JOURNAL OF DIFFERENTIAL EQUATIONS **81**, 221–233 (1989)

On the Existence of Positive Radial Solutions for Nonlinear Elliptic Equations in Annular Domains

SONG-SUN LIN[1]

Department of Applied Mathematics, National Chiao-Tung University, Hsin-chu, Taiwan, Republic of China

Received February 24, 1988

We study the existence of positive radially symmetric solutions of $\Delta u + g(|x|) f(u) = 0$ in annulus with Dirichlet (Dirichlet/Neumann) boundary conditions. We show that the equation has a positive radial solution on any annulus if f and g are positive and f is superlinear at 0 and ∞.

1. INTRODUCTION

In this paper we consider the existence of positive radial solutions of the equation

$$\Delta u + g(|x|) f(u) = 0 \qquad \text{in} \quad R < |x| < \bar{R}, \tag{1.1}$$

$x \in \mathbb{R}^n$, $n \geqslant 2$, with one of the following sets of boundary conditions:

$$u = 0 \text{ on } |x| = R \qquad \text{and} \qquad u = 0 \text{ on } |x| = \bar{R}, \tag{1.2a}$$

$$u = 0 \text{ on } |x| = R \qquad \text{and} \qquad \frac{\partial u}{\partial r} = 0 \text{ on } |x| = \bar{R}, \tag{1.2b}$$

$$\frac{\partial u}{\partial r} = 0 \text{ on } |x| = R \qquad \text{and} \qquad u = 0 \text{ on } |x| = \bar{R}. \tag{1.2c}$$

Here $r = |x|$ and $\partial/\partial r$ denotes differentiation in the radial direction.

This paper was motivated by the recent works of Bandle, Coffman, and Marcus [1] and Garaizar [6]. When $g(r) \equiv 1$, Bandle *et al.* [1] proved that the problems (1.1), (1.2) have positive radial solutions for any annulus in $\mathbb{R}^n$, $n \geqslant 3$, under the assumptions

[1] Work partially supported by the National Science Council of the Republic of China.

0022-0396/89 $3.00

(A-0) $f \in C^1(R)$, $f(t) > 0$ for $t > 0$, and $f(0) = 0$,

(A-1) f is nondecreasing in $(0, \infty)$,

(A-2) $\lim_{u \to \infty} f(u)/u = \infty$,

(A-3) $\lim_{u \to 0} f(u)/u = 0$.

In [6], Garaizar proved various existence and nonexistence results. Among them, existence of positive radial solutions on any annulus was proved by assuming (A-0), (A-3), and

$(\text{A-2})_k$ there are $b > 0$, constants $d_1, d_2 > 0$, and $k > 1$ such that $d_1 u^k \leqslant f(u) \leqslant d_2 u^k$ for $u \geqslant b$.

It is easy to see that Conditions (A-2) and (A-3) are in a sense necessary for the existence of positive radial solutions on all annuli (see, e.g., [1]). In this paper we will show that they are sufficient too. In fact, we prove the existence of positive solutions on any annulus for (1.1), (1.2) when f satisfies (A-2), (A-3), and

(A-0)′ (i) $f \in C^1(R)$, $f(u) \geqslant 0$ *for* $u > 0$,

(ii) $g \in C^1((0, \infty))$, $g(r) \geqslant 0$ for $r > 0$ and is not identically zero in any finite subinterval of $(0, \infty)$.

The method used here is the shooting method combined with the Sturm Comparison Theorem.

Various uniqueness and nonuniqueness results for Problem (1.1), (1.2a) have been obtained by many authors (see, e.g. [3, 9, 10]). Existence results for positive non-radially symmetric solutions were observed in [2, 4]. The problems of non-radially symmetric bifurcation from the radial solutions were studied by Dancer [5], Smoller and Wasserman [11], and Lin [8].

2.

