國立交通大學

資訊科學與工程研究所

碩 士 論 文

建構有優美標號或α標號的圖

On the construction of graphs with graceful labeling and α-labeling

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中 華 民 國 九 十 八 年 十二 月

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摘要

另 G 為一個簡單圖(simple graph)，G 上的端點標號(vertex labeling)所指的是一個 vertex 的函數 f 對應到一些數值，而 G 上的每一個 edge (u,v) 被指定個由 $f(x)$ 和 $f(y)$ 所決定的數值。如果這個 f：$V(G) \rightarrow \{0, 1, ...,m\}$為單射，所指定 edge (u,v) 的數值為$|f(x)-f(y)|$，並且所有的 edge 都被指定不同的數值，則 f 被稱做是優美標號。如果還另外存在一個邊界數值(boundary value) k，使每一個 edge(u,v) 都能滿足 $f(u) \leq k < f(v)$或 $f(v) \leq k < f(u)$的條件，我們就稱 f 叫做是 α 標號。

我們定義兩種圖型 $P_n^{(G_1,\ldots,G_n)}$ 以及 $C_n^{(G_1,\ldots,G_n)}$，並使用建構的方法去建造他們。我們的研究結果也包含了一些目前已知的結果。

On the construction of graphs with graceful labeling and α-labeling

Student：Mu-Yin Chi Advisor：Dr. Shi-Chun Tsai

Institute of Computer Science and Engineering

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ABSTRACT

Let G be a simple graph with m edges and let $f : V(G) \rightarrow \{0,1, ...,m\}$ be an injection. The vertex labeling is called a graceful labeling if every edge (u,v) is assigned an edge label $|f(x) - f(y)|$ and the resulting edge labels are mutually distinct. A graph possessing a graceful labeling is called a graceful graph. With an additional property that there exists an boundary value k so that for each edge (u,v) either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$, the graceful labeling is called an α−labeling.

One approach about graph labeling is to construct larger graphs from smaller graphs which have some required properties. For this, starting with a graph that possesses α- labeling is a common approach. In this thesis, we define new families of graphs and prove that they have graceful labelings or α-labelings, ex : $P_n^{(G_1,\ldots,G_n)}$ and $C_n^{(G_1,\ldots,G_n)}$. Moreover, our results generalize some previous results.

Acknowledgements

I am grateful to my advisor, Dr. Shi-Chun Tsai, for his guidance and encouragement. I also thank my family and my friends for their spiritual support. Also, I am obliged to all members in CCIS lab. Specially, I deeply appreciate Chia-Jong's help and discussion.

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Chapter 1

Introduction

Let $G = (V, E)$ be a simple graph. A *vertex labeling* of G is an assignment f of labels to the vertices which induces for each edge $uv \in E(G)$ a label, depending on the vertex labels $f(u)$ and $f(v)$. Suppose that G has m edges. Let $f : V(G) \to \{0,1,\ldots,m\}$ be an injection. The vertex labeling is called a *graceful labeling* (or *β-labeling*) if each edge uv receives a distinct absolute value $|f(u) - f(v)|$ as its label. A graph possessing a graceful labeling is called a *graceful graph*.

The concept of graceful graphs was first studied by Ringel [20] and then by Rosa [19]. Rosa was working on Ringel's conjecture, which says that K_{2n+1} (the complete graph with $2n+1$ vertices) can be decomposed into $2n+1$ subgraphs isomorphic to a tree with n edges. Rosa showed that if every tree has a graceful labeling then Ringel's conjecture is true. Golomb [13] provided a precise definition of graceful graphs when he addressed the problem of numbering a graph.

In Figure 1.1, we consider a complete graph K_4, with vertices labeled $\{0,1,4,6\}$ and edges labeled $\{1,2,3,4,5,6\}$. Since these edge labels are distinct and K_4 has six edges, this labeling is graceful and K_4 is a graceful graph.

In 1966 Rosa [19] defined *α-labeling* to be a graceful labeling with an additional property that there exists an integer k so that for each edge uv either $f(u) \le k < f(v)$ or $f(v) \le k < f(u)$. Some people also named such labeling *balanced labeling* or *interlaced labeling*. The integer k with the property that for any edge uv either $f(u) \le k < f(v)$ or $f(v) \le k < f(u)$ is called the *boundary value* of f. It follows that such k must be

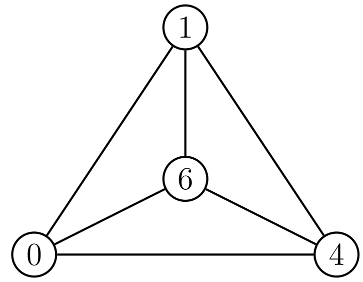

Figure 1.1: An example of graceful graph.

the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an $\alpha-$labeling is necessarily bipartite and therefore can not contain cycles of odd length. Figure 1.2 is an example of α-labeling with k=1.

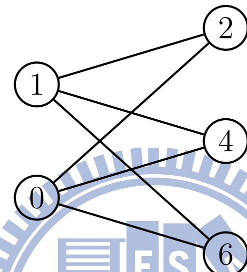

Figure 1.2: An example of an α-labeling with k=1.

Graph labeling has been proved useful in the development of the theory of graph decompositions. Especially graphs with α-labeling yield broader graph decomposition applications than other labelings. Let $G_1, G_2, ..., G_n$ be the subset of H, we say $\{G_1, G_2, ..., G_n\}$ is a decompositions of H, if it satisfies the following three conditions: (1) $V(G_i) = V_i \subseteq V(H)$, for all $i \in \{1, 2, ..., n\}$; (2) $E(G_1) \cup E(G_2) \cup ... \cup E(G_n) = E(H)$; (3) $E(G_i) \cap E(G_j) = \varnothing$, for $1 \leq i \neq j \leq n$. If $\{G_1, G_2, ..., G_n\}$ is the decomposition of H and $G_i \cong G$, for $i = 1, 2, ..., n$, then H has *G-decomposition*. By a decomposition R of the complete graph K_n we say that R is an *edge-disjoint* decomposition, if R is a set of subgraphs such that any edge of the graph K_n, belongs to exactly one of the subgraphs of R. A decomposition R of a graph K_n is said to be *cyclic*, if the following holds: if R contains a graph G, then it contains also the graph G' obtained by turning G. Rosa [19], for instance, showed that if a graph G with n edges has an α-labeling, then there exists a cyclic decomposition of K_{2kn+1} into subgraphs isomorphic to G, where k is an arbitrary natural number.

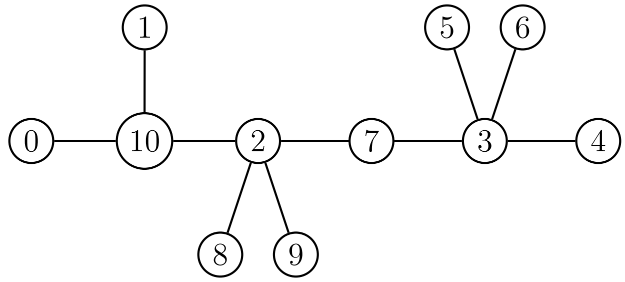

Figure 1.3: An graceful labeling on a caterpillar.

Graceful Trees. The statement "every tree has a graceful labeling" is well known as *Graceful Tree Conjecture* or *Ringel-Kotzig-Rosa Conjecture*, which has been conjectured by Rosa in 1967 [19]. To date, no proof or disproof of the conjecture is found, but several classes of trees are shown to be graceful. *Caterpillars*, as in Figure 1.3, defined to be a tree such that if all leaf vertices and their incident edges are removed, the remainder of the graph forms a path and were shown to be graceful early on by Rosa [19], it can be labeled using a similar strategy as for paths. *Balanced Trees*, which is obtained if we attach to every node of T a tree which is a copy of T', and *complete binary trees* are also proved to be graceful in 1973 by Stanton and Zarnke [23]. Chen, Lü and Yeh [8] showed that *firecrackers* (one end vertex from every stars connected in a path) are graceful. They conjectured that all *banana trees*, a graph obtained by connecting a vertex v to one leaf of each of any number of stars, are graceful. Hrnciar and Monoszová defined a *generalized banana tree*, as in Figure 1.4, which include banana trees and proved that generalized banana trees are graceful. An *Olive tree*, a collection of i paths joined in a vertex, where the ith path is of length i, is also proved to be graceful by Pastel and Raynaud [18] in 1978 as in Figure 1.5. In addition, trees of diameter at most 5 [15] and other special classes of trees have been shown to be graceful.

The concept of *joint sum of graceful trees* was given by Jin et. al. [16] in 1993. Given two trees T and R, the joint sum of T and R is denoted by $\langle T + R\rangle$ and formed by connecting certain vertex of T with a proper vertex of R. They proved that the joint sum $\langle T + 2R\rangle$ is graceful. They also defined a tree called *glue tree*, which was defined earlier in 1966 by Rosa [19] to be a tree with an α-labeling, and proved that given a glue tree R' and a graceful tree T, the joint sum of this two trees $\langle T + R'\rangle$ is graceful.

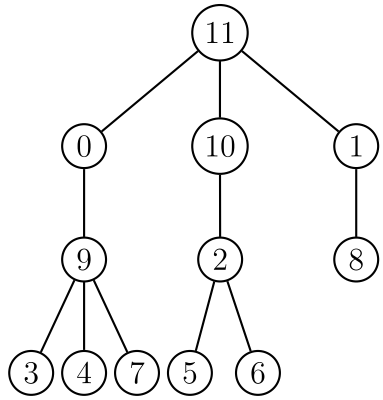

Figure 1.4: A banana tree with graceful labeling.

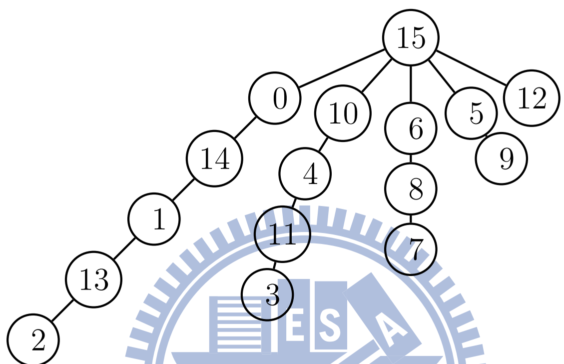

Figure 1.5: An example of olive tree.

Graceful Graphs. Several classes of graphs other than trees have been considered, and many of them have been proved to be graceful or not. Among graceful graph problems, cycle-related graphs have been the major focus of attention. Rosa [19] observed that the C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$. Abhyanker [1] brings up the idea of *unicyclic graphs*, i.e. graphs with exactly one cycle. A vertex of a graph is said to be *pendant* if its neighborhood contains exactly one vertex and an edge of a graph is said to be pendant if one of its vertices is a pendant vertex. Abhyankar proved that the result of identifying one vertex of C_4 with the root of the olive tree with $2n$ branches and the result of attaching any number of pendant edges to the union point are both graceful graph. Abhyankar also proved that by identifying an adjacent vertex on C_4 with the end point of the path P_{2n-2} is graceful. Given a graph G with n vertices and a graph H, a *corona graph* $G \odot H$ is obtained from one copy of G and n copies of H, by connecting the k^{th} vertex of G

with every vertex in the k^{th} copy of H. Frucht [11] proved that any cycle with a pendant edge attached at each vertex, i.e. the corona $C_n \odot K_1$, is graceful. Figure 1.6 shows the example of $C_8 \odot K_1$. Bu, Zhang and He [2] proved that $C_n \odot K_n$ is graceful. Barrientos [4] also defined *hairy cycle* as a unicyclic graph other than a cycle in which the deletion of any edge of the cycle results in a caterpillar and proved that all hairy cycles are graceful. Truszczyński [24] proved that dragon, which formed by joining the end point of a path to a cycle, is graceful and conjectured that all unicyclic graphs except C_n, $n \equiv 1$ or 2 (mod 4), are graceful. Wu [25] proved that, if G is a bipartite graceful graph, then P_n^G, for any n, has a graceful labeling and if G_i, for all i, has an α-labeling with the same edge number and each pair of G_{2i-1} and G_{2i}, for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $P_n^{(G_1, G_2, ..., G_n)}$.

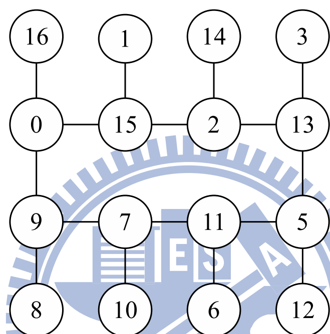

Figure 1.6: $C_8 \odot K_1$.

Graphs with an α-labeling. Rosa [19] observed that cycle C_n with $n \equiv 0$ (mod 4), caterpillar and P_n for all n both have α-labeling. Rosa also showed that $K_{m,n}$ has an α-labeling for all positive integers m and n. Figueroa-Centeno et. al. [12] showed that the one point union of 2, 3, or 4 copies of C_m, for $m \equiv 0$ (mod 4) and the one point union of 2 or 4 copies of C_m, for $m \equiv 2$ (mod 4) admits an α-labeling. They conjecture that the one point union of n copies of C_m admits an α-labeling if and only if $mn \equiv 0$ (mod 4). Snevily [22] defined $C_n^{P_m}$ to be a graph formed by adding a pendant path P_m to each vertex of the cycle C_n and prove that all graphs of the form $C_{4n}^{P_m}$ have α-labeling. Figure 1.7 shows an example of $C_8^{P_2}$.

