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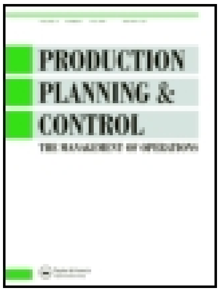

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The TOC-based algorithm for solving product mix problems

TIEN-CHUN HSU and SHU-HSING CHUNG

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Keywords theory of constraints (TOC), product mix problems, dual-simplex method with bounded variables

Abstract. The five steps of the theory of constraints (TOC) emphasize exploiting constraints in order to increase the throughput of a system. The product mix decision is one application of the TOC five steps. However, these steps were considered to be implicit or incomplete, the criticism being that they result in deriving an infeasible solution when a plant has multiple resource constraints. This paper follows the essence of these five steps and presents an explicit algorithm to address the problem. When testing its effectiveness by using a dual-simplex method with bounded variables, this algorithm gives the same result in each iteration.

1. Introduction

The theory of constraints (TOC, Goldratt 1990) is an effective approach to production planning and control. TOC is based on a different concept from other approaches such as JIT and MRP. One key idea of TOC is that the system's constraints (capacity constraint resources, or CCRs) determine the system's throughput and should be the focus of management attention. Constraints are defined as the most limited resources in a system. To increase the throughput of a system, TOC has developed five general steps related to the constraints, as follows:

(1) identify the constraint;
(2) decide how to exploit the constraint;
(3) subordinate everything else to the above decision;
(4) elevate the constraint;
(5) if in the previous steps, a constraint has been broken, go back to step 1. Do not let inertia become the constraint.

The product mix decision problem is one important application of the TOC five steps. It decides the product type and the corresponding quantity to produce a given

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market potential. The objective of this decision is to maximize throughput. Throughput in TOC terminology is defined as the rate at which the system generates money through sales (Goldratt 1990).

The product mix decision problem solved using TOC can be formulated as an LP model (Ronen and Starr 1990, Patterson 1992). The content of the TOC five steps and a comparison between TOC and LP were investigated by Luebbe and Finch (1992). In their description, only steps 1 and 2 of the above TOC five general steps are applied to derive the product mix solution. Other TOC steps focused on taking managerial actions based on the generated solution of steps 1 and 2. There is no iteration supplied for the solution, unless the constraint has been broken when using step 4. As for the comparison between TOC and LP, they concluded that LP was not as specific as TOC. One reason was that TOC could tell what the contribution per constraint time ('$/constraint-time') was for every product. The shadow prices in LP had the same meaning as '$/constraint-time', however, the shadow prices were derived from a few tightened LP constraints. In this respect, TOC outperforms LP. Unfortunately, Plenert (1993) found that TOC could not come up with the optimal solution in a multiple constrained resources situation. He used two examples to demonstrate this.

From the above papers, it appears that even though the TOC approach is more useful for management than the LP approach, the approach seems either incomplete or implicit when solving the product mix decision problem. This paper addresses the problem.

2. TOC approach

The example of Figure 1 (modified from Plenert 1993) attempts to show again how the TOC approach was considered to be incomplete for solving the product mix problem in a multiple-constraint case. Four product types, R, S, T and U are produced in seven different resources, A–G, each of which has 2400 min capacity available. The load on each resource for producing one unit of product R, S, T and U can be collected to generate the load calculation formula as shown in Table 1. The LP model is then formulated accordingly.

$$\text{maximize } 80R + 60S + 50t + 30U \quad (1)$$

subject to

$$20R + 10S + 10T + 5U \leqq 2400 \quad \text{(for resource } A) \quad (2)$$

$$5R + 10S + 5T + 15U \leqq 2400 \quad \text{(for resource } B) \quad (3)$$

$$10R + 5S + 10T + 10U \leqq 2400 \quad \text{(for resource } C) \quad (4)$$

Table 1. Load calculation formula for each resource.

	Load for a unit product				
Resource	R	S	T	U	Load calculation formula
A	20	10	10	5	$20R + 10S + 10T + 5U$
B	5	10	5	15	$5R + 10S + 5T + 15U$
C	10	5	10	10	$10R + 5S + 10T + 10U$
D	0	30	15	5	$0R + 30S + 15T + 5U$
E	5	5	20	5	$5R + 5S + 20T + 5U$
F	5	5	5	15	$5R + 5S + 5T + 15U$
G	20	5	10	0	$20R + 5S + 10T + 0U$

$$0R + 30S + 15T + 5U \leqq 2400 \quad \text{(for resource } D) \quad (5)$$

$$5R + 5S + 20T + 5U \leqq 2400 \quad \text{(for resource } E) \quad (6)$$

$$5R + 5S + 5T + 15U \leqq 2400 \quad \text{(for resource } F) \quad (7)$$

$$20R + 5S + 10T + 0U \leqq 2400 \quad \text{(for resource } G) \quad (8)$$

$$R \leqq 70 \quad (9)$$

$$S \leqq 60 \quad (10)$$

$$T \leqq 50 \quad (11)$$

$$U \leqq 150 \quad (12)$$

The following is the procedure of solving this problem using TOC.

