TABLE IV

ρ_x COMPARED WITH ρ. AMPLITUDE DISTORTION ONLY. ρ IS THE IMAGE CORRELATION PREDICTED BY THE THEORY. ρ_x AND ρ_x' ARE THE MEASURED CORRELATIONS WHEN THE QUANTIZER DESIGN WAS OPTIMIZED FOR EACH APERTURE DISTRIBUTION, AND FOR THE RAYLEIGH DISTRIBUTION, RESPECTIVELY. THE VALUES IN THE BOTTOM ROW ARE OBTAINED BASED ON SIMULATION DATA HAVING A RAYLEIGH DISTRIBUTION

	H_A (bits)								
	0			1			2		
Run	$\rho = K\rho_A$	ρ_x	ρ_x'	$\rho = K\rho_A$	ρ_x	ρ_x'	$\rho = K\rho_A$	ρ_x	ρ_x'
301	0.70	0.70	0.69	0.80	0.81	0.84	0.88	0.90	0.89
303	0.79	0.77	0.77	0.92	0.91	0.92	0.97	0.97	0.96
308	0.88	0.89	0.89	0.95	0.95	0.95	0.96	0.97	0.97
309	0.78	0.79	0.79	0.89	0.89	0.91	0.93	0.95	0.95
310	0.80	0.81	0.81	0.92	0.90	0.91	0.95	0.95	0.94
312	0.85	0.86	0.86	0.93	0.94	0.94	0.96	0.97	0.96
Rayleigh	0.95	0.94	0.94	0.99	0.98	0.98	0.99	0.99	0.99

tizer. The estimate of ρ is not a scene-free quantity in general. However, if we classify scenes into different groups (e.g., residential housing, industrial, farmland, etc.) according to the nature of the scenes, such that an amplitude distribution can be identified with a group, then the scene dependence of ρ can be eliminated. Under this condition, the theory can be considered scene-free.

5) Capability of Real-Time Processing: In many image processing situations, real-time processing is preferred. The optimum design requires knowledge of the amplitude distribution of the input data which, in principle, requires that data be acquired and analyzed. Such a process precludes real-time operation. However, the nearly scene-free performance observed in this study allows the designer to avoid the distribution-estimation steps and to base the design upon assumptions of uniform phase and Rayleigh amplitude distributions. By making these *a priori* assumptions regarding the data, the quantizing operation becomes real time.

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Spatial Pseudorandom Array Processing

CHENG-YUAN LIOU AND RUEY-MING LIOU

***Abstract*—A pseudorandom permuting procedure along with its array signal processing is introduced to resolve multiple coherent signal sources. Conventional adaptive beamforming algorithms fail to operate in such a situation or their performance will degrade. In addition, when applied to an irregularly spaced array or when background noise is colored, most of the existing adaptive algorithms are not capable of working. The important contribution of this work is that, by introducing a new procedure to the conventional processing algorithms, they can overcome the many difficulties which occur. We test this method with computer simulations, and their results are consistent with our prediction.**

I. INTRODUCTION

In radar, sonar, and seismology, one is frequently interested in estimating the directions of arrival and the spectral densities of radiating sources from measurements provided by a passive array of sensors. The problem of simultaneous estimation of the directions of arrival and the spectral densities of the impinging sources can be regarded as a two-dimensional spectral estimation problem. Given spatial and temporal samples of the received signals, the problem is to determine the 2-D spectrum or the energy distribution in both the spatial and temporal domain. The spatial spectrum consists of point masses at different angles of arrival. The temporal spectrum may consist of point masses at different frequencies in the case of narrow-band sources or may be continuous in the case of wide-band sources.

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Since the number of samples (i.e., sensors) in the spatial domain is usually small, classical Fourier analysis yields low spatial resolution. As a result, alternative methods [1] that provide high resolution have been developed. These methods are known to yield high resolution and asymptotically unbiased estimates. Although the details differ in various applications, the main assumptions and processing algorithms are the same. In particular, the key assumption in all the previous cited works is that the interfering signals are not correlated with the desired signal. Once correlated interferences happen to occur, they completely destroy the performance of adaptive array systems. Theoretically, these methods encounter difficulties only when the signals are perfectly correlated. In practice, however, significant difficulties arise even when the signals are highly correlated. The perfect correlation case (or coherence case) serves as a good model for the highly correlated signals.