Various classes of graphs have been proved to be graceful or non-graceful. There are only some techniques for finding graceful labeling of a given graph. First is a constraint

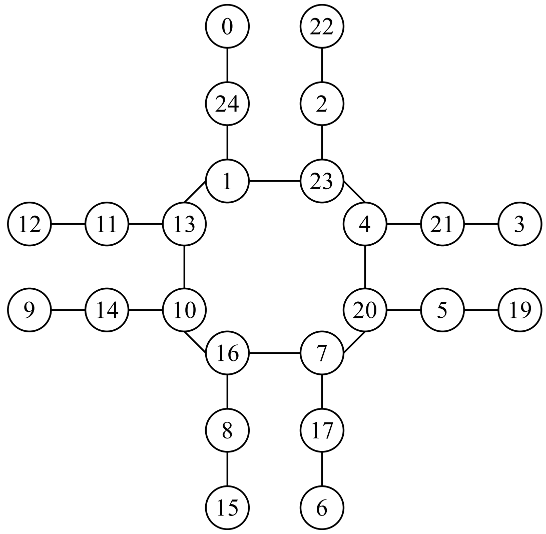

Figure 1.7: An example of $C_8^{P_2}$.

programming approach [21], second is based on integer programming [10] and the other uses a metaheuristic algorithm(Ant Colony Optimization) [17] to solved the graceful labeling problem. Many of the results about graph labeling are collected and updated regularly in a survey by Gallian [14].

Our Results. One approach in graph labeling papers is to build up graphs from smaller graphs which have desired labeling with particular properties: for instance, graph product and join of graphs. In this situations, starting with a graph which possesses an α-labeling is a common approach. Because of the particular properties of α-labeling, we also give some general ideas of constructing a larger graph. We summarize our results as follows :

1. Many trees have been proved to be graceful with root labeled zero (or maximum), such as symmetrical tree, balanced tree and trees of diameter five. We observed that the results of joining root of any of these trees with a vertex of C_n with $n \equiv 0$ (mod 4), as in Figure 2.7, is graceful. Since olive tree has also a labeling with root labeled zero, this covers the result from Abhyanker [1], which proved that a graph formed by identifying one vertex of C_4 with the root of the olive tree with $2n$ branches is graceful. The same ideas also applies to a graph formed by identifying a vertex

of C_n, $n \equiv 0 \pmod 4$, with the end point of a path P_m with any positive number m. We also answer an open problem from Cahit [6]: "Are there always graceful numbering with the largest number at the root of a rooted tree?", and prove that a graph formed by identifying one vertex of C_4 with the root the tree which does not have a graceful labeling with the number zero (or the largest number) at the root of it can still be gracefully labeled.

2. We generalize the results of Wu [25], which says that if graph G is a bipartite graceful graph, then P_n^G is graceful. We show that, given a graph $P_n^{(G_1,\ldots,G_n)}$, if G_i, $i = 1,\ldots,n$, is graceful bipartite with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, for $i = 1,\ldots,\lfloor \frac{n}{2} \rfloor$, then $P_n^{(G_1,\ldots,G_n)}$, for any n, has a graceful labeling and $P_{2n}^{(G_1,\ldots,G_{2n})}$, for any n, has an α-labeling.

3. Snevily [22] defined $C_m^{P_n}$ to be a graph formed by adding a pendant path P_n to each vertex of the cycle C_m and prove that all graphs of the form $C_{4m}^{P_n}$ have α-labeling. We define C_n^G to be a graph formed by connecting the start vertex with the end vertex of path P_n in P_n^G with an edge. In other words, a deletion of any edge in the center cycle C_n of C_n^G results in a P_n^G. We show that if G is a bipartite graceful graph, then C_{4n}^G, for any n, has an α-labeling, and C_{4n+3}^G, for any n, has a graceful labeling. We also show that, if each pair of graphs G_{2i-1} and G_{2i}, for $i = 1,\ldots,\lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $C_n^{(G_1,G_2,\ldots,G_n)}$, for any $n \equiv 0 \pmod 4$, has an α-labeling and $C_n^{(G_1,G_2,\ldots,G_n)}$, for any $n \equiv 3 \pmod 4$, is graceful. Since a path P_n has an α-labeling, our result covers the result proved by Snevily. One small corollary is that, if $n \equiv 0 \pmod 4$, $C_n \odot mK_1$, for any m, has an α-labeling and if $n \equiv 0$, 3 (mod 4), $C_n \odot mK_1$, for any m, is a graceful graph. This covers the result proved by Frucht [11], which says that $C_n \odot K_1$ is graceful.

Despite the large number of papers, there are relatively few general results or methods on constructing graceful graphs or graphs with α-labeling. Indeed, most of the results focus on particular classes of graphs or trees. In this thesis, we give some methods on constructing graphs with graceful labeling or α-labeling. Our results not only show some new families of graceful graphs and graphs with α-labeling but also covers some solved problems. Furthermore, we will summarize the open problems.

Chapter 2

Preliminaries

In this chapter, we define some new families of graphs and give some constructing methods which we will use in this thesis and also the necessary conditions of graceful labeling and α-labeling.

2.1 Definition

Definition 1 (Graceful labeling). *Let G be a simple graph with m edges and let f:$V(G)$ $\rightarrow$ $\{0,1,\ldots,m\}$ be an injection. The vertex labeling is called a graceful labeling if every edge (u,v) is assigned an edge label $|f(u) - f(v)|$ and the resulting edge labels are mutually distinct.*

Definition 2 (α-labeling and boundary value). *If G has a graceful labeling f and the vertex set $V(G)=X \cup Y$ can be properly partitioned : $E(G) \subseteq \{(u,v)|u \in X, v \in Y\}$, $X = \{x \in V(G)|\ f(x) \leq k\}$ and $Y = \{y \in V(G)|\ f(y) > k\}$ for some value k, then f is called an α-labeling or α-valuation and k is called the boundary value of graph G.*

Unlike common graceful graphs, a graceful graphs which admits an $\alpha-$labeling has an special characteristic.

Fact 1. *Let G be a graph with an $\alpha-$labeling. Since G has an $\alpha-$labeling, the vertex set $V(G)=X \cup Y$ can be properly partitioned : $E(G) \subseteq \{(u,v)|u \in X, v \in Y\}$, $X = \{x \in V(G)|$ $f(x) \leq k\}$ and $Y = \{y \in V(G)|f(y) > k\}$ for some value k. By adding a positive integer*

t to the Y part, all edge labels of G will be shifted with t and get a new graph G' with labeling Θ. Note that Θ is no more a graceful labeling.

Proof. Formally, the labeling $\Theta(v)$ is defined as follows:

$$\Theta(v) = \begin{cases} f(v), & \text{if } v \in X \text{ and } v \in V(G) \\ f(v)+t, & \text{if } v \in Y \text{ and } v \in V(G) \end{cases}$$

Consider the edge label set $F = \{f(u) - f(v) \mid u \in Y,\ v \in X,\ u,\ v \in V(G)\}$. After adding t to the Y part we have the new edge label set $F' = \{f(u)+t-f(v) \mid u \in Y,\ v \in X,\ u,\ v \in V(G)\}$. Every edge label was shifted by t as G' in Figure 2.1.

□

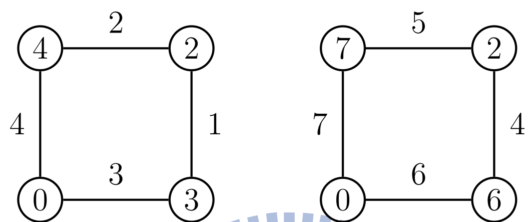

Figure 2.1: Graph G(left) and graph G'(right) with t=3.

In this thesis, we focus on the construction of graphs. Here we define some methods of constructing graphs and give some definitions for new families of graphs.

Definition 3 (One point union of two graphs)**.** *Given two graphs G and G', the one point union of G and G', $G \circ G'$ as in Figure 2.2, is to regard one vertex u in G and another vertex v in G' as the same vertex in $G \circ G'$. Notice that, there exist only two possible choices of the union pairs (u,v). If G and G' have the labeling f and f' respectively, these two pairs will be either $f(u) = 0$ and $f'(v) = k$ or $f(u) = m$ and $f'(v) = k+1$. The number of vertices $|V(G \circ G')| = |V(G)| + |V(G')| - 1$.*

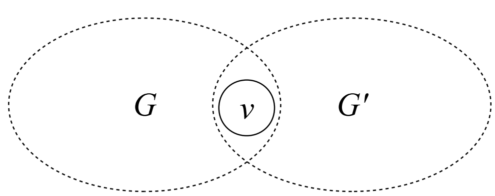

Figure 2.2: $G \circ G'$.

Definition 4 (One point union between more graphs)**.** *Given graphs* $G_1, G_2, ..., G_n$*, the one point union between graphs* G_1*,* G_2*,...,*G_n*, denoted by* $G_1 \circ G_2 \circ ... \circ G_n$*, as in Figure 2.3, is to regard one vertex in* G_i *and another vertex in* G_{i+1} *as the same vertex in the new graph, for i=1,...,n-1. Note that, the number of vertices* $|V(G_1 \circ ... \circ G_n)| = |V(G_1)| + ... + |V(G_n)| - (n-1)$*.*

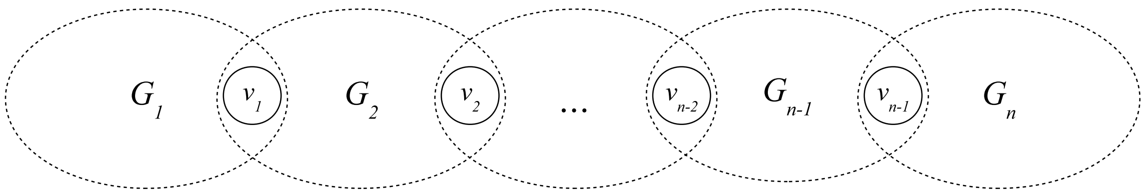

Figure 2.3: $G_1 \circ G_2 \circ ... \circ G_n$.

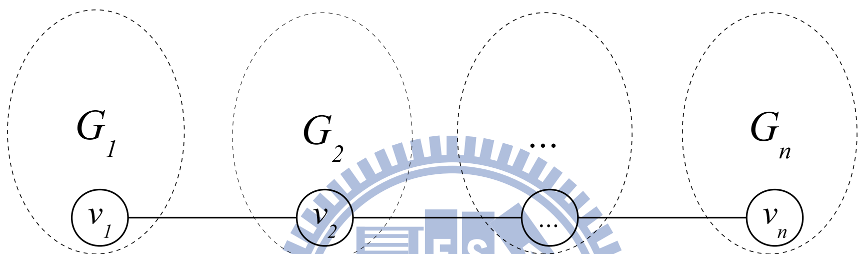

Figure 2.4: P_n^G.

Definition 5. P_n^G *is a graph formed by connecting n copies of graceful graph G by using n-1 edges as in Figure 2.4. Note that, the connecting vertex* v_i *of each copy* G_i*, for any* $i = 1, ..., n$*, must have vertex label "0" in the original graph.* $P_n^{(G_1, G_2, ..., G_n)}$ *is when the graphs* $G_1, G_2, ..., G_n$ *are instead of copies of G but n different graphs.*

Definition 6. *We define* C_n^G *to be a graph formed by connecting the start vertex with the end vertex of path* P_n *in* P_n^G *with an edge. In other words, a deletion of any edge in the center cycle* C_n *of* C_n^G *results in a* P_n^G*. See Figure 2.5.*

Definition 7 (Corona)**.** *Assume G has n vertices. The corona of G and H, denoted by* $G \odot H$*, as in Figure 2.6, is a graph obtained from one copy of G and n copies of H, by connecting the* k^{th} *vertex of G with every vertex in the* k^{th} *copy of H.*

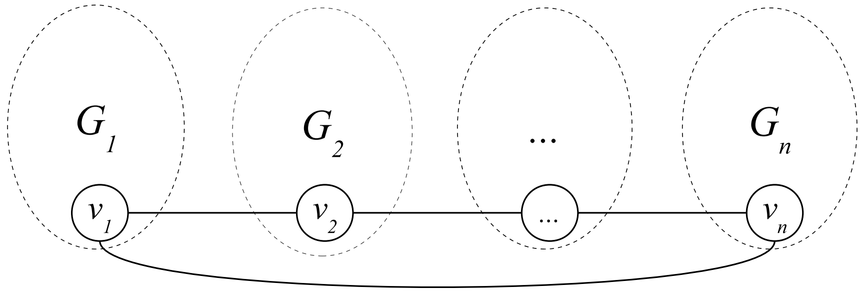

Figure 2.5: C_n^G.

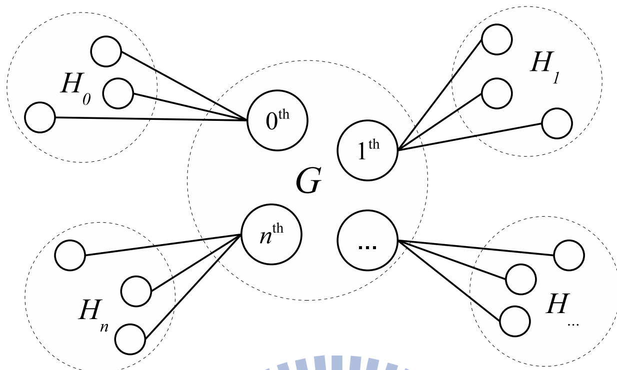

Figure 2.6: $G \odot H$.

Given a tree T, Rosa [19] defined the *base of T*, a tree obtained from T by omitting all its end vertices and end edges and a *snake*, a tree with exactly two end vertices or the tree consisting a unique vertex having no edges. If a tree T is a snake or its base is a snake, it is said to be a caterpillar. Rosa proved that all caterpillars have α-labeling.

Theorem 1 ([19])**.** *If a tree T is a snake or its base is a snake, then there exists an α-labeling of T.*

Rosa [19] showed that cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4). He observed that C_n has an α-labeling if and only if $n \equiv 0$ (mod 4). In this thesis, we use the graceful labeling of Figure 2.7 for a cycle C_n with $n \equiv 0$ (mod 4).

