國立交通大學

物理研究所

碩 士 論 文

Bianchi type I 宇宙模型的穩定性分析

Stability analysis of the Bianchi type I Model Universe

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摘　　　　要

觀測的證據顯示我們的宇宙正在膨脹，宇宙學家隨即引入帶有膨脹解的宇宙模型。根據無毛定理的猜想：只有均向膨脹的 de sitter 宇宙才是穩定的時空，因此所有非均向膨脹解必然都不穩定。在早期宇宙的研究中，宇宙模型須考慮高階修正項，修正後卻可能得到不一樣的非均向膨脹解。本論文將呈現一些已知的 Bianchi type I 膨脹解，並介紹這些非均向膨脹解是不穩定的證明，這樣的結論說明這些模型符合無毛定理預期的結果。

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ABSTRACT

According to the observation evidences, our universe is an expanding de Sitter space. As a result, many different models have been studied as a natural source of the expanding universe. In particular, higher order corrections terms may be important in the evolution of the early universe. In addition, Gibbons and Hawking come up with the No Hair theorem claiming that the de Sitter solution is the only stable expanding solution. Review on the anisotropically expanding solutions in the Bianchi type I space will be presented in this thesis along with the presentation of the proof of the instability of these expanding solutions. Hence , the result agrees with the conjecture of the no-hair theorem.

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首先我要感謝交大的許多師長，特別感謝我的指導教授高文芳老師，感謝老師在我求學的過程中不會因為遇到阻礙而放棄我，尤其在我最需要幫助的時候老師都會適時伸出援手，也很開心老師引領我學習廣義相對論和宇宙學的知識，滿足我對於這塊領域的好奇心，並拓展眼界。另外我也特別感謝林俊源教授在碩一時指導過我關於超導體方面的實驗，雖然後來我選擇宇宙論作為研究，但林老師所教的東西至今仍讓我留下深刻的印象。

其次我要感謝我的父母，在交大的這些日子以來，由於他們的辛苦和付出，我才能得以溫飽和在外求學。在求學的過程中難免會遇到許多挫折，但因為有家人的支持，在物理學這條道路上，我才能完成碩士學位，並和他們分享我的學習成果和喜悅。

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目　　　錄

Chapter 1
導論

1.1 宇宙學簡介

宇宙學(Cosmology)是一門歷史悠久，從古至今一直不斷被人們拿來討論的話題。在東方，以道德經聞名的老子早就試圖描述宇宙的起源，不僅帶給後人很多省思，甚至成爲了代表東方色彩的宇宙學資產。然而，老子的主張對科學發達的現今來說缺乏很多應有的觀測和實驗，已至於無法做出量化，這種充其量只能成爲定性的做法最終只能淪爲哲學的範疇，並不能成爲真正的科學。

二十世紀宇宙學因爲 1916 年愛因斯坦(Einstein)一篇關於重力的論文而有了重大的突破，該篇論文便是廣義相對論(General Relativity, GR) 。廣義相對論最初發展的目的是藉由等效原理將重力視爲時空遭受到質量的存在而引起彎曲的幾何效應來修改牛頓的重力觀點。這種試圖將質能和時空連結的做法剛開始備受質疑，但藉由愛丁頓等人觀測的結果證實了廣義相對論的正確性後，牛頓的重力理論至此被正式畫上句點。稍後不僅掀起了一股宇宙學熱潮，還奠定了宇宙學的理論基礎，將宇宙學的發展推至高峰，並開啓了前所未有的大門。今後，廣義相對論成了研究宇宙學不可或缺的工具。

1.2 宇宙模型與爆漲理論

愛因斯坦一戰成名後，便將廣義相對論套用在宇宙這個龐然大物身上，試圖建立一套宇宙的模型公式。但他發現宇宙本身會是個動態的方程式:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.2.1}$$

(1.2.1)的公式就是有名的愛因斯坦方程式，可說明宇宙將會處於膨漲或是收縮態，這結果讓他非常不安，也無法接受，他認爲宇宙應是處於靜態的以至於想在方程式上動點手腳來修改方程式成爲:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.2.2}$$

這便是宇宙常數Λ的由來。後來哈伯發現遠處的星系光譜有紅移的現象，證實宇宙正在擴張中，導致愛因斯坦聲稱加入宇宙常數是他的大錯誤，爲了彌補錯誤只好將Λ設定爲0。自從愛因斯坦的宇宙理論引入Λ之後，宇宙學就陷在決定Λ的困難中。根據最新的宇宙論消息告訴我們現在又發現Λ的值很小但不爲0，讓愛因斯坦原本犯的錯誤又再度戲劇性的死灰復燃。

1.3FRW 宇宙模型

自從 1929 年，哈伯發現了宇宙正在膨漲後，動態的宇宙讓科學家們投入更多的研究。直到 1965 年，Penaias 與 Wilson 兩人意外地發現 Gamow 根據大爆炸理論推算宇宙殘留下的 2.7K 宇宙背景輻射(Cosmology Microwave Background Radiation, CMB)[1]，讓大爆炸理論順理成章打敗群雄成了舞臺上的主角，也因爲宇宙背景輻射的發現，人類對於宇宙的年齡、演化方式就有了依據。

由於宇宙背景輻射是一種來自四面八方都均勻的波，這讓當時的科學家認爲宇宙模型是均勻(Homogeneous)且勻向(Isotropic)的[2-4]，因此有 Friedmann-Robertson-Walker(FRW)等人提出的 FRW 度規模型(Friedmann 模型)[5-7]:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \qquad (1.3.1)$$

其中，(t,r,θ,ϕ)是座標，$a(t)$是宇宙的尺度因子，和宇宙幾何結構的演化性質有關。而 k 的值爲:+1，-1，0。當 k=+1 空間曲率(spatial curvature)爲正的，爲開放的宇宙;k=-1 空間曲率爲負的，爲封閉的宇宙;k=0 空間曲率爲零，爲平直的宇宙。我們可以藉由計算 Christoffel symbol$\left(\Gamma^i{}_{jk}\right)$，Ricci tensor$\left(R_{ij}\right)$以及 Ricci scalar$(R)$來得到(1.1.1)式，於是我們得到非零的 Christoffel symbol$\left(\Gamma^i{}_{jk}\right)$，分別爲:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left[\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right] \tag{1.3.2}$$

$$\Gamma^t_{ij} = \frac{\dot{a}}{a} g_{ij} \tag{1.3.3}$$

$$\Gamma^i_{ti} = \frac{\dot{a}}{a} \delta^i{}_j, i = 1 \sim 3 \tag{1.3.4}$$

進一步我們可以得到 Ricci tensor$\left(R_{ij}\right)$爲:

$$R_{tt} = 3\frac{\ddot{a}}{a} \qquad R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right] g_{ij} \tag{1.3.5}$$

以及 Ricci scalar(R)爲:

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] \tag{1.3.6}$$

我們將(1.3.5)(1.3.6)代入(1.2.1)的愛因斯坦方程式可以得到微分型態的愛因斯坦方程:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{1.3.7}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G\rho + \Lambda \tag{1.3.8}$$

這裡我們將宇宙的能量密度-動量張量視爲理想流體(perfect fluid) ，並將方程式分成與時間有關的能量密度(energy density)部分(1.3.7)式和空間有關的動量張量(momentum tensor)部分(1.3.8)式 。如果只考慮能量密度-動量張量的影響，宇宙常數Λ便爲零，則(1.3.7)和(1.3.8)便可得到:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{1.3.9}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G\rho \tag{1.3.10}$$

當宇宙處於平直時(k=0) ，我們可得到 FRW 模型的真空解(de sitter solution) 為:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{vac} \tag{1.3.11}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G\rho_{vac} \tag{1.3.12}$$

其中 ρ_{vac} 爲真空能量密度(vacuum energy density) ，而 $\frac{\dot{a}}{a} \equiv H$ ， $H = \sqrt{\frac{\Lambda}{3}}$ (H 爲哈伯常數)，我們可以利用(1.3.11) (1.3.12)得到以下關係:

$$\begin{aligned} \rho_{vac} &= \frac{\Lambda}{8\pi G} = const. \\ H &= \left(\frac{8\pi G}{3}\rho_{vac}\right)^{1/2} = const. \\ a(t) &\propto \exp(Ht) \end{aligned} \tag{1.3.13}$$

由(1.3.13)式我們可以看出宇宙一開始的尺度非常的小，但會隨著時間以指數的形式擴張到無窮大，這樣的理論符合觀測到的實驗結果。

1.4 Bianchi 宇宙模型

早期的宇宙學家藉由 CMB 的觀測結果發現宇宙是均勻且勻向的，因而有了 FRW 的度規模型來描述宇宙。但隨著觀測的進步，現在我們發現原來宇宙不是那麼的均勻和勻向，因此所採用的宇宙模型必須改變，這篇論文裡我們採用數學家 Bianchi 在 1898 提出的 Bianchi I 宇宙模型[8]，當初 Bianchi 一共提出九種描述非均向但時空均勻的宇宙模型(參考 appendix A)，雖然 Bianchi 模型不一定是真正的宇宙模型，但它爲宇宙的模型採用提供了一個有用的工具和方向。

1.5 Bianchi type I 宇宙模型

Bianchi I 的度規型式爲:

$$ds^2 = -\frac{1}{B^2(t)}dt^2 + a_1^{\,2}(t)dx^2 + a_2^{\,2}(t)dy^2 + a_3^{\,2}(t)dz^2 \qquad (1.5.1)$$

其度規張量(metric tensor)可表示成:

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{B^2} & 0 & 0 & 0 \\ 0 & a_1^{\,2} & 0 & 0 \\ 0 & 0 & a_2^{\,2} & 0 \\ 0 & 0 & 0 & a_3^{\,2} \end{pmatrix} \qquad (1.5.2)$$

$$g^{\mu\nu} = \begin{pmatrix} -B^2 & 0 & 0 & 0 \\ 0 & \frac{1}{a_1^{\,2}} & 0 & 0 \\ 0 & 0 & \frac{1}{a_2^{\,2}} & 0 \\ 0 & 0 & 0 & \frac{1}{a_3^{\,2}} \end{pmatrix}$$

由度規張量我們可以算出:

$$\sqrt{-g} = \frac{a_1 a_2 a_3}{\sqrt{B}} \qquad (1.5.3)$$

接著算出非零的 Christoffel symbol$\left(\Gamma^{\alpha}_{\mu\nu}\right)$: $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$

$$\Gamma^{0}_{\mu\nu} = \begin{pmatrix} -\frac{\dot{B}}{2B} & 0 & 0 & 0 \\ 0 & Ba_1\dot{a}_1 & 0 & 0 \\ 0 & 0 & Ba_2\dot{a}_2 & 0 \\ 0 & 0 & 0 & Ba_3\dot{a}_3 \end{pmatrix} \qquad (1.5.4)$$

$$\Gamma^{1}_{\mu\nu} = \begin{pmatrix} 0 & \frac{\dot{a}_1}{a_1} & 0 & 0 \\ \frac{\dot{a}_1}{a_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.5}$$

$$\Gamma^{2}_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{\dot{a}_2}{a_2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\dot{a}_2}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.6}$$

$$\Gamma^{3}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{a}_3}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\dot{a}_3}{a_3} & 0 & 0 & 0 \end{pmatrix} \tag{1.5.7}$$