Since we are interested in a radial solution $u = u(r)$, we shall write (1.1) in the form

$$u''(r) + \frac{n-1}{r} u'(r) + g(r) f(u(r)) = 0 \qquad \text{in} \quad (R, \bar{R}). \tag{2.1*}$$

Thus, for $n \geqslant 3$, in terms of variables

$$s = \{(n-2) r^{n-2}\}^{-1} \qquad \text{and} \qquad u(s) = u(r),$$

Eq. (2.1)* can be rewritten as

$$u''(s) + \rho(s) f(u(s)) = 0 \qquad \text{in} \quad (s_0, s_1), \tag{2.1}$$

where

$$\rho(s)=[(n-2)\,s]^{-k}\,g([(n-2)\,s]^{-1/n-2}),$$
$$k=\frac{2n-2}{n-2},$$
$$s_0=\{(n-2)\,\bar{R}^{n-2}\}^{-1},$$

and

$$s_1=\{(n-2)\,R^{n-2}\}^{-1}.$$

As for $n=2$, in terms of variables

$$s=\tfrac{1}{2}-\log R+\log r \qquad \text{and} \qquad u(s)=u(r),$$

Eq. (2.1)* can also be written as (2.1) with

$$\rho(s)=R^2e^{2s-1}g(Re^{s-1/2})$$

and

$$s_0=\tfrac{1}{2} \qquad \text{and} \qquad s_1=\tfrac{1}{2}-\log R+\log \bar{R}.$$

It is clear that ρ satisfies (A-0)′(ii).

The boundary conditions become

$$u(s_0)=u(s_1)=0, \tag{2.2a}$$
$$u'(s_0)=u(s_1)=0, \tag{2.2b}$$
$$u(s_0)=u'(s_1)=0. \tag{2.2c}$$

From now on, we shall concentrate on Problem (2.1), (2.2).

Using backward shooting, we consider the family of solutions of the initial value problem

$$u''(s)+\rho(s)\,f(u(s))=0 \qquad \text{for} \quad s<s_1 \tag{2.3}$$
$$u(s_1)=0, \qquad u'(s_1)=-b, \tag{2.4}$$

where $b>0$ is the shooting parameter.

Here $s_1>0$ will be kept fixed throughout the paper.

For every $b>0$, Problem (2.3), (2.4) has a unique solution $u(\cdot)\equiv u(\cdot, b)$ with the maximal domain of existence $(s(b), s_1)$.

It is easy to check that (2.3), (2.4) is equivalent to the integral equation

$$u(s)=b(s_1-s)-\int_s^{s_1}(t-s)\,\rho(t)\,f(u(t))\,dt, \qquad s<s_1 \tag{2.5}$$

and solution u also satisfies

$$u(s) = u(\bar{s}) + u'(\bar{s})(s - \bar{s}) + \int_{\bar{s}}^{s} (t - s)\, \rho(t)\, f(u(t))\, dt \tag{2.6}$$

for $s, \bar{s} \in (s(b), s_1)$.

From (2.5), if u is positive in some interval (α, s_1) with $\alpha \geqslant 0$, then

$$u(s) \leqslant b(s_1 - s) \qquad \text{in} \quad (\alpha, s_1). \tag{2.7}$$

If u has a zero in $(s(b), s_1)$, denote $s_0(b) = \inf\{s_0 : u(s, b) > 0 \text{ in } (s_0, s_1)\}$.

By standard results in o.d.e., the functions $(s, b) \to u(s, b)$ and $(s, b) \to u'(s, b)$ are continuously differentiable in the set

$$\{(s, b) \mid b > 0 \text{ and } s \in (\bar{s}(b), s_1)\}.$$

Since $u'(s_0(b), b) > 0$, by the implicit function theorem, the set

$$I \equiv \{b > 0 : s_0(b) > 0\} \tag{2.8}$$

is open and $s_0(\cdot) \in C^1(I)$.

