A new autoregressive method for high-performance spectrum analysis

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In this paper, a new method of autoregressive (AR) spectrum estimation is presented. It shall be called two-sided autoregressive spectrum estimation, because an interpolative or smoothing model is postulated, as opposed to the predictive (one-sided) model used in AR modeling. The matrix equations arising in the estimation procedures proposed in this paper exhibit a special structure. The exploitation of these structures leads to fast solutions that reduce the total number of computations by an order of magnitude compared with straightforward approaches. Also, special attention is directed to the constrained two-sided AR model. Simulation examples show higher resolution capability of the proposed method relative to the least-squares AR method.

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INTRODUCTION

In the recent past, much attention has been given to the rational spectral model of a time series, and the result has led to the generation of a number of new algorithms (e.g., Refs. 1–5). Since there exists no single "correct" technique to calculate the spectrum in the absence of knowledge about the type of process that has generated the data, we are forced to assume that the data do satisfy some classes of representation. Once we have decided on the class, an appropriate algorithm must be selected for the calculation of the actual spectral estimate. Depending on what *a priori* assumptions we make about the underlying process and the effort we are willing to put forth, different approximate answers will be given to the problem.

So far, most parametric methods of spectrum estimation have used predictive models of AR or autoregressive moving-average (ARMA) structure to estimate the spectrum. For example, in AR modeling the mathematical structure is expressed by

$$x(t) = d_1 x(t-1) + d_2 x(t-2) + \cdots + d_n x(t-n) + a(t), \tag{1}$$

where $a(t)$ is a zero mean white noise process with variance σ_a^2.

The forward prediction errors are given by

$$a(t) = x(t) - x(t\,|\,t-1, t-2, \ldots, t-n), \tag{2}$$

where $x(t\,|\,t-1, t-2, \ldots, t-n)$ is the best estimate in a least-squares sense of $x(t)$, given $x(t-1), x(t-2), \ldots x(t-n)$. Similarly, the backward prediction errors are defined as

$$b(t) = x(t) - x(t\,|\,t+1, t+2, \ldots, t+n). \tag{3}$$

Some predictive AR schemes that minimize the forward errors, while others minimize some combination of the forward and the backward prediction errors, have been proposed in the past (e.g., see Refs. 1–4 and 6–8).

However, the linear interpolation (or smoothing) model based on past and future values has not received much attention in the past. The model is postulated that a stochastic AR process depends on both past as well as future values of $x(t)$. In consequence, the interpolative model proposed is given in the following form:

$$\begin{aligned} x(t) = {} & d_1 x(t-1) + d_2 x(t-2) + \cdots + d_n x(t-n) \\ & + d_{-1} x(t+1) + d_{-2} x(t+2) \\ & + \cdots + d_{-n} x(t+n) + a(t). \end{aligned} \tag{4}$$

Perhaps Nuttall[9] was the first researcher to work on the feasibility of this type of model. However, at that time the model was found to be unworkable with the procedures that he used. The method used to estimate the model parameters is the key to the performance of the spectrum analysis. The method used by Nuttall was based on the minimization of magnitude-squared error, which results in a set of normal equations similar to the Yule–Walker equation in AR estimates. The spectrum analysis produced by Nuttall leads to the following problems: (1) It yields the nonwhiteness of the $a(t)$ sequences; in other words, the linear filter characterized by filter coefficients is not a whitening filter. As a consequence, the standard spectral formula [see Eq. (13)] cannot be used in this case, and one is compelled to express the spectrum in another form, which in practice would yield severely biased and negative estimates of the spectrum. (2) The method technically leads to the incorrect result that the minimum-error sequence is uncorrelated with all past and future values of the $x(t)$, excluding those that take place at the same instant.

In this paper, we examine this type of model in more detail and present a new algorithm for parameter estimation. It has been shown that better results are obtained when the dependence of $x(t)$ on future and past values, rather than on past values only, is utilized in the model. Hereafter, this type of model shall be called the two-sided AR model. The key difference between the two types of models is that the AR model is based on one-sided prediction, while the two-sided AR model is based on interpolation. The method's develop-

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ment is based upon some fundamental concepts of time series analysis, which will be discussed in Sec. I.

I. PRINCIPLES

The autocovariance function and the spectral density function are interchangeable fundamental properties of a stationary time series. That is, given one of the two, the other can be found by Fourier transformation or by inverse Fourier transformation. Given one of the two functions, the statistical characteristics of the stochastic process are completely specified up to second moments. Yet, for several practical applications involving prediction, optimal control, etc., the difference equation representation of the time series proves to be more convenient. Furthermore, the difference equation representation has great appeal in data processing with digital computers. For a given time series (with a given spectral density function), there is a multiplicity of difference equation representations to choose from, depending upon the purpose and application.