2.2 Necessary Conditions

Rosa [19] identified essentially three reasons why a graphs fails to be graceful: (1) G has "too many vertices" and "not enough edges"; (2) G has "too many edges"; (3) G

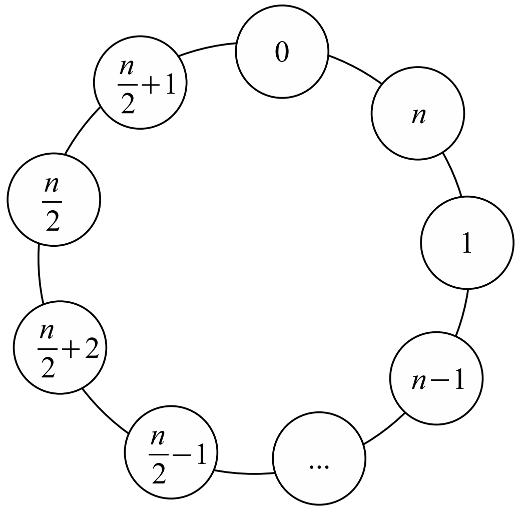

Figure 2.7: Graceful labeling for C_n with $n \equiv 0 \pmod 4$.

has the "wrong parity." As an example of the third condition Rosa showed that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Golomb [13] also brought up some necessary conditions for graceful graphs:

Theorem 2 ([13])**.** *Let G be a graph with n nodes and e edges. A necessary condition for G to be graceful is that it be possible to partition the nodes into two sets $\mathcal{E}$ and $\mathcal{O}$, such that the number of edges connecting nodes in $\mathcal{E}$ with nodes in $\mathcal{O}$ is exactly $\lfloor \frac{(e+1)}{2} \rfloor$.*

Definition 8 (Binary Labeling)**.** *If graph G has a binary labeling, then there exists a successful partition of the nodes of G into sets $\mathcal{E}$ and $\mathcal{O}$ with $\lfloor \frac{(e+1)}{2} \rfloor$ interconnecting edges.*

Theorem 3 ([13])**.** *Suppose the integers, not necessarily distinct, are assigned to the nodes of a graph G, and each edge of G is given an edge number equal to the absolute difference of the node numbers at its end points. Then the sum of the edge numbers around any circuit of G is even.*

Theorem 4 ([13])**.** *Let G be an Eulerian graph, that is, with an even number of edges at each node, with e edges. A necessary condition for G to be graceful is that $\lfloor \frac{(e+1)}{2} \rfloor$ to be even. That is, if $e \equiv 1 \pmod 4$ or $e \equiv 2 \pmod 4$, then G cannot be graceful. In fact, G cannot be binary labeled.*

Theorem 5 ([13])**.** *If $n > 4$, the complete graph K_n cannot be graceful.*

Theorem 6 ([13])**.** *Let T be a tree with n nodes and $e = n - 1$ edges. Then there exists a binary labeling of T for which $\lfloor \frac{n}{2} \rfloor$ of the nodes are odd(set $\mathcal{O}$) and $\lfloor \frac{(n+1)}{2} \rfloor$ of the nodes are even(set $\mathcal{E}$).*

Given a graph H, Golomb [13] defined $G(H)$ to be the largest integer assigned to any vertex of H and the goal is to minimize the value $G(H)$. If H has m edges, we have the general lower bound $G(H) \geq m$. A graph H for which $G(H) = m$ will be called a graceful graph, and the labeling which achieves $G(H) = m$, a graceful labeling.

Theorem 7 ([13])**.** *If H is any graph, and if H' is a subgraph of G, then $G(H') \leq G(H)$.*

Theorem 8 ([13])**.** *If H is any graph with n nodes, then $G(H) \leq G(K_n)$. This results adds further importance to the study of $G(K_n)$, which is thus the least upper bound on H for all graphs on n nodes.*

Chapter 3

Union of Graphs

In this chapter, we summarize and generalize the results from other papers and also answer an open problem.

Truszczyński [24] proved that If G is a graceful graph and G' is a graph with an $\alpha-$labeling, then the one point union of G and G', denoted by $G \circ G'$, is a graceful graph.

Theorem 9 ([24]). *Let G and H be graphs with disjoint sets of vertices. Assume that G has a graceful labeling g and $v \in V(G)$ has labeling "0". H has an $\alpha-$labeling h with boundary value "k" and $w \in V(H)$ has labeling "k". Then the graph F obtained by identifying v and w in $G \cup H$ is graceful.*

Let us explain the proof of Truszczyński [24] in our way:
Assume that G has m edges and a graceful labeling g while H has m' edges and an $\alpha-$labeling h. Since h is an α-labeling, by Definition 2 there exists a boundary value k satisfying that for each $(u,v) \in E(H)$, either $h(u) \le k < h(v)$ or $h(v) \le k < h(u)$. Then we partition the vertex set of H into two parts $V(H) = X \cup Y$ where,

$$X = \{v \in V(H) : h(v) \le k\},$$
$$Y = \{v \in V(H) : h(v) > k\}.$$

In other words, $h(X) \subseteq \{0, 1, ..., k\}$ and $h(Y) \subseteq \{k+1, ..., m'\}$. Let F be the graph $G \circ H$. If G has n vertices and H has n' vertices, then $V(F) = n + n' - 1$. Note that $|E(F)| = m + m'$.

Define the vertex labeling $f : V(F) \to \{0, 1, ..., m + m'\}$ as follows:

$$f(v) = \begin{cases} g(v) + k, & \text{if } v \in V(G) \\ h(v), & \text{if } v \in X \\ h(v) + m, & \text{if } v \in Y \end{cases}$$

Notice that, if we change the vertex labeling f by adding "$k + 1$" to $g(v)$, $v \in V(G)$, instead of k, then f stays an injective function. By sharing a common vertex $u \in G$ and $v \in H$, we get a graceful graph. There exist only two possible choices of such pairs uv. If $u \in G$ and $v \in H$, these two pairs will be either $g(u) = 0$ and $h(v) = k$ or $g(u) = m$ and $h(v) = k + 1$.

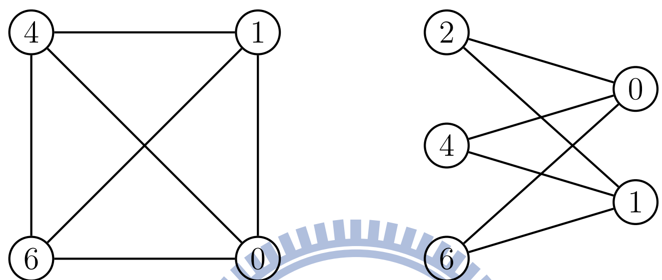

Figure 3.1: Two graceful graphs G(left) and H(right).

Example 1. *G is a complete graph with 4 vertices and 6 edges and H is a complete bipartite graph $K_{2,3}$. G is a graceful graph and H is a graceful graph with an $\alpha-$labeling. Figure 3.1 shows the labeling g and h for G and H respectively. We partition graph H into two parts and the boundary value $k = 1$ so that for each edge $(u, v) \in H'$ either $h(u) \le k < h(v)$ or $h(v) \le k < h(u)$. The labeling f for the new graph $F = G \circ H$ is defined as follows:*

$$f(v) = \begin{cases} g(v) + 1, & \text{if } v \in V(G) \\ h(v), & \text{if } v \in X \\ h(v) + 6, & \text{if } v \in Y \end{cases}$$

The new labeling $f(v)$ is a graceful labeling and $G \circ H$ is a graceful labeling as in Figure 3.2.

Given graphs $G_1, G_2, ..., G_n$, the one point union between graphs G_1, G_2,...,G_n, denoted by $G_1 \circ G_2 \circ ... \circ G_n$, is to regard one vertex in G_i and another vertex in G_{i+1} as the same vertex in the new graph, for $i = 1, ..., n - 1$. Since the union of a graceful graph and a graph with an $\alpha-$labeling is a graceful graph. If there exists a graceful graph G and

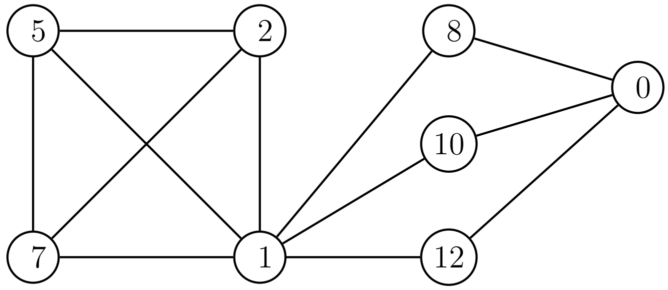

Figure 3.2: $G \circ H$.

n graphs $G_1, G_2, ..., G_n$ having α−labeling, by doing $n-1$ times of such union, we get $G \circ G_1 \circ G_2 \circ ... \circ G_n$ as a graceful graph. The above observation leads us to the following Corollary:

Corollary 1. *Given a graceful graph G and n graphs $G_1, G_2, ..., G_n$, which all admit α−labeling, then $G \circ G_1 \circ G_2 \circ ... \circ G_n$ is a graceful graph.*

Proof. Since G is graceful and G_1 has an α−labeling, by Theorem 9, we get a graceful graph $H_1 = G \circ G_1$. Since H_1 is graceful and G_2 has an α−labeling, by using the same idea, we get another graceful graph $H_2 = H_1 \circ G_2$, which can be considered as $G \circ G_1 \circ G_2$, and so on. In other words, by considering graph G as H_0, let $H_{i+1} = H_i \circ G_{i+1}$, for $i = 0, 1, ..., n-1$. We get a graceful graph $H_n = G \circ G_1 \circ G_2 \circ ... \circ G_n$. □

Jin *et. al.* [16] gives the idea about joining a graceful tree T and a graceful tree R which admits an α−labeling with an edge. Since Rosa [19] observed that caterpillar has an α-labeling, we show that, a tree formed by connecting T with R by using a caterpillar is graceful. Furthermore, instead of the restriction to trees, T and R can be any graceful graphs and one of which admits an α−labeling.

Corollary 2. *Let G_1 be a graceful graph, G_2 be a caterpillar and G_3 be a graph with an α−labeling. The graph obtained by using a caterpillar G_2 to connect graph G_1 with graph G_3, considered as $G_1 \circ G_2 \circ G_3$, is also a graceful graph.*

Proof. Since G_2 and G_3 both have α−labeling, by Corollary 1, we obtain that $G_1 \circ G_2 \circ G_3$ is graceful and G_1 and G_3 are both connected at the two ends of the caterpillar G_2.

□

Let G_i be a graph with an α−labeling and H_j be a caterpillar. If we wish to connect G_i, $i = 1 \sim n$, by using $n-1$ caterpillars H_j, $j = 1 \sim n-1$, that is $G_1 \circ H_1 \circ G_2 \circ H_2 \circ \ldots \circ H_{n-1} \circ G_n$, then the union pair uv between every $G_i \circ H_i$, $i = 1 \sim n$, will be like step1. in Corollary 2 and between every $H_j \circ G_{j+1}$, $j = 1 \sim n-1$, will be like step2. in Corollary 2.

Example 2. *Given graceful graphs G and H as in Figure 3.1, where H admits an α−labeling, and a caterpillar as in Figure 3.3. By Corollary 2, we use caterpillar to connect G and H and the result will be a graceful graph as in Figure 3.4.*

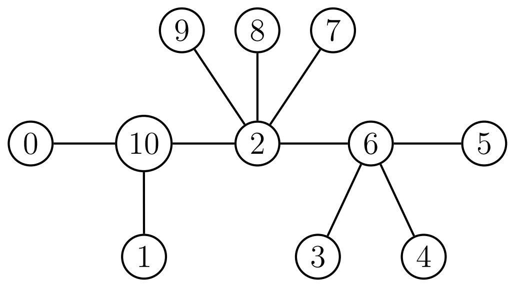

Figure 3.3: Caterpillar.

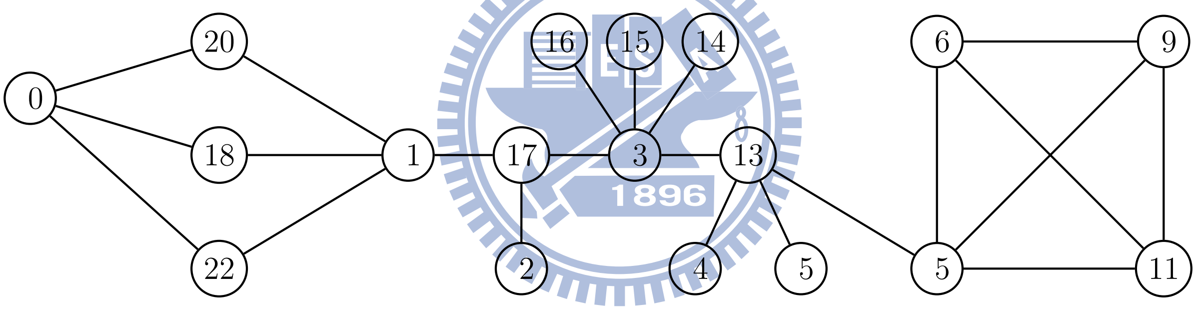

Figure 3.4: Connect two graphs with a caterpillar.

Abhyankar [1] proved that by identifying one vertex of C_4 with the root of an olive tree with $2n$ branches, it results a graceful graph. He also prove that by identifying an adjacent vertex on C_4 with the end point of a path P_{2n-2} is graceful. Since Rosa [19] showed that cycle C_n, with $n \equiv 0 \pmod 4$, has an α-labeling, we know that, given a graceful rooted tree T with root labeled zero (or maximum) and cycle C_{4n} with any positive number n, a graph formed by identifying one vertex of C_{4n} with the root of T results a graceful graph. The same idea also applies to identifying a vertex on cycle C_n, with $n \equiv 0 \pmod 4$, with the end point of a path P_m with any positive number m.

Corollary 3. *Let T be a graceful rooted tree with root labeled zero (or maximum). A graph formed by identifying one vertex of cycle C_{4n}, for any $n \geq 1$, with the root of T is a graceful graph.*

Proof. We claim that $T \circ C_{4n}$ is graceful. Since Rosa [19] proved that C_{4n} has an α-labeling, by Theorem 9 we know that $T \circ C_{4n}$ is graceful.

□

Example 3. *Given wo graphs C_4 and T with graceful labeling as in Figure 3.5. Figure 3.6 is the graph formed by identifying one vertex of C_4 with the root of T.*

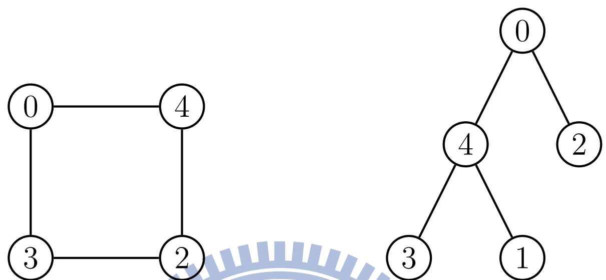

Figure 3.5: Graceful labeling of cycle C_4 and tree T.