Step 1. *Identify the system's constraint(s)*
The overload of each resource is computed, based on the market potential (Table 2). This reveals that resource B is the capacity constraint resource (CCR) (it is the most overloaded).

Step 2. *Decide how to exploit the system's constraint(s)*
We calculate the $/constraint-minute for products R, S, T and U to be 16, 6, 10 and 2, respectively (Table 3). Thus, according to the values of $/constraint-minute, the order for manufacturing preference is R, T, S and finally U. This results in the product mix solution 70R, 60S, 50T and 80U. The procedure is stopped and this solution is also the final solution using the TOC approach, according to the description in previous papers. More detail of the TOC procedure can be found in Plenert (1993).

However, the above solution is infeasible. If we go further to calculate the load for each resource according to this solution, a capacity overload situation still exists (as shown in Table 4). That is, resources A and D are still overloaded. Comparatively, the solution derived by using a simplex method is, R = 50·67, S = 38·17, T = 50, and U = 101. The throughput is 11 873·3.

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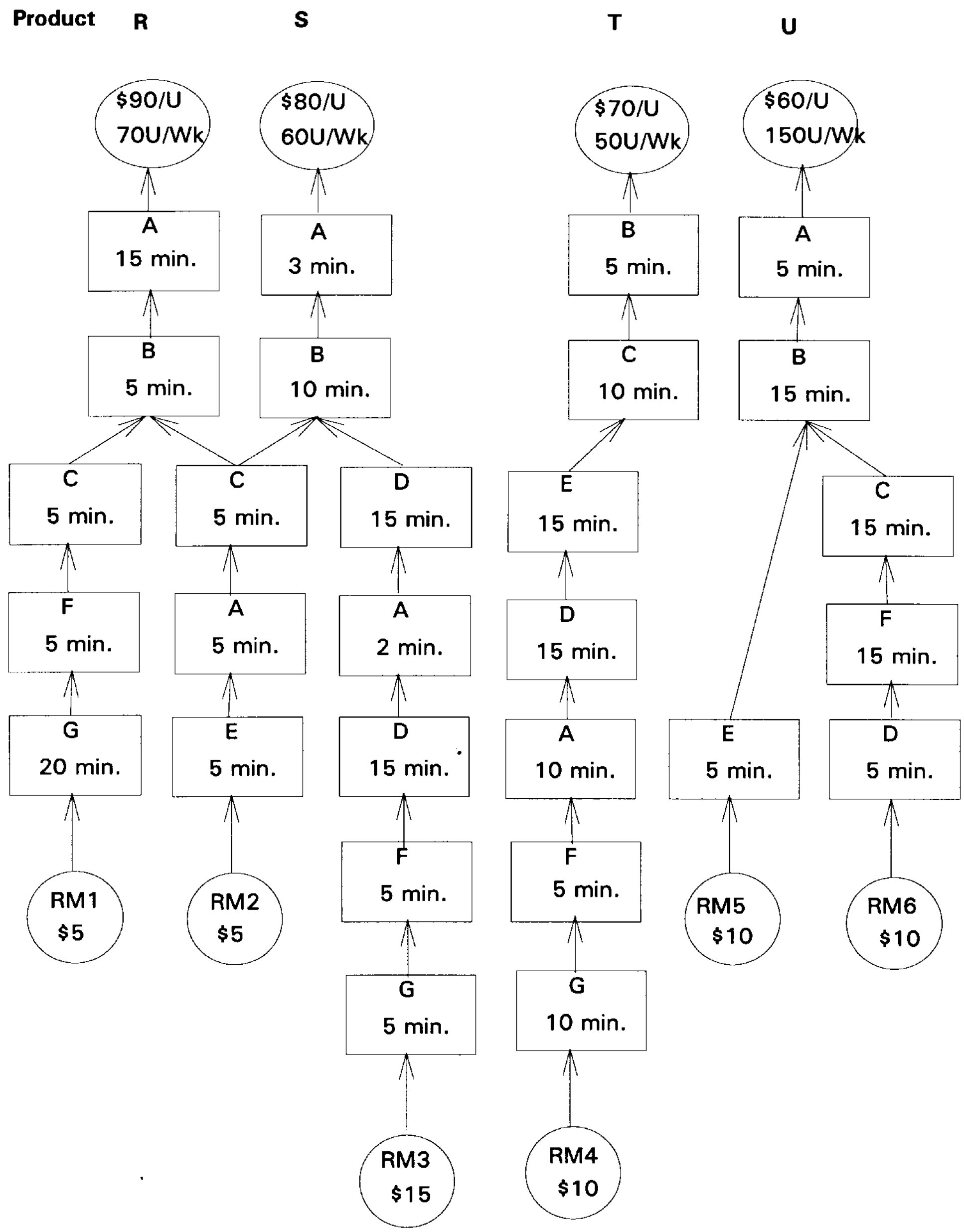

Figure 1. Data for product mix example problem.

Table 2. Capacity overloads for 70*R*, 60*S*, 50*T* and 150*U*.