Consider the factorization of the spectral density function

$$F(\omega) = 1/(2\pi)^{-1} A(e^{-i\omega}) A^*(e^{-i\omega}), \tag{5}$$

where $A(e^{-i\omega})$ is a rational function of $e^{-i\omega}$ and $A^*(e^{-i\omega})$ is the complex conjugate of $A(e^{-i\omega})$. Let $A(e^{-i\omega})$ have the convergent Fourier series expansion

$$A(e^{-i\omega}) = \sum_{l=-\infty}^{\infty} h_l e^{-i\omega l}. \tag{6}$$

Then, by Theorem 9.1 in Chap. I of Rozanov,[10] there exists a moving average representation of $x(t)$,

$$x(t) = \sum_{l=-\infty}^{\infty} h_l a(t-l), \tag{7}$$

where $a(t)$ is an uncorrelated white noise series,

$$E[a(t)a(t-l)] = \begin{cases} \sigma_a^2 & \text{if } l=0, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the backshift operator, defined by the relations $Bx(t) = x(t-1), B^{-1}x(t) = x(t+1)$. Now we define the operator

$$H(B) = \sum_{l=-\infty}^{\infty} h_l B^l. \tag{8}$$

Then, Eq. (7) may be written as

$$x(t) = H(B)a(t). \tag{9}$$

Conceptually, Eq. (9) means that the series $x(t)$ is generated from the series $a(t)$ by passing $a(t)$ through the filter $H(B)$. That is, $a(t)$ is viewed as the driving force that passes through filter $H(B)$ to yield time series $x(t)$. Here, $H(B)$ is termed the two-sided discrete filter. In general, $H(B)$ will perform a two-sided convolution operation that is noncausal in nature.

A. Relation of the filter $H(B)$ with the spectral factor $A(e^{-i\omega})$

Let $A(e^{-i\omega})$ be a rational function of $e^{-i\omega}$, i.e.,

$$A(e^{-i\omega}) = [Q(e^{-i\omega})]/[P(e^{-i\omega})],$$

where $P(Z)$ and $Q(Z)$ are polynomials in Z of orders p and q, respectively. For stationarity, the polynomial $P(Z)$ may have no root on the unit circle. From Eq. (6), the h_l's are the weights appearing in the Fourier series expansion of $A(e^{-i\omega})$. We have

$$\begin{aligned} h_l &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{-i\omega}) e^{-i\omega l}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Q(e^{-i\omega})}{P(e^{-i\omega})} e^{-i\omega l}\, d\omega \\ &= \frac{1}{2\pi i}\oint_c \frac{Q(z)}{P(z)} z^{l-1}\, dz, \end{aligned} \tag{10}$$

where $z = e^{i\omega}$ and c is the unit circle in the complex Z plane. Thus the h_l's are the same as the coefficients in the Laurent series expansion of $Q(Z)/P(Z)$ in an annulus containing the unit circle, where the function to be expanded is analytic. Now, referring to Eq. (8), and by the uniqueness of Laurent series expansions, we have

$$H(B) = [Q(B)]/[P(B)]. \tag{11}$$

Thus $H(B)$ is a rational function of B and is obtained from the spectral factor $A(e^{-i\omega})$ by substituting $B = e^{-i\omega}$. Equation (11) thus establishes the relation between the spectral factor $A(e^{-i\omega})$ and the filter $H(B)$.

Characteristics of $H(B)$ such as stationarity and causality can be associated with constraints on the pole-zero pattern and the region of convergence (ROC). For example, if a given time series is causal, then the ROC for $H(B)$ will be inside the innermost pole. If the time series is stationary, then the h_l's are absolutely summable, in which case the Fourier transformation of h_l will converge, and, consequently, the ROC of $H(B)$ must include the unit circle. For a time series that is stationary and causal, the ROC must include the unit circle and be inside the innermost pole. For a time series that is both stationary and noncausal, the ROC must include the unit circle and be inside the outermost pole while outside the innermost pole, from which it follows that the ROC will consist of a ring in the Z plane that includes the unit circle. Corresponding to the poles outside the unit circle, we have the filter coefficients h_l for $l \geqslant 0$, and corresponding to the pole inside the unit circle, we have the coefficients h_l for $l < 0$.

For illustrative purposes, consider the following univariate time series model:

$$x(t) = (1 - \alpha B + B^2)^{-1} a(t), \tag{12}$$

where $a(t)$ is a white noise series with $\sigma_a^2 = 1$ and $\alpha = (1+\phi^2)/\phi$ for some $|\phi| < 1$, ϕ being real. Here, the poles of the model are $B = \phi^{-1}$ and $B = \phi$. By the definition of stationarity as given in Box and Jenkins,[11] for instance, one would consider the model (12) to be nonstationary. But with the introduction of two-sided filters, we will consider (12) to be a valid representation of a stationary time series that has spectral density function

$$F(\omega) = (2\pi)^{-1} |(1 - \alpha e^{-i\omega} + e^{-2i\omega})^{-1}|^2. \tag{13}$$