For Problem (2.1), (2.2b), we need to consider the set

$$I_1 \equiv \{b > 0 : u'(\tau, b) = 0 \text{ for some } \tau \in (0, s_1) \text{ and } u(s, b) > 0 \text{ in } (\tau, s_1)\}. \tag{2.9}$$

If $b \in I$, then

$$f(u(\tau, b)) > 0. \tag{2.10}$$

Otherwise, if $f(u(\tau, b)) = 0$ then the initial value problem

$$u''(s) + \rho(s)\, f(u(s)) = 0 \qquad \text{in} \quad (\tau, s_1),$$
$$u(\tau) = u(\tau, b),$$
$$u'(\tau) = 0$$

has solution $u(s) = u(\tau, b)$ for any s in $(0, s_1)$. Therefore, the uniqueness of initial value problem of o.d.e. implies $u(s, b) = u(\tau, b)$ for any s in $(0, s_1)$, a contradiction. By (A-0)′ and (2.10), if $b \in I_1$ and $u'(\tau, b) = 0$, then

$$\begin{aligned} u'(s, b) &< 0 \qquad \text{for} \quad s \in (\tau, s_1), \\ u'(s, b) &> 0 \qquad \text{for} \quad s \in (\bar{s}(b), \tau), \end{aligned} \tag{2.11}$$

where $\bar{s}(b) = s_0(b)$ if $b \in I$ and $\bar{s}(b) = 0$ if $b \notin I$. Therefore, we shall denote this unique τ by $\tau(b)$ which is also the maximum point of $u(\cdot, b)$ in

$(\bar{s}(b), s_1)$. It can be verified that I_1 is an open set and $\tau(\cdot) \in C^0(I_1)$. It is also clear that

$$I \subset I_1. \tag{2.12}$$

Let

$$J = \{s_0(b) : b \in I\}$$

and

$$J_1 = \{\tau(b) : b \in I_1\}.$$

Then (2.1), (2.2a) and (2.1), (2.2b) have positive solutions if we can prove $(0, s_1) \subset J$ and $(0, s_1) \subset J_1$. We need several lemmas to achieve these results.

We begin with the study of $\tau(b)$ when b is sufficiently large.

LEMMA 2.1. *Assume conditions* (A-0)′ *and* (A-2) *are satisfied. Then* $\tau(b)$ *is defined when* b *is sufficiently large and*

$$\lim_{b \to \infty} \tau(b) = s_1. \tag{2.13}$$

Furthermore, we have

$$\lim_{b \to \infty} u(\tau(b), b) = \infty. \tag{2.14}$$

Proof. If the lemma were false there would be a point $\tau_0 \in (0, s_1)$ and a sequence $b_k \to \infty$ with

$$u_k(s) > 0 \qquad \text{and} \qquad u_k'(s) \leqslant 0 \qquad \text{in} \quad (\tau_0, s_1), \tag{2.15}$$

where $u_k(s) \equiv u(s, b_k)$.

Let $\bar{s} = (\tau_0 + s_1)/2$, we claim that

$$\limsup_{k \to \infty} u_k(\bar{s}) = \infty. \tag{2.16}$$

Suppose that this is not the case. Then there exists a constant $M > 0$ such that

$$u_k(\bar{s}) \leqslant M \qquad \text{for all} \quad k_0. \tag{2.17}$$

Now, by (2.5) and (2.17),

$$\begin{aligned} u_k(\bar{s}) &= b_k \left(\frac{s_1 - \tau_0}{2}\right) - \int_{\bar{s}}^{s_1} (t - \bar{s})\, \rho(t)\, f(u_k(t))\, dt \\ &\geqslant b_k \left(\frac{s_1 - \tau_0}{2}\right) - C \end{aligned}$$

for some constant $C \geqslant 0$. But, by (2.17), this is impossible. Therefore (2.16) holds. By choosing a subsequence of b_k if necessary, we may assume

$$\lim_{k \to \infty} u_k(\bar{s}) = \infty. \tag{2.18}$$

By (A-0)′, there exists a subinterval (s_0', s_1') of $(\tau_0, \bar{s})$ such that

$$\rho(s) \geqslant \rho_0 > 0 \qquad \text{in} \quad (s_0', s_1'). \tag{2.19}$$

Denote

$$M_k = \inf \left\{ \frac{f(u_k(s))}{u_k(s)} : s \in (s_0', s_1') \right\};$$

then

$$M_k \geqslant \inf \left\{ \frac{f(u)}{u} : u \geqslant u_k(\bar{s}) \right\}.$$