再算出 Ricci tensor $\left(R^{\mu}{}_{\nu}\right)$:

$$R^{\mu}{}_{\nu} = R^{\mu\alpha}{}_{\nu\alpha} = g^{\alpha\sigma}\left(\partial_{\nu}\Gamma^{\mu}{}_{\alpha\sigma} - \partial_{\alpha}\Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\alpha\sigma} - \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\lambda}{}_{\nu\alpha}\right)$$

$$R^{0}{}_{0} = \frac{\dot{B}\dot{a}_1}{2a_1} + \frac{\dot{B}\dot{a}_2}{2a_2} + \frac{\dot{B}\dot{a}_3}{2a_3} + \frac{B\ddot{a}_1}{a_1} + \frac{B\ddot{a}_2}{a_2} + \frac{B\ddot{a}_3}{a_3} \tag{1.5.8}$$

$$R^{1}{}_{1} = \frac{B\ddot{a}_1}{a_1} + \frac{\dot{B}\dot{a}_1}{2a_1} + \frac{B\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{B\dot{a}_1\dot{a}_3}{a_1a_3} \tag{1.5.9}$$

$$R^{2}{}_{2} = \frac{B\ddot{a}_2}{a_2} + \frac{\dot{B}\dot{a}_2}{2a_2} + \frac{B\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{B\dot{a}_2\dot{a}_3}{a_1a_3} \tag{1.5.10}$$

$$R^{3}{}_{3}=\frac{B\ddot{a}_3}{a_3}+\frac{\dot{B}\dot{a}_3}{2a_3}+\frac{B\dot{a}_1\dot{a}_3}{a_1a_3}+\frac{B\dot{a}_2\dot{a}_3}{a_2a_3} \tag{1.5.11}$$

之後得到 Ricci scalar(R)爲:

$$\begin{aligned}R=R^{\mu}{}_{\mu}&=\dot{B}\left(\frac{\dot{a}_1}{a_1}+\frac{\dot{a}_2}{a_2}+\frac{\dot{a}_3}{a_3}\right)\\&+2B\left(\frac{\ddot{a}_1}{a_1}+\frac{\ddot{a}_2}{a_2}+\frac{\ddot{a}_3}{a_3}+\frac{\dot{a}_1\dot{a}_2}{a_1a_2}+\frac{\dot{a}_1\dot{a}_3}{a_1a_3}+\frac{\dot{a}_2\dot{a}_3}{a_2a_3}\right)\\&=\dot{B}\left(H_1+H_2+H_3\right)+2B\left(\frac{\ddot{a}_1}{a_1}+\frac{\ddot{a}_2}{a_2}+\frac{\ddot{a}_3}{a_3}+H_1H_2+H_1H_3+H_2H_3\right)\end{aligned} \tag{1.5.12}$$

其中:

$$H_1=\frac{\dot{a}_1}{a_1}$$

$$H_2=\frac{\dot{a}_2}{a_2}$$

$$H_3=\frac{\dot{a}_3}{a_3}$$

根據(1.2.1)的愛因斯坦方程式:

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi GT_{\mu\nu}$$

我們可以得到:

$$\begin{aligned}G^{0}{}_{0}&=R^{0}{}_{0}-\frac{1}{2}g^{0}{}_{0}R\\&=-B\left(H_1H_2+H_1H_3+H_2H_3\right)\\&=-\left(H_1H_2+H_1H_3+H_2H_3\right)\end{aligned} \tag{1.5.13}$$

$$\begin{aligned}G^{1}{}_{1}&=R^{1}{}_{1}-\frac{1}{2}g^{1}{}_{1}R\\&=-\frac{1}{2}\dot{B}\left(H_2+H_3\right)-B\left(\dot{H}_2+\dot{H}_3+H_2^2+H_3^2+H_2H_3\right)\\&=-\left(\dot{H}_2+\dot{H}_3+H_2^2+H_3^2+H_2H_3\right)\end{aligned} \tag{1.5.14}$$

$$G^2{}_2 = R^2{}_2 - \frac{1}{2} g^2{}_2 R$$
$$= -\frac{1}{2}\dot{B}\left(H_1 + H_3\right) - B\left(\dot{H}_1 + \dot{H}_3 + H_1^2 + H_3^2 + H_1 H_3\right) \quad (1.5.15)$$
$$= -\left(\dot{H}_1 + \dot{H}_3 + H_1^2 + H_3^2 + H_1 H_3\right)$$

$$G^3{}_3 = R^3{}_3 - \frac{1}{2} g^3{}_3 R$$
$$= -\frac{1}{2}\dot{B}\left(H_1 + H_2\right) - B\left(\dot{H}_2 + \dot{H}_1 + H_1^2 + H_2^2 + H_1 H_2\right) \quad (1.5.16)$$
$$= -\left(\dot{H}_2 + \dot{H}_1 + H_1^2 + H_2^2 + H_1 H_2\right)$$

其中選定 B=1

以上為 Bianchi I 的空間幾何性質。

Chapter 2
變分與場方程式

2.1 Action Principle

愛因斯坦的場方程式當初是由作用量定理(Action Principle)中變分得到的，首先我們先來了解作用量 action S 的形式為:

$$S = -\int dx^4 \sqrt{g} \left(\frac{R}{16\pi G} + L_{matter} \right) \qquad (2.1.1)$$

其中 R 是黎曼曲率張量(Riemann Curvature Tensor)，G 是重力常數，L_{matter} 是物質場的 Lagrangian。

2.2 變分與 Friedmann equation

這一節我們將介紹變分以求得 Bianchi type I 高階場方程式，之所以介紹這一套方法的理由是不論系統的 Lagrangian 有多簡單或複雜，它都能有效地運算出場方程式。假設給定系統的 Lagrangian 為:

$L\left(H_i, \dot{H}_i\right) = \frac{1}{2}\left(R + \alpha R^2 + \beta R_{\mu\beta} R^{\mu\beta} - 2\Lambda\right)$，令 $\tilde{L} = \sqrt{-g} L$，可定義其 action 為:

$S = \int d^4 x \tilde{L}$，由古典力學的觀點對 S 作變分，分別可得到 Bianchi type I 裡時間項和空間項變分後的 equation of motion。

我們從尤拉變分可得到:

$$\begin{aligned} &\frac{\partial \tilde{L}}{\partial B} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{B}} = 0 \\ &\Rightarrow -\frac{1}{2}\frac{L}{B} + \frac{\partial L}{\partial B} - \left(\frac{d}{dt} + 3H - \frac{1}{2}\frac{\dot{B}}{B}\right)\frac{\partial L}{\partial \dot{B}} = 0 \end{aligned} \qquad (2.2.1)$$

其中 L 和 B、H 根據 chain rule 有以下關係:

$$
\begin{aligned}
\frac{\partial L}{\partial B} &\Rightarrow \frac{H_i}{2}\frac{\partial L}{\partial H_i}+\dot{H}_i\frac{\partial L}{\partial \dot{H}_i} \\
\frac{\partial L}{\partial \dot{B}} &\Rightarrow \frac{H_i}{2}\frac{\partial L}{\partial \dot{H}_i}
\end{aligned}
\qquad (2.2.2)
$$

將(2.2.2)代入(2.2.1)可得到時間項的變分結果為:

$$
D_0 L = L + H_i\left(\partial_0 + 3H\right)L^i - H_i L_i - \dot{H}_i L^i = 0 \qquad (2.2.3)
$$

(2.2.3)式稱為 Friedmann equation[9-12]，經計算可得:

$$
\begin{aligned}
D_0 L &= L + H_i\left(\partial_0 + 3H\right)L^i - H_i L_i - \dot{H}_i L^i \\
&= H_1H_2 + H_1H_3 + H_2H_3 \\
&\quad +\alpha\begin{bmatrix}
-2\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3+H_1H_2+H_1H_3+H_2H_3\right) \\
\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3-H_1H_2-H_1H_3-H_2H_3\right) \\
+4\left(H_1+H_2+H_3\right)\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3+H_1H_2+H_1H_3+H_2H_3\right)'
\end{bmatrix} \\
&\quad +\beta\begin{bmatrix}
-\frac{1}{2}\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3\right) \\
\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3-4H_1H_2-4H_1H_3-4H_2H_3\right) \\
-\frac{1}{2}\left(H_1^2+\dot{H}_1+H_1H_2+H_1H_3\right)^2-\frac{1}{2}\left(H_2^2+\dot{H}_2+H_1H_2+H_2H_3\right)^2 \\
-\frac{1}{2}\left(H_3^2+\dot{H}_3+H_1H_3+H_2H_3\right)^2 \\
+\left(H_1+H_2+H_3\right)\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3\right)'+H_1\left(H_1^2+\dot{H}_1+H_1H_2+H_1H_3\right)' \\
+H_2\left(H_2^2+\dot{H}_2+H_1H_2+H_2H_3\right)'+H_3\left(H_3^2+\dot{H}_3+H_1H_3+H_2H_3\right)'
\end{bmatrix}-\Lambda \\
&= 0
\end{aligned}
$$

(2.2.4)

再從尤拉變分可得:

$$\frac{\partial\tilde{L}}{\partial a_i}-\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{a}_i}+\frac{d^2}{dt^2}\frac{\partial\tilde{L}}{\partial\ddot{a}_i}=0$$
$$\Rightarrow\frac{L}{a_i}+\frac{\partial L}{\partial a_i}-\left(\frac{d}{dt}+3H\right)\frac{\partial L}{\partial\dot{a}_i}+\left(\frac{d}{dt}+3H\right)^2\frac{\partial L}{\partial\ddot{a}_i}=0 \tag{2.2.5}$$

經以下的 chain rule 可得:

$$\frac{\partial L}{\partial a_i}=\frac{\partial L}{\partial H_i}\frac{\partial H_i}{\partial a_i}+\frac{\partial L}{\partial\dot{H}_i}\frac{\partial\dot{H}_i}{\partial a_i}=-\frac{1}{a_i}H_iL_i-\frac{1}{a}\left(\dot{H}_i-H_i^2\right)L^i$$
$$\frac{\partial L}{\partial\dot{a}_i}=\frac{\partial L}{\partial H_i}\frac{\partial H_i}{\partial\dot{a}_i}+\frac{\partial L}{\partial\dot{H}_i}\frac{\partial\dot{H}_i}{\partial\dot{a}_i}=\frac{1}{a_i}L_i-\frac{2}{a_i}H_iL^i \tag{2.2.6}$$
$$\frac{\partial L}{\partial\ddot{a}_i}=\frac{\partial L}{\partial H_i}\frac{\partial H_i}{\partial\ddot{a}_i}+\frac{\partial L}{\partial\dot{H}_i}\frac{\partial\dot{H}_i}{\partial\ddot{a}_i}=\frac{1}{a_i}L^i$$

將(2.2.6)代入(2.2.5)可得到空間項的變分結果爲:

$$D_iL=L+\left(\partial_0+3H\right)^2L^i-\left(\partial_0+3H\right)L_i=0 \tag{2.2.7}$$