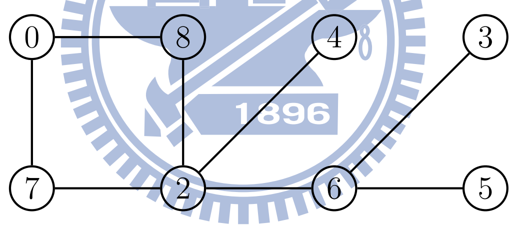

Figure 3.6: The union of C_4 and tree T.

Many rooted trees have been proved to be graceful with root labeled zero (or maximum), for example: symmetrical trees, balanced trees and trees of diameter five. By Corollary 3, we conclude that by identifying the root of any of these trees with a vertex on cycle C_{4n}, for any $n \geq 1$, results a graceful graph. Since we know the importance of graceful trees which root can be labeled zero, the following open problem is asked.

In 1976 Cahit [6] brings up the **Open Problem:**
Are there always graceful numberings with the largest number at the root of a rooted tree?
We answer this question here:

Theorem 10. *There exists a rooted tree T, which does not have a graceful labeling with the largest number at its root.*

Proof. To prove it, we give an counterexample as in Figure 3.7, a tree T rooted in v_0. Here we prove that T does not have such graceful labeling. Since tree T has 6 vertices and 5 edges, we consider the labeling $f : V(T) \to \{0, 1, ..., 5\}$.

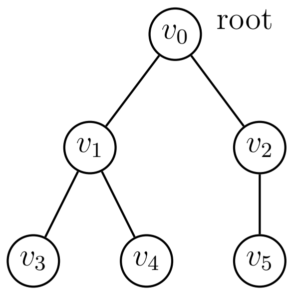

Figure 3.7: A tree T rooted at v_0.

First, let $f(v_0) = 5$. Note that an edge $(u, v) \in E(T)$ with edge label $f((u, v)) = 5$ can only be induced by $f(u) = 5$ and $f(v) = 0$ and edge label $f((u, v)) = 4$ can only be induced by $f(u) = 5$ and $f(v) = 1$ or $f(u) = 4$ and $f(v) = 0$. Hence we have the following cases:

Case 1. $f(v_1) = 0$. (see the left tree in Figure 3.8)

Case 1.1. Let $f(v_3) = 4$ and get the edge label $f((v_1, v_3)) = 4$.
We let $f(v_4) = 1$, then $f((v_1, v_4)) = 1$ and we will get $f((v_2, v_5)) = 1$ by labeling either $f(v_2) = 3$ and $f(v_5) = 2$ or $f(v_2) = 2$ and $f(v_5) = 3$, which implies f is not a graceful labeling.
Since v_2 cannot be labeled as 1, we labeled $f(v_5) = 1$. Then we let $f(v_2) = 2$ and $f(v_4) = 3$ or let $f(v_2) = 3$ and $f(v_4) = 2$. But it both contradict the definition of graceful labeling.

Case 1.2. Let $f(v_2) = 1$ and get the edge label $f((v_0, v_2)) = 4$.
Since $f(v_3)$ and $f(v_4)$ cannot be 4, we let $f(v_5) = 4$ and get $f((v_2, v_5)) = 3$. One of v_3 and v_4 will be labeled by 3, so we will get $f((v_1, v_3)) = 3$ or $f((v_1, v_4)) = 3$, which still contradicts the graceful definition.

Case 2. $f(v_2) = 0$. (see the right tree in Figure 3.8)

Case 2.1. Let $f(v_5) = 4$ and get the edge label $f((v_2, v_5)) = 4$.

Since $f(v_1)$ cannot be 1, we let $f(v_1) = 2$ and get $f((v_0, v_1)) = 3$, By labeling $f(v_3) = 1$ and $f(v_4) = 3$, we have $f((v_1, v_3)) = 1$ and $f((v_1, v_4)) = 1$, which f is not a graceful labeling.

If we let $f(v_1) = 3$, we have $f((v_0, v_1)) = 2$. By labeling $f(v_3) = 1$, we get $f((v_1, v_3)) = 2$, so does $f(v_4) = 1$. It implies f is not a graceful labeling.

Case 2.2. Let $f(v_1) = 1$ and get the edge label $f((v_0, v_1)) = 4$.

Since $f(v_5)$ cannot be 4, we let $f(v_3) = 4$ and get $f((v_1, v_3)) = 3$. If we label $f(v_5) = 3$, then $f((v_2, v_5)) = 3$, which contradicts to the definition.

If we label $f(v_5) = 2$ and $f(v_4) = 3$, then $f((v_2, v_5)) = 2$ and $f((v_1, v_4)) = 2$, which still contradicts the graceful definition.

□

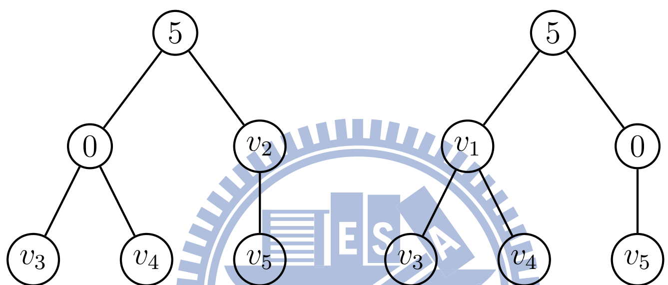

Figure 3.8: Case1(left) and Case2(right) for Theorem 10.

Note that although the tree in Figure 3.7 do not have a graceful labeling with root labeled zero (or maximum), a graph G formed by identifying the root of tree in Figure 3.7 with one vertex on the cycle C_4 can still be gracefully labeled:

Example 4. *Given a tree T as in in Figure 3.7 and a cycle C_4. Figure 3.9 is a graph formed by identifying the root of T with one vertex on the C_4.*

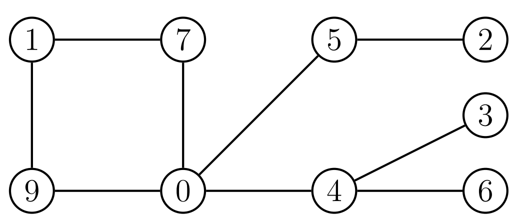

Figure 3.9: The union of C_4 and tree T.

Chapter 4

Graphs with α-labeling

In Chapter 3 we mention that Truszczyński [24] proved that the one point union of a graceful graph G and a graph H with an α-labeling results in a graceful graph. Here we prove that if G also has an α-labeling, then the result will admit an α-labeling. In other words, given two α-labeling graphs G and H, the one point union of G and H, denoted by $G \circ H$, is a graph with an α-labeling.

Theorem 11. *If G and H both have α-labeling, then $G \circ H$ is a graph with an α-labeling.*

Proof. Let F be the one point union of G and H with a graceful labeling f in Theorem 9. We obtain that F can be gracefully labeled. Here we prove that f is also an α-labeling. Let G and H have α-labeling g and h and edge number m and m', respectively. Since G and H both admit α-labeling, we partition the vertex sets of G and H, with boundary value k and k', into two parts $V(G) = X \cup Y$ and $V(G') = X' \cup Y'$, where.

$$X = \{v \in V(G) : g(v) \leq k\},\ Y = \{v \in V(G) : g(v) > k\},$$
$$X' = \{v \in V(G') : h(v) \leq k'\},\ Y' = \{v \in V(G') : h(v) > k'\}.$$

We have $g(X) \subseteq \{0, 1, ..., k\}$, $g(Y) \subseteq \{k+1, ..., m\}$, $h(X') \subseteq \{0, 1, ..., k'\}$ and $h(Y') \subseteq \{k'+1, ..., m'\}$. By Theorem 9, we add m' to every vertex in Y and add k to every vertex in $V(H)$. We get $f(X) \subseteq \{0, 1, ..., k\}$, $f(Y) \subseteq \{k+m'+1, ..., m'+m\}$, $f(X') \subseteq \{k, k+1, ..., k+k'\}$, $f(Y') \subseteq \{k+k'+1, ..., k+m'\}$. The union pair $u \in X$ and $v \in X'$ will be $f(u) = f(v) = k$, where $g(u) = k$ and $h(v) = 0$. We partition $V(H)$ into two parts $V(H) = A \cup B$, where.

$$A = X \cup X' \; ; \; f(X) \cup f(X') \subseteq \{0, 1, ..., k' + k\},$$
$$B = Y \cup Y' \; ; \; f(Y) \cup f(Y') \subseteq \{k' + k + 1, ..., m + m'\}.$$

Let $r = k' + k$. Since there exists no edge between X and Y' or between X' and Y, it satisfies the condition of α-labeling: for each edge $(u, v) \in V(F)$, either $f(u) \le r < f(v)$ or $f(v) \le r < f(u)$. As a result, $G \circ H$ is a graph with an α-labeling.

□

Example 5. *Given two graphs C_4 and $K_{2,3}$ as in Figure 4.1, which both admit α-labeling. We show $C_4 \circ K_{2,3}$ also has an α-labeling in Figure 4.2, with $X = \{0, 1, 2\}$, $Y = \{3, 5, 7, 8, 10\}$.*

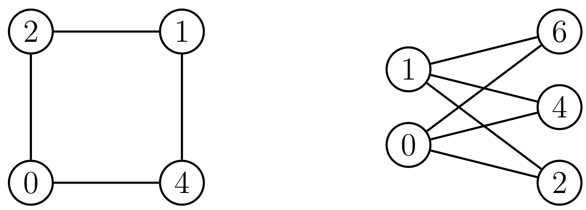

Figure 4.1: C_4 and $K_{2,3}$.

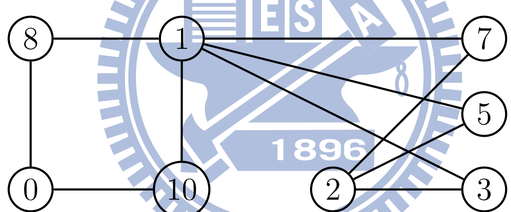

Figure 4.2: $C_4 \circ K_{2,3}$.

The above observation leads us to the following Corollary:

Corollary 4. *If graphs $G_1, ..., G_n$ all admit α-labeling, then $G_1 \circ G_2 \circ ... \circ G_n$ has an α-labeling.*

Proof. First, we obtain that $H_2 = G_1 \circ G_2$ admits an α−labeling by Theorem 11. Since H_2 has an α−labeling, using the same idea, $H_3 = H_2 \circ G_3$ considered as $G_1 \circ G_2 \circ G_3$, and so on. In other words, by considering graph G_1 as H_1, we have $H_{i+1} = H_i \circ G_{i+1}$, for $i = 1, 2, ..., n - 1$. At the end, we get $H_{n-1} = G \circ G_1 \circ G_2 \circ ... \circ G_n$ and it is a graph with an α−labeling.

□

4.1 Generalization of Other Papers

Wu [25] proved that, if G is a bipartite graceful graph, then P_n^G, for any n, has a graceful labeling and if G_i, for all i, has an α-labeling with the same edge number and each pair of G_{2i-1} and G_{2i}, for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $P_n^{(G_1,G_2,...,G_n)}$ is graceful.

Theorem 12 ([25])**.** *If G is bipartite graceful, then P_n^G, for any n, is a graceful graph.*

Theorem 13 ([25])**.** *If G_i, for all i, has an α-labeling with the same edge number and G_{2i-1} and each pair of G_{2i}, for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $P_n^{(G_1,G_2,...,G_n)}$ is graceful.*

It is clearly that if a graph has an α-labeling then it is a bipartite graceful graph. Notice that a bipartite graceful graph is not necessary to have an α-labeling. In Figure 4.3, Rosa [19] showed the minimal tree, which is bipartite but not $\alpha-$labeling.

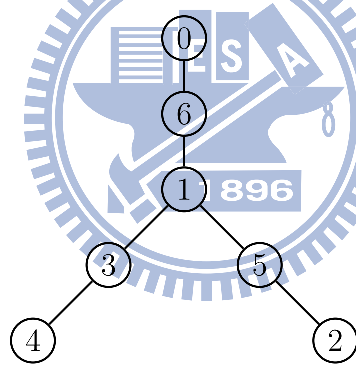

Figure 4.3: A tree has no $\alpha-$labeling.

By extending Wu's results, we prove the following theorem:

Theorem 14. *If G_i, for every i, is bipartite with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, then $P_n^{(G_1,...,G_n)}$, for any n, has a graceful labeling.*

We prove it by induction. First we know that P_1^G , which is exactly the graph G, is graceful. Then we show the following two claims, which we need for the proof of Theorem 14.

1. We prove that P_2^G has an α-labeling.

2. Assume $P_n^{(G_1,\ldots,G_n)}$ is graceful and prove that $P_{2+n}^{(G_1,\ldots,G_n)}$ is graceful.

Claim 1. *If G is a bipartite graceful graph, then P_2^G has an α-labeling.*

Proof. Assume that G has m edges. Let f and f' be the graceful labeling of graph G and its copy G'. Notice that $f = f'$. Since G is bipartite, we partition the vertex set $V(G)$ into two parts X and Y and vertex set $V(G')$ into X' and Y' respectively, with $f(X) = f'(X')$, $f(Y) = f'(Y')$, and $E(G) \subseteq \{(u,v)|u \in X, v \in Y\}$. In other words, $f(X) \cup f'(Y') \subseteq \{0,1,\ldots,m\}$ and $f'(X') \cup f(Y) \subseteq \{0,1,\ldots,m\}$. Note that $|E(P_2^G)| = 2m+1$. Define the vertex labeling $f_2 : V(P_2^G) \to \{0,1,\ldots,2m+1\}$ as follows:

$$f_2(v) = \begin{cases} f(v), & \text{if } v \in X \\ f(v) + m + 1, & \text{if } v \in Y \\ f'(v) + m + 1, & \text{if } v \in X' \\ f'(v), & \text{if } v \in Y' \end{cases}$$

Next, we show that f_2 is a graceful labeling. First we claim that f_2 is an injective function. Since $f(X) \cup f'(Y') \subseteq \{0,1,\ldots,m\}$ and $f'(X') \cup f(Y) \subseteq \{0,1,\ldots,m\}$, we have $f_2(X) \cup f_2(Y') \subseteq \{0,1,\ldots,m\}$ and $f_2(X') \cup f_2(Y) \subseteq \{m+1, m+2, \ldots, 2m+1\}$. Moreover, since f and f' are injective, the vertex labeling f_2 is an injective function.