By (2.18) and (A-2),

$$\lim_{k \to \infty} M_k = \infty. \tag{2.20}$$

By (2.3), u_k satisfies

$$u''(s) + \rho(s)\, h_k(s)\, u(s) = 0 \qquad \text{in} \quad (s_0', s_1'),$$

where

$$h_k(s) \equiv \frac{f(u_k(s))}{u_k(s)} \qquad \text{and} \qquad \rho(s)\, h_k(s) \geqslant \rho_0 M_k \qquad \text{in} \quad (s_0', s_1'). \tag{2.21}$$

Now, let v_k be a solution of

$$v''(s) + \rho_0 M_k v(s) = 0 \qquad \text{in} \quad (s_0', s_1').$$

(2.19) and (2.20) imply that v_k has at least two zeros in (s_0', s_1') when k is sufficiently large. By (2.21) and the Sturm Comparison Theorem, u_k has at least on zero in (s_0', s_1'). But, by (2.15), this impossible. Hence, (2.13) holds.

Next, we will prove (2.14). Suppose that (2.14) does not hold. Then there exist a sequence $b_k \to \infty$ and a constant $M > 0$ such that

$$u_k(\tau_k) \leqslant M \text{ for all } k, \tag{2.22}$$

where

$$u_k(s) = u(s, b_k) \qquad \text{and} \qquad \tau_k = \tau(b_k). \tag{2.23}$$

Denote

$$F(u) = \int_0^u f(s)\, ds \tag{2.24}$$

and define

$$V(s) \equiv V(s, b) \equiv \tfrac{1}{2} u'^2(s) + \rho(s)\, F(u(s)). \tag{2.25}$$

Since

$$\begin{aligned} &V'(s) = \rho'(s)\, F(u(s)), \\ &V(s_1) = V(\tau(b)) + \int_{\tau(b)}^{s_1} \rho'(t)\, F(u(t))\, dt. \end{aligned} \tag{2.26}$$

Therefore, we have

$$\frac{b_k^2}{2} = \rho(\tau_k)\, F(u_k(\tau_k)) + \int_{\tau_k}^{s_1} \rho'(t)\, F(u_k(t))\, dt. \tag{2.27}$$

(2.22) implies that the right hand side of (2.27) is bounded; this is impossible. Therefore (2.14) holds. This completes the proof.

When $\rho(r)$ is decreasing, as in the case $g \equiv 1$ and $n \geqslant 3$ in [1], by the result of Gidas *et al.* [7], $\tau(b) \leqslant (s_0(b) + s_1)/2$, which implies $\lim_{b \to \infty} s_0(b) = s_1$. But we are not assuming g is decreasing, so we need a similar argument as in the previous lemma to prove $s_0(b) \to s_1$ as $b \to \infty$.

LEMMA 2.2. *Assume conditions* (A-0)′ *and* (A-2) *are satisfied. Then* $s_0(b)$ *is well-defined when* b *is sufficiently large and*

$$\lim_{b \to \infty} s_0(b) = s_1. \tag{2.28}$$

Proof. If (2.28) were false there would be a point $s_0 \in (0, s_1)$ and a sequence $b_k \to \infty$ with

$$u_k(s) > 0 \qquad \text{and} \qquad u_k'(s) \geqslant 0 \qquad \text{in} \quad (s_0, \tau_k), \tag{2.29}$$

where u_k and τ_k are as in (2.23).