In spite of its practical importances, the coherence case did not receive considerable attention until recently. The earliest pioneer works on this problem are that of Gabriel [2] and Widrow [3]; they described two similar approaches, both aimed at "decorrelating" the coherent signals. The scheme by Widrow *et al.*, called "spatial dither," is based on mechanical movement of the array elements in some way. However, this technique does not provide a clear general procedure. Gabriel's scheme is based on "Doppler smoothing;" he also mentioned that for the so-called "single snapshot" case, a solution is sometimes possible via synthetic motion of a smaller sampling subaperature along the single snapshot data samples. Evans *et al.* [4] presented an attractive solution to the problem for the case of a uniformly spaced linear array. Their solution is based on a preprocessing scheme referred to as "spatial smoothing" that essentially decorrelates the signals and thus eliminates the difficulties encountered with coherent signals. Shan *et al.* [5] have done a complete analysis for the "subaperature sampling" or "spatial smoothing" preprocessing scheme. They provided an algorithm that can be applied to an on-line adaptive beamformer, and its performance is good even when coherent signals are presented. Su *et al.* [6] modified the above method by a so-called "parallel spatial smoothing" algorithm, using a parallel structure with a spatial averaging effect to overcome coherent jamming. Their conclusion is that a spatially smoothed maximum likelihood estimate of the desired signal can be obtained when the adaptive beamformer converges.

In parameter estimation using eigenstructure techniques, the noise covariance must be known explicitly [7]. A method based on some sort of translation or rotation of the array to overcome this problem has been developed in [7]. It is our intent to propose a solution which does not need mechanical movements to the problem of estimation of direction of arrival for a broad class of unknown noise fields.

In Section II, some investigations of current adaptive arrays will be reviewed, and several difficulties with them will be pointed out, i.e., the signal cancellation phenomenon that arises when coherent interferences present. Current approaches that solve this problem will also be examined. In Section III, a random smoothing method will be introduced plus its rationales. A high-resolution technique for spectral estimation is reviewed for later use. In Section IV, we present computer simulations of our method. And the results of simulations are analyzed and compared to other methods. In Section V, this work is concluded with several remarks.

II. Problem Statements

In this section, we will review the problem briefly. Then we will use a method by Su to point out the difference between his design and ours.

Consider a simple two-element Frost beamformer. Generally, the constraint of the Frost algorithm is to let the array have unit gain and zero phase shift over a certain frequency band in the looking direction, which can be preselected by time-delay steering of the array elements, while it eliminates all the off-looking direction jammers by means of minimizing its own output power. Suppose the desired sinusoidal signal S arrives from the looking direction and the jammer J with the same frequency as the desired signal is arriving from an off-looking direction and thus keeps a fixed phase shift with the desired signal. Denote signal S and jammer J as $S = A \cdot e^{j\omega t}$ and $J = B \cdot e^{j\omega t + j\phi}$ where A and B are the corresponding amplitudes of S and J, ϕ is the constant phase shift between desired signal S and jammer J, and ω is the angular frequency. Suppose an array has two sensor elements which are placed in the plane parallel to the propagation direction of the waves. Element 1 and element 2 receive both the desired signal and jammer as

$$X_1 = A \cdot e^{j\omega t} + B \cdot e^{j\omega t + j\phi} \quad \text{and}$$

$$X_2 = A \cdot e^{j\omega t} + B \cdot e^{j\omega t + j\phi - j\omega\Delta}$$

where $\Delta = d \cdot \sin\theta / c$, d: the interelement spacing, c: propagation speed of the waveform, θ: the jammer's incident angle from broadside. Let the received vector and the weight vector to be denoted as $\boldsymbol{X} = [X_1, X_2]^T$, $\boldsymbol{W} = [W_1, W_2]^T$; then the output of the beamformer is given by $y = \boldsymbol{X}^T \cdot \boldsymbol{W} = X_1 W_1 + X_2 W_2$. Note that all the mathematical representatives we used here are complex and the complex algorithm of linearly constrained adaptive beamformer by Su [8] is used. Typically, the Frost constraint is set to cause the receiving array having a unit gain and zero phase shift in the looking direction. Thus, we have the following expressions:

$$\min_W |y|^2 \qquad \text{subject to } W_1 + W_2 = 1.$$

Substituting X_1 and X_2 by using (1) and (2), and remembering that $W_2 = 1 - W_1$, we have

$$\min_W \left| e^{j\omega t} \right| \left| A + B \cdot e^{j\phi} (W_1 + e^{-j\omega\Delta} - W_1 e^{-j\omega\Delta}) \right|.$$