Resource	Load calculation formula	Load for *R* = 70, *S* = 60 *T* = 50, *U* = 150	Capacity limit (min)	Deficiencies (min)	
				TOC approach	TOC-based algorithm
A	20*R* + 10*S* + 10*T* + 5*U*	3250	2400	- 850	- 850
B	5*R* + 10*S* + 5*T* + 15*U*	3450	2400	- 1050	- 1050
C	10*R* + 5*S* + 10*T* + 10*U*	3000	2400	- 600	- 600
D	0*R* + 30*S* + 15*T* + 5*U*	3300	2400	- 900	- 900
E	5*R* + 5*S* + 20*T* + 5*U*	2400	2400	0	skipped
F	5*R* + 5*S* + 5*T* + 15*U*	3150	2400	- 750	skipped
G	20*R* + 5*S* + 10*T* + 0*U*	2380	2400	20	skipped

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Table 3. Computing the profit contribution per constraint *B* minute.

Parameter	Product *R*	*S*	*T*	*U*
Selling price ($)	90	80	70	60
Raw material cost ($)	10	20	20	30
Contribution ($)	80	60	50	30
Constraint time (resource *B*)	5	10	5	15
$/constraint-minute	16	6	10	2

The remaining sections are organized as follows. The next section describes the basic work for exploiting all CCR(s). This includes the categories of non-CCR(s), the assumption of real value domain, and the way of fully utilizing all CCR(s). Section 4 presents the algorithm. The above example will be solved again in the following section for further illustration. Finally, Section 6 draws conclusions.

3. Basic work for exploiting all CCR(s)

3.1. *Categorizing non-CCR(s)*

A resource is an internal constraint if and only if the output of the resource is less than the market demand (Fogarty *et al.* 1991). A production system may face the single-constraint situation, in which only one resource is overloaded according to the market potential. In this situation, the CCR does not alter no matter what the product mix decision is. On the other hand, if a plant faces a multiple constraints situation, the CCR(s) may change as the product mix decision is changed. When solving the product mix problem, a single-constraint case is much easier to handle than a multiple-constraint case.

In the multiple-constraints case, non-CCRs can be divided into three groups—first-level non-CCR(s), second-level non-CCR(s) and the third-level non-CCR(s). First, second and third are named according to the order of finding them. Figure 2 shows the relationship.

A non-CCR(s) is said to be a first-level non-CCR(s) when this resource is dominated by any other resource without considering what the product mix solution is. 'A is dominated by B' is defined as: for every product type the unit processing time required for B is always larger than that for A under the same level of capacity limit. Since this resource does not have the chance to have more load than any other resource, it is always non-CCR, unless the routeing of a product is changed.

The second-level non-CCR(s) is derived when we find that its load calculated according to the market potential is lower than its capacity limit. This kind of non-CCR is also always fixed. One additional reason to change the resource type is the change of market potential.

When the final product mix solution is derived, a resource which is found to be underloaded is called a third-level non-CCR. In this situation, different product mix solutions (product type and its corresponding quantity) yield different third-level non-CCR(s).

When solving the product mix problem for one planning period, the first and second-level non-CCR(s) are fixed and easily identified. They can be neglected to simplify the problem. Identification of the third-level non-CCR(s) becomes our focus for solving the product mix problem.

3.2. *The real-valued product mix solution*

One assumption needed for this problem is that the solution of product mix decision is real-valued: this assumption we considered to be reasonable. The explanation is as follows: the product mix solution can be viewed as the production plan or target in a planning horizon. The remaining work in this planning horizon can be planned and implemented in the next horizon. Nevertheless, many factors will make the actual result deviate from the product mix plan. Factors include inter-

Table 4. Capacity overloads for 70*R*, 60*S*, 50*T* and 80*U*.

Resource	Load calculation formula	Load for *R* = 70, *S* = 60 *T* = 50, *U* = 80	Capacity limit (min)	Deficiencies (min) TOC approach	TOC-based algorithm
A	20*R* + 10*S* + 10*T* + 5*U*	2900	2400	- 500	- 500
B	5*R* + 10*S* + 5*T* + 15*U*	2400	2400	0	0
C	10*R* + 5*S* + 10*T* + 10*U*	2300	2400	100	100
D	0*R* + 30*S* + 15*T* + 5*U*	2950	2400	- 550	- 550
E	5*R* + 5*S* + 20*T* + 5*U*	2050	2400	350	skipped
F	5*R* + 5*S* + 5*T* + 15*U*	2100	2400	300	skipped
G	20*R* + 5*S* + 10*T* + 0*U*	2380	2400	20	skipped