The filter $H(B)$ of model (12) will stand for

$$H(B) = \frac{1}{1-\alpha B+B^2} = \frac{-\phi}{(1-\phi B)(B-\phi)}$$
$$= \frac{-\phi^2}{1-\phi^2}\frac{1}{1-\phi B} - \frac{\phi}{1-\phi^2}\frac{1}{B-\phi}$$
$$= \frac{-\phi}{1-\phi^2}[\cdots + \phi B^{-2} + B^{-1} + \phi$$
$$+ \phi^2 B + \phi^3 B^2 + \cdots], \quad (14)$$

so that $H(B)a(t)$ is a well-defined random number (has finite variance) for all t. Obviously, the representation (12) of a stationary time series with spectral density function (13) is not a useful one for the purpose of prediction or forecasting. The predictive model for a time series with spectral density function (13) would be

$$x(t) = [(1-\phi B)^2]^{-1} b(t), \quad (15)$$

where $b(t) = a(t)/\phi^2$ is a white noise series with variance ϕ^2.

The two-sided filter $H(B)$ is not just a mathematical artifice, but may have meaning in terms of real world systems, for instance, space filters. However, when t refers to real time, the two-sided filter will just be a mathematical contrivance. In this paper, the generating mechanism of a given time series is assumed to be noncausal. A finite dimensional, two-sided AR filter with input $a(t)$ and output $x(t)$, which results in high-resolution spectral estimates, will be considered.

II. THE ESTIMATION OF THE TWO-SIDED AR MODEL

In principle, the specified model can be obtained by maximum likelihood (ML) estimation, being defined as the parameter set D, which maximizes the conditional probability density function $P(x|D)$ for the observed data $x(t)$. However, the maximization involves a difficult nonlinear least-squares problem. Thus, although the variance of the ML estimate asymptotically approaches the Cramer–Rao bound, it may be unattractive in many applications because of the computational burden. Instead, the approach of using an easy-to-implement suboptimal estimator will be pursued.

Let a finite-order, two-sided AR model be expressed as

$$d_0 x(t) = d_1 x(t-1) + d_2 x(t-2) + \cdots + d_n x(t-n)$$
$$+ d_{-1} x(t+1) + d_{-2} x(t+2)$$
$$+ \cdots + d_{-n} x(t+n) + a(t). \quad (16)$$

The orders of the dependence of $x(t)$ on the future and past values are restricted to be equal to n in our discussion. Note that we may, without loss of generality, assume that $d_0 = 1$.

The two-sided AR model in (16) with $d_0 = 1$ may be written as $D(B)x(t) = a(t)$ and, alternatively,

$$x(t) = \frac{a(t)}{D(B)} = \sum_{l=-\infty}^{\infty} h_l B^l a(t) = \sum_{i=-\infty}^{\infty} h_i a(t-i), \quad (17)$$

where the coefficients h_i were calculated from the expansion of $1/D(B)$ in terms of an infinite series in positive and negative powers of B. The relation between h_i and d_i can be expressed as

$$\sum_{i=-n}^{n} h_i d_{j-i} = 0 \text{ or } \sum_{\substack{i=-n \\ i\neq 0}}^{n} h_i d_{j-i} = -d_j, \quad (18)$$

where $d_0 = 1$, $h_0 = 1$, and $d_{|j-i|} = 0$ for $|j-i| > n$.

Consider the cross-correlation function betwen the $a(t)$ series and the $x(t)$ series

$$r_{ax}(k) = E[a(t)x(t+k)]$$
$$= E\left[a(t)\left(\sum_{i=-\infty}^{\infty} h_i a(t+k-i)\right)\right]. \quad (19)$$

Since $E[a(t)a(t+i)]$ is zero whenever $i \neq 0$, we have

$$r_{ax}(k) = h_k \sigma_a^2. \quad (20)$$

To obtain the estimation equations that will allow us to solve for the parameters, we shall have to employ a certain rearrangement of the unknown parameters. This rearrangement will be essential in our quest to find fast algorithms of order n^2 in computational complexity.

Equation (17) is rewritten as

$$x(t) = d_1 x(t-1) + d_{-n} x(t+n)$$
$$+ d_2 x(t-2) + d_{-(n-1)}$$
$$\times x(t+n-1) + \cdots + d_n x(t-n)$$
$$+ d_{-1} x(t+1) + a(t). \quad (21)$$

Now let us multiply each side of this equation alternatively by $[x(t-1),\ x(t+n),\ x(t-2),\ x(t+n-1),\ \cdots,\ x(t-n), x(t+1)]$ and take the expectations on each side. We then obtain the following $2n$ simultaneous equations:

$$\begin{bmatrix} R(0) & R(-1) & R(-2) & \cdots & R(-n+1) \\ R(1) & R(0) & R(-1) & \cdots & R(-n+2) \\ \vdots & \vdots & \vdots & & \vdots \\ R(n-1) & \cdots & \cdots & & R(0) \end{bmatrix}$$
$$\times \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \underline{p}_n \end{bmatrix} = \begin{bmatrix} \underline{r}(1) \\ \underline{r}(2) \\ \vdots \\ \underline{4}(n) \end{bmatrix} - \begin{bmatrix} \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_n \end{bmatrix} \sigma_a^2,$$

where

$$R(i,j) = R(i-j) = \begin{bmatrix} r(i-j) & r(n+1+j-i) \\ r(i-j-n-1) & r(j-i) \end{bmatrix} \quad (22)$$

$r(i) = E[x(t)x(t-i)]$,

$\underline{p}_i^T = (d_i\ d_{(-n+i-1)})$, $\underline{r}^T(i) = [r(i)r(-n+i-1)]$,

$\underline{h}_i^T = (h_i\ h_{(-n+i-1)})$.

