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On Non-radially Symmetric Bifurcation in the Annulus

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We discuss the radially symmetric solutions and the non-radially symmetric bifurcation of the semilinear elliptic equation $\Delta u + 2\delta e^u = 0$ in Ω and $u = 0$ on $\partial\Omega$, where $\Omega = \{x \in \mathbb{R}^2 : a^2 < |x| < 1\}$. We prove that, for each $a \in (0, 1)$, there exists a decreasing sequence $\{\delta^*(k, a)\}_{k=0}^{\infty}$ with $\delta^*(k, a) \to 0$ as $k \to \infty$ such that the equation has exactly two radial solutions for $\delta \in (0, \delta^*(0, a))$, exactly one for $\delta = \delta^*(0, a)$, and none for $\delta > \delta^*(0, a)$. The upper branch of radial solutions has a non-radially symmetric bifurcation (symmetry breaking) at each $\delta^*(k, a)$, $k \geqslant 1$. As $a \to 0$, the radial solutions will tend to the radial solutions on the disk and $\delta^*(0, a) \to \delta^* = 1$, the critical number on the disk. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper we study the multiplicity of radially symmetric positive solutions and the non-radially symmetric bifurcation of these solutions of the following (Gel'fand) equation:

$$\Delta u(x) + 2\lambda e^{u(x)} = 0, \qquad x \in \Omega, \tag{1.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where Ω is the annulus

$$\Omega = \Omega_a = \left\{ x = (x_1, x_2) \in R^2 : a^2 < x_1^2 + x_2^2 < \frac{1}{a^2} \right\},$$

$$a \in (0, 1),\ \lambda > 0, \text{ and } \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

If Ω is the unit disk, by the well-known theorem of Gidas, Ni, and

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Nirenberg [8], any positive solution of (1.1), (1.2) must be radially symmetric. Therefore, (1.1), (1.2) are reduced to

$$u''(r)+\frac{1}{r}u'(r)+2\lambda e^{u(r)}=0, \qquad r\in(0,1), \tag{1.3}$$

$$u'(0)=0=u(1), \tag{1.4}$$

where $r^2=x_1^2+x_2^2$. In [7], Gel'fand found that (1.3) is invariant with respect to the group of transformations

$$u(r,\alpha)\equiv\alpha+u_0(re^{\alpha/2}), \tag{1.5}$$

i.e., if $u_0(r)$ is a solution of (1.3), then for any $\alpha\in\mathbb{R}^1$, $u(r,\alpha)$ is also a solution of (1.3). (Note that the boundary condition $u'(0)=0$ is also invariant under (1.5).) By using this property, Gel'fand proved that there exist exactly two solutions for $\lambda\in(0,1)$, one for $\lambda=1$ and none for $\lambda>1$. In the case of annulus, using the same property, we are able to obtain a similar result for (1.1), (1.2) in the class of radially symmetric functions, i.e., there exists $\lambda^*(a)>1$ such that there exist exactly two radially symmetric solutions for $\lambda\in(0,\lambda^*(a))$, one for $\lambda=\lambda^*(a)$ and none for $\lambda>\lambda^*(a)$. These solutions can be written explicitly and $\lambda^*(a)$ is computable.

The existence of positive radial solutions on the n-annulus was also studied by Bandle *et al.* [2] and Garaizar [6].

In a series of papers, Smoller and Wasserman [12, 13] considered the possibility of the non-radially symmetric bifurcation of the equation

$$\Delta u(x)+\lambda f(u(x))=0, \qquad x\in B^n, \tag{1.6}$$

with Dirichlet or Neumann boundary conditions, where B^n is the unit ball in $\mathbb{R}^n$. They showed that, for a certain class of functions $f(u)$, an asymmetric solution bifurcates from a radially symmetric solution. In the case of (1.1), (1.2), taking advantage of knowing the explicit formula of radially symmetric solutions $u_\lambda(r)$ (upper branch of solutions), we are able to understand its linearized problem

$$\Delta w(x)+2\lambda e^{u_\lambda(r)}w(x)=0, \qquad x\in\Omega, \tag{1.7}$$

$$w(x)=0, \qquad x\in\partial\Omega. \tag{1.8}$$

More precisely, we prove that there exists a decreasing sequence $\{\lambda^*(k,a)\}_{k=1}^{\infty}$ with $\lambda^*(k,a)\to 0$ as $k\to\infty$, such that the equation

$$\varphi''(r)+\frac{1}{r}\varphi'(r)+\left(2\lambda e^{u_\lambda(r)}-\frac{k^2}{r^2}\right)\varphi(r)=0, \qquad r\in\left(a,\frac{1}{a}\right), \tag{1.9}$$

$$\varphi(a)=0=\varphi\left(\frac{1}{a}\right), \tag{1.10}$$

has a non-trivial solution $\varphi_k(r)$ if and only if $\lambda = \lambda^*(k, a)$, $k = 1, 2, 3, \ldots$. For these $\lambda^*(k, a)$, the solution set of (1.7), (1.8) is spanned by $\varphi_k(r) \cos k\theta$ and $\varphi_k(r) \sin k\theta$. $\varphi_k(r)$ can also be written explicitly.

To obtain the local non-radially symmetric bifurcation results at $\lambda^*(k, a)$, we have to verify a Crandall–Rabinowitz type transversality condition [4]. This is a crucial and sometimes difficult part in the study of local bifurcation problems. In the case of (1.1), (1.2), the transversality condition is

$$\int_a^{1/a} r\varphi_k^2(r) \frac{\partial}{\partial \lambda} \{\lambda e^{u_\lambda(r)}\}\bigg|_{\lambda = \lambda^*(k)} dr \neq 0. \tag{1.11}$$

It is hard to check (1.11) directly even we know u_λ and φ_k explicitly. However, by taking k (>0) as a parameter and considering the linearized eigenvalue problem

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + \left(2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2}\right)\varphi(r) = -\mu(\lambda, k)\,\varphi(r), \qquad r \in \left(a, \frac{1}{a}\right),$$

$$\varphi(a) = 0 = \varphi\left(\frac{1}{a}\right),$$

where $\mu(\lambda, k)$ is the principal eigenvalue, we obtain

$$\frac{\partial \mu}{\partial \lambda}(\lambda, k) = -\int_a^{1/a} r\varphi^2(r) \frac{\partial}{\partial \lambda}(\lambda e^{u_\lambda(r)})\, dr$$

and

$$\frac{\partial \mu}{\partial k}(\lambda, k) = \int_a^{1/a} \frac{2k}{r} \varphi^2(r)\, dr;$$

here the associated eigenfunction $\varphi(r) = \varphi(r, \lambda, k)$ has been normalized with $\int_a^{1/a} r\varphi^2(r)\, dr = 1$. After a careful study of $\partial\mu/\partial\lambda$ and $\partial\mu/\partial k$, we are able to verify that (1.11) holds. A global bifurcation result can also be obtained by using the well-known theorem of Rabinowitz [11].

This paper is organized as follows: In Section 2, we study the radially symmetric solutions. In Section 3, we study the linearized problem (1.9), (1.10). In Section 4, a Crandall–Rabinowitz type transversality condition (1.11) is verified. Finally, in Section 5, we show that if the outer boundaries of the annuli are fixed and the inner boundaries tend to zero, then the radially symmetric solutions on the annuli will tend to the radially symmetric solution on the disk.

2. Radially Symmetric Solutions

In this section we shall study the existence and multiplicity problems of (1.1), (1.2) in the class of radially symmetric solutions; i.e., we consider the equation

$$u''(r)+\frac{1}{r}u'(r)+2\lambda e^{u(r)}=0, \qquad r\in\left(a, \frac{1}{a}\right), \tag{2.1}$$

$$u(a)=0=u\left(\frac{1}{a}\right), \tag{2.2}$$

where $a\in(0, 1)$.

Since for any interval $[A, B]\subset(0, \infty)$, $[A, B]$ is transformed into $[a, 1/a]$ via the transformation $s=(AB)^{-1/2}r$, where $a=A^{1/2}B^{-1/2}\in(0, 1)$. Therefore, the problem on $[A, B]$ is equivalent to the problem on $[a, 1/a]$ with $a=A^{1/2}B^{-1/2}$. Hence, our study of the problem on $(a, 1/a)$ applies to all cases.

Problem (2.1), (2.2) has been considered by Crandall and Rabinowitz [5]. They showed that there exists $\lambda^*(a)>0$ such that there exist at least two solutions for $\lambda\in(0, \lambda^*(a))$, and exactly one for $\lambda=\lambda^*(a)$ and none for $\lambda>\lambda^*(a)$. In this section, we shall prove that there exist exactly two solutions for $\lambda\in(0, \lambda^*(a))$ and obtain explicit formulas for $\lambda^*(a)$ and these solutions.