Then, we claim that the labels of edges are distinct. The edge labels of G is denoted as $|f(u) - f(v)|$, $u,v \in V(G)$ and the edge labels of G' is denoted as $|f'(u) - f'(v)|$, $u,v \in V(G')$. We partition the edge set $E(G)$ into two sets A and B and edge set $E(G')$ as two sets C and D:

$$\begin{aligned} A &= \{(u,v) : f(u) > f(v), u \in X, v \in Y\} \\ B &= \{(u,v) : f(v) > f(u), u \in X, v \in Y\} \\ C &= \{(u,v) : f'(u) > f'(v), u \in X', v \in Y'\} \\ D &= \{(u,v) : f'(v) > f'(u), u \in X', v \in Y'\} \end{aligned}$$

Because $f(X) = f'(X')$ and $f(Y) = f'(Y')$, we know that the edge labels of $A \cup D$ are 1,...,m and of $B \cup C$ are 1,...,m. By the definition of f_2, we add $m+1$ to the vertex label of every vertex in set Y and X'. Then we have that the edge labels of $B \cup C$ are shifted by $m+1$, that is $m+2$,...,$2m+1$. As for the edge label of $A \cup D$, we get $|f(u) - f(v) - (m+1)| = m+1-(f(u)-f(v))$ for every $(u,v) \in A$ and $|f'(v)-f'(u)-(m+1)| = m+1-(f'(v)-f'(u))$

for every $(u,v) \in D$. Consider Figure 4.4, we connect vertex $v \in G$ with $v' \in G'$, where $f(v) = f'(v')$. Since v and v' will be either $v \in X$, $v' \in X'$ or $v \in Y$, $v' \in Y'$, we have $|f_2(u) - f_2(v)| = m + 1$.

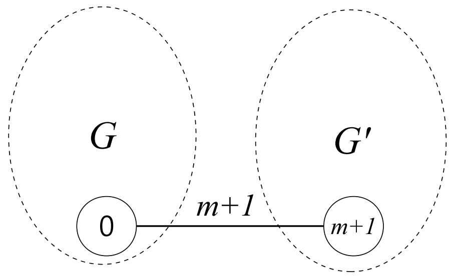

Figure 4.4: P_2^G.

Finally, we show that the labeling f_2 satisfies the α−labeling condition in Definition 2. We successfully partition the vertex set of P_2^G into two parts H and I, where $H = X \cup Y'$ and $I = X' \cup Y$. Then, for any $v \in H$, $f_2(v) \leq m$ and for any $v \in I$, $f_2(v) > m$. We get the boundary value $k_2 = m$.

□

Example 6. *Let G be a bipartite graceful graph with the graceful labeling f as in Figure 4.3. Figure 4.5 shows P_2^G with an α−labeling with $E(P_2^G) \subseteq \{(u,v) \mid u \in X, v \in Y\}$, $X = \{0,1,2,3,4,5,6\}$ and $Y = \{7,8,9,10,11,12,13\}$. Note that, G has 6 edges and $k_2 = 6$.*

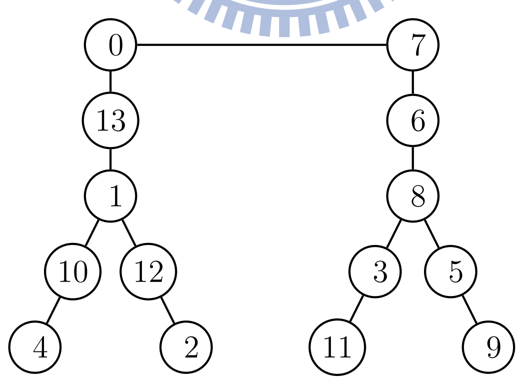

Figure 4.5: A tree with an α−labeling.

Note that, we can construct $P_{2+n}^{(G_1,\ldots,G_{2+n})}$ by connecting P_2^G with $P_n^{(G_1,\ldots,G_n)}$. Let f_2 be the α-labeling of P_2^G. Since f_2 is an α-labeling, P_2^G has a boundary value k_2. The connecting vertex between P_2^G and $P_n^{(G_1,\dots,G_n)}$ will be $u \in P_2^G$, where $f_2(u) = k_2 + 1$ and $v \in P_n^{(G_1,\dots,G_n)}$, where $f_n(v) = 0$, as in Figure 4.6.

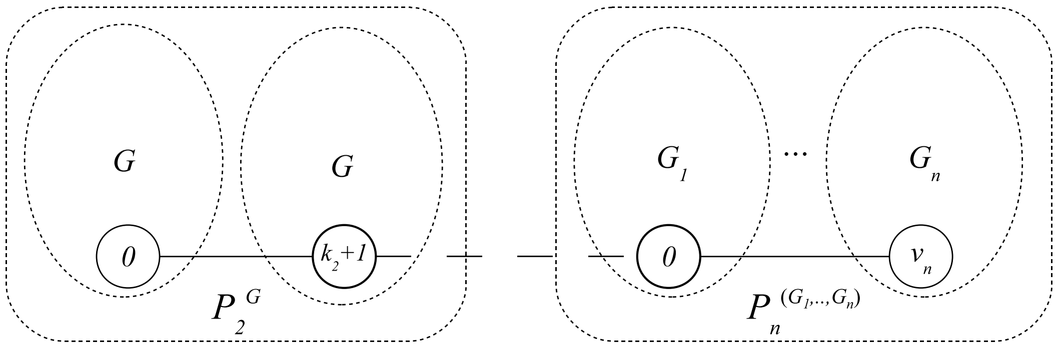

Figure 4.6: P_2^G and $P_n^{(G_1,\dots,G_n)}$.

Claim 2. *Assume that P_n^G is a bipartite graceful graph, then $P_{2+n}^{(G,G,G_1,\dots,G_n)}$, for any n, has a graceful labeling.*

Proof. Fixed any i, let $G*$ denote $G_1, G_2, \dots, G_i$.

Let m_2 be the edge number of P_2^G and m_n be the edge number of P_n^{G*}. P_2^G has an α-labeling f_2 and P_n^{G*} has a graceful labeling f_n. Thus P_2^G has a boundary value k_2 satisfying that for each $(u,v) \in E(P_2^G)$, either $f_2(u) \le k_2 < f_2(v)$ or $f_2(v) \le k_2 < f_2(u)$. We partition the vertex set $V(P_2^G) = X_2 \cup Y_2$ into two parts respectively, where:

$$X_2 = \{v \in V(P_2^G) : f_2(v) \le k_2\}$$
$$Y_2 = \{v \in V(P_2^G) : f_2(v) > k_2\}$$

Note that $|E(P_{2+n}^{(G,G,G*)})| = m_2 + m_n + 1$. We define the vertex labeling $f_{2+n} : V(P_{2+n}^{(G,G,G*)}) \to \{0, 1, \dots, m_2 + m_n + 1\}$ as follows:

$$f_{2+n}(v) = \begin{cases} f_2(v), & \text{if } v \in X_2 \\ f_2(v) + m_n + 1, & \text{if } v \in Y_2 \\ f_n(v) + k_2 + 1, & \text{if } v \in V(P_n^{G*}) \end{cases}$$

Next, we show that f_{2+n} is a graceful labeling.

First we claim that f_{2+n} is an injective function. Since $f_2(X_2) \subseteq \{0, 1, \dots, k_2\}$, $f_2(Y_2) \subseteq \{k_2 + 1, \dots, m_2\}$ and $f_n(V(P_n^{G*})) \subseteq \{0, 1, \dots, m_n\}$, we have that $f_{2+n}(X_2) \subseteq \{0, 1, \dots, k_2\}$,

$f_{2+n}(Y_2) \subseteq \{k_2 + m_n + 2, ..., m_2 + m_n + 1\}$ and $f_{2+n}(V(P_n^{G*})) \subseteq \{k_2 + 1, ..., k_2 + m_n + 1\}$. Moreover, since f_2 and f_n are injective, we have that the vertex labeling f_{2+n} is an injective function.

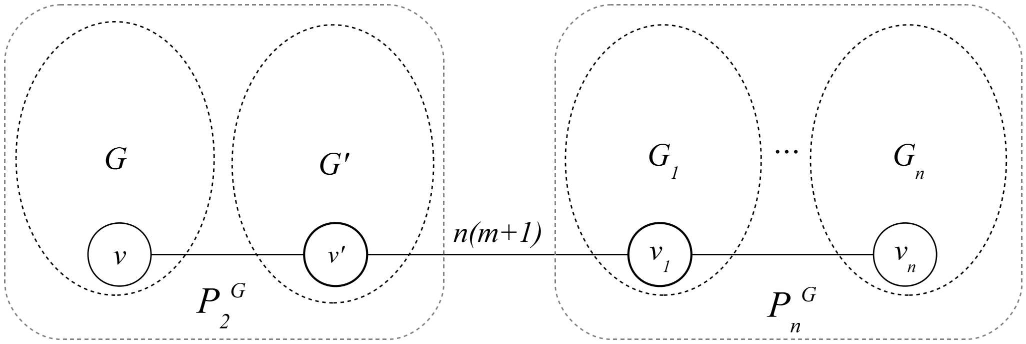

Figure 4.7: $P_{2+n}^{(G,G,G_1,...,G_n)}$.

Then, we claim that the labels of edges are distinct. Note that $E(P_{2+n}^{(G,G,G*)}) = E(P_2^G) \cup E(P_n^{G*}) \cup \{\text{connecting edge}\}$. Fix an edge $(u,v) \in E(P_2^G)$. Without loss of generality, assume that $u \in X_2$ and $v \in Y_2$. Then we have $|f_{2+n}(u) - f_{2+n}(v)| = |f_2(u) - f_2(v) - m_n - 1| = m_n + 1 + f_2(v) - f_2(u)$, where the last equality is due to $f_2(u) \leq k < f_2(v)$. Hence, the new edge labels of $E(P_2^G)$ are $\{m_n + 2, ..., m_2 + m_n + 1\}$. On the other hand, for any edge $(u,v) \in E(P_n^{G*})$, we have $|f_{2+n}(u) - f_{2+n}(v)| = |f_n(u) - f_n(v)|$, so we get the new edge labels of $E(P_n^{G*}) = \{1, 2, ..., m_n\}$. Consider Figure 4.7. By connecting vertices $v' \in P_2^G$, $f_2(v') = k_2 + 1$ and $v_1 \in P_n^{G*}$, $f_n(v_1) = 0$, where $f_{2+n}(v') = k_2 + m_n + 2$ and $f_{2+n}(v_1) = k_2 + 1$, we get "$m_n + 1$" as the edge label for (v', v_1). Since every graph G_i has the same edge number m, we have $m_n = nm + (n-1)$ and $m_n + 1 = n(m+1)$. Hence the edge labels of $E(P_{2+n}^{(G,G,G*)})$ are $\{1, 2, ..., m_2 + m_n + 1\}$. We conclude that $P_{2+n}^{(G,G,G*)}$ is a graceful graph.

□

Proof. (of Theorem 14)

We obtain that $P_n^{(G_1,...,G_n)}$ is graceful for any n, since we can construct $P_3^{(G_1,G_2,G_3)}$ by connecting $P_3^{(G_1,G_2)}$ with $P_1^{G_3}$, construct $P_4^{(G_1,G_2,G_3,G_4)}$ by connecting $P_2^{(G_1,G_2)}$ with $P_2^{(G_3,G_4)}$ and so on.

□

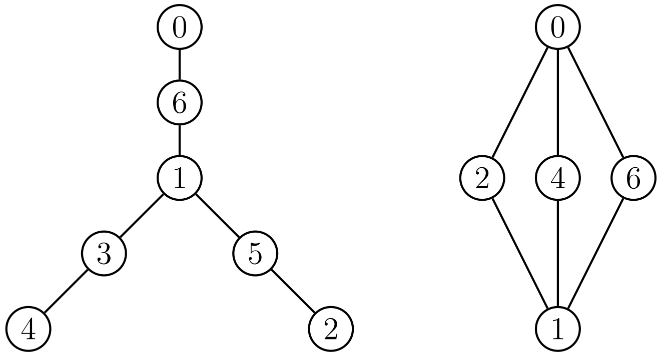

Figure 4.8: A bipartite graceful graph G and G'.

Example 7. *Given two bipartite graceful graphs G and G' as in Figure 4.8, by Theorem 14 $P_4^{(G,G,G',G')}$ has an α-labeling, as in Figure 4.9, with $E(P_4^{(G,G,G',G')}) \subseteq \{(u,v) \mid u \in X, v \in Y\}$, $X \subseteq \{0,1,...,13\}$ and $Y \subseteq \{14,...,27\}$.*

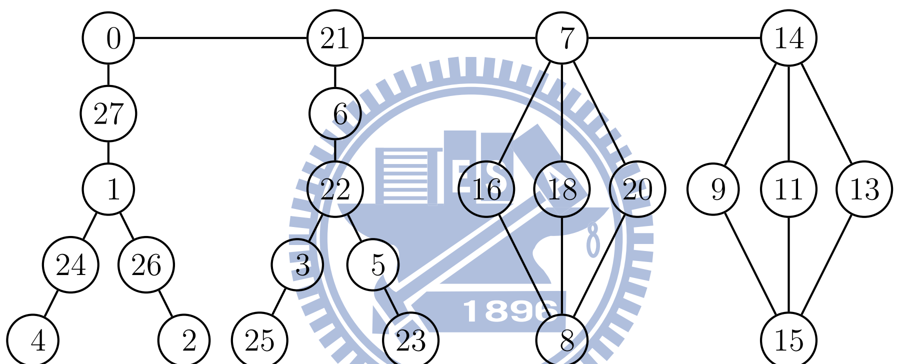

Figure 4.9: $P_4^{(G,G,G',G')}$.