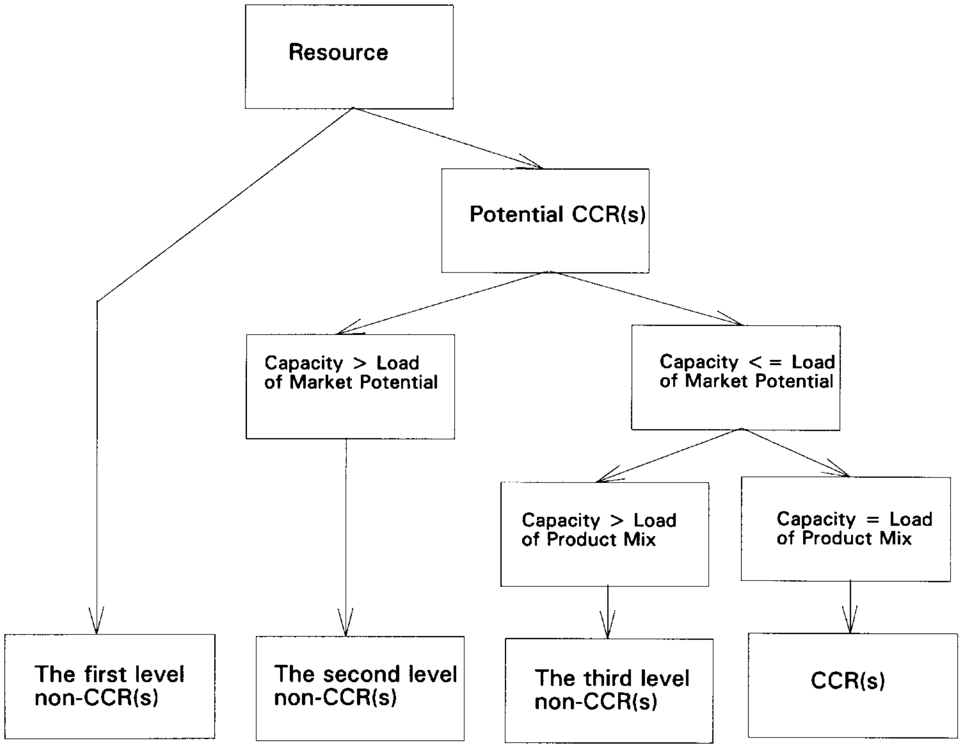

Figure 2. Relationship among non-CCR(s) and CCR(s).

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actions between process plans, inefficient scheduling, absence of materials, breakdown of resources, tool failure, and so forth. The next new plan for a time horizon will be made at the end of every scheduling period by using the rolling schedule concept. It seems unnecessary to limit the product mix solution to be integer.

3.3. *Identifying and fully utilizing all CCR(s)*

The TOC approach is based on the CCR to give products an order of manufacturing preference. It was considered that it derived only the first CCR and made other constraint resources untreated in a multiple-constraints situation. The iterative process included in the TOC five steps was applied only when the constraints were broken (in step 4). However, the iteration process is still necessary for exploiting the constraints (in step 2).

Thus, one thing that should be explicit in the TOC approach is clarification of the iteration process. For each iteration, we have to identify the new CCR. The way to identify the next CCR is to choose the resource with the most capacity overload calculated according to the current product mix solution (conforming to the concept of TOC).

Once a new CCR is identified, we have to decrease the quantity of the product type with the lowest priority to eliminate the overload. The priority order is derived according to the '\$/constraint-time' value for each product (still conforming to the concept of TOC). However, there is a key problem: How do we give the values of the 'S' (contribution) and the 'constraint-time' in each iteration? The analysis is depicted below. For convenience, the following notations are defined.

- n the iteration number,
- CCR_n the nth constraint identified by the nth iteration,
- P_n the quantity-adjustable product type identified in the nth iteration for balancing the load of CCR_n

First, the formulation of the product mix decision problem is given as follows.

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j \tag{13}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \quad i = 1, 2, \ldots, m \tag{14}$$

$$0 \leqq x_j \leqq u_j \quad j = 1, 2, \ldots, n \tag{15}$$

where c_j is given as the contribution of product type j; a_{ij} denotes the time unit required for resource i to produce a type j product; x_j (bounded by the amount u_j) is the

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decision variable which represents the quantity of the product type j; b_i represents the capacity limit of resource i.

Suppose that resource r is the first CCR, and there exists the next CCR—resource s. The following fact is observed to update the new values of the contribution margin '\$' and the 'constraint time' for a product type.

Fact 1. Suppose that in the first iteration, resource r has been the CCR and product u has been the chosen product (it had the least value of '\$/resource-r-time' among all products). Thus, the quantity of product u has been cut down to be equal to its upper capacity bound. In the second iteration, resources s are recognized as the second CCR. For any product denoted product j, if it is chosen to be cut down in quantity, then, cutting down one unit of product j will reduce

(1) the load of resource s by $(a_{sj} - a_{su} * a_{rj}/a_{ru})$;
(2) the contribution of the objective equation by $(c_j - c_u * a_{rj}/a_{ru})$.

(An explanation is given in the Appendix.)

From the fact, the '\$/constraint-time' for any product j is

$$\left(c_j - c_u * a_{rj}/a_{ru}\right) / \left(a_{sj} - a_{su} * a_{rj}/a_{ru}\right)$$

There arises another key problem. Once the quantity P_n is changed in the nth iteration, all previous CCR(s) will lose the balance between load and capacity limit. Thus, to return all CCR equations to be 'equality' equations again in the current (nth) iteration, the quantities of the chosen products (which are derived in previous iterations) must be adjusted. That is, if the load of $CCR_i (i < n)$ becomes over (under) its capacity limit, then it should be decreased (increased) by reducing (increasing) the quantity of product P_i.

On the basis of the preceding description and concept, the explicit TOC algorithm is developed.