(2.2.7)經計算可得:

$$
\begin{aligned}
D_iL &= L+\left(\partial_0+3H\right)^2L^i-\left(\partial_0+3H\right)L_i \\
&= 2H_1^2+2H_2^2+2H_3^2+2\dot{H}_1+2\dot{H}_2+2\dot{H}_3+H_1H_2+H_1H_3+H_2H_3 \\
&\quad +\alpha\left[\begin{array}{l}
2\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3+H_1H_2+H_1H_3+H_2H_3\right) \\
\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3-H_1H_2-H_1H_3-H_2H_3\right) \\
+8\left(H_1+H_2+H_3\right)\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3+H_1H_2+H_1H_3+H_2H_3\right)' \\
+12\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3+H_1H_2+H_1H_3+H_2H_3\right)''
\end{array}\right] \\
&\quad +\beta\left[\begin{array}{l}
\frac{1}{2}\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3\right) \\
\left(5H_1^2+5\dot{H}_1+5H_2^2+5\dot{H}_2+5H_3^2+5\dot{H}_3+4H_1H_2+4H_1H_3+4H_2H_3\right) \\
-\frac{3}{2}\left(H_1^2+\dot{H}_1+H_1H_2+H_1H_3\right)^2-\frac{3}{2}\left(H_2^2+\dot{H}_2+H_1H_2+H_2H_3\right)^2 \\
-\frac{3}{2}\left(H_3^2+\dot{H}_3+H_1H_3+H_2H_3\right)^2 \\
+4\left(H_1+H_2+H_3\right)\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3\right)' \\
+\left(H_2+H_3-2H_1\right)\left(H_1^2+\dot{H}_1+H_1H_2+H_1H_3\right)' \\
+\left(H_1+H_3-2H_2\right)\left(H_2^2+\dot{H}_2+H_1H_2+H_2H_3\right)' \\
+\left(H_1+H_2-2H_3\right)\left(H_3^2+\dot{H}_3+H_1H_3+H_2H_3\right)' \\
+3\left(H_1^2+\dot{H}_1+H_2^2+\dot{H}_2+H_3^2+\dot{H}_3\right)''+\left(H_1^2+\dot{H}_1+H_1H_2+H_1H_3\right)'' \\
+\left(H_2^2+\dot{H}_2+H_1H_2+H_2H_3\right)''+\left(H_3^2+\dot{H}_3+H_1H_3+H_2H_3\right)''
\end{array}\right]-\Lambda \\
&= 0
\end{aligned}
$$

(2.2.8)

2.3 inflation universe parameters

上一節中我們已經將 Friedmann equation 得到的場方程式計算出來了，只要再對其求解就行了。但由於求解過程非常複雜和困難，所以我們引進了一些變數作爲方便求解的數學手法，但在求解前我們先確保這些變數是可被引用的。根據 J.D. Barrow 在一篇 2006 年的論文[13-14]以及一些相關書籍裡有提到一套變數，首先介紹它們的定義如下:

$$
\begin{gathered}
B=\frac{1}{(3\alpha+\beta)H^2},\chi=\frac{\beta}{3\alpha+\beta},Q=\frac{\dot{H}}{H^2},\\
Q_2=\frac{\ddot{H}}{H^3},\Omega_\Lambda=\frac{\Lambda}{3H^2},N=\frac{n_{11}}{\sqrt{3H}},\\
\Sigma_\pm=\frac{\sigma_\pm}{H},\Sigma_{\pm1}=\frac{\dot{\sigma}_\pm}{H^2},\Sigma_{\pm2}=\frac{\ddot{\sigma}_\pm}{H^3}\sqrt{}
\end{gathered}
\tag{2.3.1}
$$

其中，χ 爲常數;Λ 爲宇宙常數，且我們假設 $\Lambda>0$，所以 $\Omega_\Lambda>0$。
在引進這些變數的同時，系統的時間也成爲變數，所以我們導入一個關係:

$$\frac{d\tau}{dt}=H \tag{2.3.2}$$

利用 chain rule 我們將以上變數對時間微分可得以下 equations of motion:

$$B'=-2QB \tag{2.3.3}$$

$$\Omega'_\Lambda=-2Q\Omega_\Lambda \tag{2.3.4}$$

$$N'=-(Q+1+4\Sigma_+)N \tag{2.3.5}$$

$$Q'=-2Q^2+Q_2 \tag{2.3.6}$$

$$Q_2' = -3(Q+2)Q_2 - \frac{9}{2}(Q+2)Q - \frac{3}{4}B\left(1+\Sigma^2 - \Omega_\Lambda + \frac{2}{3}Q - \frac{1}{3}N^2\right) - \frac{3}{2}(1+2\chi)\Sigma^4$$
$$-\frac{1}{4}(8+\chi)\Sigma_1^2 - (4-\chi)(\Sigma\cdot\Sigma_1) - \frac{1}{4}(4-\chi)\left(3\Sigma^2 + 2\Sigma\cdot\Sigma_2 + 2Q\Sigma^2\right) - (1+2Q)N^2$$
$$+N^2\left[\frac{1}{2}(1+8\chi)N^2 + 5(13+3\chi)\Sigma_+^2 + 8(2\Sigma_+ - \Sigma_{+1}) + (1-\chi)\Sigma_-^2\right]$$

(2.3.7)

$$\Sigma_\pm' = -Q\Sigma_\pm + \Sigma_{\pm1} \qquad (2.3.8)$$

$$\Sigma_{\pm1}' = -2Q\Sigma_{\pm1} + \Sigma_{\pm2} \qquad (2.3.9)$$

$$\Sigma_{+2}' = -3(Q+2)\Sigma_{+2} + \frac{\Sigma_{+1}}{\chi}\left[B - (11\chi - 8) + 4Q(1-\chi) + 4\Sigma^2(1+2\chi)\right]$$
$$+\frac{\Sigma_{+1}}{\chi}\left[3B + (4-\chi)(6+Q_2+7Q) + 4(1+2\chi)\left(3\Sigma^2 + 2\Sigma\cdot\Sigma_1\right)\right]$$
$$-\frac{4}{\chi}N^2\left[B + 8 + 4Q - 4(1+8\chi)N^2 + (1+15\chi)\left(\Sigma_+ + \Sigma_{+1} - 4\Sigma_+^2\right) + 4(1-\chi)\Sigma_-^2\right]$$

(2.3.10)

$$\Sigma_{-2}' = -3(Q+2)\Sigma_{-2} + \frac{\Sigma_{-1}}{\chi}\left[B - (11\chi - 8) + 4Q(1-\chi) + 4\Sigma^2(1+2\chi)\right]$$
$$+\frac{\Sigma_-}{\chi}\left[3B + (4-\chi)(6+Q_2+7Q) + 4(1+2\chi)\left(3\Sigma^2 + 2\Sigma\cdot\Sigma_1\right)\right] \qquad (2.3.11)$$
$$-\frac{4(1-\chi)}{\chi}N^2\left(\Sigma_- + \Sigma_{-1} - 8\Sigma_-\Sigma_+\right)$$

經由(2.3.3)到(2.3.11)式我們還可以得到一個 constraint:

$$0 = B\left(1 - \Omega_\Lambda - \Sigma^2 - N^2\right) + 12Q - 2Q^2 + 4Q_2 - (4-\chi)(3+2Q)\Sigma^2 - 6(1+2\chi)\Sigma^4$$
$$-\chi\left(\Sigma_1^2 - 2\Sigma\cdot\Sigma_2\right) + 4(2+\chi)(\Sigma\cdot\Sigma_1)$$
$$+4N^2\left[\frac{1}{2}(1+8\chi)N^2 + 1 + (1+15\chi)\Sigma_+^2 + 8\Sigma_+ + (1-\chi)\Sigma_-^2\right]$$

(2.3.12)

對照 J.D. Barrow 論文裡的符號定義可知:

$$U^{\mu}(1,0,0,0) \tag{2.3.13}$$

$$H_i \equiv \frac{\dot{a}_i}{a_i} \tag{2.3.14}$$

$$H \equiv \frac{1}{3}\nabla_{\mu}U^{\mu} = \frac{H_1+H_2+H_3}{3} \tag{2.3.15}$$

$$\sigma^{a}{}_{b} \equiv \frac{\nabla_b U^a + \nabla^a U_b}{2} - H\delta_b^a = \begin{pmatrix} H_1-H & 0 & 0 \\ 0 & H_2-H & 0 \\ 0 & 0 & H_3-H \end{pmatrix} \tag{2.3.16}$$

$$\sigma^{a}{}_{b} = \begin{pmatrix} -2\sigma_+ & 0 & 0 \\ 0 & \sigma_+ + \sqrt{3}\sigma_- & 0 \\ 0 & 0 & \sigma_+ - \sqrt{3}\sigma_- \end{pmatrix} \tag{2.3.17}$$

$$\sigma_+ \equiv -\frac{H_1}{3} + \frac{H_2+H_3}{6}$$
$$\sigma_- \equiv \frac{H_2-H_3}{2\sqrt{3}} \tag{2.3.18}$$

$$\Sigma^2 = \Sigma_+{}^2 + \Sigma_-{}^2 = \frac{1}{9H^2}\left(H_1{}^2 + H_2{}^2 + H_3{}^2 - H_1H_2 - H_2H_3 - H_1H_3\right) \tag{2.3.19}$$

將(2.3.13)到(2.3.19)式代入(2.3.12)式可得:

$$
\begin{aligned}
0 = & \frac{1}{(9\alpha+3\beta)H^4}\left(-\Lambda + H_1H_2 + H_2H_3 + H_1H_3\right) \\
& - \frac{2\alpha}{(9\alpha+3\beta)H^4}\left[\begin{array}{l}\left(H_1^2+H_2^2+H_3^2+H_1H_2+H_2H_3+H_1H_3\right)\left(H_1^2+H_2^2+H_3^2-H_1H_2-H_2H_3-H_1H_3\right) \\ +3\dot{H}\left(2H_1^2+2H_2^2+2H_3^2+3\dot{H}\right)-3H\dot{R}\end{array}\right] \\
& - \frac{\beta}{(9\alpha+3\beta)H^4}\left[\begin{array}{l}\left(H_1^2+H_2^2+H_3^2\right)\left(H_1^2+H_2^2+H_3^2-H_1H_2-H_2H_3-H_1H_3\right) \\ -6H\left(\dot{H}_1H_1+\dot{H}_2H_2+\dot{H}_3H_3\right)-2\dot{H}\left(H_1H_2+H_2H_3+H_1H_3\right) \\ +\left(\dot{H}_1^2+\dot{H}_2^2+\dot{H}_3^2+\dot{H}_1\dot{H}_2+\dot{H}_2\dot{H}_3+\dot{H}_1\dot{H}_3\right) \\ -\ddot{H}_1\left(2H_1+H_2+H_3\right)-\ddot{H}_2\left(H_1+2H_2+H_3\right)-\ddot{H}_3\left(H_1+H_2+2H_3\right)\end{array}\right]
\end{aligned}
$$

(2.3.20)

(2.3.20)式經整理便是 Friedmann equation，即使用 variables 和變分所得的結果是一樣的。

Chapter 3
求解

3.1 equations of motion

在這章中我們將對場方程式求解，關於 Bianchi type I metric:

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2 \tag{3.1.1}$$

Friedmann equation:

$$\begin{aligned} D_0 L &= L + H_i\left(\partial_0 + 3H\right)L^i - H_i L_i - \dot{H}_i L^i = 0 \\ &\Rightarrow L + H_i\left(\frac{d}{dt} + 3H\right)L^i - H_i L_i - \dot{H}_i L^i = 0 \end{aligned} \tag{3.1.2}$$