Theorem 15. *If G_i, for every i, is bipartite with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, $i = 1,...,n$, then $P_{2n}^{(G_1,...,G_{2n})}$, for any n, has an α-labeling.*

We prove it by induction. First, we know that P_2^G has an α-labeling by Lemma 1. Then we show the following claim, which we need for the proof of Theorem 15. We assume $P_{2n}^{(G_1,...,G_{2n})}$ has an α-labeling and prove that $P_{2+2n}^{(G,G,G_1,...,G_{2n})}$ has an α-labeling.

Claim 3. *If $P_{2n}^{(G_1,...,G_{2n})}$ has an α-labeling, then $P_{2+2n}^{(G,G,G_1,...,G_{2n})}$, for any n, has an α-labeling.*

Proof. Fixed any i, let $G*$ denote $G_1, G_2, ..., G_i$.

Since if G is a bipartite graceful graph, P_n^{G*} is graceful. We only have to show that if $n \equiv 0 \pmod 2$, the vertex set $V(P_{2+n}^{G,G,G*})$ can be partitioned into two parts with a boundary value k_n and satisfy the condition of α-labeling. Since P_2^G and P_n^{G*} have an α-labeling, we partition the vertex set $V(P_2^G) = X_2 \cup Y_2$ and $V(P_n^{G*}) = X_n \cup Y_n$ into two parts, respectively.

Since $f_2(X_2) \subseteq \{0,1,...,k_2\}$, $f_2(Y_2) \subseteq \{k_2+1,...,m_2\}$, $f_n(X_n) \subseteq \{0,1,...,k_n\}$ and $f_n(Y_n) \subseteq \{k_n+1,...,m_n\}$, we have that $f_{2+n}(X_2) \subseteq \{0,1,...,k_2\}$, $f_{2+n}(Y_2) \subseteq \{k_2+m_n+2,...,m_2+m_n+1\}$, $f_{2+n}(X_n) \subseteq \{k_2+1,...,k_n+k_2+1\}$ and $f_{2+n}(Y_n) \subseteq \{k_2+k_n+2,...,k_2+m_n+1\}$. Then, we show that the labeling f_{2+n} satisfies the condition of $\alpha-$labeling. We partition the vertex set $V(P_{2+n}^G)$ into two parts X_{2+n} and Y_{2+n}, where $X_{2+n} = X_2 \cup X_n$, $Y_{2+n} = Y_2 \cup Y_n$, $f_{2+n}(X_{2+n}) = \{1,2,...,k_2+k_n+1\}$ and $f_{2+n}(Y_{2+n}) = \{k_n+k_2+2,...,m_2+m_n+1\}$, with $E(P_{2+n}^G) \subseteq \{(u,v) | u \in X_{2+n}, v \in Y_{2+n}\}$ and boundary value $k_{2+n} = k_2+k_n+1$. □

Proof. (of Theorem 15)

We obtain that $P_n^{(G_1,...,G_n)}$ is graceful for any n, since we can construct $P_4^{(G_1,G_2,G_3,G_4)}$ by connecting $P_2^{(G_1,G_2)}$ with $P_2^{(G_3,G_4)}$ and so on.

□

Wu [25] proved that, if G_i, for all i, has an α-labeling with the same edge number and G_{2i-1} and each pair of G_{2i}, for $i = 1,...,\lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $P_n^{(G_1,G_2,...,G_n)}$ is graceful. We give a new proof for it and show that $P_n^{(G_1,G_2,...,G_n)}$ is not only graceful but also has an α-labeling.

Theorem 16. *If G_i, for every i, has an α-labeling with the same edge number and each pair of graphs G_{2i-1} and G_{2i}, $i = 1,...,\lfloor \frac{n}{2} \rfloor$, has the same boundary value, then $P_n^{(G_1,...,G_n)}$, for any n, has an α-labeling.*

We prove it by induction. First we know that P_1^G has an α-labeling. Then we show the following two claims, which we need for the proof of Theorem 16.

1. We prove that $P_2^{G,G'}$ has an α-labeling.

2. Assume $P_{2n}^{(G_1,...,G_{2n})}$ has an α-labeling and prove that $P_{2+2n}^{(G,G',G_1,...,G_{2n})}$ has an α-labeling.

Claim 4. *If G and G' has the same boundary value and edge number m, then $P_2^{(G,G')}$ has an α-labeling.*

Proof. Let f and f' be the α-labeling of graph G and G', respectively.

According to the boundary value, we partition the vertex set $V(G)$ into two parts X and Y and vertex set $V(G')$ into X' and Y' respectively. Since G and G' have the same boundary value and edge number m, we know $f(X) = f'(X')$ and $f(Y) = f'(Y')$. In other words, $f(X) \cup f'(Y') \subseteq \{0, 1, ..., m\}$ and $f'(X') \cup f(Y) \subseteq \{0, 1, ..., m\}$. Using a similar proof of Lemma 1, we can prove that $P_2^{(G,G')}$ has an α-labeling with the boundary value $k_2 = m$.

□

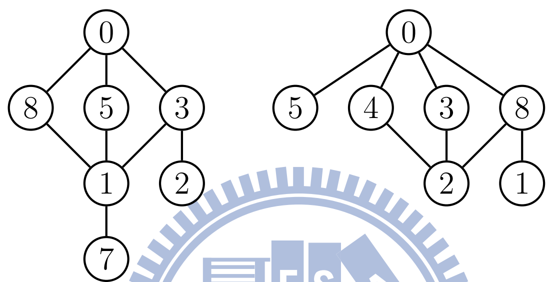

Figure 4.10: G_1(left), G_2(right).

Example 8. *Given two α-labeling graphs G_1, G_2, with the same edge number, as in Figure 4.10. G_1 and G_2 has the same boundary value 2. We show that $P_4^{(G_1,G_2)}$ has an α-labeling, as in Figure 4.11, with $E(P_4^{(G_1,G_2)}) \subseteq \{(u, v) \mid u \in X, v \in Y\}$, $X \subseteq \{0, 1, ..., 8\}$ and $Y \subseteq \{9, ..., 16\}$.*

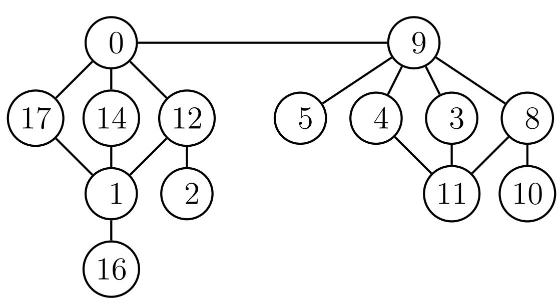

Figure 4.11: An α-labeling for $P_4^{(G_1,G_2)}$.

Claim 5. *If $P_n^{(G_1,\dots,G_n)}$ has an α-labeling, then $P_{2+n}^{(G,G',G_1,\dots,G_n)}$ has an α-labeling.*

Proof. Fixed any i, let $G*$ denote $G_1, G_2, \dots, G_i$.

Let m_2 be the edge number of $P_2^{(G,G')}$ and m_n be the edge number of P_n^{G*}. Let f_2 and f_n be the α-labelings of $P_2^{(G,G')}$ and P_n^{G*}, respectively. Thus there exists an boundary value k_2 for P_2 and k_n for P_n^{G*}. We partition the vertex set $V(P_2^{(G,G')}) = X_2 \cup Y_2$ and $V(P_n^{G*}) = X_n \cup Y_n$ into two parts respectively, with:

$$\begin{aligned}
X_2 &= \{v \in V(P_2^{(G,G')}) : f_2(v) \leq k_2\} \\
Y_2 &= \{v \in V(P_2^{(G,G')}) : f_2(v) > k_2\} \\
X_n &= \{v \in V(P_n^{G*}) : f_n(v) \leq k_n\} \\
Y_n &= \{v \in V(P_n^{G*}) : f_n(v) > k_n\}
\end{aligned}$$

Since $f_2(X_2) \subseteq \{0, 1, \dots, k_2\}$, $f_2(Y_2) \subseteq \{k_2 + 1, \dots, m_2\}$, $f_n(X_n) \subseteq \{0, 1, \dots, k_n\}$ and $f_n(Y_n) \subseteq \{k_n + 1, \dots, m_n\}$, following the proof of Theorem 14, we can proved that $P_{2+n}^{(G,G',G*)}$ has an α-labeling with a boundary value $k_{2+n} = k_2 + k_n + 1$.

□

Proof. We obtain that $P_n^{(G_1,\dots,G_n)}$, for any n, has α-labeling, since we can construct $P_3^{(G_1,G_2,G_3)}$ by connecting $P_3^{(G_1,G_2}$ with $P_1^{G_3}$, construct $P_4^{(G_1,G_2,G_3,G_4)}$ by connecting $P_2^{(G_1,G_2)}$ with $P_2^{G_3,G_4)}$ and so on. □

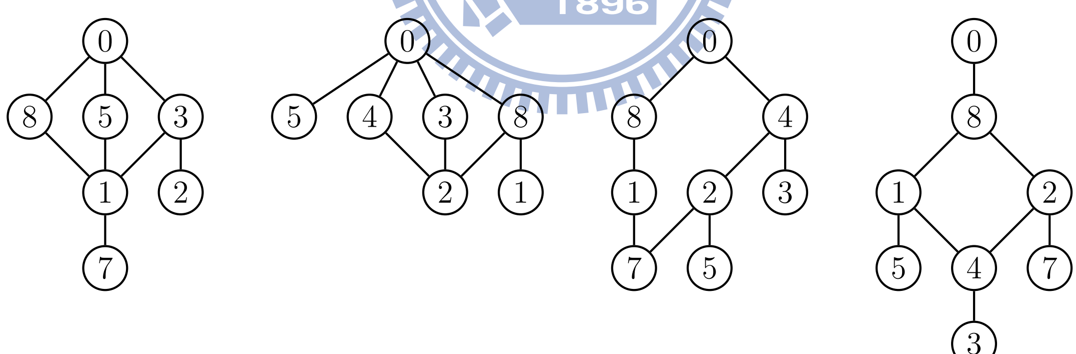

Figure 4.12: G_1, G_2, G_3, G_4 (left to right).

Example 9. *Given four α-labeling graphs G_1, G_2, G_3, G_4, with the same edge number, as in Figure 4.12. G_1 and G_2 has the same boundary value 2 and G_3 and G_4 has the same boundary value 3. We show that $P_4^{G_1,G_2,G_3,G_4}$ has an α-labeling, as in Figure 4.13, with $E(P_4^{G_1,G_2,G_3,G_4}) \subseteq \{(u,v) \mid u \in X, v \in Y\}$, $X \subseteq \{0, 1, \dots, 17\}$ and $Y \subseteq \{18, \dots, 35\}$.*

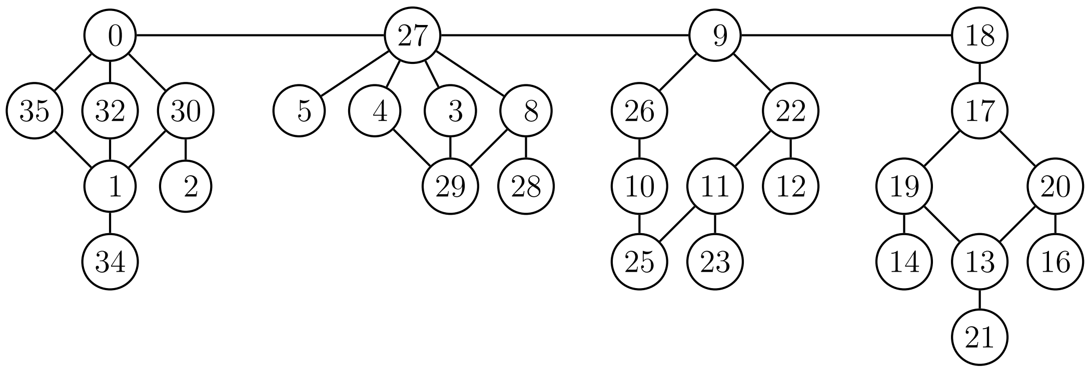

Figure 4.13: An α-labeling for $P_4^{(G_1,G_2,G_3,G_4)}$.

4.2 New Families of Graceful Graphs

Recall that we define C_n^G in Definition 6. We generalize the results of Wu [25], and show that if G_i, for every i, is bipartite and with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, then $C_n^{(G_1,...,G_n)}$, for any $n \equiv 0 \pmod 4$, has an α-labeling, and for any $n \equiv 3 \pmod 4$, has a graceful labeling. Note that since a graph with an α-labeling cannot contain an odd cycle, we also show that, if G_i, for every i, has an α-labeling and each pair of graphs G_{2i-1} and G_{2i}, $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, has the same boundary value and edge number, then $C_n^{(G_1,...,G_n)}$, for any $n \equiv 0 \pmod 4$, has an α-labeling and for any $n \equiv 3 \pmod 4$, has a graceful labeling. Since a path P_n has an α-labeling, our results covers the result proved by Snevily [22].

First, we show that for both cases, C_{4n}^{G*}, for any n, admits an α-labeling.

Theorem 17. *If G_i, for $1 \le i \le 4n$, is bipartite and with the same edge number m and each pair of graphs $G_{2i-1} = G_{2i}$, $i = 1, 2, ..., 2n$, then $C_{4n}^{(G_1,...,G_{4n})}$, for any $n \ge 1$, has an α-labeling.*

Proof. Fixed any i, let $G*$ denote $G_1, G_2, ..., G_i$.

Let f_{4n} be the labeling of P_{4n}^{G*} as in Theorem 15. Consider Figure 4.14. We give two proofs latter:

1. the edge label of (v_{2n}, v_{2n+1}) is $2n(m+1)$.

2. the vertex label of v_{4n} is $2n(m+1)$.

According the labeling of $P_n^{(G_1,\ldots,G_n)}$ in Theorem 15, it satisfies the following three conditions:

1. The edge labels of $E({P'}_{2n}^{G'*})$ are $1,\ldots,2n(m+1)-1$.

2. Because that P_{2n}^{G*} has an α-labeling, we can successfully partition the vertex set of $V(P_{2n}^{G*})$ into two parts X_{2n} and Y_{2n}, satisfying that any label of the vertices in X_{2n} is smaller than every vertex label in $V({P'}_{2n}^{G'*})$ and any label of the vertices in Y_{2n} is larger than every vertex label in $V({P'}_{2n}^{G'*})$.