By a classical transformation

$$x=\log r \qquad \text{and} \qquad v(x)=u(r)+2\log r, \tag{2.3}$$

(2.1), (2.2) are transformed into

$$v''(x)+2\lambda e^{v(x)}=0, \qquad x\in(-A, A) \tag{2.4}$$

$$v(-A)=-2A \qquad \text{and} \qquad v(A)=2A, \tag{2.5}$$

where

$$A=\log\frac{1}{a}>0. \tag{2.6}$$

Moreover, by

$$Z(x)=e^{v(x)} \qquad \text{and} \qquad Y(x)=v'(x), \tag{2.7}$$

(2.4) can be written as a dynamic system

$$\begin{cases} Z'(x)=ZY, \\ Y'(x)=-2\lambda Z, \end{cases} \tag{2.8}$$

and the corresponding boundary conditions as

$$Z(-A)=e^{-2A} \qquad \text{and} \qquad Z(A)=e^{2A}. \tag{2.9}$$

Since (2.8) is an autonomous system, (2.9) can be replaced by

$$Z(0)=e^{-2A} \qquad \text{and} \qquad Z(2A)=e^{2A}. \tag{2.10}$$

Hence we have the following equivalent problems:

(A) (2.1), (2.2),
(B) (2.4), (2.5),
(C) (2.8), (2.9) with $Z>0$,
(D) (2.8), (2.10) with $Z>0$.

We will work on any one of them.

The following lemma characterizes the solutions of the problem.

LEMMA 2.1. *The solutions of* (2.4), (2.5) *are given by*

$$v_{K,\beta}(x)=\log\frac{\beta^2\lambda^{-1}Km^{\beta/2}e^{\beta x}}{(1+Km^{\beta/2}e^{\beta x})^2}, \tag{2.11}$$

where $K>0$, $\beta>0$ *satisfies*

$$\frac{\beta^2\lambda^{-1}K}{(1+K)^2}=\frac{1}{m}, \tag{2.12}$$

$$\frac{\beta^2\lambda^{-1}Km^\beta}{(1+Km^\beta)^2}=m, \tag{2.13}$$

and

$$m=a^{-2}. \tag{2.14}$$

Proof. We first study the effects of the invariance property of solutions of (2.1). If $u_0(r)$ is a solution of (2.1), then for any $\alpha\in\mathbb{R}^1$,

$$u_\alpha(r)\equiv\alpha+u_0(re^{\alpha/2}) \tag{2.15}$$

is also its solution. According to (2.3), we may set $v_0(x)=u_0(r)+2x$ and $v_\alpha(x)=u_\alpha(r)+2x$. Then $v_\alpha(x)=u_0(re^{\alpha/2})+2(x+\alpha/2)=v_0(x+\alpha/2)$. This implies that $Z_\alpha(x)=Z_0(x+\alpha/2)$ and $Y_\alpha(x)=Y_0(x+\alpha/2)$, i.e., (Z_α, Y_α) lies on the same trajectory with different phase on the $Z-Y$ phase plane. This is also consistent with the following considerations:

It has been known, since Liouville [10], that for any $K>0$,

$$v_K(x) \equiv \log Z_K(x) \equiv \log \frac{\lambda^{-1}Ke^x}{(1+Ke^x)^2} \tag{2.16}$$

is a solution of (2.4). Now, for any $K_1>0$, let $x_1 = \log(K_1/K)$. Then we have

$$Z_{K_1}(x) = \frac{\lambda^{-1}K_1 e^x}{(1+K_1e^x)^2} = \frac{\lambda^{-1}Ke^{x+x_1}}{(1+Ke^{x+x_1})^2} = Z_K(x+x_1);$$

i.e., different K's in Z_K change the phase only.

On the other hand, there is also an invariance property of solutions for (2.4): If $v_0(x)$ is a solution of (2.4), then for any $\alpha \in \mathbb{R}^1$,

$$\bar{v}_\alpha(x) \equiv \alpha + v_0(xe^{\alpha/2}) \tag{2.17}$$

is also a solution of (2.4). Now, for different α's, the corresponding $(\bar{Z}_\alpha, \bar{Y}_\alpha)$'s will lie on different trajectories of the $Z-Y$ phase plane. Hence, by (2.16) and (2.17), we have a one-parameter family of solutions $\bar{Z}_{K,\alpha}$ given by $\bar{Z}_{K,\alpha}(x) \equiv \exp\{\bar{v}_{K,\alpha}(x)\} \equiv \exp\{\alpha + v_K(xe^{\alpha/2})\} = \beta^2\lambda^{-1}Ke^{\beta x}/(1+Ke^{\beta x})^2$, where $\beta = e^{\alpha/2}$. For any $K>0$, $\beta>0$, denote by

$$Z_{K,\beta}(x) = \frac{\beta^2\lambda^{-1}Ke^{\beta x}}{(1+Ke^{\beta x})^2}.$$

It is easy to check that $\{(Z_{K,\beta}(x), Y_{K,\beta}(x)): K>0, \beta>0, x\in\mathbb{R}^1\}$ covers the right half-plane $\mathbb{R}^2_+ = \{(z, y): z>0, y\in\mathbb{R}^1\}$. Hence, any solution $v(x)$ of (2.4) will be of the form $v(x) = \log Z_{K,\beta}(x)$ for some $K>0$ and $\beta>0$. It is clear that boundary condition (2.5) is transformed into (2.12), (2.13). This completes the proof.

It is easier to solve the transcendental equation (2.12), (2.13) than (2.5). To solve (2.12), (2.13), we need some simple facts as follows:

LEMMA 2.2. *If λ, K, β solve* (2.12), (2.13), *then* $0<K<1$.

Proof. If $g(s) = s/(1+s)^2$ for $s>0$, then $g'(s) = (1-s)/(1+s)^3$. Hence $g(s)$ is strictly increasing in $(0, 1)$ and strictly decreasing in $(1, \infty)$. If λ, K, β solve (2.12), (2.13), then $\beta^2\lambda^{-1}g(K) = 1/m < m = \beta^2\lambda^{-1}g(Km^\beta)$. Hence $g(K) < g(Km^\beta)$ and $K < Km^\beta$, which imply that $0<K<1$.

LEMMA 2.3. *If $b \geqslant 4c > 0$, then the solutions of*

$$\frac{bs}{(1+s)^2} = c$$

are

$$s_{\pm}=\frac{b-2c\pm\sqrt{b^2-4bc}}{2c}.$$

Furthermore, if $b>4c>0$, *then*

$$0<s_-<1<s_+.$$

If we set

$$t=\beta^2\lambda^{-1}, \tag{2.18}$$

then (2.12), (2.13) are transformed into more compact forms

$$\frac{tK}{(1+K)^2}=\frac{1}{m}, \tag{2.19}$$

$$\frac{tKm^\beta}{(1+Km^\beta)^2}=m. \tag{2.20}$$

Now, taking $t>0$ as a parameter, we can solve (2.19), (2.20) in terms of t as follows:

LEMMA 2.4. *The solutions of* (2.19), (2.20) *are given by two functions* $\lambda_\pm(\cdot)\colon [4m,\infty)\to(0,\infty)$,

$$\lambda_+(t)=t^{-1}\left(\frac{1}{\log m}\log\frac{P_+(t)}{4m}\right)^2, \tag{2.21}$$

$$\lambda_-(t)=t^{-1}\left(\frac{1}{\log m}\log\frac{P_-(t)}{4m}\right)^2, \tag{2.22}$$

where

$$P_+(t)=P_1(t)\,P_2(t), \qquad P_-(t)=\bar{P}_1(t)\,P_2(t), \tag{2.23}$$

$$P_1(t)=(t-2m)+(t^2-4mt)^{1/2}, \tag{2.24}$$

$$\bar{P}_1(t)=(t-2m)-(t^2-4mt)^{1/2}, \tag{2.25}$$

$$P_2(t)=(mt-2)+(m^2t^2-4mt)^{1/2}. \tag{2.26}$$

Proof. By Lemmas 2.3 and 2.4, we have

$$K=\tfrac{1}{2}\{(mt-2)-(m^2t^2-4mt)^{1/2}\}, \tag{2.27}$$

and

$$Km^{\beta} = \frac{1}{2m}\{(t-2m) \pm (t^2 - 4mt)^{1/2}\} \tag{2.28}$$

for $t \geqslant 4m$. After canceling out K in (2.27), (2.28), we obtain

$$m^{\beta} = \frac{1}{m}\{(t-2m) \pm (t^2-4mt)^{1/2}\}\{(mt-2)-(m^2t^2-4mt)^{1/2}\}^{-1}$$

$$= \frac{1}{4m}\{(t-2m) \pm (t^2-4mt)^{1/2}\}\{(mt-2)+(m^2t^2-4mt)^{1/2}\},$$

and then (2.21), (2.22) follows.

We list some properties of $P_{\pm}$ and $\lambda_{\pm}$ which are useful.

LEMMA 2.5. (i) $P'_{+}(t) > 0$ *and* $P'_{-}(t) < 0$ *for* $t > 4m$,

(ii) $P_{+}(4m) = P_{-}(4m) = 2m\{(4m^2-2) + 4m(m^2-1)^{1/2}\}$,

(iii) $\lim_{t \to \infty} P_{-}(t) = 4m^3$,

(iv) $\lambda_{+}(t) > \lambda_{-}(t)$ *in* $(4m, \infty)$ *and* $\lambda_{*}(m) \equiv \lambda_{+}(4m) = \lambda_{-}(4m)$,

(v) $\lim_{t \to \infty} \lambda_{\pm}(t) = 0$.