空間項運動方程:

$$\begin{aligned} D_i L &= L + \left(\partial_0 + 3H\right)^2 L^i - \left(\partial_0 + 3H\right)L_i = 0 \\ &\Rightarrow 3L + \left(\frac{d}{dt} + 3H\right)^2\left(L^1 + L^2 + L^3\right) - \left(\frac{d}{dt} + 3H\right)\left(L_1 + L_2 + L_3\right) = 0 \end{aligned} \tag{3.1.3}$$

其中 $L \equiv L\left(H_i, \dot{H}_i\right); L_i = \dfrac{\delta L}{\delta H_i}; L^i = \dfrac{\delta L}{\delta \dot{H}_i}; H = \dfrac{H_1 + H_2 + H_3}{3}; H_i = \dfrac{\dot{a}_i}{a_i}; i = 1,2,3$

(3.1.4)

我們分別求得:

$$\begin{aligned} L = &\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\right) \\ &+2\alpha\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\right)^2 \\ &+\frac{\beta}{2}\Big[\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right)^2 + \left(\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3\right)^2 + \left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right)^2 \\ &+\left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right)^2 + \left(\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2\right)^2\Big] - \Lambda \end{aligned}$$

(3.1.5)

$$L_1 = 2H_1 + H_2 + H_3 + 4\alpha\left(2H_1 + H_2 + H_3\right)\Big(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\Big) + \beta\Big[4H_1\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(2H_1 + H_2 + H_3\right)\left(\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3\right) + H_2\left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right) + H_3\left(\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2\right)\Big]$$

(3.1.6)

$$L_2 = H_1 + 2H_2 + H_3 + 4\alpha\left(H_1 + 2H_2 + H_3\right)\Big(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\Big) + \beta\Big[4H_2\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(H_1 + 2H_2 + H_3\right)\left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right) + H_1\left(\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3\right) + H_3\left(\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2\right)\Big]$$

(3.1.7)

$$L_3 = H_1 + H_2 + 2H_3 + 4\alpha\left(H_1 + H_2 + 2H_3\right)\Big(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\Big) + \beta\Big[4H_3\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(H_1 + H_2 + 2H_3\right)\left(\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2\right) + H_1\left(\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3\right) + H_2\left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right)\Big]$$

(3.1.8)

$$L^1 = 1 + 4\alpha\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\right) + \beta\Big[\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(\dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3\right)\Big] \quad (3.1.9)$$

$$L^2 = 1 + 4\alpha\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\right) + \beta\Big[\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(\dot{H}_2 + H_2^2 + H_2H_1 + H_2H_3\right)\Big] \quad (3.1.10)$$

$$L^3 = 1 + 4\alpha\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_3H_2\right) + \beta\Big[\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right) + \left(\dot{H}_3 + H_3^2 + H_3H_1 + H_3H_2\right)\Big] \quad (3.1.11)$$

3.2 求解

Bianchi type I metric 系統的解可假設為:

$$
\begin{aligned}
&a_1(t) = e^{at} \\
&a_2(t) = e^{bt} \\
&a_3(t) = e^{ct} \\
&\forall a, b, c \in R
\end{aligned}
\tag{3.2.1}
$$

將(3.1.4)到(3.1.11)式以及(3.2.1)式代入(3.1.2)和(3.1.3)式分別可得:

$$
\begin{aligned}
&-(2\alpha+\beta)\left(a^4+b^4+c^4\right)+\beta\left[ab\left(a^2+b^2\right)+bc\left(b^2+c^2\right)+ac\left(a^2+c^2\right)\right] \\
&-2(\alpha+\beta)\left(a^2b^2+b^2c^2+a^2c^2\right)+(4\alpha+\beta)abc(a+b+c)+ab+bc+ca-\Lambda=0
\end{aligned}
\tag{3.2.2}
$$

$$
\begin{aligned}
&(2\alpha+\beta)\left(a^4+b^4+c^4\right)-\beta\left[ab\left(a^2+b^2\right)+bc\left(b^2+c^2\right)+ac\left(a^2+c^2\right)\right] \\
&+2(\alpha+\beta)\left(a^2b^2+b^2c^2+a^2c^2\right)-(4\alpha+\beta)abc(a+b+c)+2\left(a^2+b^2+c^2\right) \\
&+ab+bc+ca-3\Lambda=0
\end{aligned}
\tag{3.2.3}
$$

(3.2.2)和(3.2.3)兩式中可得兩組解:

第一組:

$$
\begin{aligned}
a^2+b^2+c^2 &= \Lambda \\
ab+bc+ca &= \Lambda
\end{aligned}
\tag{3.2.4}
$$

第二組:

$$
\begin{aligned}
a^2+b^2+c^2 &= -\frac{1+8\alpha\Lambda}{2\beta} \\
ab+bc+ca &= \frac{1+8\alpha\Lambda+4\beta\Lambda}{2\beta}
\end{aligned}
\tag{3.2.5}
$$

由第一組解(3.2.4)我們可得到 de Sitter solution:

$$a = H_1 = \sqrt{\frac{\Lambda}{3}}$$
$$b = H_2 = \sqrt{\frac{\Lambda}{3}}$$
$$c = H_3 = \sqrt{\frac{\Lambda}{3}}$$

由第二組解(3.2.5)兩式可看出三個未知數只有兩個方程式，我們設定 c=0 將系統簡化成:

$$a^2 + b^2 = -\frac{1+8\alpha\Lambda}{2\beta}$$
$$ab = \frac{1+8\alpha\Lambda+4\beta\Lambda}{2\beta} \qquad (3.2.6)$$

並求出:

$$a^2 = \frac{1}{4}\left(\frac{-1}{\beta} - \frac{8\alpha\Lambda}{\beta} \mp \frac{\sqrt{-3\beta^2 - 48\alpha\beta^2\Lambda - 32\beta^3\Lambda - 192\alpha^2\beta^2\Lambda^2 - 256\alpha\beta^3\Lambda^2 - 64\beta^4\Lambda^2}}{\beta^2}\right)$$

(3.2.7)

$$b^2 = \frac{1}{4}\left(\frac{-1}{\beta} - \frac{8\alpha\Lambda}{\beta} \pm \frac{\sqrt{-3\beta^2 - 48\alpha\beta^2\Lambda - 32\beta^3\Lambda - 192\alpha^2\beta^2\Lambda^2 - 256\alpha\beta^3\Lambda^2 - 64\beta^4\Lambda^2}}{\beta^2}\right)$$

(3.2.8)

由於 Bianchi type I metric 爲 anisotropic，透過先前的 variables 可被寫成:

$$ds^2 = -dt^2 + e^{2bt}\left[e^{-4\sigma_+ t}dx^2 + e^{2\left(\sigma_+ + \sqrt{3}\sigma_-\right)t}dy^2 + e^{2\left(\sigma_+ - \sqrt{3}\sigma_-\right)t}dz^2\right] \quad (3.2.9)$$

(3.2.9)式和(3.1.1)式對應的結果可假設:

$$\begin{aligned} a &= b_0 - 2\sigma_+ \\ b &= b_0 + \sigma_+ + \sqrt{3}\sigma_- \\ c &= b_0 + \sigma_+ - \sqrt{3}\sigma_- \end{aligned} \tag{3.2.10}$$

(3.2.10)式帶回(3.2.2)和(3.2.3)式可得:

$$-3b_0^2\left[-1+9(4\alpha+\beta)\left(\sigma_+^2+\sigma_-^2\right)\right]-3\left(\sigma_+^2+\sigma_-^2\right)\left[1+18(\alpha+\beta)\left(\sigma_+^2+\sigma_-^2\right)\right]=\Lambda$$

(3.2.11)

$$3\left(\sigma_+^2+\sigma_-^2\right)\left[1+6(\alpha+\beta)\left(\sigma_+^2+\sigma_-^2\right)\right]+b_0^2\left[3+9(4\alpha+\beta)\left(\sigma_+^2+\sigma_-^2\right)\right]=\Lambda$$

(3.2.12)

由(3.2.11)和(3.2.12)式可解出:

$$\begin{aligned} b_0^2 &= \frac{1+8\Lambda(\alpha+\beta)}{18\beta} \\ \sigma_+^2+\sigma_-^2 &= -\frac{1+2\Lambda(4\alpha+\beta)}{9\beta} \end{aligned} \tag{3.2.13}$$

而(3.2.13)這組解將不會是一個 de Sitter solution。

Chapter 4
穩定性分析

上一章我們已得到 equation of motion 並求其解，現在要進一步作平衡點展開(一階線性化)的動作，以分析解的穩定性問題[14-15]。這和物理學家們經常對於所研究的系統作微擾的動作是同樣的手法。我們再次從 inflation universe parameters 出發。

已知:

$$B=\frac{1}{(3\alpha+\beta)H^2},\chi=\frac{\beta}{3\alpha+\beta},Q=\frac{\dot{H}}{H^2},$$

$$Q_2=\frac{\ddot{H}}{H^3},\Omega_\Lambda=\frac{\Lambda}{3H^2},N=\frac{n_{11}}{\sqrt{3}H},$$

$$\Sigma_\pm=\frac{\sigma_\pm}{H},\Sigma_{\pm1}=\frac{\dot{\sigma}_\pm}{H^2},\Sigma_{\pm2}=\frac{\ddot{\sigma}_\pm}{H^3}\sqrt{\ }$$

已知 equations of motion 將其定義如下:

$$B'=-2QB\equiv f_1 \tag{4.1}$$

$$\Omega_\Lambda{}'=-2Q\Omega_\Lambda\equiv f_2 \tag{4.2}$$

$$N'=-\left(Q+1+4\mathrm{E}_+\right)N\equiv f_3 \tag{4.3}$$

$$Q'=-2Q^2+Q_2\equiv f_4 \tag{4.4}$$

$$\begin{aligned}Q_2{}'=&-3(Q+2)Q_2-\frac{9}{2}(Q+2)Q-\frac{3}{4}B\left(1+\Sigma^2-\Omega_\Lambda+\frac{2}{3}Q-\frac{1}{3}N^2\right)-\frac{3}{2}(1+2\chi)\Sigma^4-\frac{1}{4}(8+\chi)\Sigma_1^2\\&-(4-\chi)(\Sigma\cdot\Sigma_1)-\frac{1}{4}(4-\chi)\left(3\Sigma^2+2\Sigma\cdot\Sigma_2+2Q\Sigma^2\right)-(1+2Q)N^2\\&+N^2\left[\frac{1}{2}(1+8\chi)N^2+5(13+3\chi)\Sigma_+^2+8(2\Sigma_+-\Sigma_{+1})+(1-\chi)\Sigma_-^2\right]\end{aligned}$$

(4.5)

$$\Sigma'_{+} = -Q\Sigma_{+} + \Sigma_{+1} \equiv f_6 \qquad (4.6)$$

$$\Sigma'_{-} = -Q\Sigma_{-} + \Sigma_{-1} \equiv f_7 \qquad (4.7)$$

$$\Sigma'_{+1} = -Q\Sigma_{+1} + \Sigma_{+2} \equiv f_8 \qquad (4.8)$$

$$\Sigma'_{-1} = -Q\Sigma_{-1} + \Sigma_{-2} \qquad (4.9)$$

$$\begin{aligned}\Sigma'_{+2} &= -3(Q+2)\Sigma_{+2} + \frac{\Sigma_{+1}}{\chi}\left[B-(11\chi-8)+4Q(1-\chi)+4\Sigma^2(1+2\chi)\right] \\ &\quad + \frac{\Sigma_{+}}{\chi}\left[3B+(4-\chi)(6+Q_2+7Q)+4(1+2\chi)\left(3\Sigma^2+2\Sigma\cdot\Sigma_1\right)\right] \\ &\quad - \frac{4}{\chi}N^2\left[B+8+4Q-4(1+8\chi)N^2\right] - \frac{4}{\chi}N^2\left[(1+1\ 5\chi)\left(\Sigma_{+}+\Sigma_{+1}-4\Sigma_{+}^2\right)+4(1-\chi)\Sigma_{-}^2\right] \\ &\equiv f_{10}\end{aligned}$$