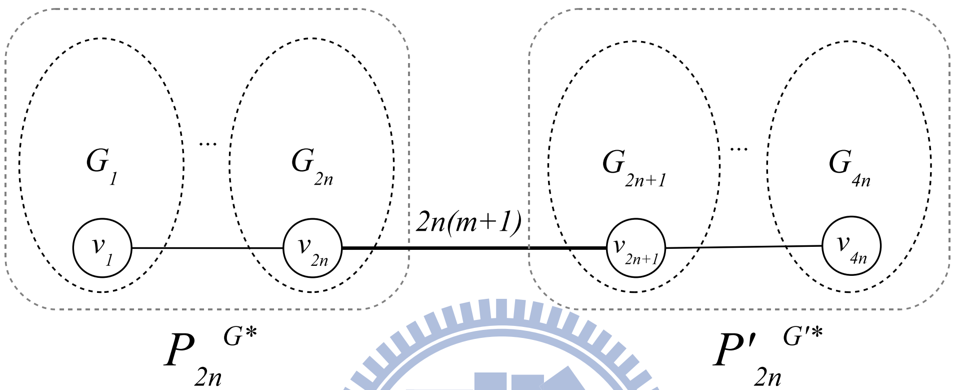

Figure 4.14: $P_{4n}^{(G*,G'*)}$.

Since $|E(C_{4n}^{G*})| = m_{4n}+1$, we define the vertex labeling $\Theta : V(C_{4n}^{G*}) \to \{0,1,\ldots,m_{4n}+1\}$ as follows:

$$\Theta(v) = \begin{cases} f_{4n}(v), & \text{if } v \in X_{2n} \\ f_{4n}(v)+1, & \text{if } v \in Y_{2n} \\ f_{4n}(v)+1, & \text{if } v \in V({P'}_{2n}^{G'*}) \end{cases}$$

By the third condition which mentioned, we obtain that the labeling Θ is injective.

Then, we claim that the edge labels of C_{4n}^{G*} are distinct. The edge labels of C_{4n}^{G*} are $1,2,\ldots,m_{4n}+1$. Note that, $E(C_{4n}^{G*}) = E(P_{2n}^{G*}) \cup E({P'}_{2n}^{G'*}) \cup (v_{2n},v_{2n+1}) \cup (v_1,v_{4n})$. The edge labels of $E({P'}_{2n}^{G'*})$ stay unchanged, that are $1,2,\ldots,2n(m+1)-1$. Since the original edge labels of $E(P_{2n}^{G*})$ were $2n(m+1)+1,\ldots,m_{4n}$, the new edge labels of $E(P_{2n}^{G*})$ are $2n(m+1)+2,\ldots,m_{4n}+1$. Since the path from v_1 to v_{4n} is a connected path, if $v_1 \in X_{2n}$ then $v_{2n} \in Y_{2n}$. The edge label of (v_{2n},v_{2n+1}) stays $2n(m+1)$. Since $\Theta(v_{4n}) = 2n(m+1)+1$, the edge label of (v_1,v_{4n}) is $2n(m+1)+1$. We conclude that the new edge labels of $E(C_{4n}^{G*})$ are $1,2,\ldots,m_{4n}+1$.

Now we prove the two assumptions are true. First, the edge label of (v_{2n}, v_{2n+1}) is $2n(m+1)$. The time we connect a $P_n^{(G_1,\dots,G_n)}$ with a P_2^G, the labels of edge are settled. In other words, edge labels will not change in $P_n^{(G_1,\dots,G_n)}$. By the definition of f_{2+n}, the connecting edge between $P_n^{(G_1,\dots,G_n)}$ and P_2^G will be labeled $m_n + 1 = n(m+1)$, with m the edge number of every graph G_i. Since (v_{2n}, v_{2n+1}) is the edge connecting $P'^{G'*}_{2n}$ and a P_2^G, the edge label of (v_{2n}, v_{2n+1}) is $2n(m+1)$. Second, we show that the vertex label of v_{4n} is $2n(m+1)$. Since by every connection of the graph P_2^G, we add k_2+1, also said to be "$m+1$", to vertices in $P_n^{(G_1,\dots,G_n)}$, the label for v_{4n} is "$(m+1)$" in the very beggining while we construct a P_2^G. After we finish the construction of $P_{4n}^{(G*,G'*)}$, we have the vertex v_{4n} labeled $2n(m+1)$.

Finally, we show that Θ is an α–labeling. Since $P_{4n}^{(G*,G'*)}$ has an α-labeling, by the definition of Θ, the vertex set $\mathrm{V}(C_{4n}^{(G*,G'*)})$ can be partitioned into two parts X_{4n} and Y_{4n}, and the edge $(v_1, v_{4n}) \in E(C_{4n}^{(G*,G'*)})$ satisfies that $v_1 \in X_{4n}$ and $v_{4n} \in Y_{4n}$. $C_{4n}^{(G*,G'*)}$ has a boundary value $k_{4n}+1$, such that the labeling $\Theta(X_{4n}) \subseteq \{0, \dots, k_{4n}+1\}$, $\Theta(Y_{4n}) \subseteq \{k_{4n}+2, \dots, m_{4n}+1\}$ and with no edge between vertices in X_{4n} and between vertices in Y_{4n}. We conclude that $C_{4n}^{(G*,G'*)}$ has an α–labeling.

□

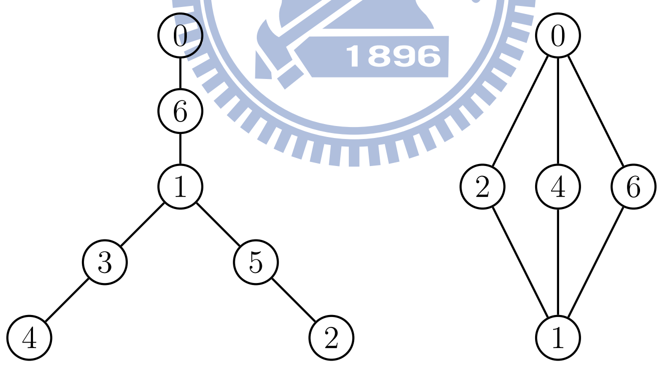

Figure 4.15: Bipartite graceful graphs G and G'.

Example 10. *Given two bipartite graceful graphs G and G' as in Figure 4.15, we show that* $C_4^{(G,G,G',G')}$ *has an* α*-labeling, as in Figure 4.16, with* $E(C_4^{(G,G,G',G')}) \subseteq \{(u,v) \mid u \in X, v \in Y\}$, $X = \{0,1,\dots,14\}$ *and* $Y = \{15,\dots,28\}$.

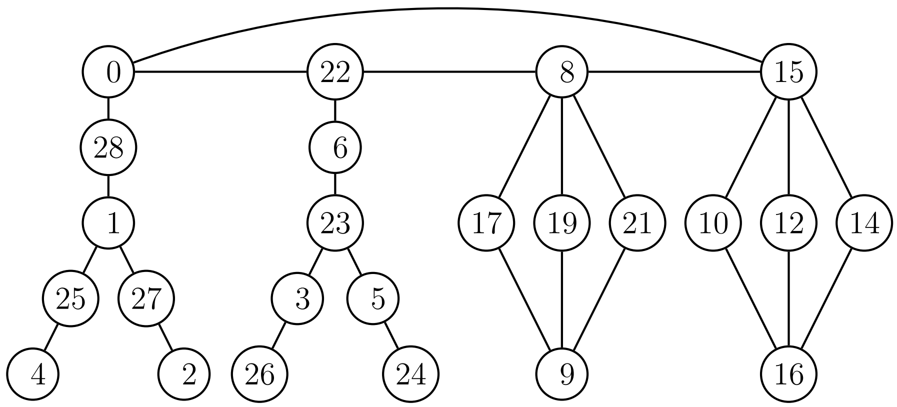

Figure 4.16: $C_4^{(G,G,G',G')}$.

Theorem 18. *If G_i, for every i, has an α-labeling and each pair of graphs G_{2i-1} and G_{2i}, $i = 1, ..., 2n$, has the same boundary value and edge number m, then $C_{4n}^{(G_1,...,G_{4n})}$, for any n, has an α-labeling.*

Proof. In Lemma 4, we show that if G and G' has the same boundary value and edge number m, then $P_2^{(G,G')}$ has an α-labeling with the boundary value $k_2 = m$. In Theorem 16, we show that if G_i, for every i, has an α-labeling and each pair of graphs G_{2i-1} and G_{2i}, $i = 1, ..., 2n$, has the same boundary value and edge number, then $P_n^{(G_1,...,G_n)}$, for any $n \geq 1$, admits an α-labeling. We can follow the proof idea of Theorem 17 to prove that $C_{4n}^{(G_1,...,G_{4n})}$, for any $n \geq 1$, admits an α-labeling. □

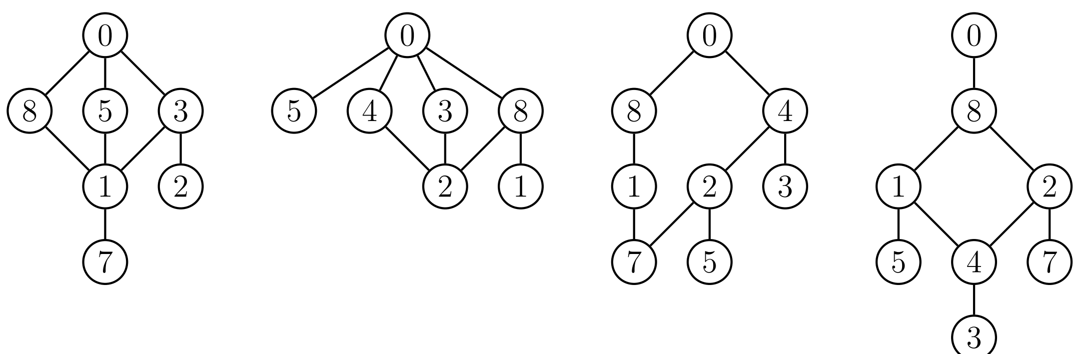

Figure 4.17: G_1, G_2, G_3, G_4 (left to right).

Example 11. *Given four α-labeling graphs G_1, G_2, G_3, G_4, with the same edge number, as in Figure 4.22. G_1 and G_2 has the same boundary value 2 and G_3 and G_4 has the same*

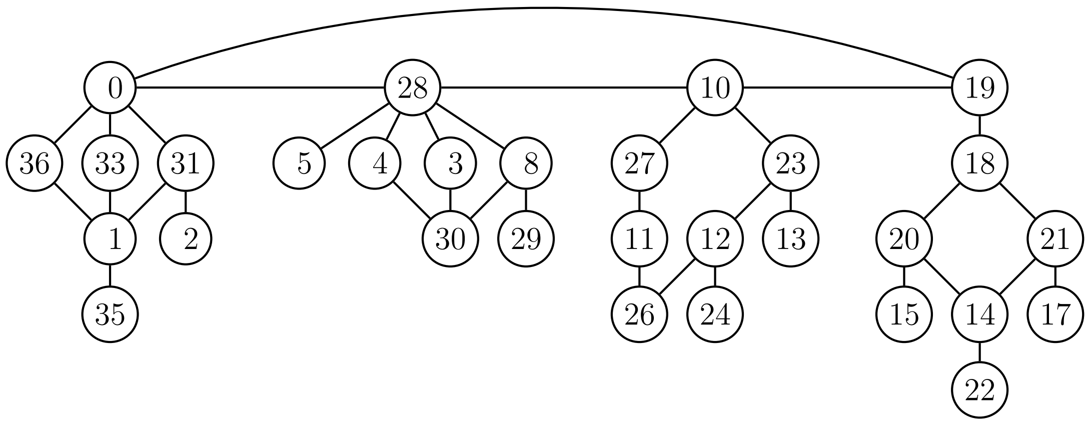

Figure 4.18: An α-labeling for $C_4^{(G_1,G_2,G_3,G_4)}$.

boundary value 3. *By Theorem 18,* $C_4^{(G_1,G_2,G_3,G_4)}$*, as in Figure 4.13, has an* α*-labeling with* $E(C_4^{(G_1,G_2,G_3,G_4)}) \subseteq \{(u,v) \mid u \in X, v \in Y\}$*,* $X \subseteq \{0,1,...,18\}$ *and* $Y \subseteq \{19,...,36\}$*.*

Theorem 19. *If* G_i*, for every* i*, is bipartite and with the same edge number* m *and each pair of graphs* $G_{2i-1} = G_{2i}$*,* $i = 1,2,...,\lfloor \frac{n}{2} \rfloor$*, then* $C_n^{(G_1,...,G_n)}$*, for* $n \equiv$ *0,3 (mod 4), is graceful.*

Proof. In Theorem 17, we prove that $C_{4n}^{(G_1,...,G_{4n})}$, for any n, has an α-labeling. Now we show that $C_{4n+3}^{(G_1,...,G_{4n+3})}$, for any n, is graceful.

Let f_{4n+3} be the labeling of $P_{4n+3}^{(G_1,...,G_{4n+3})}$ in Theorem 14. Consider Figure 4.19. We give two proofs latter.

1. the edge label of (v_{2n+2}, v_{2n+3}) is $(2n+1)(m+1)$.

2. the vertex label of v_{4n+3} is $(2n+1)(m+1)$.

According the previous construction of $P_n^{(G_1,...,G_n)}$ in Theorem 14, it satisfies the following conditions:

1. The edge labels of $E(P_{2n+1}^{G'*})$ are 1,...,$(2n+1)(m+1)-1$.

2. Because that P_{2n+2}^{G*} has an α-labeling, we can successfully partition the vertex set of $V(P_{2n+2}^{G*})$ into two parts X_{2n+2} and Y_{2n+2}, satisfying any label of the vertices in X_{2n+2} is smaller than every vertex label in $V(P_{2n+1}^{G'*})$ and any label of the vertices in Y_{2n+2} is larger than every vertex label in $V(P_{2n+1}^{G'*})$.