4. The TOC-based algorithm for product mix determination

Two lemmas, discussed in Section 3.1 are utilized to reduce the complexity of the given problem.

Lemma 1. Dominance rule. In a product mix decision problem, the capacity constraint for two different types of resources i and k can be expressed as:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leqq b_0 \qquad (16)$$

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \leqq b_0 \qquad (17)$$

Here $x_1, x_2, \ldots, x_n$ are the decision variables representing the quantities of products $1, 2, \ldots, n$. a_{i1} is the time required to produce product 1 using resource i, while a_{k1} is the time required to produce product 1 using resource k. If $a_{i1} \leqq a_{k1}$, $a_{i2} \leqq a_{k2}, \ldots, a_{in} \leqq a_{kn}$, then resource i which is dominated by resource k, is a non-CCR. Here, we classify it as the first-level non-CCR.

Lemma 2. In a product mix decision problem, if we substitute the market potential values of $x_1, x_2, \ldots, x_n$ into the resource constraint expressed as equation (16) or (17), and the inequality still holds, then this resource is also a non-CCR in the planning period. It is classified as a second-level non-CCR in this paper.

The first-level and second-level non-CCR(s) never become CCR(s) for a given market potential. When deciding the product mix, they can be neglected first. Hence, the TOC-based algorithm is as follows.

Preparation step. Delete the first-level and the second-level non-CCR(s) using Lemmas 1 and 2. The current product mix solution is the market potential of each product type. Set $n = 1$, and no previous CCR_i and P_i exist $(i < n)$.

Step 1. Identify the system's constraint.
Calculate the load of each resource based on the current product mix solution. Then compare the load of each resource with its capacity limit. The resource with the highest overload amount is identified as the CCR_n. If no CCR exists, then stop, and the current solution is the final solution.

Step 2. Decide how to exploit the system's constraint.

Step 2a. Treat the CCR_n resource constraint as an equality equation. Delete previous $P_i (i < n)$ terms from this equation by manipulating the row operations between CCR_n and $CCR_i (i < n)$ equation.

Step 2b. Also make the objective equation to be the new one without the previous P_{n-1} term. It also can be done by substituting the value of P_{n-1}, which is derived from the CCR_{n-1} equality equation, into the previous objective equation.

Step 2c. Calculate the '\$/constraint-time' for all product types that have positive coefficients in CCR_n equality equation, according to the new objective equation (step 2b) and the new CCR_n equality equation (step 2a).

Step 2d. Choose the P_n which has the smallest '\$/constraint-time' value for CCR_n. Reduce the quantity of P_n until the load of the CCR_n just equals its capacity limit.

Step 2e. Starting from CCR_{n-1} through CCR_1, adjust the quantity of previous P_i $(i < n)$ to make the load of CCR_i (which is changed after performing step 2d) equal to its capacity limit again. Then, the new product mix is derived. Let $n = n + 1$ and go to step 1.

5. Example and verification

To provide a better understanding of the algorithm, the example discussed in Section 2 is used.

In the preparation step, we determine the first and the second-level non-CCR(s) by using Lemmas 1 and 2, and then ignore them. We observe that the time required on resource F for every product is less than that on resource B. That is, $5 \leqq 5$, $5 \leqq 10$, $5 \leqq 5$ and $15 \leqq 15$. Thus, resource F is dominated by resource B. Also, resource G is dominated by resource A, since $20 \leqq 20$, $5 \leqq 10$, $10 \leqq 10$ and $0 \leq 5$. Resources F and G are first-level non-CCRs. We then substitute the market potential quantities into the left constraints of the LP model. The results show that the inequality condition still holds for resource E (i.e. $5*70 + 5*60 + 20*50 + 5*150 \leqq 2400$). According to Lemma 2, resource E is a second-level non-CCR.

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After deleting the LP constraints of resources F, G and E, the LP formulation becomes:

$$\text{maximize } Z = 80R + 60S + 50T + 30U \qquad (1)$$

subject to

$$20R + 10S + 10T + 5U \leqq 2400 \quad (\text{for resource } A) \qquad (2)$$

$$5R + 10S + 5T + 15U \leqq 2400 \quad (\text{for resource } B) \qquad (3)$$

$$10R + 5S + 10T + 10U \leqq 2400 \quad (\text{for resource } C) \qquad (4)$$

$$0R + 30S + 15T + 5U \leqq 2400 \quad (\text{for resource } D) \qquad (5)$$

$$R \leqq 70 \qquad (9)$$

$$S \leqq 60 \qquad (10)$$

$$T \leqq 50 \qquad (11)$$

$$U \leqq 150 \qquad (12)$$

The current product mix solution now is the market potential—$R = 70$, $S = 60$, $T = 50$ and $U = 150$. The first iteration ($n = 1$) begins.

Iteration 1

Step 1. Identify the system's constraint
The product mix $R = 70$, $S = 60$, $T = 50$ and $U = 150$ is substituted into each resource constraint. This reveals that resources A, B, C and D are overloaded, as shown in Table 2. Among them, resource B is identified as CCR_1, since it has the most overload.

Step 2. Decide how to exploit the system's constraint.

