

# STEP-DRAWDOWN DATA ANALYSIS

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## INTRODUCTION

The total drawdown  $sw$  in a pumping well consists of the formation loss and the well loss. The equation of the total drawdown  $sw$  may be written as (Todd 1980)

$$sw = BQ + CQ^p \dots\dots\dots (1)$$

where  $Q$  = the pumping rate;  $B$  = the formation constant;  $p$  = a constant and greater than one; and  $C$  = the well-loss constant. The formation loss  $BQ$  is due to head loss in the aquifer through which the groundwater flows towards the pumping well. The well loss  $CQ^p$  is caused by the frictional loss of flow through the well screen and the flow of the water inside the well bore.

For a steady flow to a well in a confined aquifer, the formation constant can be expressed as

$$B = \frac{\ln\left(\frac{R}{r_w}\right)}{2\pi T} \dots\dots\dots (2)$$

where  $R$  = the radius of influence;  $r_w$  = the radius of the pumping well; and  $T$  = the aquifer transmissivity. The value of  $C$  depends on the well radius, well construction, and the condition of the well. Based on field experience, Walton (1962) suggested a relationship between the loss constant,  $C$ , and the well condition. Since the flow through the well screen and inside the well will usually be turbulent, the well loss is proportional to a  $p$ th power of the pumping rate, as shown in Eq. 1. Jacob (1946) used a value that  $p = 2$ , while Rorabaugh (1953) concluded that values of  $p$  are consistently in the range near 2.5. In a number of field cases, Lennox (1966) pointed out that the value of  $p$  may range as high as 3.5.

## EARLIER WORKS FOR ESTIMATING LOSS CONSTANTS

For  $p = 2$ , the specific drawdown equation can be obtained from Eq. 1 as

$$\frac{sw}{Q} = B + CQ \dots\dots\dots (3)$$

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The values of  $B$  and  $C$  may be determined by plotting  $sw/Q$  versus  $Q$  and fitting a straight line through the data points (Todd 1980). A linear regression approach was suggested by Nahm (1980) to estimate  $B$  and  $C$  using Eq. 3. Rorabaugh (1953) rewrote Eq. 1 as

$$\log \left( \frac{sw}{Q} - B \right) = \log C + (p - 1) \log Q \dots\dots\dots (4)$$

Then a straight line with slope  $(p - 1)$  and intercept  $C$  will be shown in a logarithmic plot of  $(sw/Q - B)$  versus  $Q$ . By assuming different values of  $B$ , a plot of  $sw/Q - B$  versus  $Q$  on logarithmic paper will be repeated until a straight line is reached. Then values of  $p$  and  $C$  can be obtained from the slope and the intercept of the fitted straight line, respectively. This is essentially a graphic trial-and-error procedure. Sheahan (1971) developed a method for direct analysis of step-drawdown data using type curves. Both Rorabaugh's method and Sheahan's method are graphic approaches. Rorabaugh's method is more accurate but more time consuming, whereas Sheahan's method is quicker but slightly less accurate (Labadie and Helweg 1975).

A computer code was developed for the step-drawdown test analysis by Labadie and Helweg (1975) using optimization techniques. The best value of  $p$  is found by a standard Fibonacci search approach, then the values of  $B$  and  $C$  are obtained by solving two simultaneous linear equations.

**PRESENT METHOD**

The purpose of this paper is to present a method which does not require graphic procedures to analyze the step-drawdown data. The method is based on a nonlinear least-squares approach and the finite-difference Newton's method to determine the loss constants of the aquifer formation and the pumping well. This method has been successfully used by Yeh (1987) to determine the values of aquifer transmissivity and storage coefficient.