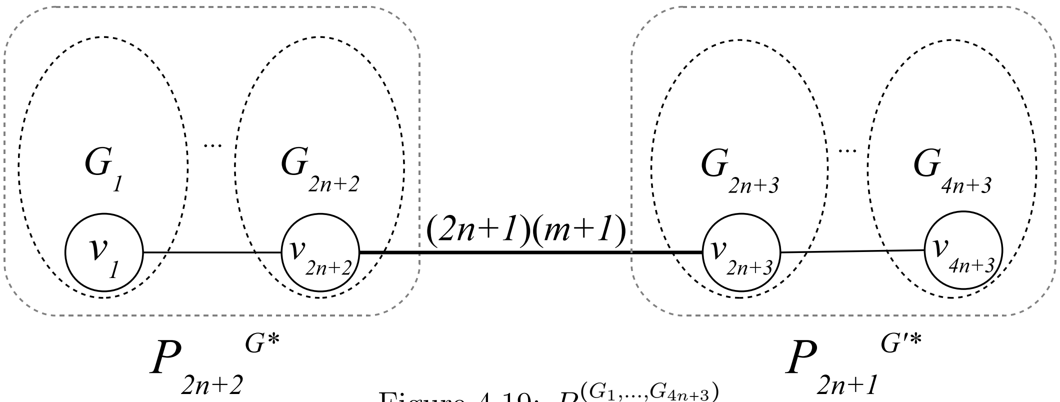

Figure 4.19: $P_{4n+3}^{(G_1,\ldots,G_{4n+3})}$.

Since $|E(C_{4n+3}^{G*})| = m_{4n+3}+1$, we define the vertex labeling $\Theta : V(C_{4n+3}^{G*}) \to \{0, 1, ..., m_{4n+3}+1\}$ as follows:

$$\Theta(v) = \begin{cases} f_{4n}(v), & \text{if } v \in X_{2n+2} \\ f_{4n}(v)+1, & \text{if } v \in Y_{2n+2} \\ f_{4n}(v)+1, & \text{if } v \in V(P_{2n+1}^{G'*}) \end{cases}$$

By the third condition which mentioned, we claim that the labeling Θ is injective.

Then, we claim that the edge labels are distinct. The edge labels of $E(C_{4n+3}^{G*})$ is $\{1, 2, ..., m_{4n+3}+1\}$. Note that, $E(C_{4n+3}^{(G_1,\ldots,G_{4n+3})}) = E(P_{2n+2}^{G*}) \cup E(P_{2n+1}^{G'*}) \cup (v_{2n+2}, v_{2n+3}) \cup (v_1, v_{4n+3})$. The edge labels of $E(P_{2n+1}^{G'*})$ stay unchanged, that is $\{1, 2, ..., 2n+1(m+1)-1\}$ and the original edge labels of $E(P_{2n+2}^{G*})$ were $(2n+1)(m+1)+1, ..., m_{4n+3}$, we have that the new edge labels of $E(P_{2n+2}^{G*})$ are $(2n+1)(m+1)+2, ..., m_{4n+3}+1$. Since the path from v_1 to v_{4n+3} is a connected path, if $v_1 \in X_{2n+2}$ then $v_{2n} \in Y_{2n+2}$. The edge label of (v_{2n}, v_{2n+1}) stays $(2n+1)(m+1)$. Since $\Theta(v_{4n+3}) = (2n+1)(m+1)+1$, the edge label of (v_1, v_{4n}) is $(2n+1)(m+1)+1$. We conclude that the new edge labels of $E(C_{4n+3}^{G*})$ are $1, 2, ..., m_{4n+3}+1$.

Now we prove the two assumptions are true, which says that the edge label of (v_{2n+2}, v_{2n+3}) is $(2n+1)(m+1)$. The time we connect a $P_n^{(G_1,\ldots,G_n)}$ with a P_2^G, the labels of edge are settled. In other words, edge label will not change in $P_n^{(G_1,\ldots,G_n)}$. By the definition of f_{2+n}, the edge label of the connecting edge between $P_n^{(G_1,\ldots,G_n)}$ and P_2^G will be $m_n+1 = n(m+1)$, with m the edge number of every graph G_i. Since (v_{2n+2}, v_{2n+3}) appears when we connect $P_{2n+1}^{G'*}$ and a P_2^G, the edge label of (v_{2n+2}, v_{2n+3}) is $(2n+1)(m+1)$. Next, we show that the vertex label of v_{4+3} is $(2n+1)(m+1)$. Since by every connection of the graph P_2^G, we add k_2+1, also said to be "$m+1$", to vertices in $P_n^{(G_1,\ldots,G_n)}$, the label for v_{4n+3} is

"$(m+1)$" in the very begging while we construct a P_3^G. After we finish the construction of $P_{4n+3}^{(G_1,\ldots,G_{4n+3})}$, we have the vertex v_{4n+3} labeled $(2n+1)(m+1)$. □

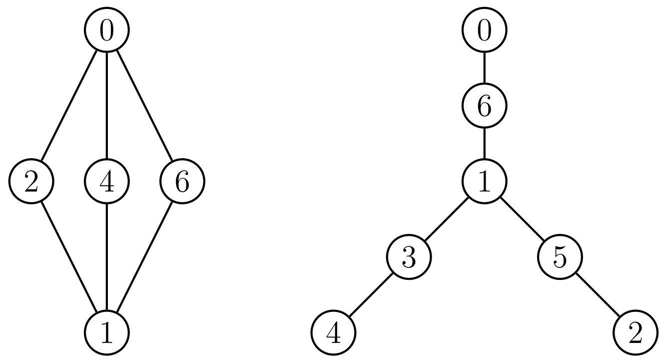

Figure 4.20: Bipartite graceful graphs G and G'.

Example 12. *Given two bipartite graceful graphs G and G' as in Figure 4.15, by Theorem 19, $C_3^{(G,G,G')}$ is graceful, as in Figure 4.21.*

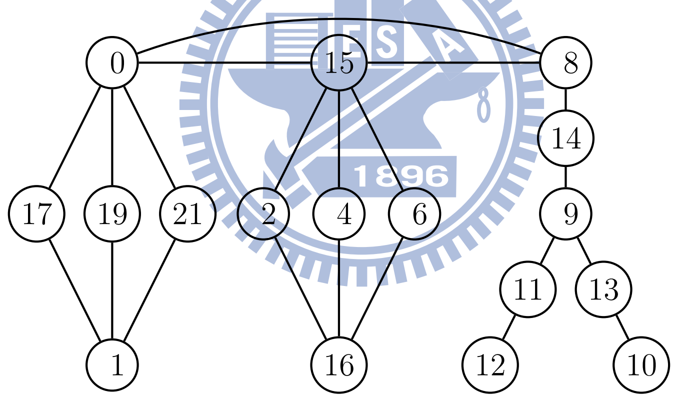

Figure 4.21: $C_3^{(G,G,G')}$.

Theorem 20. *Given a graph $P_n^{(G_1,\ldots,G_n)}$, if G_i, $1 \le i \le n$, has an α-labeling and each pair of graphs G_i and G_{i+1}, $i \equiv 0$ (mod 2), has the same boundary value and edge number, $C_n^{(G_1,\ldots,G_n)}$, $n \equiv 0$, 3 (mod 4), is graceful.*

Proof. In Theorem 18, we prove that $C_{4n}^{(G_1,\ldots,G_{4n})}$, for any n, has an α-labeling. Now we show that $C_n^{(G_1,\ldots,G_n)}$, for $n \equiv 3$ (mod 4), is graceful.

Since in Lemma 4, we show that if G and G' has the same boundary value and edge number m, then $P_2^{(G,G')}$ has an α-labeling with the boundary value $k_2 = m$. By Theorem 16, we show that if G_i, $1 \le i \le n$, has an α-labeling and each pair of graphs G_i and G_{i+1}, $i \equiv 0 \pmod 2$, has the same boundary value and edge number, then $P_n^{(G_1,\ldots,G_n)}$, for any $n \ge 1$, admits an α-labeling. We can follow the proof idea of Theorem 17 to prove that $C_{4n+3}^{(G_1,\ldots,G_{4n+3})}$, for any $n \ge 1$, is graceful.

□

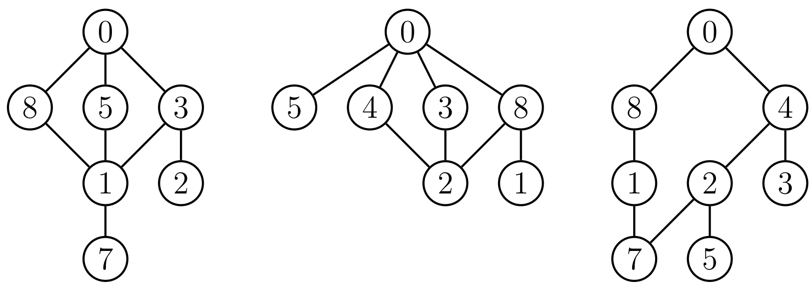

Figure 4.22: G_1, G_2, G_3 (left to right).

Example 13. *Given α-labeling graphs G_1, G_2, G_3, and G_4 with the same edge number, as in Figure 4.12. G_1 and G_2 has the same boundary value* 2 *and G_3 and G_4 has the same boundary value* 3*. By Theorem 20, $C_4^{(G_1,G_2,G_3,G_4)}$, as in Figure 4.13, is graceful.*

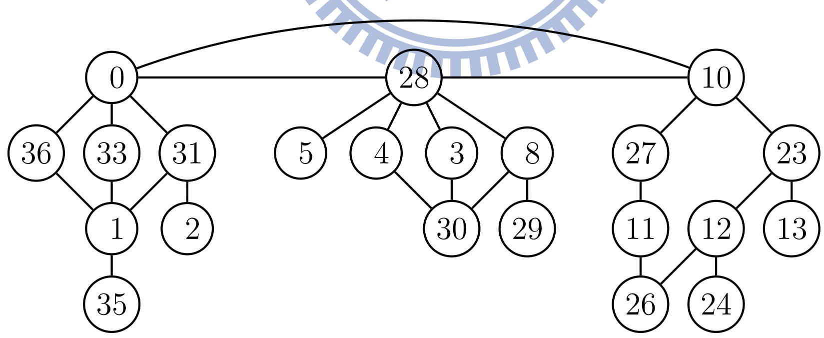

Figure 4.23: An α-labeling for $C_3^{(G_1,G_2,G_3)}$

Corona $C_n \odot mK_1$ is a cycle C_n with m pendant edge attached at each vertex. Frucht [11] proved that $C_n \odot K_1$ is graceful. By Theorem 17 we obtain the following corollary which shows that if $n \equiv 0 \pmod 4$, $C_n \odot mK_1$, for any m, is not only graceful but also a graph with an α-labeling.

Corollary 5. *If* $n \equiv 0$ *(mod 4),* $C_n \odot mK_1$*, for any* m*, has an* α*-labeling.*

Proof. Since mK_1 is a graph with an α-labeling, by Theorem 17 we prove that if $n \equiv 0$ (mod 4), then $C_n \odot mK_1$ has an α-labeling.

□

Example 14. $C_4 \odot 2K_1$*, as in Figure 4.24, is a graceful graph with an* α*-labeling, with* $E(C_4 \odot 2K_1) \subseteq \{(u,v) | u \in X, v \in Y\}$*,* $X = \{0,1,2,4,5,6\}$ *and* $Y = \{7,8,9,10,11,12\}$*.*

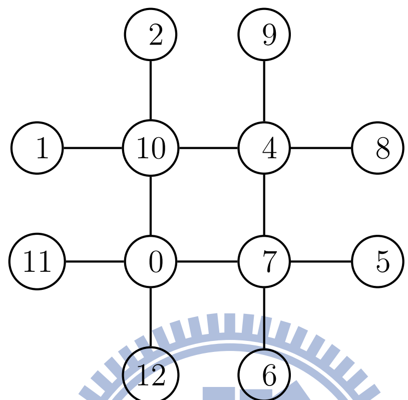

Figure 4.24: $C_4 \odot 2K_1$

Chapter 5

Conclusion

By extending Wu's results [25], we prove that if G_i, for $1 \leq i \leq n$, is a bipartite graceful graph with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, then $P_n^{(G_1,...,G_n)}$, for any n, has a graceful labeling and $P_{2n}^{(G_1,...,G_{2n})}$ has an α-labeling. We also show that if G_i, for all i, has an α-labeling and each pair of graphs G_{2i-1} and G_{2i}, for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$, has the same boundary value and edge number, then $P_n^{(G_1,...,G_n)}$, for any $n \geq 1$, admits an α-labeling.

We also define C_n^G and show that if G_i, for every i, is bipartite and with the same edge number and each pair of graphs $G_{2i-1} = G_{2i}$, $\lfloor \frac{n}{2} \rfloor$, then $C_{4n}^{(G_1,...,G_{4n})}$, for any n, has an α-labeling, and $C_{4n+3}^{(G_1,...,G_{4n+3})}$, for any n, has a graceful labeling. We show that, if G_i, for every i, has an α-labeling and each pair of graphs G_{2i-1} and G_{2i}, $\lfloor \frac{n}{2} \rfloor$, has the same boundary value and edge number, then $C_{4n}^{(G_1,...,G_{4n})}$, for any n, has an α-labeling and $C_{4n+3}^{(G_1,...,G_{4n+3})}$, for any n, is graceful. This result covers the result proved by Snevily [22].

Chapter 6

Open Problems

Various classes of graphs have been proven to be graceful or not. We summarize some open problems here:

1. Given n different graphs $G_1, ..., G_n$, which all admit α-labeling, then $P_n^{G_1,...,G_n}$ has an α-labeling.

2. Given n different graphs $G_1, ..., G_n$, which all admit α-labeling, then $C_n^{G_1,...,G_n}$, $n \equiv 0$, 3 (mod 4), is graceful. If $n \equiv 0$ (mod 4), then $C_n^{G_1,...,G_n}$ has an α-labeling.

3. Every tree is graceful.

4. Take away one of the edge or vertex of a complete bipartite graph, it stays a graceful or α-labeling graph.

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