Proof. The proof is elementary which we omit.

LEMMA 2.6. $\lambda'_{-}(t) < 0$ *in* $(4m, \infty)$.

Proof. It is easy to check that

$$\lambda'_{-}(t) = \lambda_{-}(t)^{1/2}\, t^{-3/2} Q_{-}(t)(\log m)^{-1},$$

where

$$Q_{-}(t) = 2t\frac{P'_{-}(t)}{P_{-}(t)} - \log\frac{P_{-}(t)}{4m}.$$

By Lemma 2.5 (i), (iii), we have $Q_{-}(t) < 0$ in $(4m, \infty)$, so the result follows.

LEMMA 2.7.

$$\lambda'_{+}(t) = \lambda_{+}^{1/2}(t)\, t^{-3/2} Q_{+}(t)(\log m)^{-1}, \tag{2.29}$$

where

$$Q_{+}(t) = 2t\frac{P'_{+}(t)}{P_{+}(t)} - \log\frac{P_{+}(t)}{4m}. \tag{2.30}$$

Moreover, we have

(i) $Q'_+(t) < 0$ *for* $t > 4m$,

(ii) $\lim_{t \to 4m} Q_+(t) = \infty$ *and* $\lim_{t \to \infty} Q_+(t) = -\infty$.

Proof. The derivation of (2.29) and (2.30) is elementary, so we omit it. It remains to show (i), (ii). Since

$$\frac{P'_+}{P_+} = \frac{(P_1 P_2)'}{P_1 P_2} = \frac{P'_1}{P_1} + \frac{P'_2}{p_2}, \quad Q'_+ = \frac{P'_1}{P_1} + \frac{P'_2}{p_2} + 2t\left\{\left(\frac{P'_1}{P_1}\right)' + \left(\frac{P'_2}{p_2}\right)'\right\},$$

by a straightforward computation, we have

$$\frac{P'_1}{P_1} = (t^2 - 4mt)^{-1/2}, \tag{2.31}$$

$$\frac{P'_2}{p_2} = m(m^2t^2 - 4mt)^{-1/2}, \tag{2.32}$$

$$\left(\frac{P'_1}{P_1}\right)' = -(t - 2m)(t^2 - 4mt)^{-3/2},$$

$$\left(\frac{P'_2}{p_2}\right)' = -m^2(mt - 2)(m^2t^2 - 4mt)^{-3/2}.$$

After simplification, we obtain $Q'_+(t) = -t^2(t^2 - 4mt)^{-3/2} - m^3t^2(m^2t^2 - 4mt)^{-3/2}$, which proves (i). By (2.23), (2.24), (2.26), (2.31), and (2.32), (ii) follows.

An immediate consequence of Lemma 2.7 is

LEMMA 2.8. *Let* $t^* = t^*(m)$ *be the unique solution of*

$$2t\frac{P'_+(t)}{P_+(t)} - \log\frac{P_+(t)}{4m} = 0. \tag{2.33}$$

Then

$$\lambda'_+(t) > 0 \ \text{in}\ (4m, t^*), \qquad \text{and} \qquad \lambda'_+(t) < 0 \ \text{in}\ (t^*, \infty), \tag{2.34}$$

and $\lambda_+(t)$ *attains its maximum*

$$\lambda^* = \lambda^*(m) = (t^*)^{-1}\left\{\frac{1}{\log m}\log\frac{P_+(t^*)}{4m}\right\}^2 \tag{2.35}$$

at $t = t^*$.

Combining the results of Lemmas 2.1, 2.4, 2.6, and 2.8, we have the following theorem.

THEOREM 2.9. (i) *For any* $a \in (0, 1)$, *there exists a number* $\lambda^*(a)$ $(=\lambda^*(m))$, *which is given by* (2.35), *such that* (2.1) *has exactly two solutions for* $\lambda \in (0, \lambda^*(a))$, *exactly one at* $\lambda = \lambda^*(a)$, *and none for* $\lambda > \lambda^*(a)$.

(ii) *The solutions are of the form*

$$u(r) = \log \frac{\beta^2 \lambda^{-1} K m^{\beta/2} r^\beta}{(1 + K m^{\beta/2} r^\beta)^2 \, r^2}, \tag{2.36}$$

where

$$K = K(t) = \frac{2}{P_2(t)}, \tag{2.37}$$

$$\lambda = \lambda_\pm(t) = \beta_\pm^2(t)\, t^{-1}, \tag{2.38}$$

$$\beta = \beta_\pm(t) = \frac{1}{\log m} \log \frac{P_\pm(t)}{4m}, \tag{2.39}$$

$t \geq 4m$, *and* $P_2(t)$, $P_\pm(t)$ *are given in* (2.23)–(2.26).

(iii) *The upper branch of solutions* u_λ, $\lambda \in (0, \lambda^*(m))$, *is given by* (2.36) *with* $\beta = \beta_+(t)$, $\lambda = \lambda_+(t)$, *and* $t \geq t^*(m)$, *where* $t^*(m)$ *is the solution of* (2.33), *and the lower branch of solutions* $\underline{u}_\lambda$ *is also given by* (2.36) *but consists of two pieces: for* $\lambda \in [\lambda_*(m), \lambda^*(m)]$, $\beta = \beta_+(t)$, $\lambda = \lambda_+(t)$, *and* $t \in [4m, t^*(m)]$, *for* $\lambda \in (0, \lambda_*(m)]$, $\beta = \beta_-(t)$, $\lambda = \lambda_-(t)$, *and* $t \geq 4m$, *where* $\lambda_*(m) = \lambda_+(4m) = \lambda_-(4m)$.

3. LINEARIZED EIGENVALUE PROBLEMS

From the last section, we know that for any $m\ (=1/a^2) > 1$, there are two smooth branches of radially symmetric solutions of (2.1), (2.2) in $(0, \lambda^*(m))$, namely, the upper (maximal) branch u_λ and the lower (minimal) branch $\underline{u}_\lambda$. It is well-known that the minimal branch $\underline{u}_\lambda$ can be obtained by a monotone iteration starting from 0 (see, e.g., [9]), and $\underline{u}_\lambda(r) < u_\lambda(r)$ in $(a, 1/a)$ for any $\lambda \in (0, \lambda^*(m))$.

Let $\mu_1(\lambda)$ and $\underline{\mu}_1(\lambda)$ be the principal eigenvalues of linearized eigenvalue problem of (1.1), (1.2) at u_λ and $\underline{u}_\lambda$, respectively; i.e., let $\mu_1(\lambda)$ be the least eigenvalue of

$$\Delta w(x) + 2\lambda e^{u_\lambda(r)} w(x) = -\mu w(x), \qquad x \in \Omega, \tag{3.1}$$

$$w(x) = 0, \qquad x \in \partial\Omega, \tag{3.2}$$

and $\underline{\mu}_1(\lambda)$ be the least eigenvalue of

$$\Delta w(x) + 2\lambda e^{\underline{u}_\lambda(r)} w(x) = -\mu w(x), \qquad x \in \Omega,$$
$$w(x) = 0, \qquad x \in \partial\Omega.$$

It is known that $\underline{\mu}_1(\lambda) > 0$ for any $\lambda \in (0, \lambda^*(m))$ and $\underline{\mu}_1(\lambda^*(m)) = 0$ (see, e.g., [9]). Therefore, the minimal branch $\underline{u}_\lambda$ cannot bifurcate. On the other hand, due to the convexity of e^u, it has been shown by Crandall and Rabinowitz [5] that $\mu_1(\lambda) < 0$ for any $\lambda \in (0, \lambda^*(m))$. Therefore it is possible that there is a bifurcation from the upper branch u_λ. In this section, we shall investigate (3.1), (3.2) in detail.

By the method of separation of variables in polar coordinates, (3.1), (3.2) can be reduced to

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + \left(2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2}\right)\varphi(r) = -\mu_{k,l}(\lambda)\,\varphi(r), \qquad r \in \left(a, \frac{1}{a}\right), \tag{3.3}$$

$$\varphi(a) = 0 = \varphi\left(\frac{1}{a}\right), \tag{3.4}$$

$k = 0, 1, 2, \ldots, l = 1, 2, \ldots$. Let $\varphi_{k,l}$ be the eigenfunction of (3.3), (3.4) associated with the eigenvalue $\mu_{k,l}$; then the eigenfunction $w_{k,l}$ of (3.1), (3.2) is

$$w_{k,l}(r, \theta) = \varphi_{k,l}(r)(a_k \cos k\theta + b_k \sin k\theta),$$

where a_k and b_k are constants.

By several changes of variables, we can bring (3.3), (3.4) into a more desirable form.

First, if we set

$$x = \log r,\ \psi(x) = \varphi(r), \qquad \text{and} \qquad \bar{v}(x) = u(r), \tag{3.5}$$

then (3.3), (3.4) are transformed into

$$\psi''(x) + (2\lambda e^{\bar{v}(x)} - k^2)\,\psi(x) = -\mu_{k,l}(\lambda)\, e^{2x}\psi(x), \qquad x \in (-A, A), \tag{3.6}$$

$$\psi(-A) = 0 = \psi(A), \tag{3.7}$$

where $A = \log(1/a)$.