(4.10)

$$\begin{aligned}\Sigma'_{-2} &= -3(Q+2)\Sigma_{-2} + \frac{\Sigma_{-1}}{\chi}\left[B-(11\chi-8)+4Q(1-\chi)+4\Sigma^2(1+2\chi)\right] \\ &\quad + \frac{\Sigma_{-}}{\chi}\left[3B+(4-\chi)(6+Q_2+7Q)+4(1+2\chi)\left(3\Sigma^2+2\Sigma\cdot\Sigma_1\right)\right] - \frac{4(1-\chi)}{\chi}N^2\left(\Sigma_{-}+\Sigma_{-1}-8\Sigma_{-}\Sigma_{+}\right) \\ &\equiv f_{11}\end{aligned}$$

(4.11)

以及 constraint:

$$\begin{aligned}0 &= B\left(1-\Omega_\Lambda-\Sigma^2-N^2\right)+12Q-2Q^2+4Q_2-(4-\chi)(3+2Q)\Sigma^2-6(1+2\chi)\Sigma^4-\chi\left(\Sigma_1^2-2\Sigma\cdot\Sigma_2\right) \\ &\quad +4(2+\chi)(\Sigma\cdot\Sigma_1)+4N^2\left[\frac{1}{2}(1+8\chi)N^2+1+(1+5\chi)\Sigma_{+}^2+8\Sigma_{+}+(1-\chi)\Sigma_{-}^2\right]\end{aligned}$$

(4.12)

由 constraint(4.12)解出Σ_{-2}：

$$\Sigma_{-2} = -\frac{1}{2\Sigma_-}\left(\frac{1}{\chi}\begin{pmatrix} -B\Omega_\Lambda - BN^2 - B\Sigma_+^2 - B\Sigma_-^2 + B + 2N^4 + 4N^2\Sigma_+^2 + 32N^2\Sigma_+ \\ +4N^2\Sigma_-^2 + 4N^2 - 2Q^2 - 8Q\Sigma_+^2 - 8Q\Sigma_-^2 + 12Q + 4Q2 - 6\Sigma_+^4 \\ -12\Sigma_+^2\Sigma_-^2 - 12\Sigma_+^2 + 8\Sigma_+\Sigma_{+1} - 6\Sigma_-^4 - 12\Sigma_-^2 + 8\Sigma_-\Sigma_{-1} \end{pmatrix} + \begin{pmatrix} 16N^4 + 60N^2\Sigma_+^2 - 4N^2\Sigma_-^2 + 2Q\Sigma_+^2 + 2Q\Sigma_-^2 - 12\Sigma_+^4 - 24\Sigma_+^2\Sigma_-^2 \\ +3\Sigma_+^2 + 4\Sigma_+\Sigma_{+1} + 2\Sigma_+\Sigma_{+2} - 1\ \Sigma_-^4 + 23\Sigma_-^2 + 4\Sigma_-\Sigma_{-1} - \Sigma_{+1}^2 - \Sigma_{-1}^2 \end{pmatrix}\right)$$

(4.13)

(4.13)代入(4.1)到(4.11)消去Σ_{-2}，其中只有(4.5)與(4.9)有Σ_{-2}，所以 (4.5)與(4.9)改爲：

$$Q'_2 = \frac{1}{\chi}\begin{pmatrix} -B\Omega_\Lambda - BN^2 - B\Sigma_+^2 - B\Sigma_-^2 + B + 2N^4 + 4N^2\Sigma_+^2 + 32N^2\Sigma_+ \\ +4N^2\Sigma_-^2 + 4N^2 - 2Q^2 - 8Q\Sigma_+^2 - 8Q\Sigma_-^2 + 12Q + 4Q_2 - 6\Sigma_+^4 \\ -12\Sigma_+^2\Sigma_-^2 - 12\Sigma_+^2 + 8\Sigma_+\Sigma_{+1} - 6\Sigma_-^4 - 12\Sigma_-^2 + 8\Sigma_-\Sigma_{-1} \end{pmatrix}$$
$$+B\Omega_\Lambda + 1/2BN^2 - 1/2BQ - 1/2B\Sigma_+^2 - 1/2B\Sigma_-^2 - B + 16N^4 - 2N^2Q + 124N^2\Sigma_+^2 + 8N^2\Sigma_+$$
$$-4N^2\Sigma_-^2 - 8N^2\Sigma_{+1} - 2N^2 - 4Q^2 - 3QQ_2 + 2Q\Sigma_+^2 + 2Q\Sigma_-^2 - 12Q - 7Q_2 - 12\Sigma_+^4 - 24\Sigma_+^2\Sigma_-^2$$
$$+3\Sigma_+^2 - 2\Sigma_+\Sigma_{+1} - 12\Sigma_-^4 + 3\Sigma_-^2 - 2\Sigma_-\Sigma_{-1} - 3\Sigma_{+1}^2 - 3\Sigma_{-1}^2 \equiv f_5$$

(4.14)

$$\Sigma'_{-1} = \frac{1}{\Sigma_-}\left(\frac{1}{\chi}\begin{pmatrix} 1/2B\Omega_\Lambda + 1/2BN^2 + 1/2B\Sigma_+^2 + 1/2B\Sigma_-^2 - 1/2B - N^4 - 2N^2\Sigma_+^2 \\ -16N^2\Sigma_+ - 2N^2\Sigma_-^2 - 2N^2 + Q^2 + 4Q\Sigma_+^2 + 4Q\Sigma_-^2 - 6Q - 2Q_2 \\ +3\Sigma_+^4 + 6\Sigma_+^2\Sigma_-^2 + 6\Sigma_+^2 - 4\Sigma_+\Sigma_{+1} + 3\Sigma_-^4 + 6\Sigma_-^2 - 4\Sigma_-\Sigma_{-1} \end{pmatrix} + \begin{pmatrix} -8N^4 - 30N^2\Sigma_+^2 + 2N^2\Sigma_-^2 - Q\Sigma_+^2 - Q\Sigma_-^2 - 2Q\Sigma_-\Sigma_{-1} + 6\Sigma_+^4 + 12\Sigma_+^2\Sigma_-^2 \\ -3/2\Sigma_+^2 - 2\Sigma_+\Sigma_{+1} - \Sigma_+\Sigma_{+2} + 6\Sigma_-^4 - 3/2\Sigma_-^2 - 2\Sigma_-\Sigma_{-1} + 1/2\Sigma_{+1}^2 + 1/2\Sigma_{-1}^2 \end{pmatrix}\right) \equiv f_9$$

(4.15)

聯立求解平衡點(零點)可得：

$$
\begin{cases}
f_1 = 0 \\
f_2 = 0 \\
f_3 = 0 \\
f_4 = 0 \\
f_5 = 0 \\
f_6 = 0 \\
f_7 = 0 \\
f_8 = 0 \\
f_9 = 0 \\
f_{10} = 0 \\
\text{constraint} = 0
\end{cases}
\Rightarrow
\begin{cases}
B = 2(\chi - 4) - 4\left(\Sigma_+^2 + \Sigma_-^2\right)(1 + 2\chi) \\
\Omega_\Lambda = 1 + \frac{1}{2}\left(\Sigma_+^2 + \Sigma_-^2\right) \\
N = 0 \\
Q = 0 \\
Q_2 = 0 \\
\Sigma_+ = \Sigma_+ \\
\Sigma_- = \Sigma_- \\
\Sigma_{+1} = 0 \\
\Sigma_{-1} = 0 \\
\Sigma_{+2} = 0
\end{cases}
\tag{4.16}
$$

作一階線性化可得:

$$
\begin{pmatrix} B' \\ \Omega'_\Lambda \\ N' \\ Q' \\ Q'_2 \\ \Sigma'_+ \\ \Sigma'_- \\ \Sigma'_{+1} \\ \Sigma'_{-1} \\ \Sigma'_{+2} \end{pmatrix}
=
\begin{pmatrix}
\left(\frac{\partial f_1}{\partial B}\right)_0 & \left(\frac{\partial f_1}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_1}{\partial N}\right)_0 & \left(\frac{\partial f_1}{\partial Q}\right)_0 & \left(\frac{\partial f_1}{\partial Q_2}\right)_0 & \left(\frac{\partial f_1}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_1}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_1}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_1}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_1}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_2}{\partial B}\right)_0 & \left(\frac{\partial f_2}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_2}{\partial N}\right)_0 & \left(\frac{\partial f_2}{\partial Q}\right)_0 & \left(\frac{\partial f_2}{\partial Q_2}\right)_0 & \left(\frac{\partial f_2}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_2}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_2}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_2}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_2}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_3}{\partial B}\right)_0 & \left(\frac{\partial f_3}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_3}{\partial N}\right)_0 & \left(\frac{\partial f_3}{\partial Q}\right)_0 & \left(\frac{\partial f_3}{\partial Q_2}\right)_0 & \left(\frac{\partial f_3}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_3}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_3}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_3}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_3}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_4}{\partial B}\right)_0 & \left(\frac{\partial f_4}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_4}{\partial N}\right)_0 & \left(\frac{\partial f_4}{\partial Q}\right)_0 & \left(\frac{\partial f_4}{\partial Q_2}\right)_0 & \left(\frac{\partial f_4}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_4}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_4}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_4}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_4}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_5}{\partial B}\right)_0 & \left(\frac{\partial f_5}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_5}{\partial N}\right)_0 & \left(\frac{\partial f_5}{\partial Q}\right)_0 & \left(\frac{\partial f_5}{\partial Q_2}\right)_0 & \left(\frac{\partial f_5}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_5}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_5}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_5}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_5}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_6}{\partial B}\right)_0 & \left(\frac{\partial f_6}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_6}{\partial N}\right)_0 & \left(\frac{\partial f_6}{\partial Q}\right)_0 & \left(\frac{\partial f_6}{\partial Q_2}\right)_0 & \left(\frac{\partial f_6}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_6}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_6}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_6}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_6}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_7}{\partial B}\right)_0 & \left(\frac{\partial f_7}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_7}{\partial N}\right)_0 & \left(\frac{\partial f_7}{\partial Q}\right)_0 & \left(\frac{\partial f_7}{\partial Q_2}\right)_0 & \left(\frac{\partial f_7}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_7}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_7}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_7}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_7}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_8}{\partial B}\right)_0 & \left(\frac{\partial f_8}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_8}{\partial N}\right)_0 & \left(\frac{\partial f_8}{\partial Q}\right)_0 & \left(\frac{\partial f_8}{\partial Q_2}\right)_0 & \left(\frac{\partial f_8}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_8}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_8}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_8}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_8}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_9}{\partial B}\right)_0 & \left(\frac{\partial f_9}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_9}{\partial N}\right)_0 & \left(\frac{\partial f_9}{\partial Q}\right)_0 & \left(\frac{\partial f_9}{\partial Q_2}\right)_0 & \left(\frac{\partial f_9}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_9}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_9}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_9}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_9}{\partial \Sigma_{+2}}\right)_0 \\
\left(\frac{\partial f_{10}}{\partial B}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial f_{10}}{\partial N}\right)_0 & \left(\frac{\partial f_{10}}{\partial Q}\right)_0 & \left(\frac{\partial f_{10}}{\partial Q_2}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial f_{10}}{\partial \Sigma_{+2}}\right)_0
\end{pmatrix}
\begin{pmatrix} B \\ \Omega_\Lambda \\ N \\ Q \\ Q_2 \\ \Sigma_+ \\ \Sigma_- \\ \Sigma_{+1} \\ \Sigma_{-1} \\ \Sigma_{+2} \end{pmatrix}
$$