**Nonlinear Least-Squares**

The nonlinear least-squares approach is commonly used for the curve-fitting problems, where one is attempting to fit the data  $(x_i, y_i)$ ,  $i = 1, 2, \dots, n$  with a relationship  $y = f(x)$  that is nonlinear in  $x$ . That approach is used to find the values of loss constants such that the sum of the squares of differences between the predicted drawdowns and observed drawdowns at the pumping well is minimized. The sum of the square errors is defined as

$$\sum_{i=1}^n e_i^2 = \sum_{i=1}^n (s_{oi} - s_{pi})^2 \dots\dots\dots (5)$$

where  $e_i$  = the prediction error;  $s_{oi}$  = the observed drawdown at the pumping well; and  $s_{pi} = BQ_i + CQ_i^p$  = the predicted drawdown at the pumping well.

If  $\sum_{i=1}^n e_i^2$  is to be a minimum, the first partial derivatives of  $\sum_{i=1}^n e_i^2$  with respect to each loss constant must be zero. Thus

$$\frac{\partial}{\partial B} \sum_{i=1}^n e_i^2 = -2 \sum_{i=1}^n (s_{oi} - s_{pi})Q_i = 0 \dots\dots\dots (6)$$

$$\frac{\partial}{\partial C} \sum_{i=1}^n e_i^2 = -2 \sum_{i=1}^n (so_i - sp_i) Q_i^p = 0 \dots\dots\dots (7)$$

$$\frac{\partial}{\partial p} \sum_{i=1}^n e_i^2 = -2 \sum_{i=1}^n (so_i - sp_i) C Q_i^p \ln Q_i = 0 \dots\dots\dots (8)$$

**Finite-Difference Newton's Method**

Eqs. 6–8 are nonlinear in terms of the unknown constants  $B$ ,  $C$ , and  $p$ . Suppose that variables  $x_1$ ,  $x_2$ , and  $x_3$  represent  $B$ ,  $C$ , and  $p$ , respectively; Eqs. 6–8 may be written as one vector equation:

$$\mathbf{F}(\mathbf{X}) = \mathbf{0} \dots\dots\dots (9)$$

where  $\mathbf{X}$  = the vector with component  $(x_1, x_2, x_3)$ ; and  $\mathbf{F}$  = the vector with component  $(f_1, f_2, f_3)$ .

The Jacobian matrix  $\mathbf{J}$  of partial derivatives of  $f_i$  with respect to  $x_j$ , where  $i, j = 1, 2, 3$ , is defined as

$$\mathbf{J}(\mathbf{X}) = \frac{\partial f_i}{\partial x_j} \dots\dots\dots (10)$$

Suppose that  $\mathbf{X}$  is the exact solution of Eq. 9, in the neighborhood of  $\mathbf{X}$ ;  $\mathbf{F}$  can be expanded in Taylor series as

$$\mathbf{F}(\mathbf{X} + \delta\mathbf{X}) = \mathbf{F}(\mathbf{X}) + \mathbf{J}(\mathbf{X})\delta\mathbf{X} + O(\delta\mathbf{X}^2) \dots\dots\dots (11)$$

With the terms of order  $\delta\mathbf{X}^2$  and higher neglected, a set of linear equations is obtained as follows

$$\delta\mathbf{X} = \mathbf{J}^{-1}(\mathbf{X})\mathbf{F}(\mathbf{X} + \delta\mathbf{X}) \dots\dots\dots (12)$$

where  $\mathbf{J}$  is supposed to be nonsingular (Conte and de Boor 1980). Eq. 12 is a linear system and can be solved for the unknown vector  $\delta\mathbf{X}$  as long as the Jacobian matrix has been evaluated. Then an iterative formula can be formed to solve the system of nonlinear equations of Eqs. 6–8 as

$$\mathbf{X}^{(k)} = \mathbf{X}^{(k-1)} - \delta\mathbf{X} \dots\dots\dots (13)$$

where  $k$  = the number of iteration.