By (2.36), (3.6) can be written as

$$\psi''(x) + \left\{\frac{2\beta^2 K_1 e^{\beta x}}{(1 + K_1 e^{\beta x})^2} - k^2\right\}\psi(x) = -\mu_{k,l}(x)\, e^{2x}\psi^{(x)}, \qquad x \in (-A, A), \tag{3.8}$$

where $K_1 = Km^{\beta/2}$ and K, $\beta = \beta_+$ are given in (2.37), (2.39).

Next, with

$$y = \beta x \qquad \text{and} \qquad \Psi(y) = \psi(x), \tag{3.9}$$

(3.7), (3.8) can be transformed further into

$$\Psi''(y) + \left\{ \frac{2K_1 e^y}{(1+K_1 e^y)^2} - \frac{k^2}{\beta^2} \right\} \Psi(y)$$
$$= -\mu_{k,l}(\lambda)\, e^{(2/\beta) y} \Psi(y), \qquad y \in (-\beta A, \beta A), \tag{3.10}$$

$$\Psi(-\beta A) = 0 = \Psi(\beta A). \tag{3.11}$$

Since the bifurcation only occurs at $\lambda \in (0, \lambda^*(m))$ which satisfies $\mu_{k,l}(\lambda) = 0$, it is necessary to identify these λ. We begin with the case $k = 0$.

Suppose that $\lambda \in (0, \lambda^*(m))$ satisfies $\mu_{0,l}(\lambda) = 0$. Setting

$$X = K_1 e^y \qquad \text{and} \qquad \Phi(X) = \Psi(y), \tag{3.12}$$

we can transform (3.10), (3.11) into

$$\Phi''(X) + \frac{1}{X} \Phi'(X) + \frac{2}{X(1+X)^2} \Phi(X) = 0, \qquad X \in (K_1 e^{-\beta A}, K_1 e^{\beta A}), \tag{3.13}$$

$$\Phi(K_1 e^{-\beta A}) = 0 = \Phi(K_1 e^{\beta A}). \tag{3.14}$$

Now, the linear equation (3.13) can be solved as follows:

LEMMA 3.1. *The general solution of* (3.13) *is* $\Phi(X) = b\Phi_1(X) + d\Phi_2(X)$, *where* $\Phi_1(X) = (X-1)/(X+1)$, $\Phi_2(X) = y\Phi_1(X) - 4/(X+1) = (y(X-1) - 4)/(X+1)$, *and* b, d *are constants.*

Proof. It is easy to check that Φ_1 is a solution of (3.13), and then Φ_2 is obtained from Φ_1 by the method of variation of parameters.

Knowing the general solution of (3.13), we can prove the following theorem.

THEOREM 3.2. (i) $\mu_{0,l}(\lambda) = 0$ *if and only if* $\lambda = \lambda^*(m)$ *and* $l = 1$.

(ii) *For* $l \geq 2$, $\mu_{k,l}(\lambda) > 0$ *for any* $\lambda \in (0, \lambda^*(m)]$ *and* $k = 0, 1, 2, \ldots$.

Proof. By Lemma 3.1, (3.14) is equivalent to

$$b(K_1 e^{-\beta A} - 1) + d\{-\beta A(K_1 e^{-\beta A} - 1) - 4\} = 0, \tag{3.15}$$

$$b(K_1 e^{\beta A} - 1) + d\{\beta A(K_1 e^{\beta A} - 1) - 4\} = 0. \tag{3.16}$$

System (3.15), (3.16) has non-trivial solutions if and only if it has zero determinant, i.e.,

$$\begin{aligned}0 &= (K_1 e^{-\beta A} - 1)\{\beta A(K_1 e^{\beta A} - 1) - 4\} \\ &\quad + (K_1^{\beta A} - 1)\{\beta A(K_1 e^{-\beta A} - 1) + 4\} \\ &= 2\beta A(K_1 e^{-\beta A} - 1)(K_1 e^{\beta A} - 1) + 4(K_1 e^{\beta A} - K_1 e^{-\beta A}).\end{aligned}$$

Using $2\beta A = \beta \log m$ and $e^{\beta A} = m^{\beta/2}$, we can write the last equation as

$$(\beta \log m)(K_1 m^{-\beta/2} - 1)(K_1 m^{m/2} - 1) + 4K_1(m^{\beta/2} - m^{-\beta/2}) = 0.$$

By (2.39) and $K_1 = Km^{\beta/2}$, this becomes

$$\log \frac{P_+(t)}{4m} = \frac{4K(m^\beta - 1)}{(Km^\beta - 1)(1-K)}.$$

By (2.37), $m^\beta = P_1 P_2/4m$, and a lengthy but straightforward computation, we can show that

$$\frac{2K(m^\beta - 1)}{(Km^\beta - 1)(1-k)} = t\,\frac{P'_+(t)}{P_+(t)},$$

(we omit the detail here). Hence, $\mu_{0,l}(\lambda) = 0$ if and only if t satisfies (2.33), i.e., $t = t^*(m)$ and then $\lambda = \lambda^*(m)$. In this case, it is clear that $l = 1$. This proves (i).

To prove (ii), we note that

$$\mu_{0,2}(\lambda) > 0 \qquad \text{for any } \lambda \in (0, \lambda^*(m)]. \tag{3.17}$$

Since $\mu_{0,2}(\lambda^*(m)) > \mu_{0,1}(\lambda^*(m)) = 0$, (3.17) follows by (i) and the continuous dependence of $\mu_{0,2}(\lambda)$ with respect to λ.

Now, by the mini-max principle of eigenvalues (see, e.g., [3]), $\mu_{k,l}(\lambda)$ can be characterized by

$$\mu_{k,l}(\lambda) = \sup_{\substack{\varphi_1,\ldots,\varphi_{l-1} \\ \varphi_i \in X_0}} \inf_{\substack{(\varphi,\varphi_i)=0 \\ i=1,\ldots,l-1}} \frac{D_{k,\lambda}(\varphi)}{\|\varphi\|_2^2},$$

where

$$(\varphi, \psi) = \int_a^{1/a} r\varphi(r)\,\psi(r)\,dr, \qquad \|\varphi\|_2^2 = \int_a^{1/a} r\varphi^2(r)\,dr,$$

$$D_{k,\lambda}(\varphi) = \int_a^{1/a} r\left\{\varphi'^2(r) - 2\lambda e^{u_\lambda(r)}\varphi^2(r) + \frac{k^2}{r^2}\varphi^2(r)\right\} dr,$$

and

$$X_0 = C_0^1\left(\left[a, \frac{1}{a}\right]\right),$$

the set of continuously differentiable functions on $[a, 1/a]$ which vanish at $r=a$ and $r=1/a$. Since, for $k_1>k_2$ and any $\varphi \in X_0$, $D_{k_1,\lambda}(\varphi)>D_{k_2,\lambda}(\varphi)$ holds, we have $\mu_{k_1,l}(\lambda) \geqslant \mu_{k_2,l}(\lambda)$ for $k_1>k_2$, $\lambda \in (0, \lambda^*(m)]$, and $l=1, 2, \ldots$. This proves (ii).

An immediate consequence of Theorem 3.2 is that $\mu_{k,l}(\lambda)=0$ implies $l=1$. Therefore, we shall take k as a parameter which varies in $(0, \infty)$ and search for $\lambda \in (0, \lambda^*(m))$ which satisfies $\mu_{k,1}(\lambda)=0$.

Set

$$c=\frac{-k}{\beta(t)}, \qquad R(y)=\frac{2K_1 e^y}{(1+K_1 e^y)^2}, \tag{3.18}$$

and

$$\tilde{\Psi}(y)=e^{-cy}\Psi(y),$$

and let $\mu_{k,1}(\lambda)=0$. Then (3.10), (3.11) can be transformed into

$$\tilde{\Psi}''(y)+2c\tilde{\Psi}'(y)+R(y)\,\tilde{\Psi}(y)=0, \tag{3.19}$$

$$\tilde{\Psi}(-\beta A)=0=\tilde{\Psi}(\beta A). \tag{3.20}$$

Set

$$X=K_1 e^y \qquad \text{and} \qquad \Phi(X)=\tilde{\Psi}(y). \tag{3.21}$$

Then (3.19), (3.20) are transformed into

$$\Phi''(X)+\frac{1+2c}{X}\Phi'(X)+\frac{1}{X(1+X)^2}\Phi(X)=0, \qquad X\in(L, R) \tag{3.22}$$

$$\Phi(L)=0=\Phi(R), \tag{3.23}$$

where

$$L=K_1 e^{-\beta A}=K \qquad \text{and} \qquad R=K_1 e^{\beta A}=Km^{\beta}. \tag{3.24}$$

Denote by

$$\Phi_1(X)=\frac{X-X_c}{X+1}, \tag{3.25}$$

where

$$X_c=\frac{1+2c}{1-2c}. \tag{3.26}$$

Then it can be checked that Φ_1 is a solution of (3.22). By the method of variation of parameters, i.e., by assuming

$$\Phi_2(X) = C(X)\,\Phi_1(X) \tag{3.27}$$

is a solution of (3.22), we obtain

$$C'(X) = g(X)(X - X_c)^{-2}, \tag{3.28}$$

where

$$g(X) = (1 + X)\,X^{-1-2c}. \tag{3.29}$$

Therefore, the general solution of (3.22) is given by

$$\Phi(X) = b\Phi_1(X) + d\Phi_2(X)$$

and the boundary conditions (3.23) are

$$b\Phi_1(L) + d\Phi_2(L) = 0, \tag{3.30}$$

$$b\Phi_1(R) + d\Phi_2(R) = 0. \tag{3.31}$$

We first prove the following lemma.