By multiplying both sides of Eq. (21) by $x(t)$ and using Eq. (20), we may also get the estimated residual energy

$$\sigma_a^2 = E[x(t)a(t)] = r(0) - d_1 r(-1) - d_{-n} r(n)$$
$$- \cdots - d_n r(-n) - d_{-1} r(1). \quad (23)$$

Proceeding in the same manner, Eq. (18) can be rearranged

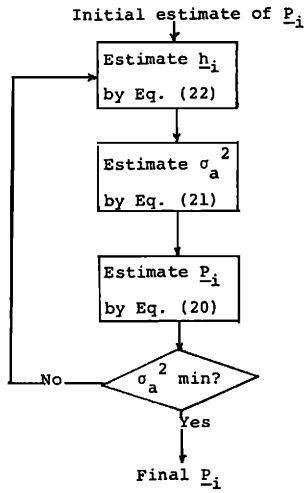

FIG. 1. An algorithm for two-sided AR parameter estimation.

to obtain the following $2n$ simultaneous equations:

$$\begin{bmatrix} D(0) & D(-1)D(-2)\cdots D(-n+1) \\ D(1) & D(0)\ D(-1)\ D(-n+2) \\ \vdots & \vdots \quad\quad \vdots \\ D(n+1) & \cdots \quad\quad \cdots \quad\quad D(0) \end{bmatrix} \times \begin{bmatrix} \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_n \end{bmatrix} = \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \underline{p}_n \end{bmatrix}, \tag{24}$$

where

$$D(i,j) = D(i-j) = \begin{bmatrix} d_{i-j} & d_{i-j+n+1} \\ d_{i-j-n-1} & d_{i-j} \end{bmatrix}.$$

In practice, we do not have the exact values of the covariance function available, but, instead, we are only given a finite segment of a time series $x(t)$. So the estimates have to be obtained from a finite number of data. One method of obtaining unbiased estimates of the correlation values is to define estimators,

$$\hat{r}(i) = \frac{1}{N-i} \sum_{t=i+1}^{N} x(t)x(t-i). \tag{25}$$

There are, of course, a number of different ways of estimating the autocorrelation sequence from a time series $x(t)$. For example, see Jenkins and Watts.[12]

The parameter estimation procedure relies on Eqs. (22), (23), and (24) to minimize the value of σ_a^2. The algorithm is an iterative procedure in which we begin with some initial estimate of $\underline{p}_i$, obtained from Eq. (22) by dropping the second term of the right-hand member in (22), acquire the corresponding impulse response function by using Eq. (24), and estimate the residual power by using Eq. (23). This completes one iteration, and by applying (22) again, a new estimate of the $\underline{p}_i$'s is obtained. This algorithm is illustrated in Fig. 1.

The block Toeplitz equations shown in Eqs. (22) and (24) may be solved by ordinary matrix solution methods, such as Gauss elimination. These methods require computational time of $O(n^3)$, where n is the number of unknowns. However, due to the special rearrangement of the unknowns, the resulting equations have a regularity that may be exploited in order to reduce the number of computations by an order of magnitude. The block-Levinson[13] algorithm, which has a computational complexity of $O(n^2)$, is presented in Appendix A, and may be used for that purpose.

III. CONSTRAINED TWO-SIDED AR MODEL

In this section, we constrain the two-sided AR model to have identical parameters in the forward and backward directions. The model will now be

$$\begin{aligned} x(t) = {} & d_1x(t-1) + d_2x(t-2) + \cdots + d_nx(t-n) \\ & + d_1x(t+1) + d_2x(t+2) \\ & + \cdots + d_nx(t+n) + a(t). \end{aligned} \tag{26}$$

The assumption that the forward and backward parameters are identical implies that the process is stationary. That is,

$$r(i) = r(-i)$$

and the symmetric impulse response function $h_i = h_{-i}$.