Some specified tolerances commonly applied to terminate the iteration are

$$|\delta\mathbf{X}| < XTOL \dots\dots\dots (14)$$

or

$$|\mathbf{F}[\mathbf{X}^{(k)}]| < FTOL \dots\dots\dots (15)$$

where the values of  $XTOL$  and  $FTOL$  are predefined depending on the required accuracy of the solutions.

Those partial derivative terms in Eqs. 10 and 12 can be approximated by a finite-difference formula such as

$$\frac{\partial f_2}{\partial x_1} = \frac{f_2(x_1 + \Delta x_1, x_2, x_3) - f_2(x_1, x_2, x_3)}{\Delta x_1} \dots\dots\dots (16)$$

where  $\Delta x_1$  = a small increment to the variable  $x_1$ .

TABLE 1. Step-Drawdown Test Data (Todd 1980)

Pumping rate (m <sup>3</sup> /d) (1)	Drawdown (m) (2)
500	1.0
1,000	2.6
2,000	8.9
2,500	14.0
2,750	18.6

### EXAMPLES AND ANALYSIS

Since there are many approaches to estimate the values of  $B$ ,  $C$ , and  $p$ , error criteria such as the standard error of estimate (SEE), the mean error (ME), and the mean absolute error (MAE) are chosen to assess the performance for different methods. The use of those error criteria has been described in detail by Yeh (1987).

A pumping drawdown data shown in Table 1 was taken from Todd (1980) for a step-drawdown pumping test. Based on Jacob's suggestion  $p = 2$ , the estimated drawdown equation is

$$sw = 8.0 \times 10^{-4}Q + 2.0 \times 10^{-6}Q^2 \dots \dots \dots (17)$$

determined by fitting a straight line from a plot of  $sw/Q$  versus  $Q$ .

A reasonably good initial guess may be made for the loss constants based on knowledge of the in situ aquifer formation and well condition. The initial guess values for analyzing Todd's step-drawdown data are made as follows:  $1.0 \times 10^{-4}$  and  $1.0$  for  $B$ ,  $1.0 \times 10^{-4}$  and  $10$  for  $C$ , and  $2.0$  and  $3.0$  for  $p$ . By choosing  $1.0 \times 10^{-4}$  for  $XTOL$  and  $FTOL$ , the proposed method using the iterative formula, Eqs. 12, 13, and 16 for the system of nonlinear equations (Eqs. 6–8) converges to a result of

$$sw = 2.158 \times 10^{-3}Q + 4.811 \times 10^{-11}Q^{3.318} \dots \dots \dots (18)$$

It takes no more than 10 iterations and less than 0.13 central-processing-unit (CPU) sec on a CDC-Cyber 170-720 computer at National Chiao Tung University to obtain this result by the proposed method. Prediction errors by the graphic method using  $p = 2$  and the proposed method are given in Table 2. Obviously, the proposed method yields less prediction errors and better fits Todd's step-drawdown data than the graphic method using  $p = 2$ . The estimated formation losses and well losses by the graphic method using Eq. 17 and the proposed method using Eq. 18 are given in Table 3. For formation losses the predicted values by Eq. 17 are less than half of the values

TABLE 2. Prediction Errors for Todd's Step-Drawdown Data

Prediction error (1)	Graphic method using $p = 2$ (2)	Yeh's method (3)
SEE	1.10	0.42
ME	-0.01	-0.01
MAE	0.56	0.22

**TABLE 3. Predicted Formation Loss and Well Loss for Todd's Step-Drawdown Data**

Observed drawdown (m) (1)	Graphic Method Using $p = 2$		Yeh's Method	
	Formation loss (m) (2)	Well loss (m) (3)	Formation loss (m) (4)	Well loss (m) (5)
1.0	0.4	0.5	1.08	0.04
2.6	0.8	2.0	2.16	0.43
8.9	1.6	8.0	4.32	4.32
14.0	2.0	12.5	5.40	9.05
18.6	2.2	15.1	5.94	12.42