LEMMA 3.3. *If* (3.30), (3.31) *has a non-trivial solution, then*

$$c \in \left(\frac{-1}{2}, 0\right) \qquad \textit{and} \qquad X_c \in (L, R). \tag{3.32}$$

Proof. If $c \leqslant -1/2$, then $X_c \leqslant 0$. Hence $(X - X_c)^2 > X_c^2 > 0$ for $X > 0$. Therefore $C(X)$ is smooth and $C'(X) > 0$ for $X > 0$. Since

$$C(R) > C(L) \qquad \text{and} \qquad \Phi_1(R) > \Phi_1(L) > 0,$$

we have

$$\Phi_1(L)\,\Phi_2(R) - \Phi_1(R)\,\Phi_2(L) = \Phi_1(L)\,\Phi_1(R)(C(R) - C(L)) > 0.$$

This implies that (3.30), (3.31) has no non-trivial solution if $c \leqslant -1/2$.

Next, we shall prove $X_c \in (L, R)$ if (3.30), (3.31) has a non-trivial solution.

Since

$$g'(X) = (1 - 2c)(1 + X)^{-2-2c}\,(X - X_c), \tag{3.33}$$

it is clear that we can define

$$C(X) = -g(X_c)(X - X_c)^{-1} + \int_1^X (z - X_c)^{-2}\,(g(z) - g(X_c))\,dz$$

for $X>0$. Hence, we may write $\Phi_2(X)=\tilde{C}(X)\,\Phi_1(X)+\tilde{\Phi}_2(X)$, where

$$\tilde{C}(X)=\int_1^X (z-X_c)^{-2}\,(g(z)-g(X_c))\,dz, \qquad \text{and} \qquad \tilde{\Phi}_2(X)=\frac{g(X_c)}{X+1}.$$

Therefore, we obtain

$$\begin{aligned}
&\Phi_1(L)\,\Phi_2(R)-\Phi_1(R)\,\Phi_1(L)\\
&\quad=\Phi_1(L)\,\Phi_1(R)(\tilde{C}(R)-\tilde{C}(L))+\Phi_1(L)\,\tilde{\Phi}_2(R)-\Phi_1(R)\,\tilde{\Phi}_2(L)\\
&\quad=\frac{1}{(L+1)(R+1)}\Big\{(R-L)\,g(X_c)+(R-X_c)(L-X_c)\\
&\qquad\times\int_L^R (z-X_c)^{-2}\,(g(z)-g(X_c))\,dz\Big\}.
\end{aligned}$$

For $c\in(-1/2,0)$, (3.33) implies

$$g(X)\geqslant g(X_c) \text{ if } X\geqslant X_c \qquad \text{and} \qquad g(X)\leqslant g(X_c) \text{ if } X\leqslant X_c.$$

Therefore, it is easy to see that $\Phi_1(L)\,\Phi_2(R)-\Phi_1(R)\,\Phi_2(L)>0$ whenever $X_c\leqslant L$ or $R\leqslant X_c$. Hence, if (3.30), (3.31) has a non-trivial solution then $X_c\in(L,R)$. This completes the proof.

To have an explicit expression for Φ_2, it is necessary to integrate $C'(x)$. Fortunately, this can be done as follows:

Set

$$s=-2c \qquad \text{and} \qquad X_s=X_c=\frac{1-s}{1+s}. \tag{3.34}$$

Then $s\in(0,1)$ and

$$C'(X)=(1+X)^2\,(X-X_s)^{-2}\,X^{s-1}.$$

$C'(X)$ can be integrated as

$$C(X)=\frac{s+1}{s}\,X^s+\frac{s+1}{S-1}\,X^{s-1}-(1+X)^2\,(X-X_s)^{-1}\,X^{s-1}.$$

Therefore

$$\Phi_1(X)=\frac{X-X_s}{X+1}, \qquad \text{and} \qquad \Phi_2(X)=\frac{X^s}{sX_s(1+X)}\,(XX_s-1). \tag{3.35}$$

Knowing these two linearly independent solutions Φ_1 and Φ_2 of (3.22), we have

LEMMA 3.4. *Problem* (3.30), (3.31) *has a non-trivial solution if and only if*

$$(L-X_s)(RX_s-1)\,R^s-(R-X_s)(LX_s-1)\,L^s=0. \tag{3.36}$$

Furthermore, (3.36) *is equivalent to the system*

$$H(t,s,k)=0, \tag{3.37}$$

$$s\beta(t)=2k, \tag{3.38}$$

where

$$H(t,s,k)=(L-X_s)(RX_s-1)\,m^{2k}-(R-X_s)(LX_s-1). \tag{3.39}$$

The corresponding eigenfunction can be taken as

$$\Phi(X)=\frac{1}{sX_s(L+1)(X+1)}\{(L-X_s)(XX_s-1)\,X^s-(X-X_s)(LX_s-1)\,L^s\}. \tag{3.40}$$

Proof. By (3.35), we have

$$\begin{aligned}&\Phi_1(L)\,\Phi_2(R)-\Phi_1(R)\,\Phi_2(L)\\&\quad=\frac{1}{sX_s(L+1)(R+1)}\{(L-X_s)(RX_s-1)\,R^s-(R-X_s)(LX_s-1)\,L^s\}.\end{aligned}$$

This gives the first part of our lemma.

Next, by (3.18), (3.24), and (3.34), we have

$$\left(\frac{R}{L}\right)^s=m^{s\beta}=m^{2k}.$$

Therefore, (3.36) is equivalent to (3.37), (3.38). As (3.40) can be obtained easily, we omit the details here. This completes the proof.

In the following, we try to solve t and s of (3.37), (3.38) in terms of k. Note that a function is said to be smooth if it belongs to C^l for some $l\geqslant 1$.

LEMMA 3.5. *For any $k>0$, there exists a unique solution $(t(k), s(k), k)$ of* (3.37), (3.38). *Furthermore, $t(k)$ and $s(k)$ are smooth in k and*

$$\lim_{k\to\infty} t(k)=\infty. \tag{3.41}$$

Proof. First, we shall solve s as a function of t and k in (3.37). Since $s=(1-X_s)/(1+X_s)$, it suffices to solve X_s as a function of t and k in (3.36).

Since

$$\beta(t)>2k, \qquad Lm^{2k}-R=K(m^{2k}-m^{\beta})<0,$$

the requirement of $X_s>0$ implies

$$\begin{aligned} X_s &= X_s(t,k) \\ &= \frac{(RL+1)(m^{2k}-1)+\{(RL+1)^2(m^{2k}-1)^2-4(Rm^{2k}-L)(Lm^{2k}-R)\}^{1/2}}{2(Rm^{2k}-L)} \end{aligned} \tag{3.42}$$

Next, we shall compute $\partial s/\partial t$ or $\partial X_s \partial t$. Since it is rather complicated to differentiate (3.42) with respect to t directly, we shall compute $\partial H/\partial t$ and $\partial H/\partial s$ instead. It is easy to obtain that

$$\frac{\partial H}{\partial t}(t,s,k)=\left(1-X_s^2\right)\left\{R'\frac{LX_s-1}{RX_s-1}-L'\frac{R-X_s}{L-X_s}\right\}$$

and

$$\frac{\partial H}{\partial s}(t,s,k)=\frac{2}{(1+s)^2}(R-L)\left\{\frac{R-X_s}{RX_s-1}+\frac{LX_s^{-1}}{L-X_s}\right\}.$$

Furthermore, if (t,s,k) is a solution of $H(t,s,k)=0$, then $L<X_s<R$ implies $LX_s<1<RX_s$. Hence, we have

$$\frac{\partial H}{\partial t}(t,s,k)<0 \qquad \text{and} \qquad \frac{\partial H}{\partial s}(t,s,k)>0 \tag{3.43}$$

on $\{(t,s,k)\colon H(t,s,k)=0\}$. This implies

$$\frac{\partial s}{\partial t}(t,k)=-\frac{\partial H}{\partial t}(t,s(t,k),k)\bigg/\frac{\partial H}{\partial s}(t,s(t,k),k)>0,$$

and then

$$\frac{\partial X_s(t,k)}{\partial t}=\frac{-2}{(1+s)^2}\frac{\partial s(t,k)}{\partial t}<0.$$

To prove the first part of the lemma, it suffices to show that for each $k>0$, the graph of $s(t,k)$ intersects the graph of $s=2k/\beta(t)$ exactly once in the set

$$(\bar{t}_k,\infty)\equiv\{t\in\mathbb{R}^1\colon t>4m \qquad \text{and} \qquad \beta(t)>2k\}.$$

Since $s(t, k)$ is strictly increasing in t and $2k/\beta(t)$ is strictly decreasing in t, they intersect at most once in $(\bar{t}_k, \infty)$. It remains to show that they indeed intersect in $(\bar{t}_k, \infty)$.