Step 2a. Treat the resource B constraint '$5R + 10S + 5T + 15U \leqq 2400$' as '$5R + 10S + 5T + 15U = 2400$'. Since no previous CCR_i and P_i exist, no manipulation is needed.

Step 2b. Set '$Z = 80R + 60S + 50T + 30U$' as the current objective equation. Since no previous P_i exists, no adjustment is applied.

Step 2c. Calculate the '\$/constraint-time' for all products. The result is the same as with the TOC approach (shown in Table 3).

Step 2d. Identify the product type U as P_1, since it has the smallest \$/constraint-time.
U then is cut from 150 units to 80 units to make the load of CCR_1 (resource B) meet its capacity limit—2400 min.

Step 2e. The current product mix solution is $70R$, $60S$, $50T$ and $80U$. This is also the solution given by the TOC approach. Now, we proceed to the second iteration.

Iteration 2

Step 1. Identify the system's constraint.
When substituting $70R$, $60S$, $50T$ and $80U$ into the resource constraints of the LP model, the resource overloads are shown in Table 4. Resource D is identified as CCR_2.

Step 2. Decide how to exploit the system's constraint.

Step 2a. Treat the resource D constraint $0R + 30S + 15T + 5U \leqq 2400$ as $0R + 30S + 15T + 5U = 2400$. The equation has to exclude the U term (the previous chosen product). By manipulating the row operations between the following equations, we delete the U term from the CCR_2 equation:

$$(\text{CCR}_1 \text{ equation})\ 5R + 10S + 5T + 15U = 2400 \qquad (18)$$

$$(\text{CCR}_2 \text{ equation})\ 0R + 30S + 15T + 5U = 2400 \qquad (19)$$

The updated CCR_2 equality equation is:

$$(-5/3)R + (80/3)S + (40/3)T = 1600 \qquad (20)$$

Step 2b. The original objective equation is $Z = 80R + 60S + 50T + 30U$. The equation also has to exclude the U term. By substituting the value of U ($U = 160 - (1/3)R - (2/3)S - (1/3)T$ which is derived in the first iteration) into the objective equation. The updated objective equation becomes:

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Table 5. The profit contribution per constraint D minute.

Parameter	Product		
	R	S	T
Contribution ($)	70	40	40
Constraint time (resource D)	–	80/3	40/3
$/constraint-minute	–	3/2	3

$$Z = 4800 + 70R + 40S + 40T \quad (21)$$

Step 2c. Calculate the '$/constraint-time' for products R, S and T, based on the above updated objective equation (21) and the equality equation (20). Table 5 shows the results.

Step 2d. Product S is chosen as P_2. S then has to be reduced to 315/8, in order that the CCR_2 equality equation (20) holds.

Step 2e. Since the quantity of product type S is changed, the quantity of product U has to be changed again. That is, the equality equation $5R + 10S + 5T + 15U = 2400$ has to be kept. Thus, U becomes 375/4.

Currently, the product mix solution is $R = 70$, $S = 315/8$, $T = 50$ and $U = 375/4$. Set $n = 3$ and continue the third iteration.

Iteration 3

Step 1. Identify the system's constraint.
Table 6 shows the capacity overload/deficiency situation based on the above product mix solution. Now, resource A is the only one with overloaded capacity in this iteration, and thus is CCR_3.

Step 2. Decide how to exploit the system's contraint.

Step 2a. Treat $20R + 10S + 10T + 5U \leqq 2400$ (for resource A) as CCR_3 equality equation:

$$20R + 10S + 10T + 5U = 2400 \quad (22)$$

Then, this equation, after excluding U and S terms, becomes: $(75/4)R + 5T = 1200$. This is derived by row operation between the following equations:

$$(\text{CCR}_1 \text{ equation})\ 5R + 10S + 5T + 15U = 2400 \quad (18)$$

$$(\text{CCR}_2 \text{ equation})(-5/3R) + (80/3)S + (40/3)T = 1600 \quad (20)$$

$$(\text{CCR}_3 \text{ equation})\ 20R + 10S + 10T + 5U = 2400 \quad (22)$$

Step 2b. CCR_2 LP equation cannot be rewritten as $S = 60 + (1/16)R - (1/2)T$ to represent the value of S. By substituting the S value into the previous objective equation $Z = 4800 + 70R + 40S + 40T$ (21), the current objective equation becomes $Z = 7200 + (145/2)R + 20T$.

Step 2c. Calculate $/constraint-time for the remaining product types. Table 7 shows the results.

Table 7. The profit contribution per constraint A minute.

Parameter	Product	
	R	T
Contribution ($)	145/2	20
Constraint time (resource A)	75/4	5
$/constraint-minute	3·87	4

Step 2d. Since 3·87 is smaller than 4, P_3 is product type R. R is the chosen product which has to be reduced. It becomes 50·67 (152/3) to force CCR_3 be an equality equation—$(75/4)R + 5T = 1200$.