Solving the above equation, the optimal weights are

$$W_1^{\dagger} = \frac{e^{-j\omega\Delta}}{e^{-j\omega\Delta} - 1} + \frac{Ae^{-j\phi}}{B(e^{-j\omega\Delta} - 1)}$$

$$W_2^{\dagger} = \frac{-1}{e^{-j\omega\Delta} - 1} - \frac{Ae^{-j\phi}}{B(e^{-j\omega\Delta} - 1)}.$$

When the beamformer converges, i.e., the weights reach their steady state, the optimal solution results in a zero output, $\lim_{t \to \infty} y_{\min}(t) = \boldsymbol{W}^{\dagger T} \cdot \boldsymbol{X} \to 0$. This is what we emphasized in the signal cancellation phenomenon. This adaptive beamformer has a null in the incorrect direction, even though the linear constraints in the desired looking direction have been imposed.

To preserve the desired signal while eliminating the coherent jammer, the optimal solution should be

$$W_1^{\text{opt}} = \frac{e^{-j\omega\Delta}}{e^{-j\omega\Delta} - 1}, \qquad W_2^{\text{opt}} = \frac{-1}{e^{-j\omega\Delta} - 1}.$$

Compare $W_1^{\dagger}$ and W_1^{opt}; we find that there are two ways to let $W_1^{\dagger}$ approach W_1^{opt}. The first one is to make $A \ll B$, i.e., the desired signal power is much smaller than that of the coherent jammer. But even in this case, $W_1^{\dagger} \approx W_1^{\text{opt}}$, we still get a zero output in the time domain. The second one is to set A to zero or eliminate the desired signal in the adaptation process, which means the influence over weight settings will be dominated by the jammer. Duvall [9] applied this ideal in his master–slave beamformer. But nevertheless, this method will introduce an undesirable bias term in the system output, and when there exist two or more coherent interferences, their directions of arrival will not be resolved correctly.

The method which is helpful to understanding our method is the "parallel spatial processing" algorithm. It is proposed in different ways by Shan, Evan, and Su; one way is to smooth spatially in the direction orthogonal to the desired looking direction of the beamformer. This kind of array structure can be applied in conjunction with any adaptive algorithm and any array structure. We now review it briefly.

Given a "shapshot" of N sensor outputs sampled at time instant t and sample interval τ $(t = m \cdot \tau)$, $\boldsymbol{X}(m) = [X_1(m), X_2(m),$

$\cdots, X_N(m)]^T$; let it be divided into K subarrays with each subarray having P sensors and adjacent subarrays having $P - 1$ overlapping sensors. Also suppose the desired signal S and the jammer J are impinging on the array; the desired signal is from the looking direction and the jammer is from an off-looking direction. Since the array is an equally spaced linear array, each element receives

$$X_n(m) = A \cdot e^{j\omega m\tau} + B \cdot e^{j\omega m\tau + j\phi + j(n-1)\omega\Delta}$$

$$n = 1, 2, \cdots, N = P + K - 1.$$

The linear constraint of the weights is expressed as $\Sigma_{i=1}^{P} W_i(m) = 1$ for any m. Then the overall system output is obtained as follows:

$$y(m + K - 1) \approx Ae^{j\omega m\tau} + Be^{j\omega m\tau + j\phi} \cdot \alpha(m + K)\left[\frac{1}{K}\sum_{k=1}^{K} e^{j(k-1)\omega\Delta}\right]$$

where

$$\alpha(m + K) = \sum_{p=1}^{P} W_p(m + K - 1)e^{j(p-1)\omega\Delta}.$$

In the above equation, we see that the jammer can be modified by two factors. The first factor is a function of time and subject to the least mean square criterion and the linear constraint. The second term is given as $1/K\, \Sigma_{k=1}^{K} e^{j(k-1)\omega\Delta}$, which is the summation of K uniformly spaced terms on unit circle. As K becomes large, the summation term results in a very small value. So when the adaption process reaches steady state, the coherent jammer effect will be greatly reduced by such a modification. Therefore, if a large number of subarrays are used, i.e., when K is very large, it is easy to get $\lim_{K\to\infty} y(m + K - 1) = A \cdot e^{j\omega m\tau}$ as desired.