Denote

$$\bar{s} = \frac{s_0 + s_1}{2}.$$

By (2.13) we may assume $\bar{s} < \tau_k$ for any k. We claim that

$$\limsup_{k \to \infty} u_k(\bar{s}) < \infty. \tag{2.30}$$

Otherwise, by the Sturm Comparison Theorem again, u_k has zeros in $(\bar{s}, \tau_k)$ when k is sufficiently large, which is impossible by (2.29). (Note that $\tau_k \to s_1$ as $k \to \infty$.)

Using (2.6), we have

$$u_k(\tau_k) = u_k(\bar{s}) + u_k'(\bar{s})(\tau_k - \bar{s}) + \int_{\bar{s}}^{\tau_k} (t - \tau_k)\, \rho(t)\, f(u_k(t))\, dt$$

$$\leqslant u_k(\bar{s}) + u_k'(\bar{s})(\tau_k - \bar{s}).$$

Therefore, (2.14) and (2.30) imply

$$\lim_{k \to \infty} u_k'(\bar{s}) = \infty.$$

Since $u'' \leqslant 0$,

$$\lim_{k \to \infty} u_k'(s_0) = \infty. \tag{2.31}$$

By (2.6) and (2.30),

$$u_k(\bar{s}) = u_k(s_0) + u_k'(s_0)(\bar{s} - s_0) + \int_{s_0}^{\bar{s}} (t - s')\, \rho(t)\, f(u_k(t))\, dt$$

$$\geqslant u_k'(s_0)\, \frac{(s_1 - s_0)}{2} - C$$

for some constant C. Hence, (2.31) implies

$$\lim_{k \to \infty} u_k(\bar{s}) = \infty,$$

but this is impossible by (2.30). This completes the proof.

Next, we will study the behaviour of $\tau(b)$ and $s_0(b)$ as $b \to 0^+$. The results depend on the integral $\int_0^{s_1} t\rho(t)\, dt$. We shall investigate the problem in the following two cases:

Case 1. $\int_0^{s_1} t\rho(t)\, dt = \infty$, (2.32)

Case 2. $\int_0^{s_1} t\rho(t)\, dt < \infty$. (2.33)

Case 1 occurs when $n \geqslant 3$ and $g(0) > 0$ or $g(r) \to 0$ slowly as $r \to 0^+$. Part of Case 1 has been studied in [1]. Case 2 occurs when $n = 2$ or $n \geqslant 3$ with $g(r) \to 0$ rapidly as $r \to 0^+$.

LEMMA 2.3. *Assume conditions* (A-0)$'$ *and* (A-3) *are satisfied. Then for b sufficiently small we have*

(α) $s_0(b) \geqslant 0$ *if* (2.32) *holds and* $f(u) > 0$ *in* $(0, u_0)$ *for some* $u_0 > 0$;

(β) $\lim_{s \to 0} u(s, b) > 0$ *if* (2.33) *holds*;

(γ) $b \notin I_1$ *if* ρ *satisfies the condition*

$$\int_0^{s_1} \rho(t)\, dt < \infty. \tag{2.34}$$

Proof. To prove (α), it suffices to show

(α') if b is sufficiently small and $u(\cdot, b) > 0$ in $(0, s_1)$, then $\lim_{s \to 0} u(s, b) = 0$.

Proof of (α'). Since $u > 0$ and concave in $(0, s_1)$, $\lim_{s \to 0^+} u(s, b) = u_1 \geqslant 0$. If $u_1 > 0$, then there exists a constant $c > 0$ such that $f(u(s)) > c$ for $s \in (0, s_1/2)$. Therefore, by (2.5), for any $s \in (0, s_1/2)$ we have

$$u(s) \leqslant b(s_1 - s) - \int_s^{s_1/2} (t - s)\, \rho(t)\, f(u(t))\, dt$$

$$\leqslant b(s_1 - s) - c \int_s^{s_1/2} (t - s)\, \rho(t)\, dt.$$

Hence, for any $s \in (0, s_1/2)$

$$c \int_s^{s_1/2} (t - s)\, \rho(t)\, dt \leqslant b s_1.$$

But this is impossible in view of (2.32). Therefore $u_1 = 0$ and (α') holds.