(4.17)

(4.17)計算結果可得:

$$
\left(\left(\frac{\partial f_1}{\partial B}\right)_0 \left(\frac{\partial f_1}{\partial \Omega_\Lambda}\right)_0 \left(\frac{\partial f_1}{\partial N}\right)_0 \left(\frac{\partial f_1}{\partial Q}\right)_0 \left(\frac{\partial f_1}{\partial Q_2}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_+}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_-}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_{+1}}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_{-1}}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_{+2}}\right)_0 \left(\frac{\partial f_1}{\partial \Sigma_{-2}}\right)_0\right)
$$
$$
= \left(0,\ 0,\ 0,\ \text{-2B},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ \right)
$$

(4.18)

$$
\left(\left(\frac{\partial f_2}{\partial B}\right)_0 \left(\frac{\partial f_2}{\partial \Omega_\Lambda}\right)_0 \left(\frac{\partial f_2}{\partial N}\right)_0 \left(\frac{\partial f_2}{\partial Q}\right)_0 \left(\frac{f_2}{\partial Q_2}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_+}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_-}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_{+1}}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_{-1}}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_{+2}}\right)_0 \left(\frac{\partial f_2}{\partial \Sigma_{-2}}\right)_0\right)
$$
$$
= \left(0,0,0,-\Sigma_-^2 - \Sigma_+^2 - 2,0,0,0,0,0,0\right)
$$

(4.19)

$$\left(\left(\frac{\partial f_3}{\partial B}\right)_0\left(\frac{\partial f_3}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_3}{\partial N}\right)_0\left(\frac{\partial f_3}{\partial Q}\right)_0\left(\frac{\partial f_3}{\partial Q_2}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_3}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\left(0,\ 0,\ \text{-}1\text{-}4\Sigma_+,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right)$$

(4.20)

$$\left(\left(\frac{f_4}{\partial B}\right)_0\left(\frac{\partial f_4}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_4}{\partial N}\right)_0\left(\frac{\partial f_4}{\partial Q}\right)_0\left(\frac{\partial f_4}{\partial Q_2}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_4}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\left(0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\right)$$

(4.21)

$$\left(\left(\frac{\partial f_5}{\partial B}\right)_0\left(\frac{\partial f_5}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_5}{\partial N}\right)_0\left(\frac{\partial f_5}{\partial Q}\right)_0\left(\frac{\partial f_5}{\partial Q_2}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_5}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\begin{pmatrix} \dfrac{3\left(4\Sigma_+^4+8\Sigma_+^2\Sigma_-^2-\Sigma_+^2+4\Sigma_-^4-\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},\dfrac{B\left(B+12\Sigma_+^2+12\Sigma_-^2+6\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},0, \\ -\dfrac{B^2+32B-144\Sigma_+^4-288\Sigma_+^2\Sigma_-^2+288\Sigma_+^2-144\Sigma_-^4+288\Sigma_-^2+144}{2\left(B+4\Sigma_+^2+4\Sigma_-^2+8\right)}, \\ -\dfrac{7B+60\Sigma_+^2+60\Sigma_-^2+48}{B+4\Sigma_+^2+4\Sigma_-^2+8},-\dfrac{\Sigma_+\left(B^2+36B\Sigma_+^2+36B\Sigma_-^2+6B+216\Sigma_+^2+216\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8}, \\ -\dfrac{\Sigma_-\left(B^2+36B\Sigma_+^2+36B\Sigma_-^2+6B+216\Sigma_+^2+216\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8}, \\ C_{5,8}=-\dfrac{2\Sigma_+\left(B+36\Sigma_+^2+36\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},-\dfrac{2\Sigma_-\left(B+36\Sigma_+^2+36\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},0 \end{pmatrix}$$

(4.22)

$$\left(\left(\frac{\partial f_6}{\partial B}\right)_0\left(\frac{\partial f_6}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_6}{\partial N}\right)_0\left(\frac{\partial f_6}{\partial Q}\right)_0\left(\frac{\partial f_6}{\partial Q_2}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_6}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\left(0,\ 0,\ 0,\ -\Sigma_+,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\right)$$

(4.23)

$$\left(\left(\frac{\partial f_7}{\partial B}\right)_0\left(\frac{\partial f_7}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_7}{\partial N}\right)_0\left(\frac{\partial f_7}{\partial Q}\right)_0\left(\frac{\partial f_7}{\partial Q_2}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_7}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\left(0,\ 0,\ 0,-\Sigma_-,\ 0,\ 0,\ 0,\ 0,1,\ 0\right)$$

(4.24)

$$\left(\left(\frac{\partial f_8}{\partial B}\right)_0\left(\frac{\partial f_8}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_8}{\partial N}\right)_0\left(\frac{\partial f_8}{\partial Q}\right)_0\left(\frac{\partial f_8}{\partial Q_2}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_8}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\left(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\right)$$

(4.25)

$$\left(\left(\frac{\partial f_9}{\partial B}\right)_0\left(\frac{\partial f_9}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_9}{\partial N}\right)_0\left(\frac{\partial f_9}{\partial Q}\right)_0\left(\frac{\partial f_9}{\partial Q_2}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_9}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\begin{pmatrix} -\frac{3}{2}\frac{4\Sigma_+^4+8\Sigma_+^2\Sigma_-^2-\Sigma_+^2+4\Sigma_-^4-\Sigma_-^2}{\Sigma_-\left(B+4\Sigma_+^2+4\Sigma_-^2+8\right)},-\frac{B\left(4\Sigma_+^2+4\Sigma_-^2-1\right)}{\Sigma_-\left(B+4\Sigma_+^2+4\Sigma_-^2+8\right)}, \\ 0,-\frac{B\Sigma_+^2+B\Sigma_-^2+36\Sigma_+^4+72\Sigma_+^2\Sigma_-^2-48\Sigma_+^2+36\Sigma_-^4-48\Sigma_-^2+12}{\Sigma_-\left(B+4\Sigma_+^2+4\Sigma_-^2+8\right)}, \\ \frac{16B\Sigma_+^2+16B\Sigma_-^2-B+108\Sigma_+^2+108\Sigma_-^2}{B+4\Sigma_+^2+4\Sigma_-^2+8},-\frac{2\Sigma_+\left(B-12\Sigma_+^2-12\Sigma_-^2+12\right)}{\Sigma_-\left(B+4\Sigma_+^2+4\Sigma_-^2+8\right)} \\ -\frac{2\left(B-12\Sigma_+^2-12\Sigma_-^2+12\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},-\frac{\Sigma_+}{\Sigma_-} \end{pmatrix}$$

(4.26)

$$\left(\left(\frac{\partial f_{10}}{\partial B}\right)_0\left(\frac{\partial f_{10}}{\partial \Omega_\Lambda}\right)_0\left(\frac{\partial f_{10}}{\partial N}\right)_0\left(\frac{\partial f_{10}}{\partial Q}\right)_0\left(\frac{\partial f_{10}}{\partial Q_2}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_+}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_-}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_{+1}}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_{-1}}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_{+2}}\right)_0\left(\frac{\partial f_{10}}{\partial \Sigma_{-2}}\right)_0\right)$$
$$=\begin{pmatrix} -\frac{6\Sigma_+\left(4\Sigma_+^2+4\Sigma_-^2-1\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},0,0,-\frac{7\Sigma_+\left(B+36\Sigma_+^2+36\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8}, \\ -\frac{\Sigma_+\left(B+36\Sigma_+^2+36\Sigma_-^2\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},\frac{48\Sigma_+^2\left(B+9\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},\frac{48\Sigma_-\Sigma_+\left(B+9\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8}, \\ \frac{16B\Sigma_+^2-9B+108\Sigma_+^2-36\Sigma_-^2-72}{B+4\Sigma_+^2+4\Sigma_-^2+8},\frac{16\Sigma_-\Sigma_+\left(B+9\right)}{B+4\Sigma_+^2+4\Sigma_-^2+8},-6 \end{pmatrix}$$

(4.27)

特徵方程爲：

$$
\begin{aligned}
&\lambda^{10}+\left(4\Sigma_{-}+16\right)\lambda^{9}+\left(1/2\,B-18\Sigma_{-}^{2}+60\Sigma_{-}-18\Sigma_{+1}^{2}+105\right)\lambda^{8}\\
&+\left(2B\Sigma_{-}+13/2\,B-72\Sigma_{-}^{3}-234\Sigma_{-}^{2}-72\Sigma_{-}\Sigma_{+1}^{2}+360\Sigma_{-}-234\Sigma_{+1}^{2}+360\right)\lambda^{7}\\
&+\begin{pmatrix}-9B\Sigma_{-}^{2}+24B\Sigma_{-}-9B\Sigma_{+1}^{2}+33B-864\Sigma_{-}^{3}-1188\Sigma_{-}^{2}\\ -864\Sigma_{-}\Sigma_{+1}^{2}+1080\Sigma_{-}-1188\Sigma_{+1}^{2}+675\end{pmatrix}\lambda^{6}\\
&+\begin{pmatrix}-36B\Sigma_{-}^{3}-90B\Sigma_{-}^{2}-36B\Sigma_{-}\Sigma_{+1}^{2}+108B\Sigma_{-}-90B\Sigma_{+1}^{2}+81B\\ -3888\Sigma_{-}^{3}-2916\Sigma_{-}^{2}-3888\Sigma_{-}\Sigma_{+1}^{2}+1620\Sigma_{-}-2916\Sigma_{+1}^{2}+648\end{pmatrix}\lambda^{5}\\
&+\begin{pmatrix}-324B\Sigma_{-}^{3}-324B\Sigma_{-}^{2}-324B\Sigma_{-}\Sigma_{+1}^{2}+216B\Sigma_{-}-324B\Sigma_{+1}^{2}+\dfrac{189}{2}B\\ -7776\Sigma_{-}^{3}-3402\Sigma_{-}^{2}-7776\Sigma_{-}\Sigma_{+1}^{2}+972\Sigma_{-}-3402\Sigma_{+1}^{2}+243\end{pmatrix}\lambda^{4}\\
&+\begin{pmatrix}-972B\Sigma_{-}^{3}-486B\Sigma_{-}^{2}-972B\Sigma_{-}\Sigma_{+1}^{2}+162B\Sigma_{-}-486B\Sigma_{+1}^{2}\\ +\dfrac{81}{2}B-5832\Sigma_{-}^{3}-1458\Sigma_{-}^{2}-5832\Sigma_{-}\Sigma_{+1}^{2}-1458\Sigma_{+1}^{2}\end{pmatrix}\lambda^{3}\\
&+\left(-972B\Sigma_{-}^{3}-243B\Sigma_{-}^{2}-972B\Sigma_{-}\Sigma_{+1}^{2}-243B\Sigma_{+1}^{2}\right)\lambda^{2}=0
\end{aligned}
$$