Let us multiply both sides of (26) by $x(t-i)$ for $i = 1,2,\ldots,n$ and take expectations. The first equation gives the noise variance in terms of the parameters of the model and the covariances of the process as

$$\sigma_a^2 = r(0) - 2d_1r(1) - 2d_2r(2) - \cdots - 2d_nr(n). \tag{27}$$

The next n equations may be written in matrix form as

$$\begin{bmatrix} r(0) & r(1)\cdots r(n-1) \\ r(1) & r(0)\cdots r(n-2) \\ \vdots & \vdots \quad\quad \vdots \\ r(n-1) & \cdots \quad\quad r(0) \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} + \begin{bmatrix} r(2) & r(3)\cdots r(n+1) \\ r(3) & r(4)\cdots r(n+2) \\ \vdots & \vdots \quad\quad \vdots \\ r(n+1) & \cdots \quad\quad r(2n) \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{bmatrix} r'(1) \\ r'(2) \\ \vdots \\ r'(n) \end{bmatrix}, \tag{28}$$

where $r'(i) = r(i) - h_i\sigma_a^2$. Also, from Eq. (24), we obtain

$$\begin{bmatrix} 1 & d_1\cdots d_{n-1} \\ d_1 & 1\cdots d_{n-2} \\ \vdots & \vdots\cdots\ \vdots \\ d_{n-1} & \cdots\cdots\ 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} + \begin{bmatrix} d_2 & d_3\cdots d_{n+1} \\ d_3 & d_4\cdots d_{n+2} \\ \vdots & \vdots \quad\quad \vdots \\ d_{n+1} & \cdots\cdots\ d_{2n} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = -\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}, \tag{29}$$

where $d_{|i+j|} = 0$ for $|i+j| > n$.

It should be noted that the same equations arise when we multiply both sides of Eq. (26) by $x(t-i)$ for $i = 0,1,...,n$ and take expectations, due to the stationarity relation and the symmetric impulse response function.

The system of equations in (28) and (29) involves the sum of a Toeplitz plus a Hankel matrix. It was shown in Ref. 14 that this type of equation may be transformed to a block Toeplitz form, and a fast solution can be obtained (see Appendix B).

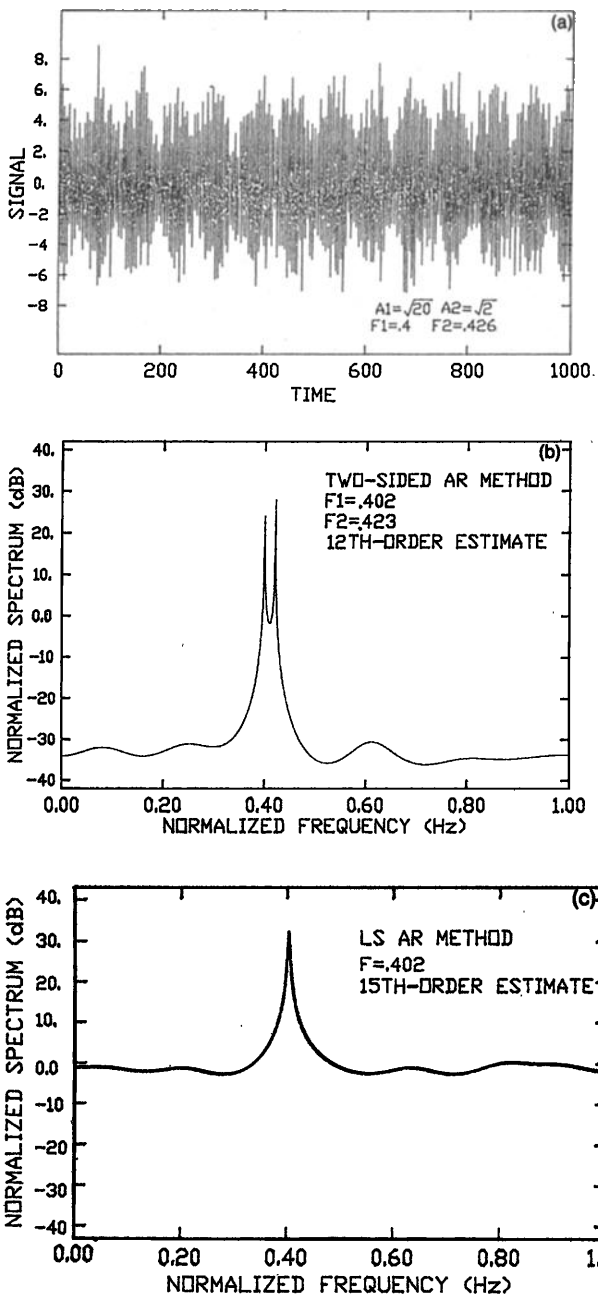

FIG. 2. Spectral estimates of the time series $x(n) = \sqrt{20}\cos(0.4\pi n) + \sqrt{2}\times\cos(0.426\pi n) + a(n)$, in which $[a(n)]$ is a white Gaussian random process with variance one. (a) The time series, and spectral estimates obtained by using (b) the two-sided AR method and (c) the least-squares AR method.