Step 2e. First, the quantity of product type S has to be adjusted to balance the CCR_2 equality equation—$(5/3)R + (80/3)S + (40/3)T = 1600$. Hence, S becomes 229/6 (38·17). Second, adjust the quantity of product U to keep the CCR_1 equality equation—$20R + 10S + 10T + 5U = 2400$. U becomes 101. Thus, the new product mix solution is $R = 50·67$, $S = 38·17$, $T = 50$ and $U = 101$.

Table 6. Capacity overload for $70R$, $(315/8)S$, $50T$ and $(375/4)U$.

Resource	Load calculation formula	Load for $R = 70$, $S = 315/8$, $T = 50$, $U = 375/4$	Capacity limit (min)	Deficiencies (min)
A	$20R + 10S + 10T + 5U$	2762·5	2400	−725/2
B	$5R + 10S + 5T + 15U$	2400	2400	0
C	$10R + 5S + 10T + 10U$	2334·375	2400	525/8
D	$0R + 30S + 15T + 5U$	2400	2400	0

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Table 8. Capacity overload for 50·67R, 38·17S, 50T and 101U.

Resource	Load calculation formula	Load for R = 50·67, S = 38·17, T = 50, U = 101	Capacity limit (min)	Deficiencies (min)
A	$20R + 10S + 10T + 5U$	2400	2400	0
B	$5R + 10S + 5T + 15U$	2400	2400	0
C	$10R + 5S + 10T + 10U$	2207·55	2400	192·45
D	$0R + 30S + 15T + 5U$	2400	2400	0

Table 9. Summary of results.

n	OR*	CCR_n	Objective equation	CCR_n constraint	\$/constraint-time	P_n	Product mix (R, S, T, U)	Z
1	A, B, C, D	B	$Z = 80R + 60S + 50T + 30U$	$5R + 10S + 5T + 15U = 2400$	80/5, 60/10, 50/5, 30/15	U	70,60,50,80	14 100
2	A, D	D	$Z = 4800 + 70R + 40S + 40T$	$(-5/3)R + (80/3)S + (40/3)T = 1600$	–, 120/80, 120/40, –	S	70, 315/8, 50, 375/4	13 275
3	A	A	$Z = 7200 + (145/2)R + 20T$	$(75/4)R + 5T = 1200$	290/75, –20/5, –	R	50·67, 38·17, 50, 101	11 873·3

*OR: Overload resource.

The fourth iteration now begins. However, there is no CCR_4 found in step 1 (see Table 8). This means that the current product mix solution is the final solution. We summarize the execution process of this algorithm in Table 9.

The dual-simplex method with bounded variables is very similar to our algorithm. Therefore, we use this method to test the effectiveness of our algorithm using the same example. The results, shown in Table 10 where $R' = 70 - R$, $S' = 60 - S$, $T' = 50 - T$ and $U' = 150 - U$, are equal to those derived using our algorithm. The algorithm and the dual-simplex method give the same product mix solution in iteration 1, 2 and 3 and the same final throughput.

Observing the contents of Table 9 and Table 10, it is verified that the design idea of the TOC approach is correct. However, the TOC approach is implicit and was considered to act as the first iteration of the presented algorithm.

6. Conclusions

The TOC method has been applied to the product mix problem in two areas. In one area it is used as input for scheduling (Schragenheim 1991), and is called drum–buffer–rope scheduling (DBR). Other areas include master production planning and marketing (Goldratt 1990). However, the TOC approach to this problem is so implicit that it was considered to be an incomplete approach. That is, it derives infeasible solutions when the plant is in a multiple constraint situation.

In this paper, we make the TOC approach explicit. This includes successive iterations for deriving all CCR(s), a method to update the value of '\$/constraint-time', and quantity adjustment for previously chosen products. The presented algorithm derives the same result at each iteration as the dual-simplex method with bounded variables.

Some models, such as the market prices determination model (Eden and Ronen 1990) and the strategic master production scheduling (SMPS) model (Ronen and Rozen 1992), have been developed only for the single-constraint situation. However, because the situation of multiple constraint in a plant may appear in practice, the behaviour of these models may be distorted. To make a more accurate decision, there are two aspects that should be studied in future research:

(1) Modify current models to fit the situation of multiple constraints.
(2) Transfer the multiple-constraint situation to the single-constraint situation, and prevent it returning to the original status. The original model can then be applied.

Table 10. Solution tableau using the dual-simplex method with bounded variables.