To make the spatial summation factor close to zero in the time domain is the key of this method to solve coherence cases. Since the signature (or envelope) part of the target's spectrum is much more important than the flat white noise part, can we reduce the effect of the second term in resolving the directional spectrum by making the jammer's spectrum into a white-noise-like spectrum? The following section will give a solution to this question when $K = 1$.

III. Description of the Random Smoothing Algorithm

We now describe a preprocessing method for the sensor outputs that will preserve all the information of the desired signal from the steered looking direction while stirring all the off-looking direction signals into white noise.

This scheme is based on permuting the measurements of the array sensors. Let

$$X(m) = [X_1(m), X_2(m), \cdots, X_N(m)]^T \tag{1}$$

denote the steered $N \times 1$ vector sequence received from the sensors where N is the number of sensors that the array contains and m represents the ordered time. The N data within each $X(m)$, i.e., $[X_1(m), X_2(m), \cdots, X_N(m)]$, will be *randomly permuted* and *independent* permutations are applied for different m value. Thus, we obtain a new permuted $N \times 1$ vector sequence, namely,

$$Y(m) = [Y_1(m), Y_2(m), \cdots, Y_N(m)]^T. \tag{2}$$

Note that there are $N!$ ways to permute the data within each $X(m)$. Each outcome $Y(m)$ has a probability of occurrence $1/N!$ when $X(m)$ has N different data. Then we estimate the autocorrelation function $R_n(i)$ for each permuted data sequence $\{Y_n(m), m = 1, 2, \cdots, M\}$ using any conventional methods, e.g.,

$$R_n(i) = \frac{1}{M - i} \cdot \sum_{m=1}^{M-i} Y_n(m) \cdot Y_n^*(m + i)$$

$$0 \le i \le L - 1 \le M \tag{3}$$

where M is the total number of "snapshots," L is the number of autocorrelation functions we want, and the asterisk denotes complex conjugate. The average autocorrelation function $R(i)$, which resembles the ensemble mean, can be estimated by averaging $R_n(i)$ over $n = 1, 2, \cdots, N$, i.e.,

$$R(i) = \frac{1}{N} \cdot \sum_{n=1}^{N} R_n(i). \tag{4}$$

This $R(i)$ contains all the spectral information of the looking direction. Now the target's spectrum can be estimated using any high-resolution spectral analysis methods based on $R(i)$. Finally, changing the time-delayed steering vector degree by degree and doing the same procedures, the estimated target spatiotemporal spectrum can be obtained. Fig. 1 illustrates a general block diagram of this random smoothing algorithm.

To carry out our scheme, the high-resolution method for power spectral estimation we will use in simulations is the maximum likelihood method (MLM) developed by Capon [10]. The MLM spectrum is given by

$$P(\omega) = \frac{1}{A^H(\omega) \cdot R^{-1} \cdot A(\omega)} \qquad 0 \le \omega \le 2\pi,$$

$$R = \begin{pmatrix} R(0) & R(1) & \dots R(L-1) \\ R^*(1) & R(0) & \dots R(L-2) \\ \vdots & \vdots & \ddots \vdots \\ R^*(L-1) & R^*(L-2) & \dots R(0) \end{pmatrix} \tag{5}$$

where $A(\omega) = [1, e^{-j\omega\tau}, e^{-j\omega\cdot 2\tau}, \cdots, e^{-j\omega\cdot(L-1)\tau}]^T$ is the steering vector at temporal angular frequency ω and $\{R(i), i = 0, 1, 2, \cdots, L - 1\}$ is the averaged autocorrelation function obtained from (4). The temporal power spectra of the signals coming from each steered direction can be obtained using the above formula. This is different from the conventional beamforming method which focuses on one frequency for all direction and obtains the directional spectrum for that frequency. In our method, we obtain the whole temporal power spectrum of all frequencies for one direction, and then steer to another direction.