IV. NUMERICAL EXAMPLES

The constrained two-sided AR model was implemented in order to test the effectiveness of the proposed method. In the simulated case, we try to resolve two closely spaced sinusoids in the presence of white noise. Specifically, we investigate the time series generated by

$$x(n) = A_1\cos(\pi f_1 n + \phi_1) + A_2\cos(\pi f_2 n + \phi_2) + a(n) \quad \text{for } 1 \leqslant n \leqslant N, \tag{30}$$

where $\phi_1 = \phi_2 = 0$, $a(n)$ is a white Gaussian sequence with variance one, and the sinusoidal frequencies are normalized so that $f = 1$ corresponds to the Nyquist rate. The individual sinusoidal signal-to-noise ratios (SNRs) for this time series are given by $20\log(A_k/\sqrt{2})$ for $k = 1,2$, where use of the fact that the noise $a(n)$ has variance one has been made. Two cases will be considered in order to test the performance of the proposed spectral estimator in different noise environments. These cases have been examined in Refs. 5 and 15, where the performances of many modern spectral estimators were empirically compared.

A. Case I: $\begin{cases} A_1 = \sqrt{20}, & f_1 = 0.4 \\ A_2 = \sqrt{2}, & f_2 = 0.426 \end{cases}$

In this example, the stronger sinusoid has an SNR of 10 dB while the weaker sinusoid has 0-dB SNR. The data generated by using relationship (30) with $N = 1024$ are displayed in Fig. 2(a). The AIC criteria provided by Akaike[16] is adopted for the selection of the two-sided AR model order. The result obtained from the proposed method is a 12th-order model and is shown in Fig. 2(b). An AR spectral estimator originated by Nuttall[9] and Ulrych and Clayton[8] independently and modified by Marple[4] is used for comparison, since it has been reported to have good behavior at low SNR and insensitivity to initial phase. This recursive algorithm, called the "unconstrained least-squares" estimator by them, applies the modified covariance technique to obtain the model coefficients by minimizing the sum of the forward and backward prediction error energies. The result is shown in Fig. 2(c) with an order selection of 15.

As can be seen from Fig. 2(b) and (c), and two-sided

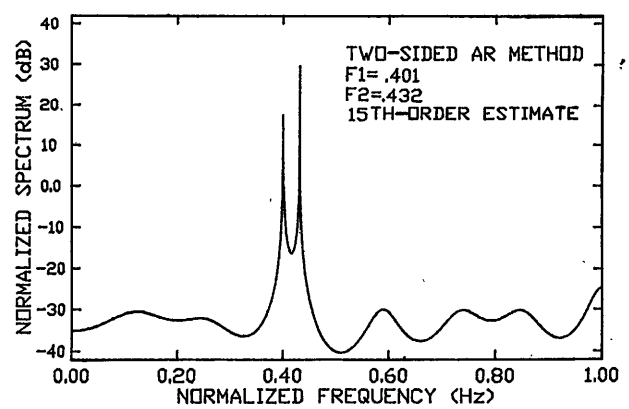

FIG. 3. Two-sided AR spectral estimate obtained using the first 128 data points of case I.

AR spectral estimator provides excellent results, with two sharp peaks occurring at $\hat{f}_1 = 0.402$ and $\hat{f}_2 = 0.423$, while the AR estimator is unable to resolve the two frequencies in the low SNR environment.

To further demonstrate the ability of the new method, the first 128 points of the data sequence in case I were used to generate a spectral estimate. The resultant 15th-order, two-sided AR spectral estimate obtained is shown in Fig. 3, where the ability to resolve the two closely spaced sinusoids ($\hat{f}_1 = 0.401$ and $\hat{f}_2 = 0.432$) is again evident.