I*	BV*	z	R'	S'	T'	U'	s_1	s_2	s_3	s_4	s_5	s_6	s_7	RH*
0	z	1	80	60	50	30	0	0	0	0	0	0	0	16 200
	s_1	0	- 20	- 10	- 10	- 5	1	0	0	0	0	0	0	- 850
	s_2	0	- 5	- 10	- 5	- 15	0	1	0	0	0	0	0	- 1 050
	s_3	0	- 10	- 5	- 10	- 10	0	0	1	0	0	0	0	- 600
	s_4	0	0	- 30	- 15	- 5	0	0	0	1	0	0	0	- 900
	s_5	0	- 5	- 5	- 20	- 5	0	0	0	0	1	0	0	0
	s_6	0	- 5	- 5	- 5	- 15	0	0	0	0	0	1	0	- 750
	s_7	0	- 20	- 5	- 10	0	0	0	0	0	0	0	1	20
1	z	1	70	40	40	0	0	2	0	0	0	0	0	14 100
	s_1	0	- 55/3	- 20/3	- 25/3	0	1	- 1/3	0	0	0	0	0	- 500
	U'	0	1/3	2/3	1/3	1	0	- 1/15	0	0	0	0	0	70
	s_3	0	- 20/3	5/3	- 20/3	0	0	- 2/3	1	0	0	0	0	100
	s_4	0	5/3	- 80/3	- 40/3	0	0	- 1/3	0	1	0	0	0	- 550
	s_5	0	- 10/3	- 5/3	- 55/3	0	0	- 1/3	0	0	1	0	0	350
	s_6	0	0	5	0	0	0	- 1	0	0	0	1	0	300
	s_7	0	- 20	- 5	- 10	0	0	0	0	0	0	0	1	20
2	z	1	145/2	0	20	0	0	3/2	0	3/2	0	0	0	13 275
	s_1	0	- 75/4	0	- 5	0	1	- 1/4	0	- 1/4	0	0	0	- 725/2
	U'	0	3/8	0	0	1	0	- 3/40	0	1/40	0	0	0	225/4
	s_3	0	- 105/16	0	- 15/2	0	0	- 11/16	1	1/16	0	0	0	525/8
	S'	0	- 1/16	1	1/2	0	0	1/80	0	- 3/80	0	0	0	165/8
	s_5	0	- 55/16	0	- 35/2	0	0	- 5/16	0	- 1/16	1	0	0	3 075/8
	s_6	0	1/16	0	- 5/2	0	0	- 17/16	0	3/16	0	1	0	1575/8
	s_7	0	- 325/16	0	- 15/2	0	0	1/16	0	- 3/16	0	0	1	985/8
3	z	1	0	0	2/3	0	58/13	8/15	0	8/15	0	0	0	11 873·3
	R'	0	1	0	4/15	0	- 4/75	1/75	0	1/75	0	0	0	58/3
	U'	0	0	0	- 1/10	1	1/50	- 2/25	0	1/50	0	0	0	49
	s_3	0	0	0	- 11/2	0	- 7/20	- 3/5	1	3/20	0	0	0	385/2
	S'	0	0	1	31/60	0	- 1/300	13/600	0	- 11/300	0	0	0	131/6
	s_5	0	0	0	- 199/12	0	- 11/60	- 4/15	0	- 1/60	1	0	0	2 705/6
	s_6	0	0	0	- 151/60	0	1/300	- 319/300	0	14/75	0	1	0	587/3
	s_7	0	0	0	- 25/12	0	- 13/12	1/3	0	1/12	0	0	1	3 095/6

I*: iteration, BV*: basic variable, RH*: right-hand side.

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Appendix: Explanation of Fact 1

For the first CCR (resource r), based on the current product mix solution, the LP constraint of resource r must be an 'equality' equation. That is,

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{ru}x_u + \ldots + a_{rn}x_n = b_r \qquad \text{(A1)}$$

To make the resource r fully utilized and the above equation hold when x_j reduces by one unit, x_u needs to increase by $a_{rj}a/_{ru}$ unit(s).

For the second CCR (resource s), its current load is denoted by the left-hand side of its constraint $a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{su}x_u + \cdots + a_{sn}x_n$. Since x_j reduces by one unit and x_u dependently increases by a_{rj}/a_{ru} unit(s), this causes the load of resource s be reduced by $a_{sj} - a_{su} * a_{rj}/a_{ru}$.

The substitution is also applied to the objective function $(c_1x_1 + c_2x_2 + \cdots + c_ux_u + \cdots + c_nx_n)$. When x_j reduces by one unit, x_u dependently increases by a_{rj}/a_{ru} unit(s). This causes the objective value (contribution) to be reduced by $c_j - c_u * a_{rj}/a_{ru}$

The above result can also be derived by the following row operations using the Gaussian elimination method. The new objective equation without the x_u term is

$$z' = (c_1 - a_{r1} * c_u/a_{ru}) * x_1 + (c_2 - a_{r2} * c_u/a_{ru}) * x_2 + \cdots + (c_u - a_{ru} * c_u/a_{ru}) * x_u + \cdots + (c_n - a_{rn} * c_u/a_{ru}) * x_n \quad (A2)$$

That is done in the same way as the previous objective equation $- (c_u/a_{ru}) *$equality equation of resource r.

Similarly, by row operation the new resource s LP constraint without the x_u term is its original constraint $- a_{su}/a_{ru} *$resource r equality equation. It becomes

$$(a_{s1} - a_{r1} * a_{su}/a_{ru}) * x_1 + (a_{s2} - a_{r2} * a_{su}/a_{ru}) * x_2 + \cdots + (a_{su} - a_{ru} * a_{su}/a_{ru}) * x_u + \cdots + (a_{sn} - a_{rn} * a_{su}/a_{ru}) * x_n \leqq b_s - b_r * a_{su}/a_{ru} \quad (A3)$$

Thus, '$/constraint-time' for any product type j is $(c_j - c_u * a_{rj}/a_{ru})/(a_{sj} - a_{su} * a_{rj}/a_{ru})$.

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