The Rationales

The rationale for our algorithm is that the random permuting procedure can whiten the coherent signals coming from off-looking directions. With this kind of permutation, the correlation properties of the signal sources coming from all directions, except the steered one, are totally or partially destroyed. The spectral energy of those off-looking directions is whitened by the random permutation. In other words, the desired signal's spectrum cannot be contaminated by coherence jammers from a different direction. The jammer's signals will add a white-noise spectrum or thermal-noise spectrum to the baseline of the desired signal's spectrum.

According to our scheme, the signals coming from the off-looking direction are stirred by the random permuting procedure and their $\{Y_n(m), m = 1, \cdots, M\}$ should be close to random noise or thermal noise. And their entropy should be increased by this random permutation, too. In order to see this point and the resolution ability in the direction spectrum, we employ the entropy rate function h [11] to show this effect. The definition of h for a Gaussian-distributed random variables of zero mean is given by

$$h = \lim_{L\to\infty} \frac{H}{L} = \lim_{L\to\infty} \frac{1}{2} \cdot \ln\,[\det(R)]^{1/L} \tag{6}$$

where $H = \frac{1}{2} \cdot \ln[\det(R)]$ and L is the order of correlation matrix. Theoretically, when a signal is a pure tone, its R is one eigenvector dominant and h tends to $-\infty$. When a signal is close to thermal noise (or white noise), it has a very large value of h. We test this idea by simulation of a very narrow-band signal $s(t)$ which is impinging on an array from the normal direction of the array. Applying our algorithm shown in Fig. 1 and using (1)-(4), we can estimate the R matrix for each steered direction. Then we substitute this R in (6). The result is plotted in Fig. 2. It is clear that this

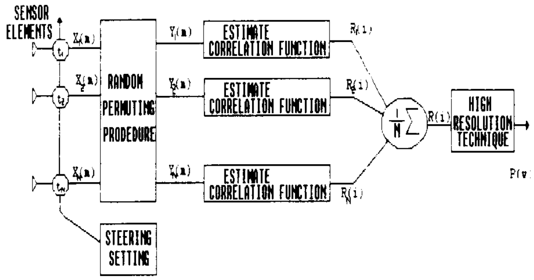

Fig. 1. Block diagram of the random smoothing algorithm.

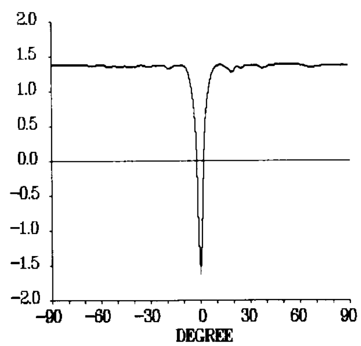

Fig. 2. Entropy analysis of the effect of the random smoothing algorithm on a tonal signal coming from all directions using a uniformly spaced linear array with $\frac{1}{2}$ wavelength interelement spacing.

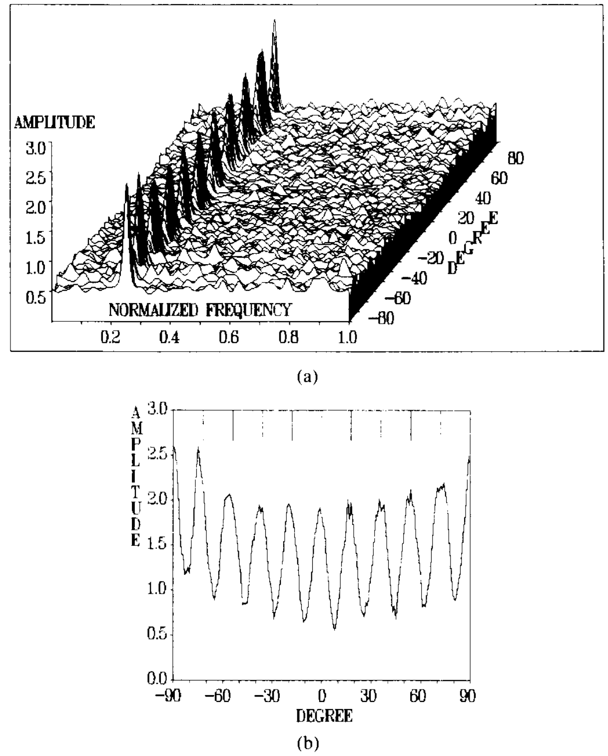

Fig. 3. (a) Spatiotemporal spectrum by the random smoothing method with a uniformly spaced linear array. (b) Profile of Fig. 3(a) at $f = 0.25$ (or $\omega = 0.5\pi$).