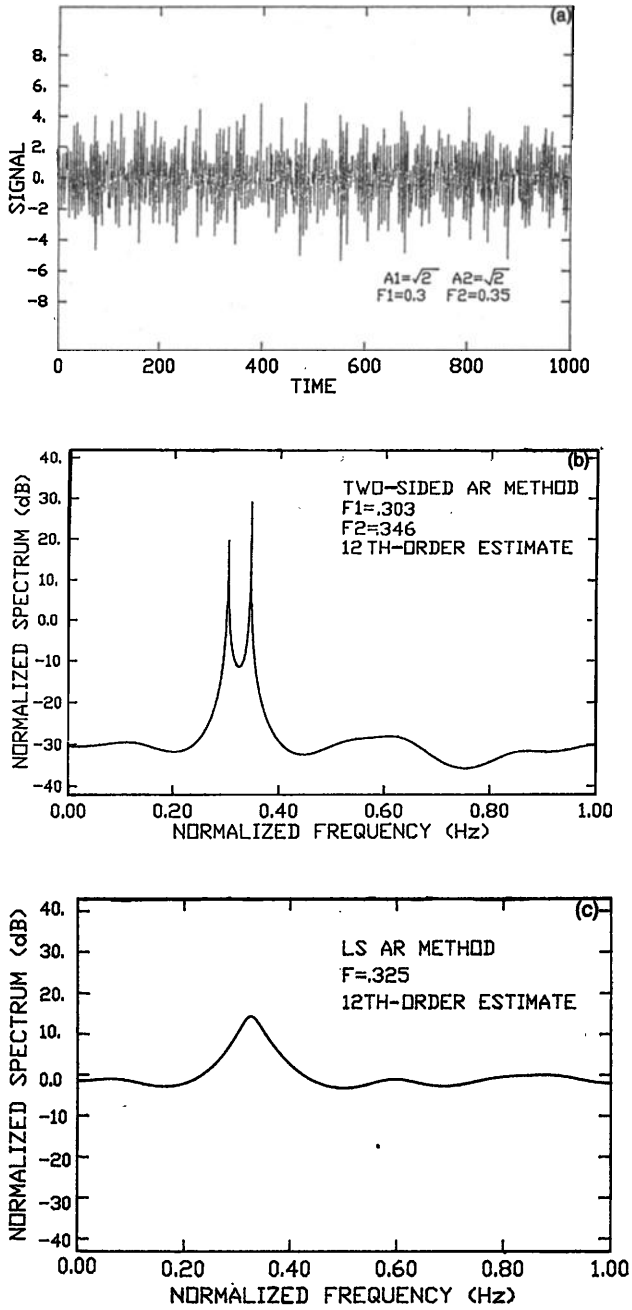

FIG. 4. Spectral estimates of the time series $x(n) = \sqrt{2}\cos(0.3\pi n) + \sqrt{2}\cos(0.35\pi n) + a(n)$, in which $[a(n)]$ is a white Gaussian random process with variance one. (a) The time series, and spectral estimates obtained by using (b) the two-sided AR method and (c) the least-squares AR method.

B. Case II: $\begin{cases} A_1=\sqrt{2}, & f_1=0.3 \\ A_2=\sqrt{2}, & f_2=0.35 \end{cases}$

In this example, the ability to detect more widely separated sinusoids in a low SNR environment (i.e., 0 dB) is examined. The data generated with $N = 1024$ are displayed in Fig. 4(a). The same behavior is obtained in this case [shown in Fig. 4(b)], where the resolution of frequencies is also sharper, and good quality frequency estimates $\hat{f}_1 = 0.303$ and $\hat{f}_2 = 0.346$ are obtained in this low SNR environment. For the purpose of comparison, an AR spectral estimator of 12th order generated by using this data is displayed in Fig. 4(c). It is apparent that the AR estimator is unable to resolve the two frequencies under such a low SNR condition.

V. CONCLUSION

A new autoregressive model called the two-sided AR model based on interpolation (smoothing) rather than prediction has been postulated; based on such a model, spectral estimation has been done. Simulation examples show higher resolution capability of the proposed method when compared with AR spectral estimation. The matrix equations arising in the estimation procedures proposed in this paper exhibit a special structure. The exploitation of these structures leads to fast solutions that reduce the total number of computations by an order of magnitude compared with straightforward approaches.

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APPENDIX A: FAST SOLUTION OF TWO-SIDED AR EQUATIONS

In this appendix, all the matrices involved will be of (2×2) dimension, whereas the vectors will be of length 2. Algorithm:

(1) $x_0 = y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{P}_1 = R^{-1}(0)\underline{r}_1'$,

$V_x = V_y = R(0)$.

(2) For $i = 1,2,\ldots,n-1$, do the following:

(a) $E_x = \sum_{j=0}^{i-1} R(i-j)x_j, \quad E_y = \sum_{j=1}^{i} R(i-j)y_{i-j}$,

(b) $\underline{e} = \sum_{j=0}^{i-1} R(i-j)\underline{p}_{j+1}$,

(c) $B_x = V_y^{-1}E_x, \quad B_y = V_x^{-1}E_y$,

(d) $$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_i \end{bmatrix} \leftarrow \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{i-1} \\ x_0 \end{bmatrix} - \begin{bmatrix} 0 \\ y_{i-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} B_x,$$

$$\begin{bmatrix} y_i \\ y_{i-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} \leftarrow \begin{bmatrix} 0 \\ y_{i-1} \\ \vdots \\ y^1 \\ y_0 \end{bmatrix} - \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{i-1} \\ 0 \end{bmatrix} B_y ,$$

$$V_x \leftarrow V_x - E_y B_x, \quad V_y \leftarrow V_y - E_x B_y,$$

(e) $g = V_x^{-1}[\underline{r}'(i) - \underline{e}]$, where $\underline{r}'(i) = \underline{r}(i) - \underline{h}_i \sigma_a^2$,

(f) $$\begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \underline{p}_i \\ \underline{p}_{i+1} \end{bmatrix} \leftarrow \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \underline{p}_i \\ 0 \end{bmatrix} + \begin{bmatrix} y_i \\ y_{i-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} g.$$

(3) The solution vector is given by

$$(\underline{p}_1^T, \underline{p}_2^T, \ldots, \underline{p}_n^T) = (d_1\, d_{-n}\, d_2\, d_{-(n-1)} \cdots d_n\, d_{-1}).$$