Proof of (β). Assumption (A-3) implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(u) \leqslant \varepsilon u \qquad \text{and} \qquad F(u) \leqslant \frac{\varepsilon}{2} u^2 \tag{2.35}$$

whenever $0 \leqslant u \leqslant \delta$.

Let $M = \int_0^{s_1} t\rho(t)\, dt$.

Using (2.5), for $b \leqslant \delta/s_1$ and $s \in (0, s_1/2)$, we have

$$u(s) \geqslant b(s_1 - s) - \varepsilon \int_s^{s_1} (t - s)\, \rho(t)\, u(t)\, dt$$

$$\geqslant b(s_1 - s) - \varepsilon M b s_1$$

$$\geqslant b s_1 \left(\frac{1}{2} - \varepsilon M\right).$$

Hence, (β) holds when ε (and so b) is small.

Proof of (γ). Let $M_1 = \int_0^{s_1} \rho(t)\,dt$. Then, for $b \leqslant \delta/s_1$,

$$\begin{aligned} u'(s) &= -b + \int_s^{s_1} \rho(t)\, f(u(t))\, dt \\ &\leqslant -b + \varepsilon M_1 b s_1 \\ &= -b(1 - \varepsilon M_1 s_1). \end{aligned}$$

Therefore, $u'(s) < 0$ in $(0, s_1)$ if ε is small. This completes the proof.

LEMMA 2.4. *Assume conditions* (A-0)′ *and* (A-3) *are satisfied. Then we have*

(α) *if* $(0, b_1) \subset I_1$ *for some* $b_1 > 0$, *then*

$$\lim_{b \to 0} \tau(b) = 0; \tag{2.36}$$

(β) *if* $(0, b') \subset I$ *for some* $b' > 0$, *then*

$$\lim_{b \to 0} s_0(b) = 0. \tag{2.37}$$

Proof. In the case $f(u) > 0$ for $u > 0$, (α) and (β) can be proved by using estimates for eigenvalue problems in o.d.e. as in [1]. Here, we give another proof. By (2.25), (2.26), and (2.35), for $b \leqslant \delta/s_1$, we have

$$\begin{aligned} \rho(\tau(b))\, F(u(\tau(b))) &= \frac{b^2}{2} - \int_{\tau(b)}^{s_1} \rho'(t)\, F(u(t))\, dt \\ &\geqslant \frac{b^2}{2} \left\{ 1 - \varepsilon s_1^2 \int_{\tau(b)}^{s_1} (\rho'(t))^+\, dt \right\}, \end{aligned} \tag{2.38}$$

where $v^+ = \max(v, 0)$.

If (2.36) were false there would be a point $\tau_0 > 0$ and a sequence $b_k \to 0$ with $\tau(b_k) \geqslant \tau_0$. Then

$$\bar{\rho} \equiv \sup_k \rho(\tau(b_k)) < +\infty$$

and

$$\int_{\tau(b_k)}^{s_1} (\rho'(t))^+\, dt \leqslant \int_{\tau_0}^{s_1} (\rho'(t))^+\, dt < \infty.$$

Therefore

$$F(u(\tau(b_k))) \geqslant \frac{b_k^2}{3\bar{\rho}} \tag{2.39}$$

by choosing an ε which is sufficiently small in (2.38).

On the other hand, (2.35) implies

$$F(u(\tau(b_k))) \leqslant F(s_1 b_k) \leqslant \frac{\varepsilon}{2} s_1^2 b_k^2,$$

a contradiction to (2.39). This proves (α).

Since $I \subset I_1$ and $s_0(b) < \tau(b)$, (2.37) follows from (2.36). This completes the proof.