By (3.42), it can be checked that $\lim_{t \to +\infty} X_s(t, k) = 1/m^k < 1$.

There are two cases to be considered;

Case 1. $\bar{t}_k = 4m$, i.e., $\beta(t) > 2k$ for $t \geq 4m$.

Case 2. $\bar{t}_k > 4m$, i.e., there exists a $\bar{t}_k > 4m$ such that $\beta(\bar{t}_k) = 2k$.

In Case 1, since $R(4m) = P_1(4m)/2m = 1$, after a straightforward but lengthy computation, it can be proved $X_s(4m, k) = 1$, i.e., $s(4m, k) = 0$. Hence, there exists a unique $t(k) \in (4m, \infty)$ such that $s(t(k), k) = 2k/\beta(t(k))$.

In Case 2, we have

$$1 - X_s(\bar{t}_k, k) = 1 - \frac{K^2 \mathrm{m}^{2k} + 1}{K(m^{2k} + 1)} = \frac{(1 - K)}{K(m^{2k} + 1)} (Km^{\beta} - 1),$$

where K and β are evaluated at $t = \bar{t}_k$.

Since

$$Km^{\beta} = \frac{P_1(\bar{t}_k)}{2m} = \frac{(\bar{t}_k - 2m) + (\bar{t}_k^2 - 4m\bar{t}_k)^{1/2}}{2m} > 1,$$

we have $X_s(\bar{t}_k, k) \in (0, 1)$; i.e., $s(\bar{t}_k, k) \in (0, 1)$. But $2k/\beta(\bar{t}_k) = 1$, which implies that there exists a unique $t(k) \in (\bar{t}_k, \infty)$ such that $s(t(k), k) = 2k/\beta(t(k))$. This proves the first part of lemma.

By (3.43) and the implicit function theorem, $t(k)$ and $s(k)$ are smooth in k. Since $s(k) \in (0, 1)$ and $\lim_{t \to \infty} \beta(t) = \infty$, $\beta(t(k)) = 2k/s(k) > 2k$ implies (3.41). This completes the proof.

Combining the results of Lemmas 3.3, 3.4, and 3.5, we obtain the following theorem:

THEOREM 3.6. *For any* $k \in (0, \infty)$ *there exists a unique* $\lambda^*(k) > 0$ *such that* $\mu_{k,1}(\lambda^*(k)) = 0$. *The function* $\lambda^*(\cdot)$: $(0, \infty) \to (0, \lambda^*(m))$ *is smooth and has the following properties*:

(i) $\lim_{k \to 0} \lambda^*(k) = \lambda^*(m)$,

(ii) $\lim_{k \to \infty} \lambda^*(k) = 0$.

Proof. Using Lemma 3.5 and letting

$$\lambda^*(k) = t(k)^{-1} \left\{ \frac{1}{\log m} \log \frac{P_+(t(k))}{4m} \right\}^2 \tag{3.44}$$

we see that $\lambda^*(k)$ is the unique solution of $\mu_{k,1}(\lambda)=0$. Then (i), (ii) follow from (3.41) and (3.42).

Summarizing the results of Theorems 3.2 and 3.6, we have the following theorem:

THEOREM 3.7. *The linearized problems*

$$\Delta w + 2\lambda e^{u_\lambda(r)} w = 0, \qquad \text{in } \Omega$$
$$w = 0, \qquad \text{on } \Omega,$$

have a non-trivial solution if and only if $\lambda = \lambda^*(k)$, $k = 0, 1, 2, \ldots$. *Furthermore, for each* $k \geqslant 1$, *the corresponding eigenspace is spanned by* $\varphi_k(r) \cos k\theta$ *and* $\varphi_k(r) \sin k\theta$, *where* $\varphi_k(r) = \Phi_k(X)$ *and* $\Phi_k(X)$ *is given in* (3.40) *with* $X = Km^{\beta/2} r^\beta$.

4. SYMMETRY BREAKING

In this section we shall prove that there are non-radially symmetric solutions which bifurcare from the upper branch u_λ at every $\lambda^*(k)$, $k = 1, 2, \ldots$. We shall apply a bifurcation theorem of Crandall and Rabinowitz [4].

THEOREM 4.1. *Let* X, Y *be Banach spaces,* V *a neighborhood of* 0 *in* X, $\bar{\lambda}$, ε *in* $\mathbb{R}^1$, *and*

$$F\colon (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon) \times V \to Y$$

have the properties

(a) $F(\lambda, 0) = 0$ *for* $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$,

(b) *the partial derivatives* F_λ, F_u, $F_{\lambda u}$ *exist and are continuous,*

(c) $N(F_u(\bar{\lambda}, 0))$ *and* $Y/R(F_u(\bar{\lambda}, 0))$ *are one-dimensional,*

(d) $F_{\lambda u}(\bar{\lambda}, 0)\, w_0 \notin R(F_u(\bar{\lambda}, 0))$, *where* $N(F_u(\bar{\lambda}, 0)) = \operatorname{span}\{w_0\}$.

If Z *is any complement of* $N(F_u(\bar{\lambda}, 0))$ *in* X, *then there is a neighborhood* U *of* $(\bar{\lambda}, 0)$ *in* $\mathbb{R} \times X$, *an interval* $(-\delta, \delta)$, *and continuous functions*

$$\varphi\colon (-\delta, \delta) \to \mathbb{R}^1, \qquad \psi\colon (-\delta, \delta) \to Z$$

such that $\varphi(0) = 0$, $\psi(0) = 0$, *and*

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha w_0 + \alpha\psi(\alpha))\colon |\alpha| < \delta\} \cup \{(\lambda, 0)\colon (\lambda, 0) \in U\}.$$

To apply Theorem 4.1, we need to rewrite (1.1), (1.2) as a nonlinear operator equation on an appropriate function space. We shall work on Hölder spaces.

Denote by $C_0^{1+\gamma}(\bar{\Omega})$ the set of continuously differentiable functions on $\bar{\Omega}$ which vanish on $\partial\Omega$ and whose first order derivatives are Hölder continuous in $\bar{\Omega}$ with exponent $\gamma \in (0, 1)$.

$C_0^{1+\gamma}(\bar{\Omega})$ is a Banach space under the usual norm,

$$\|u\|_{1+\gamma} = \max_{x\in\Omega} |u(x)| + \max_{i=1,2} \max_{x\in\Omega} \left|\frac{\partial u}{\partial x_i}(x)\right|$$
$$+ \max_{i=1,2} \max_{\substack{x,y\in\Omega\\ x\neq y}} \left|\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y)\right| \Big/ |x-y|^{\gamma}.$$

Denote by $\tilde{C}_0^{1+\gamma}(\bar{\Omega})$ the subspace of $C_0^{1+\gamma}(\bar{\Omega})$ consisting of functions which are even with respect to the x_1-coordinate, i.e.,

$$\tilde{C}_0^{1+\gamma}(\bar{\Omega}) = \{u \in C_0^{1+\gamma}(\bar{\Omega}): u(-x_1, x_2) = u(x_1, x_2)\}.$$

Then (1.1), (1.2) is equivalent to

$$F(\lambda, u) = 0, \tag{4.1}$$

where

$$F(\lambda, u): (0, \lambda^*(m)) \times C_0^{1+\gamma}(\bar{\Omega}) \to C_0^{1+\gamma}(\bar{\Omega})$$

is defined by

$$F(\lambda, u) = u + u_\lambda + 2\lambda G f(u + u_\lambda) \tag{4.2}$$

with

$$G = (\Delta)^{-1} \qquad \text{and} \qquad f(u) = e^u. \tag{4.3}$$

It is easy to check that the linearized operator

$$F_u(\lambda, 0): C_0^{1+\gamma}(\bar{\Omega}) \to C_0^{1+\gamma}(\bar{\Omega})$$

is given by

$$F_u(\lambda, 0)\, w = w + 2\lambda G(e^{u_\lambda} w) \tag{4.4}$$

and the mixed derivative

$$F_{\lambda u}(\lambda, 0): \mathbb{R}^1 \times C_0^{1+\gamma}(\bar{\Omega}) \to C_0^{1+\gamma}(\bar{\Omega})$$

is given by

$$F_{\lambda u}(\lambda, 0)\, w = G\left\{\frac{\partial}{\partial\lambda}(2\lambda e^{u_\lambda})\, w\right\}. \tag{4.5}$$

By Theorem 3.7, the kernel of $F_u(\lambda, 0)$ is non-trivial if and only if $\lambda = \lambda^*(k)$, and for any $k \geq 1$,

$$\operatorname{Ker} F_u(\lambda^*(k), 0) = \operatorname{span}\{\varphi_k(r)\cos k\theta, \varphi_k(r)\sin k\theta\}.$$