APPENDIX B: FAST SOLUTION OF CONSTRAINED TWO-SIDED AR EQUATIONS

1. Introduction of new notation

Let A be an $(n)\times(n)$ matrix,

$$A = [\underline{a}(1), \underline{a}(2), \ldots, \underline{a}(n)]$$

and

$$\underline{b}^T = [b(1), b(2), \ldots, b(n)] = \underline{b}^T(1,n).$$

We define the $(n)\times(n)$ operator matrix

$$J = [\underline{e}(n)\underline{e}(n-1) \cdots \underline{e}(1)],$$

where $\underline{e}(i)$ is an $n\times 1$ vector with 1 at the ith position and zeros everywhere else, for $i = 1,2,\ldots,n$. In other words, J has 1's along the cross diagonal, and zeros everywhere else. The J matrix performs a reversal operation, such that

$$AJ_T = [\underline{a}(n), \ldots, \underline{a}(2), \underline{a}(1)],$$

$$J\underline{b}^T = [b(n), \ldots, b(2), b(1)] = \underline{b}^T(n,1).$$

The effect of postmultiplication of a matrix by J is thus to reverse the order of the columns. Premultiplications of a matrix A by J reverses the order of the rows of A. Notice, also, that $JJ = I$, the identity matrix.

Also define a $(2n)\times(2n)$ "interleaving" operator Q such that

$$\{Q\}_{ij} = \begin{cases} 1 & \text{for } i = 2r, j = r, r = 0,1,\ldots,n-1, \\ 1 & \text{for } i = 2r+1, j = n+r, r = 0,1,\ldots,n-1, \\ 0 & \text{for all other } i,j \text{ pairs.} \end{cases}$$

The effect of Q operating on a $(2n)\times 1$ vector is to interleave the sequence in the following way:

$$Q[b(1), b(2), \ldots, b(n), c(1), c(2), \ldots, c(n)] = [b(1), c(1), b(2), c(2), \ldots, b(n), c(n)].$$

2. Conversion of Eq. (28) to block Toeplitz form

Let us rewrite the system of equations in Eq. (28) as

$$T\underline{d}(1,n) + H\underline{d}(1,n) = \underline{r}'(1,n), \tag{B1}$$

where T is an $(n)\times(n)$ Toeplitz matrix such that $\{T\}_{ij} = r(i-j)$ for $i,j = 1,n$ and H is an $(n)\times(n)$ Hankel matrix such that $\{H\}_{ij} = r(i+j)$ for $i,j = 1,n$.

We may write (B1) in two different ways:

$$T\underline{d}(1,n) + HJJ\underline{d}(1,n) = \underline{r}'(1,n), \tag{B2}$$

$$JTJJ\underline{d}(1,n) + JH\underline{d}(1,n) = J\underline{r}'(1,n). \tag{B3}$$

Since T is persymmetric (symmetric around the main cross diagonal), it can be shown that

$$JTJ = T^T,$$

and we may rewrite (B2) and (B3) as

$$\begin{bmatrix} T & HJ \\ JH & T \end{bmatrix} = \begin{bmatrix} \underline{d}(1,n) \\ \underline{d}(n,1) \end{bmatrix} = \begin{bmatrix} \underline{r}'(1,n) \\ \underline{r}'(n,1) \end{bmatrix}. \tag{B4}$$

Define $S = HJ$ and notice that S is Toeplitz. Since a Hankel matrix is symmetric, that is, $H^T = H$ and $S^T = (HJ)^T = J^T H = JH$, we therefore conclude that the coefficient matrix in (B4) consists of four Toeplitz matrices. We now apply the interleaving operation on (B4)

$$Q\begin{bmatrix} T & S \\ S & T \end{bmatrix} Q^T Q \begin{bmatrix} \underline{d}(1,n) \\ \underline{d}(n,1) \end{bmatrix} = Q \begin{bmatrix} \underline{r}'(1,n) \\ \underline{r}'(n,1) \end{bmatrix} \tag{B5}$$

and we may group (B5) in terms of $(2)\times(2)$ matrices:

$$\begin{bmatrix} R(0) & R(-1) \cdots R(-n+1) \\ R(1) & R(0) \quad \cdots \quad \vdots \\ \vdots \quad & \vdots \quad \cdots \quad \vdots \\ R(n-1) & \cdots \quad \cdots \quad R(0) \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \underline{p}(n) \end{bmatrix} = \begin{bmatrix} r'(1) \\ r'(2) \\ \vdots \\ r'(n) \end{bmatrix},$$

where

$$R(i) = \begin{bmatrix} r(i) & r(n+1-i) \\ r(n+1-i) & r(i) \end{bmatrix}, \quad i = 0,n-1,$$

$$\underline{p}^T = [d(i)d(n-i)], \quad i = 1,n,$$

$$\underline{r}^T = [r(i)r(n-i)], \quad i = 1,n.$$

What we have done is to transform an $(n)\times(n)$ system of equations that involves the sum of a Toeplitz plus a Hankel matrix to a $(2n)\times(2n)$ system of equations that has a block Toeplitz form. This will allow us to use the fast block Levinson algorithm to solve the equations efficiently.

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