(4.28)

得到 eigenvalues 爲[16]:

$$\lambda=-\frac{3}{2}\left(1\pm\sqrt{1+8\Sigma^{2}}\right),-\frac{3}{2}\pm\frac{1}{2}\sqrt{9-2B},-1-4\Sigma_{+},0,0,-3,-3,-3 \qquad (4.29)$$

(4.27)爲一階線性化分析之結果，我們已看出所得的 eigenvalues 已明顯有負號項存在，已知:

$$
\begin{aligned}
&\Sigma^{2}>0\\
&\Rightarrow\lambda=-\frac{3}{2}\left(1-\sqrt{1+8\Sigma^{2}}\right)>0
\end{aligned}
\qquad (4.30)
$$

根據非線性動力學相關理論(參考 Appendix B)，只要再找到正號項，系統的解必爲 saddle points，即爲 unstable。

由於系統是一個 10x10 的矩陣，因此以上 eigenvalues 的計算，我們需藉由電腦程式來幫助我們做運算，有關程式的詳細內容請參考 Appendix C。

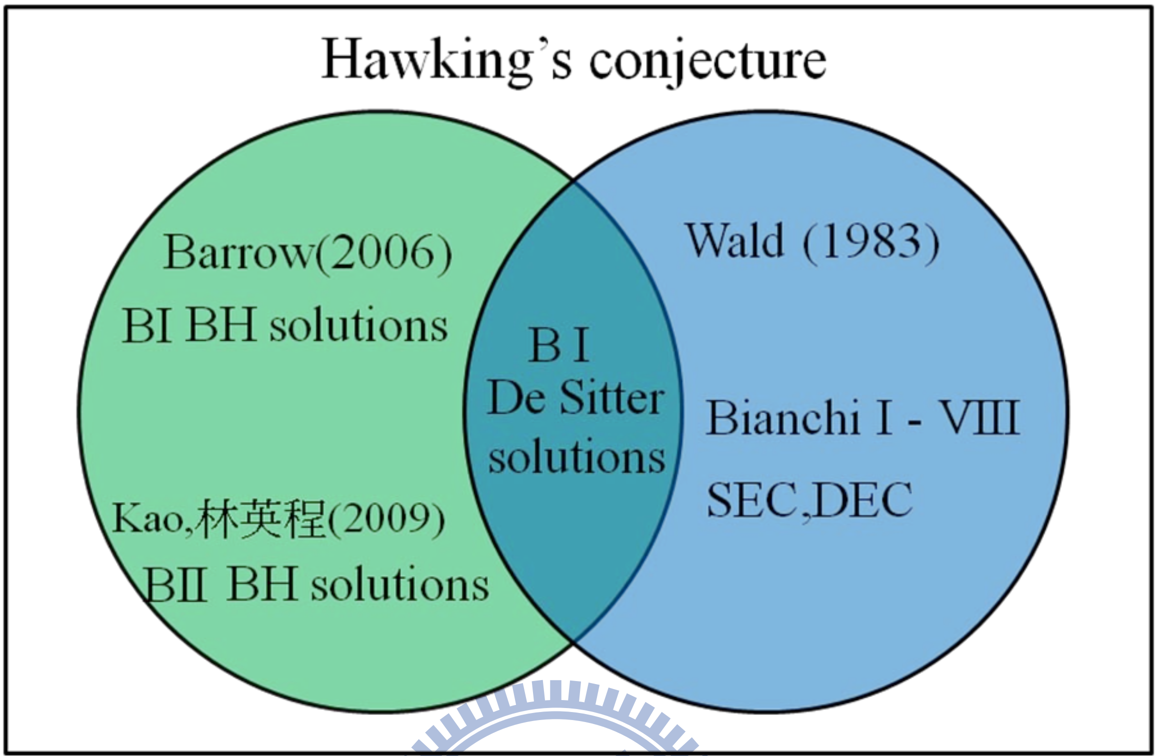
Chapter 5
結論

這篇論文首先介紹 FRW 宇宙模型，而最後得到的解形式為:

$$a(t) \propto e^{Ht}, H = \sqrt{\frac{\Lambda}{3}}$$

會是一個我們稱為 de Sitter solution 的結果，而其物理意義描述宇宙在一開始 t=0 時會是一個點，之後會隨著時間以自然指數的形式一直膨漲下去。目前為止 FRW 宇宙模型預測的結果和我們的宇宙所觀測到的現象都是吻合的，但不禁讓人提出疑問，宇宙在演化上為何具有高度均勻(Homogeneous)且勻向(isotropic)的對稱特質?難道宇宙在早期都是這麼完美的嗎?因此我們採用 Bianchi type I (BI) 這類非勻向(anisotropic)的宇宙模型，來探討宇宙演化上各種的可能性。而研究數據所呈現的結果，都符合 Hawking 和 Gibbons 等人所提出的無毛定理 [18-19]，即宇宙常數為正的情況下，宇宙模型除了 de Sitter solution 之外所得的解最後都會不穩定。無毛定理目前只是個猜測的理論，究竟最後是不是正確的推論?仍要不斷地被驗證下去。這對往後的研究者而言，提供了一個大方向。或許有一天，早期宇宙的神秘面紗，有機會在世人面前被完全掀開，讓人類一窺它的真面目。

宇宙無毛定理以及目前被驗證的進展關係圖:



說明：

1. R. Wald(1983)[3]: 不限 Lagrange,限 Bianchi I-VIII 且符合 SEC,DEC 的解,不穩定。
2. J. Barrow,S. Hervik[14],林英程(2010)：$L = R - 2\Lambda + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}$,Bianchi I,不符合能量條件的解(BH solutions),不穩定。
3. W. Kao,林英程(2009), J. Barrow,S. Hervik[14]：$L = R - 2\Lambda + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}$,Bianchi II, 不符合能量條件的解(BH solutions),不穩定。

Appendix A

九種 Bianchi space 的宇宙模型列出如下:

Bianchi Ⅰ: $ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2$

Bianchi Ⅱ: $ds^2 = -dt^2 + a_1^2(t)\left(dx + zdy\right)^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2$

Bianchi Ⅲ: $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)\left(\cosh zdx + \sinh zdy\right)^2 + a_2^2(t)\left(\sinh zdx + \cosh zdy\right)^2\right) + a_3^2(t)dz^2$

Bianchi Ⅳ: $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)\left(dx + zdy\right)^2 + a_2^2(t)dy^2\right) + a_3^2(t)dz^2$

Bianchi Ⅴ: $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)dx^2 + a_2^2(t)dy^2\right) + a_3^2(t)dz^2$

Bianchi Ⅵ: $ds^2 = -dt^2 + a_1^2(t)\left(\cosh zdx + \sinh zdy\right)^2 + a_2^2(t)\left(\sinh zdx + \cosh zdy\right)^2 + a_3^2(t)dz^2$

Bianchi Ⅶ: $ds^2 = -dt^2 + a_1^2(t)\left(\cos zdx + \sin zdy\right)^2 + a_2^2(t)\left(\sin zdx - \cos zdy\right)^2$

Bianchi Ⅷ: $ds^2 = -dt^2 + a_1^2(t)\left(\cosh zdx - \sinh z \sinh xdy\right)^2 + a_2^2(t)\left(\sinh zdx - \cosh z \sinh xdy\right)^2 + a_3^2(t)\left(dz + \cosh xdy\right)^2$

Bianchi Ⅸ: $ds^2 = -dt^2 + a_1^2(t)\left(\cos zdx + \sin z \sin xdy\right)^2 + a_2^2(t)\left(-\sin zdx + \cos z \sin xdy\right)^2 + a_3^2(t)\left(dz + \cos xdy\right)^2$

Appendix B

若我們考慮一個 homogeneous linear system 如下:

$$y' = Ay = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y$$

其中
$$y_1' = a_{11}y_1 + a_{12}y_2$$
$$y_2' = a_{21}y_1 + a_{22}y_2$$

已知:

$$Ay = \lambda y$$

我們可得:

$$\det(A - \lambda I) = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det A = 0$$

我們可以整理出一個二次方程式爲:

$$\lambda^2 - p\lambda + q = 0$$

其中
$$p = a_{11} + a_{22}$$
$$q = \det A = a_{11}a_{22} - a_{12}a_{21}$$

另外我們定義:

$$\Delta = p^2 - 4q$$

則我們可以得到:

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta})$$
$$\lambda_2 = \frac{1}{2}(p - \sqrt{\Delta})$$

代回剛才得到的二次方程式可整理出:

$$\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

我們可以很容易的發現:

$$p = \lambda_1 + \lambda_2$$
$$q = \lambda_1\lambda_2$$
$$\Delta = (\lambda_1 - \lambda_2)^2$$

由以上的關係我們可以整理出以下的關係表[17]:

Name	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q > 0$	$\Delta \geqq 0$	Real, same sign
(b) Saddle point		$q < 0$		Real, opposite sign
(c) Center	$p = 0$	$q > 0$		Pure imaginary
(d) Spiral point	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

以上四個特殊的點我們可以對照我們所處理的系統會是哪一種情形?在這篇論文的第四章中我們可以根據(4.27)所求出的 eigenvalues 可看出已確定系統有負的 eigenvalue，另外可由(4.28)確定也有正的 eigenvalue，因此系統一階線性化的結果我們可以確定會有 saddle point 的存在。

現在我們進一步探討系統的穩定性分析，可以再整理出一個關係表如下[17]:

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$
(a) Stable and attractive	$p < 0$	$q > 0$
(b) Stable	$p \leqq 0$	$q > 0$
(c) Unstable	$p > 0$ OR	$q < 0$

我們可以看出 saddle point 的存在會讓系統處於 unstable。

Appendix C

這邊附上關於論文裡需要用到的數學軟體 Mathematica 程式碼:

1.

$$Clear[Bprime, \Omega\Lambda prime, nprime, Qprime, Q2prime, \Sigma pprime, \Sigma mprime, \Sigma p1prime, \Sigma m1prime, \Sigma p2prime, \Sigma m2prime, constraint, B, \Omega\Lambda, n, Q, Q2, \Sigma p, \Sigma m, \Sigma p1, \Sigma m1, \Sigma p2, \Sigma m2, \Sigma m2sub, \Sigma, \Sigma 1, \Sigma 2, \chi, matrix, matrix2, eigenvalues, \lambda]$$

2.

$$\Sigma = \sqrt{\Sigma p^2 + \Sigma m^2}$$

$$\Sigma 1 = \sqrt{\Sigma p1^2 + \Sigma m1^2}$$

$$\Sigma 2 = \sqrt{\Sigma p2^2 + \Sigma m2^2}$$

3.