However, if we restrict (1.1), (1.2) on $\tilde{C}_0^{1+\gamma}(\bar{\Omega})$ then for any $k \geq 1$,

$$\begin{aligned} &\operatorname{Ker} F_u(\lambda^*(k), 0) \cap \tilde{C}_0^{1+\gamma}(\bar{\Omega}) \\ &\quad = \begin{cases} \operatorname{span}\{\varphi_k(r)\cos k\theta\} & \text{if } k \text{ is even} \\ \operatorname{span}\{\varphi_k(r)\sin k\theta\} & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

This is obtained from the following elementary facts:

LEMMA 4.2. *Let $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Then, $\cos k\theta$ is even (odd) in x_1 if k is even (odd), and $\sin k\theta$ is odd (even) in x_1 if k is even (odd).*

Therefore, with this setting the conditions (a), (b), (c) of Theorem 4.1 are satisfied and (d) is

$$\int_a^{1/a} r\varphi_k^2(r) \frac{\partial}{\partial\lambda} \{\lambda e^{u_\lambda(r)}\}\big|_{\lambda=\lambda^*(k)}\, dr \neq 0. \tag{4.6}$$

We shall prove

LEMMA 4.3. *For $k > 0$, we have*

$$\begin{aligned} &\text{(i)} \quad \frac{d\lambda^*(k)}{dk} < 0, \\ &\text{(ii)} \quad \int_a^{1/a} r\varphi_k^2(r) \frac{\partial}{\partial\lambda} \{\lambda e^{u_\lambda(r)}\}\big|_{\lambda=\lambda^*(k)}\, dr < 0. \end{aligned} \tag{4.7}$$

Proof. To verify (4.6) directly is rather difficult even we have explicit expressions for u_λ and φ_k. We shall verify it in the following way.

For $\lambda \in (0, \lambda^*(m))$ and $k \in (0, \infty)$, let $\mu(\lambda, k)$ and $\varphi(\lambda, k)$ be the principal eigenvalue and principal eigenfunction of linearized eigenvalue problem

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + \left(2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2}\right)\varphi(r) = -\mu\varphi(r), \qquad r \in \left(a, \frac{1}{a}\right),$$

where $\varphi(\lambda, k)$ is normalized by $\int_a^{1/a} r\varphi^2(r, \lambda, k)\, dr = 1$.

It is not difficult to verify that both $\mu(\lambda, k)$ and $\varphi(\lambda, k)$ are smooth in (λ, k).

Denote by $W(r, \lambda, k) = (\partial\varphi/\partial\lambda)(r, \lambda, k)$ and $V(r, \lambda, k) = (\partial\varphi/\partial k)(r, \lambda, k)$. Then W and V satisfy

$$W''(r)+\frac{1}{r}W'(r)+\left(2\lambda e^{u_\lambda(r)}-\frac{k^2}{r^2}\right)W(r)$$
$$=-\mu(\lambda,k)\,W(r)-\left\{\frac{\partial\mu}{\partial\lambda}+\frac{\partial}{\partial\lambda}(2\lambda e^{u_\lambda(r)})\right\}\varphi(r),\qquad r\in\left(a,\frac{1}{a}\right),$$
$$W(a,\lambda,k)=0=W\left(\frac{1}{a},\lambda,k\right),$$

and

$$V''(r)+\frac{1}{r}V'(r)+\left(2\lambda e^{u_\lambda(r)}-\frac{k^2}{r^2}\right)V(r)$$
$$=-\mu(\lambda,k)\,V(r)-\left\{\frac{\partial\mu}{\partial k}-\frac{2k}{r^2}\right\}\varphi(r),\qquad r\in\left(a,\frac{1}{a}\right),$$
$$V(a,\lambda,k)=0=V\left(\frac{1}{a},\lambda,k\right),$$

respectively. Therefore, we have

$$\int_a^{1/a} r\varphi^2(r,\lambda,k)\left\{\frac{\partial\mu}{\partial\lambda}+\frac{\partial}{\partial\lambda}(2\lambda e^{u_\lambda(r)})\right\}dr=0$$
$$\int_a^{1/a} r\varphi^2(r,\lambda,k)\left\{\frac{\partial\mu}{\partial k}-\frac{2k}{r^2}\right\}dr=0,$$

i.e.,

$$\frac{\partial\mu}{\partial\lambda}(\lambda,k)=-\int_a^{1/a} r\varphi^2(r,\lambda,k)\frac{\partial}{\partial\lambda}\{2\lambda e^{u_\lambda(r)}\}\,dr \tag{4.8}$$

and

$$\frac{\partial\mu}{\partial k}(\lambda,k)=2k\int_a^{1/a}\frac{1}{r}\varphi^2(r,\lambda,k)\,dr>0. \tag{4.9}$$

Since $\mu(\lambda^*(k),k)=0$, using Theorem 3.6, we have

$$\frac{\partial\mu}{\partial\lambda}(\lambda^*(k),k)\frac{d\lambda^*(k)}{dk}+\frac{\partial\mu}{\partial k}(\lambda^*(k),k)=0.$$

Therefore, (4.9) implies $d\lambda^*(k)/dk\neq 0$. Moreover, using Theorem 3.6, we have $d\lambda^*(k)/dk<0$. Hence

$$\frac{\partial\mu}{\partial\lambda}(\lambda^*(k),k)>0. \tag{4.10}$$

Therefore (4.7) follows by (4.8) and (4.10). This completes the proof.

Using Theorem 4.1, Lemmas 4.2 and 4.3, we obtain the following theorem:

THEOREM 4.4. *The upper branch u_λ of radially symmetric solutions of (1.1), (1.2) has a non-radially symmetric bifurcation at each $\lambda^*(k)$, $k=1, 2, \ldots$. Furthermore, in a neighborhood of $(\lambda^*(k), u_{\lambda^*(k)})$, the dimension of the set of bifurcating asymmetric solutions is two.*

Remark 4.5. By using the global bifurcation theorem of Rabinowitz [11], we can obtain the following global results:

Denote by S the solution set of (1.1), (1.2) and R the set of radial symmetric solutions of (1.1), (1.2). Let C be the closure of $\{(0, \lambda^*(m)) \times \tilde{C}_0^{1+\gamma}(\bar{\Omega})\} \cap (S \backslash R)$. Then, for any $k > 1$, the connected component C_k of $C \cup \{(\lambda^*(k), u_{\lambda^*(k)}\}$ to which $(\lambda^*(k), u_{\lambda^*(k)})$ belongs is either unbounded or meets $(\lambda^*(l), u_{\lambda^*(l)})$ for some positive integer $l \neq k$.

5. ANNULI AND DISK

In this section we shall prove that if the outer boundaries of annuli are fixed and the inner boundaries tend to zero, i.e., the annuli tend to the disk, then the radially symmetric solutions of (1.1), (1.2) will tend to the (radial) solutions of (1.1), (.2) on the disk.

We shall rewrite the equations (1.1), (1.2) on the disks as

$$u''(s) + \frac{1}{s} u'(s) + 2\delta e^{u(s)} = 0, \qquad s \in (0, 1), \tag{5.1}$$

$$u'(0) = 0 = u(1). \tag{5.2}$$

The critical number δ^* of (5.1), (5.2) is $\delta^* = 1$ and it is known (see, e.g., [1, 7]) that for any $\delta \in (0, \delta^*)$, the maximal solution $u_\delta(s)$ and minimal solution $\underline{u}_\delta(s)$ are given by

$$u_\delta(s) = \log \frac{4}{\delta} \frac{\gamma}{(1+\gamma s^2)^2} \qquad \text{and} \qquad \underline{u}_\delta(s) = \log \frac{4}{\delta} \frac{\tilde{\gamma}}{(1+\tilde{\gamma} s^2)^2},$$

where

$$\gamma = \left(\frac{2}{\delta} - 1\right) + \frac{2}{\delta}\sqrt{1-\delta} \qquad \text{and} \qquad \tilde{\gamma} = \left(\frac{2}{\delta} - 1\right) - \frac{2}{\delta}\sqrt{1-\delta}.$$

Set

$$s = ar, \ \delta = \frac{\lambda}{a^2} = \lambda m, \qquad \text{and} \qquad u(s) = u(r). \tag{5.3}$$

Then (2.1), (2.2) are transformed into

$$u''(s) + \frac{1}{s} u'(s) + 2\delta e^{u(s)} = 0, \qquad s \in (a^2, 1), \tag{5.4}$$

$$u(a^2) = 0 = u(1). \tag{5.5}$$

The critical number $\delta^*(a)$ of (5.4), (5.5) is

$$\delta^*(a) = \lambda^*(m)\, m, \tag{5.6}$$

where $\lambda^*(m)$ is given in (2.35).

For simplicity, we shall only treat the upper branch $u_{a,\lambda}$ of (5.4), (5.5); the lower branch $\underline{u}_{a,\lambda}$ can also be treated analogously.

Using (2.36) and (5.3), we write

$$u_{a,\delta}(s) \equiv u_{a,\lambda}(s) = \log \frac{t}{m} \frac{Km^\beta s^\beta}{(1 + Km^\beta s^\beta)^2\, s^2}, \tag{5.7}$$

where K and $\beta = \beta_+$ are given in (2.37), (2.39), respectively.