**TABLE 4. Step-Drawdown Test Data**

Case 1 from Sheahan (1971)		Case 2 from Rorabaugh (1953)		Case 3 from Bierschenk (1963)	
$Q$ (gpm) (1)	$sw$ (ft) (2)	$Q$ (cfs) (3)	$sw$ (ft) (4)	$Q$ (gpm) (5)	$sw$ (ft) (6)
400	8.6	1.21	25.4	100	2.5
700	20.0	2.28	50.4	200	6.5
1,200	56.6	3.12	72.2	400	22.2
1,800	141.0	3.39	80.3	500	34.9
—	—	—	—	550	46.5

Note: 1 gpm =  $6.31 \times 10^{-5}$  cms; 1 ft = 0.3048 m; and 1 cfs =  $0.028 \text{ m}^3/\text{s}$ .

**TABLE 5. Comparison of Results**

Method (1)	Loss Constant			Prediction Error		
	$B$ (2)	$C$ (3)	$p$ (4)	SEE (5)	ME (6)	MAE (7)
(a) Case 1						
Sheahan	0.017	$0.82 \times 10^{-7}$	2.8	4.173	1.678	1.678
Labadie-Helweg	0.017	$1.14 \times 10^{-7}$	2.76	0.722	0.301	0.301
Yeh	0.0173	$1.20 \times 10^{-7}$	2.753	0.435	-0.035	0.189
(b) Case 2						
Rorabaugh	20.4	0.44	2.64	0.335	-0.055	0.111
Labadie-Helweg	20.62	0.298	2.90	0.287	-0.006	0.134
Yeh	20.619	0.2979	2.900	0.287	-0.004	0.134
(c) Case 3						
Jacob	0.012	$1.2 \times 10^{-4}$	2.0	2.992	0.020	1.460
Labadie-Helweg	0.028	$1.22 \times 10^{-8}$	3.43	1.074	-0.077	0.568
Yeh	0.0283	$1.00 \times 10^{-8}$	3.460	1.073	-0.055	0.583

by Eq. 18. On the other hand, the well losses would be overestimated by Eq. 17 for the graphic method using  $p = 2$ .

Three sets of step-drawdown data listed in Table 4 were selected from the literature by Labadie and Helweg (1975). A comparison of the results by various solution techniques for analyzing these data is made in Table 5. Although these methods proposed by Labadie and Helweg and the writer give about the same results for the second and the third cases, it is found that prediction error  $MA = 0.301$  is exactly the same as MAE by Labadie and Helweg's method in the first case. It means that the predicted drawdowns at the pumping well are all underestimated by Labadie and Helweg's method in the first case.

## CONCLUSIONS AND RECOMMENDATIONS

When the Todd's step-drawdown data are analyzed, the predicted formation loss values by the graphic method using  $p = 2$  are less than half of the values by the proposed method as shown in Table 3. The ratio for the value of formation constant  $B$  in Eqs. 18 and 17 is 2.7. Consequently, based on Eq. 2, the aquifer transmissivity  $T$  estimated from the graphic method using  $p = 2$  will be 2.7 times the value of  $T$  estimated from the proposed method. For better fit of the observed step-drawdown data and good estimation of loss constants, Eq. 1 should be used.

In Eq. 1 a large value of  $p$  will give a significantly large value for the well loss. Labadie and Helweg concluded the importance for accurate value of  $p$  if the step-drawdown test is to be used to predict drawdown beyond pumping test discharge. Judging from the prediction errors shown in Table 5, it is fair to conclude that the value of  $p$  needs be carried out to three decimal places for better accuracy.

A method using nonlinear least-squares and finite-difference Newton's method is proposed to analyze the step-drawdown data. The values of SEE and ME in Tables 2 and 5 examined, the proposed method produces the least statistical errors for the predicted drawdowns at the pumping well. The proposed method has the advantages of quick convergence and high accuracy with the initial guesses of formation constant and well loss constants over a reasonable range.

## APPENDIX. REFERENCES

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