×10^{-5})	1.4939 (3.8×10^{-5})	1.4939 (3.8×10^{-5})	1.4939 (3.8×10^{-5})	1.4939 (3.8×10^{-5})	1.4938 (3.8×10^{-5})	1.4939 (3.8×10^{-5})	1.4942 (3.7×10^{-5})

Table 12. Estimates of $E(\hat{C}^{T}_{pk})$ and their standard errors (in parentheses) for some cases of $C^{T}_{pk} = 2.0$ and various n.

C_{pk1}	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372
C_{pk2}	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372	2.0372
$\frac{d_1}{\sigma_1}$	6.9846	6.1116	6.1116	9.7785	12.223	9.7785	6.9846	9.7785	6.9846	6.1116
$\frac{d_2}{\sigma_2}$	6.9846	6.1116	6.9846	9.7785	12.223	8.1488	9.7785	12.223	12.223	9.7785
$\frac{(\mu_1 - m_1)}{\sigma_1}$	0.8731	0	0	3.6669	6.1116	3.6669	0.8731	3.6669	0.8731	0
$\frac{(\mu_2 - m_2)}{\sigma_2}$	0.8731	0	0.8731	3.6669	6.1116	2.0372	3.6669	6.1116	6.1116	3.6669
n	Estimate of $E(\hat{C}^{T}_{pk})$ and its standard error (in parentheses)									
30	1.9243 (2.1×10^{-4})	1.9281 (2.0×10^{-4})	1.9260 (2.0×10^{-4})	1.9243 (2.1×10^{-4})	1.9242 (2.1×10^{-4})	1.9243 (2.1×10^{-4})	1.9243 (2.1×10^{-4})	1.9249 (2.1×10^{-4})	1.9243 (2.1×10^{-4})	1.9263 (2.0×10^{-4})
50	1.9407 (1.6×10^{-4})	1.9434 (1.6×10^{-4})	1.9417 (1.6×10^{-4})	1.9406 (1.6×10^{-4})	1.9405 (1.6×10^{-4})	1.9407 (1.6×10^{-4})	1.9402 (1.6×10^{-4})	1.9405 (1.6×10^{-4})	1.9402 (1.6×10^{-4})	1.9419 (1.6×10^{-4})
100	1.9594 (1.1×10^{-4})	1.9613 (1.1×10^{-4})	1.9603 (1.1×10^{-4})	1.9595 (1.1×10^{-4})	1.9595 (1.1×10^{-4})	1.9592 (1.1×10^{-4})	1.9595 (1.1×10^{-4})	1.9595 (1.1×10^{-4})	1.9594 (1.1×10^{-4})	1.9606 (1.1×10^{-4})
200	1.9740 (8.0×10^{-5})	1.9755 (7.8×10^{-5})	1.9747 (7.9×10^{-5})	1.9741 (8.0×10^{-5})	1.9741 (8.0×10^{-5})	1.9740 (8.0×10^{-5})	1.9741 (8.0×10^{-5})	1.9741 (8.0×10^{-5})	1.9741 (8.0×10^{-5})	1.9747 (7.9×10^{-5})
500	1.9869 (5.0×10^{-5})	1.9876 (4.8×10^{-5})	1.9873 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9869 (5.0×10^{-5})	1.9873 (5.0×10^{-5})

Table 13. Sample sizes required for a specified margin of sampling error.

C_{pk}^{T}	α \ error	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.1**
1.00	0.05	413	287	211	162	128	104
	0.025	587	408	300	230	181	147
	0.01	826	574	422	323	255	207
1.33	0.05	624	434	319	244	193	156
	0.025	886	616	452	346	274	222
	0.01	1248	867	637	488	386	312
1.50	0.05	750	521	383	293	232	188
	0.025	1065	740	544	416	329	267
	0.01	1501	1042	766	587	464	376
1.67	0.05	903	627	461	353	279	226
	0.025	1282	891	654	501	396	321
	0.01	1806	1254	922	706	558	452
2.00	0.05	1224	850	625	478	378	306
	0.025	1738	1207	887	679	537	435
	0.01	2448	1700	1249	957	756	612

Table 14. Sample mean, sample standard deviation, specifications of individual characteristics for the dual-fiber tips, and the estimated capability indices.

Characteristic	$\bar{X}$	S	LSL	USL	$\hat{C}_{pk}$
Capillary length (mm)	6.255	0.04035	6.00	6.50	2.024
Wedge (^{o})	7.99	0.0959	7.5	8.5	1.703