For any $\delta \in (0, 1]$ and $a \in (0, 1)$, the solution $\underline{u}_\delta$ of (5.1), (5.2) is a supersolution of (5.4), (5.5). Since 0 is a subsolution of (5.4), (5.5), by using the monotone iteration (starting from 0) (see, e.g., [9]), we have a positive solution for (5.4) (5.5). Hence,

$$\delta^*(a) > \delta^* = 1. \tag{5.8}$$

In the remaining part of the section, we shall adapt the following notation:

$$g(m) \sim h(m) \qquad \text{or} \qquad g \sim h \qquad \text{if } \lim_{m \to \infty} \frac{g(m)}{h(m)} = 1.$$

We first prove

LEMMA 5.1. $\lim_{a \to 0^+} \delta^*(a) = 1$.

Proof. Since $\delta^*(a) = \lambda^*(m)\, m$, it suffices to show that

$$\lim_{m \to \infty} \lambda^*(m)\, m = 1. \tag{5.9}$$

We need the asymptotic expansion of $t^*(m)$ as $m \to \infty$, where $t^*(m)$ satisfies

$$2t \frac{P'_+(t)}{P_+(t)} = \log \frac{P_+(t)}{4m}. \tag{2.33}$$

For simplicity, we shall abbreviate $t^*(m) = t$ for any $m > 1$.

Denote by $\tau = \tau(m) = t - 4m > 0$.
We shall prove

$$\tau(m) \sim \frac{4m}{(\log m)^2}. \tag{5.10}$$

Since

$$P_1(t) = (2m+\tau) + (t\tau)^{1/2}, \qquad P_2(t) = (mt-2) + mt\left(1 - \frac{4}{mt}\right)^{1/2},$$

$$\frac{P_1'(t)}{P_1(t)} = (t\tau)^{-1/2}, \qquad \frac{P_2'(t)}{P_2(t)} = t^{-1}\left(1 - \frac{4}{mt}\right)^{-1/2},$$

we have

$$\log \frac{P_+(t)}{4m} \sim \log\left\{\frac{t}{2}\left[(2m+\tau) + (t\tau)^{1/2}\right]\right\}$$

and

$$2t\frac{P'_+(t)}{P_+(t)} \sim 2\left\{\left(\frac{t}{\tau}\right)^{1/2} + 1\right\}.$$

Therefore, (2.33) implies

$$2\left\{\left(\frac{t}{\tau}\right)^{1/2} + 1\right\} \sim \log\left\{\frac{t}{2}\left[(2m+\tau) + (t\tau)^{1/2}\right]\right\}. \tag{5.11}$$

Hence $\tau/t \sim 0$, i.e., $\lim_{m\to\infty} (\tau(m)/m) = 0$.
Furthermore, using (5.11) we obtain

$$2\left(\frac{t}{\tau}\right)^{1/2} \sim \log mt \sim \log 4m^2 \sim 2\log m.$$

This proves (5.10).
Now, using (2.35) and (5.10), we have

$$m\lambda^*(m) = mt^{-1}\left\{\frac{1}{\log m}\log\frac{P_+(t)}{4m}\right\}^2 \sim \frac{1}{4}\left\{\frac{\log mt}{\log m}\right\}^2 \sim 1.$$

This completes the proof.

By (2.21) and (5.3), $\delta \in (0, 1)$ satisfies

$$\delta = \frac{m}{t}\left\{\frac{1}{\log m}\log\frac{P_+(t)}{4m}\right\}^2. \tag{5.12}$$

Using (2.34), in (2.21), we can obtain $t = t(m, \lambda)$ which is a function of $(m, \lambda) \in (1, \infty) \times (0, \lambda^*(m))$ with $t(m, \lambda) > t^*(m)$. Hence (5.3) implies that $t = t(m, \delta)$, being a function of $(m, \delta) \in (1, \infty) \times (0, 1)$, satisfies $t(m, \delta) > t^*(m)$. It is clear that $t(m, \delta)$ satisfies (5.12), for $(m, \delta) \in (1, \infty) \times (0, 1)$. By (2.18),

$$\beta^2(m, \delta) = \lambda(m, \delta)\, t(m, \delta). \tag{5.13}$$

Then, using an argument as in proving Lemma 5.1, we can prove

LEMMA 5.2. *For any $\delta \in (0, 1)$, we have*

$$\lim_{m \to \infty} \frac{t(m, \delta)}{m} = \frac{4}{\delta} \tag{5.14}$$

and

$$\lim_{m \to \infty} \beta(m, \delta) = 2. \tag{5.15}$$

Proof. The proof can be made as rigorous as the proof of Lemma 5.1; here we only sketch it. First, we find the asymptotic expansion of $t(\delta, m)$ as $m \to \infty$. Assuming $t(\delta, m) \sim \eta m$ for $\eta > 0$, we obtain

$$\delta = \frac{m}{t} \left\{ \frac{1}{\log m} \log \frac{P_+(t)}{4m} \right\}^2 \sim \frac{1}{\eta} \left\{ \frac{1}{\log m} \log m^2 \right\}^2 \sim \frac{4}{\eta}.$$

Then, using an argument as in proving Lemma 5.1, we can prove (5.14). Finally, (5.3), (5.13), and (5.14) imply (5.15). This completes the proof.

Now, we are ready to prove the main result of this section.

THEOREM 5.3. *For any $\delta \in (0, 1]$, let $S_{a,\delta}$ be the point in $(a^2, 1)$ where $u_{a,\delta}$ attains its maximum. Then*

$$\lim_{a \to 0^+} S_{a,\delta} = 0. \tag{5.16}$$

Furthermore, $u_{a,\delta}(s)$ converges uniformly to $u_\delta(s)$ on $[S_{a,\delta}, 1]$; i.e., for any $\varepsilon > 0$ there exists $a_\varepsilon = a(\varepsilon, \delta) > 0$ such that $|u_{a,\delta}(s) - u_\delta(s)| < \varepsilon$ for $s \in [S_{a,\delta}, 1]$ and $a \in (0, a_\varepsilon)$.

Proof. In (5.7), we first prove

$$\lim_{m \to \infty} K(t)\, m^{\beta(t)} = \gamma. \tag{5.17}$$

Using (2.42) and Lemma 5.2, we obtain $Km^\beta = 2m^\beta/P_2 \sim m^{\beta-2}(\delta/4)$. Now,

$$m^{\beta-2} = \exp\{(\beta-2)\log m\} = \exp\left\{\log\frac{P_+(t)}{4m} - 2\log m\right\} = \frac{P_+(t)}{4m^3}.$$

Using Lemma 5.2 again, we have

$$m^{\beta-2} \sim \frac{1}{2}\left\{\left(\frac{4}{\delta}-2\right)+\frac{4}{\delta}(1-\delta)^{1/2}\right\}\cdot\frac{4}{\delta}.$$

This proves (5.17).

Next, we shall prove (5.16). Let $r_{a,\delta} = a^{-1}S_{a,r}$ and $x_{a,\delta} = \log r_{a,\delta}$. Then $u_{a,\delta}(r_{a,s}) = \max_{r\in[a,1/a]} u_{a,\delta}(r)$. Therefore, using (2.3), we have $v'(x)-2 = u'(r)$. Hence $v'_{a,\delta}(x_{a,\delta})-2 = u'_{a,\delta}(r_{a,\delta}) = 0$. On the other hand, it is easy to check that $v'(x) = \beta((1-X)/(1+X))$, where $X = K_1e^{\beta x} = Km^\beta s^\beta$. Therefore, $X = (\beta-2)/(\beta+2)$ at $x = x_{a,\delta}$, i.e.,

$$Km^\beta S^\beta_{a,\delta} = \frac{\beta-2}{\beta+2}. \tag{5.18}$$

Using (5.15), (5.17), and (5.18), we obtain (5.16).

Finally, (5.18) implies

$$S^{\beta-2}_{a,\delta} = \{(\beta+2)\,Km^\beta\}^{-(\beta-2)/\beta}\,(\beta-2)^{(\beta-2)/\beta}.$$

Using (5.15) and (5.17) again, we obtain $\lim_{m\to\infty} S^{\beta-2}_{a,\delta} = 1$. Hence

$$\lim_{a\to 0^+} u_{a,\delta}(S_{a,\delta}) = \log\frac{4}{\delta}\gamma = u_\delta(0).$$

Since $\beta > 2$ and for any $s \in [S_{a,\delta}, 1]$, we have $S^{\beta-2}_{a,\delta} \leqslant s^{\beta-2} \leqslant 1$. Then

$$\lim_{m\to\infty} s^{\beta-2} = 1 \qquad \text{uniformly on } [S_{a,\delta}, 1]. \tag{5.19}$$

Now,

$$u_{a,\delta}(s) - u_\delta(s) = \log\frac{t}{m}\cdot\frac{\delta}{4}\frac{km^\beta}{\gamma}s^{\beta-2}\frac{(1+\gamma s^2)^2}{(1+Km^\beta s^{\beta-2}s^2)^2}.$$

Therefore, (5.14), (5.17), and (5.19) imply that $u_{a,\delta}(s)$ converges to $u_\delta(s)$ uniformly on $[S_{a,\delta}, 1]$ as $a\to 0^+$. This completes the proof.

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