LEMMA 2.5. *Assume conditions* (A-0)′, (A-2), *and* (A-3) *are satisfied. Then we have*

(i) *for any connected component* $(\bar{b}_1, \bar{b}_2)$ *of* I,

$$\lim_{b \to \bar{b}_1^+} s_0(b) = 0; \tag{2.40}$$

(ii) *for any connected component* $(\bar{\bar{b}}_1, \bar{\bar{b}}_2)$ *of* I_1,

$$\lim_{b \to \bar{\bar{b}}_1^+} \tau(b) = 0.$$

Here I *and* I_1 *are given in* (2.8) *and* (2.9).

Proof. Let $(\bar{b}_1, \bar{b}_2)$ be a connected component of I.

In Case 1, i.e., (2.32) holds, either $\bar{b}_1 = 0$ or $\bar{b}_1 > 0$. If $\bar{b}_1 = 0$ the result follows from Lemma 2.4(β). In case 2, i.e., (2.33) holds, by Lemma 2.3(β) it is necessary that $\bar{b}_1 > 0$. We shall prove (2.40) when $\bar{b}_1 > 0$. Suppose that this is not the case; then there exist a point $s_0 > 0$ and a sequence $b_k \to \bar{b}_1$ with $s_0(b_k) \to s_0$. Then $u(s_0, \bar{b}_1) = 0$, i.e., $\bar{b}_1 \in I$, a contradiction to $(\bar{b}_1, \bar{b}_2)$ is a connected component of I. This proves (i).

Next, let $(\bar{\bar{b}}_1, \bar{\bar{b}}_2)$ be a connected component of I_1.

If $\bar{\bar{b}}_1 = 0$ the result follows from Lemma 2.4(α). If $\bar{\bar{b}}_1 > 0$ and $\lim_{b \to \bar{\bar{b}}_1^+} \tau(b) \neq 0$, then there exist a $\tau_0 > 0$ and a sequence $b_k \to \bar{\bar{b}}_1$ with $\tau(b_k) \to \tau_0$. Since $u'(\tau(b_k), b_k) = 0$, $u'(\tau_0, \bar{\bar{b}}_1) = 0$ i.e., $\bar{\bar{b}}_1 \in I_1$, a contradiction. This proves (ii).

The proof is complete.

For problem (2.1), (2.2c), we can apply the result of Bandle *et al.* [1]: Suppose that (A-3) holds and Problem (2.1), (2.2a) has a positive solution for every s_0, s_1 such that $0 < s_0 < s_1 < \infty$, then (2.1), (2.2c) has a positive solution for every s_0, s_1 as above. However, we can use the forward shooting method to obtain the same result. Here, we only sketch the main steps and results.

For a fixed $s_0 > 0$, consider the family of positive solutions $u(s) \equiv u(s, a)$ of the initial value problem

$$u''(s) + \rho(s) f(u(s)) = 0 \qquad \text{for} \quad s > s_0,$$
$$u(s_0) = 0, \qquad u'(s_0) = a,$$

where $a > 0$ is the shooting parameter. Define the set

$$I_0 = \{a > 0\colon u'(\tau, a) = 0 \text{ for some } \tau > s_0 \text{ and } u(s, a) > 0 \text{ in } (s_0, \tau)\}.$$

For $a \in I_0$, denote the unique maximum point $\tau_0(a)$; then I_0 is open and $\tau_0(\cdot) \in C^0(I_0)$. By similar arguments as in Lemmas 2.1, 2.4, and 2.5, we can prove

LEMMA 2.6. *Assume conditions* (A-0)′, (A-2) *and* (A-3) *are satisfied. Then we have*

(α) *for sufficiently large* a, $\tau_0(a)$ *is defined and* $\lim_{a \to \infty} \tau_0(a) = s_0$;

(β) *let* (a_1, a_2) *be a connected component of* I_0; *then we have*

(i) $\lim_{a \to a_1^+} \tau_0(a) = \infty$;

(ii) *if* $a_2 < \infty$, *then* $\lim_{a \to a_2^-} \tau_0(a) = \infty$.