$$Bprime = -2*Q*B$$

$$\Omega\Lambda prime = -2*Q*\Omega\Lambda$$

$$nprime = -(Q+1+4*\Sigma p)*n$$

$$Qprime = -2*Q^2 + Q2$$

$$\begin{aligned} Q2prime = &-3*(Q+2)*Q2 - 9/2*(Q+2)*Q - 3/4*B*(1+\Sigma^2 - \Omega\Lambda + 2/3*Q - 1/3*n^2) \\ &-3/2*(1+2*\chi)*\Sigma^4 - 1/4*(8+\chi)*\Sigma 1^2 - (4-\chi)*(\Sigma p*\Sigma p1 + \Sigma m*\Sigma m1) \\ &-1/4*(4-\chi)*(3*\Sigma^2 + 2*(\Sigma p*\Sigma p2 + \Sigma m*\Sigma m2) + 2*Q*\Sigma^2) \\ &-(1+2*Q)*n^2 + n^2*(1/2*(1+8*\chi)*n^2 + 5*(13+3*\chi)*\Sigma p^2 \\ &+8*(2*\Sigma p - \Sigma p1) + (1-\chi)*\Sigma m^2) \end{aligned}$$

$$\Sigma pprime = -Q*\Sigma p + \Sigma p1$$

$$\Sigma mprime = -Q*\Sigma m + \Sigma m1$$

$$\Sigma p1prime = -2*Q*\Sigma p1 + \Sigma p2$$

$$\Sigma m1prime = -2*Q*\Sigma m1 + \Sigma m2$$

$$\begin{aligned} \Sigma p2prime = &-3*(Q+2)*\Sigma p2 + \Sigma p1/\chi*(B-(11*\chi-8) + 4*Q*(1-\chi) + 4*\Sigma^2*(1+2*\chi)) \\ &+\Sigma p/\chi*(3*B + (4-\chi)*(6+Q2+7*Q) + 4*(1+2*\chi)*(3*\Sigma^2 + 2*(\Sigma p*\Sigma p1 + \Sigma m*\Sigma m1))) \\ &-4/\chi*n^2*(B+8+4*Q-4*(1+8*\chi)*n^2) \\ &-4/\chi*n^2*((1+15*\chi)*(\Sigma p + \Sigma p1 - 4*\Sigma p^2) + 4*(1-\chi)*\Sigma m^2) \end{aligned}$$

$$\begin{aligned} \Sigma p2prime = &-3*(Q+2)*\Sigma p2 + \Sigma p1/\chi*(B-(11*\chi-8) + 4*Q*(1-\chi) + 4*\Sigma^2*(1+2*\chi)) + \\ &\Sigma p/\chi*(3*B + (4-\chi)*(6+Q2+7*Q) + 4*(1+2*\chi)*(3*\Sigma^2 + 2*(\Sigma p*\Sigma p1 + \Sigma m*\Sigma m1))) \\ &-4/\chi*n^2*(B+8+4*Q-4*(1+8*\chi)*n^2) - 4/\chi*n^2*((1+15*\chi)*(\Sigma p + \Sigma p1 - 4*\Sigma p^2) \\ &+4*(1-\chi)*\Sigma m^2) \end{aligned}$$

$$\begin{aligned} constraint = &B*(1-\Omega\Lambda - \Sigma^2 - n^2) + 12*Q - 2*Q^2 + 4*Q2 - (4-\chi)*(3+2*Q)*\Sigma^2 - 6*(1+2*\chi)*\Sigma^4 \\ &-\chi*(\Sigma 1^2 - 2*(\Sigma p*\Sigma p2 + \Sigma m*\Sigma m2)) + 4*(2+\chi)*(\Sigma p*\Sigma p1 + \Sigma m*\Sigma m1) \\ &+4*n^2*(1/2*(1+8*\chi)*n^2 + 1 + (1+15*\chi)*\Sigma p^2 + 8*\Sigma p + (1-\chi)*\Sigma m^2) \end{aligned}$$

$$\begin{aligned} \Sigma m2 = &\frac{1}{2\Sigma m\chi}(-B - 4n^2 + Bn^2 - 2n^4 - 12Q + 2Q^2 - 4Q2 + 12\Sigma m^2 + B\Sigma m^2 - 4n^2\Sigma m^2 \\ &+8Q\Sigma m^2 + 6\Sigma m^4 - 8\Sigma m\Sigma m1 - 32n^2\Sigma p + 12\Sigma p^2 + B\Sigma p^2 - 4n^2\Sigma p^2 + 8Q\Sigma p^2 + 12\Sigma m^2\Sigma p^2 \\ &+6\Sigma p^4 - 8\Sigma p\Sigma p1 - 16n^4\chi - 3\Sigma m^2\chi + 4n^2\Sigma m^2\chi - 2Q\Sigma m^2\chi + 12\Sigma m^4\chi - 4\Sigma m\Sigma m1\chi \\ &+\Sigma m1^2\chi - 3\Sigma p^2\chi - 60n^2\Sigma p^2\chi - 2Q\Sigma p^2\chi + 24\Sigma m^2\Sigma p^2\chi + 12\Sigma p^4\chi - 4\Sigma p\Sigma p1\chi + \\ &\Sigma p1^2\chi - 2\Sigma p\Sigma p2\chi + B\Omega\Lambda) \end{aligned}$$

4.

$$
\begin{aligned}
matrix = \{ & \{\partial_B Bprime, \partial_{\Omega\Lambda} Bprime, \partial_n Bprime, \partial_Q Bprime, \partial_{Q2} Bprime, \partial_{\Sigma p} Bprime, \\
& \partial_{\Sigma m} Bprime, \partial_{\Sigma p1} Bprime, \partial_{\Sigma m1} Bprime, \partial_{\Sigma p2} Bprime\}, \{\partial_B \Omega\Lambda prime, \partial_{\Omega\Lambda} \Omega\Lambda prime, \\
& \partial_n \Omega\Lambda prime, \partial_Q \Omega\Lambda prime, \partial_{Q2} \Omega\Lambda prime, \partial_{\Sigma p} \Omega\Lambda prime, \partial_{\Sigma m} \Omega\Lambda prime, \\
& \partial_{\Sigma p1} \Omega\Lambda prime, \partial_{\Sigma m1} \Omega\Lambda prime, \partial_{\Sigma p2} \Omega\Lambda prime\}, \{\partial_B nprime, \partial_{\Omega\Lambda} nprime, \partial_n nprime, \\
& \partial_Q nprime, \partial_{Q2} nprime, \partial_{\Sigma p} nprime, \partial_{\Sigma m} nprime, \partial_{\Sigma p1} nprime, \partial_{\Sigma m1} nprime, \partial_{\Sigma p2} nprime\}, \\
& \{\partial_B Qprime, \partial_{\Omega\Lambda} Qprime, \partial_n Qprime, \partial_Q Qprime, \partial_{Q2} Qprime, \partial_{\Sigma p} Qprime, \partial_{\Sigma m} Qprime, \\
& \partial_{\Sigma p1} Qprime, \partial_{\Sigma m1} Qprime, \partial_{\Sigma p2} Qprime\}, \{\partial_B Q2\,prime, \partial_{\Omega\Lambda} Q2\,prime, \partial_n Q2\,prime, \\
& \partial_Q Q2\,prime, \partial_{Q2} Q2\,prime, \partial_{\Sigma p} Q2\,prime, \partial_{\Sigma m} Q2\,prime, \partial_{\Sigma p1} Q2\,prime, \\
& \partial_{\Sigma m1} Q2\,prime, \partial_{\Sigma p2} Q2\,prime\}, \{\partial_B \Sigma pprime, \partial_{\Omega\Lambda} \Sigma pprime, \partial_n \Sigma pprime, \partial_Q \Sigma pprime, \\
& \partial_{Q2} \Sigma pprime, \partial_{\Sigma p} \Sigma pprime, \partial_{\Sigma m} \Sigma pprime, \partial_{\Sigma p1} \Sigma pprime, \partial_{\Sigma m1} \Sigma pprime, \partial_{\Sigma p2} \Sigma pprime\}, \\
& \{\partial_B \Sigma mprime, \partial_{\Omega\Lambda} \Sigma mprime, \partial_n \Sigma mprime, \partial_Q \Sigma mprime, \partial_{Q2} \Sigma mprime, \partial_{\Sigma p} \Sigma mprime, \\
& \partial_{\Sigma m} \Sigma mprime, \partial_{\Sigma p1} \Sigma mprime, \partial_{\Sigma m1} \Sigma mprime, \partial_{\Sigma p2} \Sigma mprime\}, \{\partial_B \Sigma p1\,prime, \\
& \partial_{\Omega\Lambda} \Sigma p1\,prime, \partial_n \Sigma p1\,prime, \partial_Q \Sigma p1\,prime, \partial_{Q2} \Sigma p1\,prime, \partial_{\Sigma p} \Sigma p1\,prime, \\
& \partial_{\Sigma m} \Sigma p1\,prime, \partial_{\Sigma p1} \Sigma p1\,prime, \partial_{\Sigma m1} \Sigma p1\,prime, \partial_{\Sigma p2} \Sigma p1\,prime\}, \{\partial_B \Sigma m1\,prime, \\
& \partial_{\Omega\Lambda} \Sigma m1\,prime, \partial_n \Sigma m1\,prime, \partial_Q \Sigma m1\,prime, \partial_{Q2} \Sigma m1\,prime, \partial_{\Sigma p} \Sigma m1\,prime, \\
& \partial_{\Sigma m} \Sigma m1\,prime, \partial_{\Sigma p1} \Sigma m1\,prime, \partial_{\Sigma m1} \Sigma m1\,prime, \partial_{\Sigma p2} \Sigma m1\,prime\}, \{\partial_B \Sigma p2\,prime, \\
& \partial_{\Omega\Lambda} \Sigma p2\,prime, \partial_n \Sigma p2\,prime, \partial_Q \Sigma p2\,prime, \partial_{Q2} \Sigma p2\,prime, \partial_{\Sigma p} \Sigma p2\,prime, \\
& \partial_{\Sigma m} \Sigma p2\,prime, \partial_{\Sigma p1} \Sigma p2\,prime, \partial_{\Sigma m1} \Sigma p2\,prime, \partial_{\Sigma p2} \Sigma p2\,prime\}\}
\end{aligned}
$$

5.

$$
\begin{aligned}
&B = B \\
&\Omega\Lambda = 1/2 * \Sigma^2 + 1 \\
&n = 0 \\
&Q = 0 \\
&Q2 = 0 \\
&\Sigma p = \Sigma p \\
&\Sigma m = \Sigma m \\
&\Sigma p1 = 0 \\
&\Sigma m1 = 0 \\
&\Sigma p2 = 0 \\
&\Sigma m2 = 0 \\
&\chi = (B + 4 * \Sigma^2 + 8) / (2 * (1 - 4 * \Sigma^2))
\end{aligned}
$$

6.

$$matrix$$

7.

$$
\begin{aligned}
matrix2 = matrix - \lambda * \{ &\{1,0,0,0,0,0,0,0,0,0\},\{0,1,0,0,0,0,0,0,0,0\}, \\
&\{0,0,1,0,0,0,0,0,0,0\},\{0,0,0,1,0,0,0,0,0,0\},\{0,0,0,0,1,0,0,0,0,0\}, \\
&\{0,0,0,0,0,1,0,0,0,0\},\{0,0,0,0,0,0,1,0,0,0\},\{0,0,0,0,0,0,0,1,0,0\}, \\
&\{0,0,0,0,0,0,0,0,1,0\},\{0,0,0,0,0,0,0,0,0,1\},\{0,0,0,0,0,0,0,0,0,1\}\}
\end{aligned}
$$

8.

$$eigenvalues = Det[matrix2]$$

9.

$$Solve[eigenvalues == 0, \lambda]$$

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