algorithm will preserve the information content of the signal only when the array is steered toward the signal. When the direction steered is not consistent with the arrival direction of the signal, the R is close to white noise, that is, its h is increased by the permutation. Instead of deriving the theoretical approach to the resolution bandwidth reached by taking the expectation values of correlations between adjacent directions, we give the following simple intuitive explanation to the solution of resolution bandwidth. *Since the whiteness (or randomness) of unwanted signals, which have no correlation property with $R(i) \approx 0$ for $i \neq 0$ after the random permutation, designates the independence between the desired (steered) signal, which has some correlation properties, and unwanted signals, according to the definition of resolution, the ability of resolution in the direction spectrum can be properly indicated by the width of the notch in Fig. 2.* And the widths of the notches can be predetermined by computer simulations for a given array. In the above analyses, we do not assume that the sensors are equally spaced. When the array is composed of irregularly spaced sensor elements, the autocorrelation functions of all permuted time sequences will still preserve the autocorrelation function properties of the signals from the steered direction except $R(0)$.

IV. Simulation Results

This section provides several computer simulations. The results of simulations support our prediction. The example we considered had ten ($Q = 10$) planar wavefronts at directions of arrival $-90 + 18 \cdot (q - 1)$ degree, $q = 1, 2, \cdots, 10$. All of the ten signal sources are perfectly coherent with the same amplitude, namely, $S_q(t) = \sin 0.5\pi t$. In the first case, the array is linear and uniformly spaced with ten sensor elements. Each element is assumed to be omnidirectional, and the interelement spacing is one-half wavelength. The ambient white noise is assumed to be negligible. Two-hundred snapshots ($M = 200$) for a steered direction are thus obtained. We then apply the algorithm shown in Fig. 1 with the

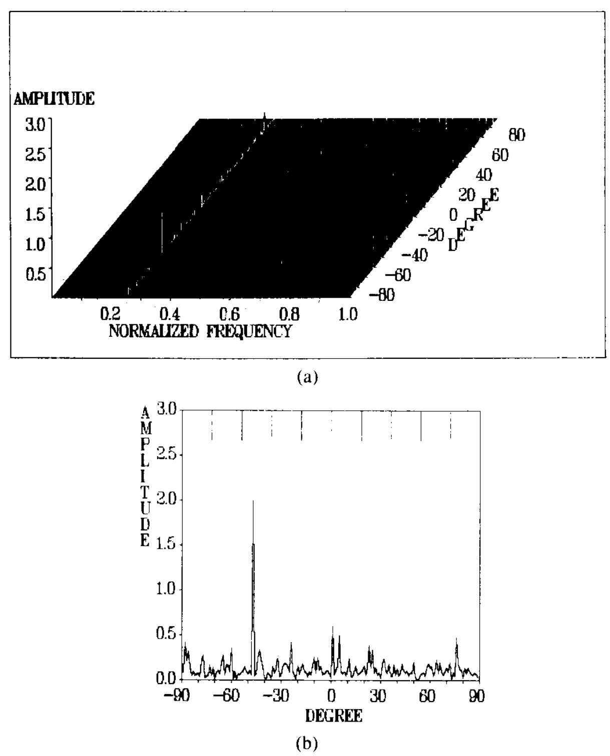

Fig. 4. (a) Spatiotemporal spectrum by the delay-and-summed beamformer with the same array as in Fig. 3. (b) Profile of Fig. 4(a) at $f = 0.25$ (or $\omega = 0.5\pi$).