As a consequence of Lemmas 2.1–2.6, we have

THEOREM 2.7. *Assume conditions* (A-0)′, (A-2), *and* (A-3) *are satisfied. Then* (1.1), (1.2a), (1, 1), (1.2b), *and* (1.1), (1.2c) *have at least one positive radial solution for all* R, $\bar{R}$ *such that* $0 < R < \bar{R} < \infty$.

Proof. Lemma 2.2 implies $I \neq \phi$ and there exists $\bar{b}_1 \geqslant 0$ such that $(\bar{b}_1, \infty) \subset 1$. Lemma 2.5(i) implies the set $\{s_0(b)\colon b \in (\bar{b}_1, \infty)\} \supset (0, s_1)$. This proves that (2.1), (2.2a) has a solution for any $s_0 < s_1$.

Lemma 2.1 implies $I_1 \neq \phi$ and there exists $\tilde{b}_1 \geqslant 0$ such that $(\tilde{b}_1, \infty) \subset I$. Lemma 2.5(ii) implies $\{\tau(b)\colon b \in (\tilde{b}_1, \infty)\} \supset (0, s_1)$. Hence, (2.1), (2.2b) has a solution for any $s_0 < s_1$.

Similarly, Lemma 2.6 implies (2.1), (2.2c) has a solution for any $s_0 < s_1$. This completes the proof.

Remark. With a slight modification of the previous arguments, we can obtain existence results for positive radial solutions of the equation

$$\Delta u + f(|x|, u) = 0 \qquad \text{in} \quad R < |x| < \bar{R}$$

with boundary condition (1.2a) (or (1.2b) or (1.2c)) on any annulus when $f(r, u)$ satisfies some appropriate conditions. Here we give assumptions as follows:

(A-0)″ $f(r, u) \in C^1((0, \infty) \times R)$, $f(r, u) > 0$ for $u > 0$ and $f(r, 0) = 0$ for $r > 0$,

(A-2)″ $\lim_{u \to \infty} f(r, u)/u = \infty$ uniformly on any finite subinterval $[R_1, R_2] \subset (0, \infty)$,

(A-3)″ there exists a positive function $\rho(\cdot) \in C^0((0, \infty))$ such that $\lim_{u \to 0^+} f(r, u)/\rho(r)\, u = 0$ uniformly on any finite subinterval $[R_1, R_2] \subset (0, \infty)$.

REFERENCES

1. C. BANDLE, C. V. COFFMAN, AND M. MARCUS, Nonlinear elliptic problems in annular domains, *J. Differential Equations* **69** (1987), 332–345.
2. H. BREZIS AND L. NIRENBERG, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **34** (1983), 437–477.
3. C. V. COFFMAN, On the positive solutions of boundary value problems for a class of nonlinear differential equations, *J. Differential Equations* **3** (1967), 92–111.
4. C. V. COFFMAN, A nonlinear boundary value problem with many positive solutions, *J. Differential Equations* **54** (1984), 429–437.
5. E. N. DANCER, On non-radially symmetric bifurcation, *J. London Math. Soc.* **20** (1979), 287–292.
6. X. GARAIZAR, Existence of positive radial solutions for semilinear elliptic equations in the annulus, *J. Differential Equations* **70** (1987), 69–92.
7. B. GIDAS, W.-M. NI, AND L. NIRENBERG, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
8. S. S. LIN, On non-radially symmetric bifurcation in the annulus, preprint.
9. W.-M. NI, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50** (1983), 289–304.
10. W.-M. NI AND R. NUSSBAUM, Uniqueness and non-uniqueness for positive radial solution of $\Delta u + f(u, r) = 0$, *Comm. Pure Appl. Math.* **38** (1985), 67–108.
11. J. SMOLLER AND A. WASSERMAN, Symmetry-breaking for positive solutions of semilinear elliptic equations, *Arch. Rational Mech. Anal.* **95** (1986) 217–225.