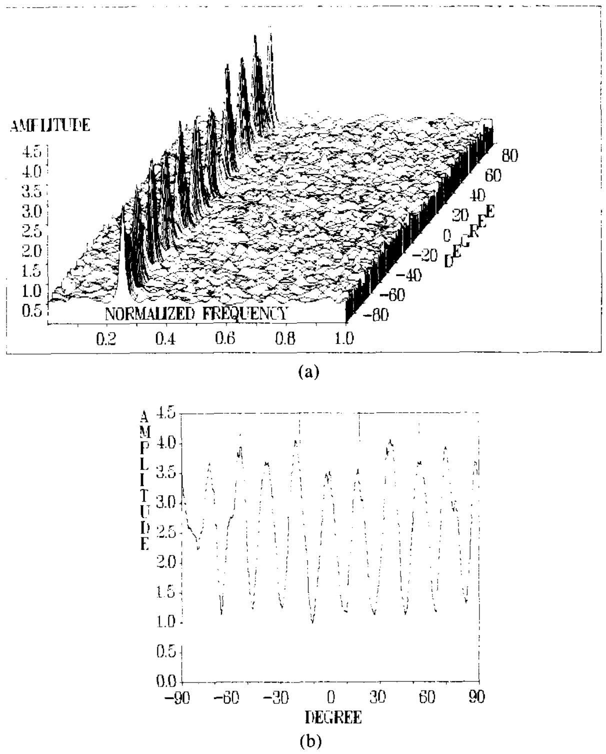

Fig. 5. (a) Spatiotemporal spectrum by the random smoothing method with irregularly spaced linear array. (b) Profile of Fig. 5(a) at $f = 0.25$ (or $\omega = 0.5\pi$).

substitution of (1)–(5) in proper places and obtain the power spectrum for that direction. Then we steer to another direction, from $-90°$ to $90°$, degree by degree. The result is shown in Fig. 3(a). The profile of Fig. 3(a) at frequency 0.25 is shown in Fig. 3(b). The ten peaks corresponding to the ten signal sources are clearly seen in these figures. The resolution bandwidth along the axis of the direction spectrum is close to the width of the notch of Fig. 2. Fig. 4(a) and (b) shows the results of simulation using conventional delay-and-summed algorithm in the same environment. In this figure, the ten coherent signals are totally lost by the coherence.

In the second case, suppose three of the ten sensors, namely, sensor number 2, 5, and 9, are damaged. Thus, the array is linear but irregularly spaced. The results, obtained from 200 "snapshots" using our algorithm are presented in Fig. 5(a). The profile of Fig. 5(a) at frequency 0.25 is presented in Fig. 5(b). We still can identify the ten peaks corresponding to the ten signal sources. There is no other method that can resolve this case efficiently as far as we know.

V. Conclusions

A "random smoothing" algorithm for array signal processing is proposed to overcome signal cancellation effects in correlated jamming environment. Our method is able to handle the particular situation when the array is irregularly spaced, especially when some of the sensors have been damaged and when the background noise is colored. The effectiveness of our method has been verified by many simulations. Due to the estimation of the ensemble average $R(i)$, our algorithm requires additional computations. Our scheme is easily extended to a multidimensional irregularly spaced array for broad-band signals and it has been verified by enormous simulations.

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Instability in the Solution of Banded Toeplitz Systems

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Abstract—**Some algorithms for the solution of banded Toeplitz systems calculate certain elements of the solution first and then the remaining elements by forward or backward substitution. We show that, for symmetric matrices, this method is almost always highly unstable. A numerical example is given to support the argument.**

I. Introduction

We consider the solution of the system of linear equations

$$T_n x = y \tag{1}$$

where T_n is a Toeplitz matrix, i.e., $(T_n)_{ij} = t_{i-j}, j = 0, \cdots, n$. We also assume that T_n is a banded matrix, i.e., there exist p, q with $1 \leq p, q < n$ such that $t_i = 0$, $i > p$ or $i < -q$.

For small values of p and q, standard linear equation solvers take essentially $O(n)$ operations to solve (1). There has, thus, been interest in adapting the general Toeplitz methods to the banded case so that $O(n)$ operations only are required.

Dickinson [1] and Trench [2] are examples of such methods. Fundamental to the efficient implementation of these methods is the observation that we need only calculate some of the elements of x. The remainder can be calculated by forward or backward substitution.

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