國立交通大學

資訊科學與工程研究所

碩士論文

一在 Coloured Petri Nets 中的錯誤找尋與校正之方法

A Methodology to Identify and Correct Faults in Coloured Petri Nets

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摘要

Coloured Petri nets 及其分析技術可應用於協助尋找工作流程的缺陷，而於一 coloured Petri net 中可能存在一個或多個錯誤，但目前的存在的演算法並無法完全地檢測出這些錯誤。於本論文中，定義了三種尚未被分析過的錯誤，分別爲 place faults、amount transition faults 及 colour transition faults。本文所提出之方法爲將一 coloured Petri net 轉爲另一 separate 的 coloured Petri net，並且在 separate 的 coloured Petri net 上之所有 marking 將維持某相同的編碼規則相對於原始的 coloured Petri net 上之所有 marking。而後，若 coloured Petri net 進入錯誤狀態時便可透過 parity check 之方式將其找尋出來並校正。在加入 $2k$ places 與其相對應的 arcs 後，最多可同時在一 coloured Petri net 上找尋並校正 k place faults、x amount transition faults 及 $k - x$ colour transition faults，其中 $0 \le x \le k$。此錯誤找尋與校正之方法的時間複雜度爲 $O(k\gamma(\alpha+\beta))$，其中 α、β 及 γ 分別代表於一 coloured Petri net 中 transitions、places 及 colours 的數量。

關鍵字: coloured Petri net、錯誤找尋、錯誤校正、parity check。

A Methodology to Identify and Correct Faults in Coloured Petri Nets

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Abstract

Coloured Petri nets and related analysis techniques can be applied to help find the defects in workflows. One or more errors may exist in a coloured Petri net, but current algorithms can not detect them completely. In this thesis, three kinds of faults are defined, namely *place faults*, *amount transition faults* and *colour transition faults*, not analyzed before. A methodology provided in this thesis translating a coloured Petri net into a *separate* one, in which all the markings will keep the same *encoding* rule with all the markings in the original coloured Petri net. If the coloured Petri net results in faulty states, they can be identified and corrected via *parity checks*. After adding $2k$ places and their relative arcs, the coloured Petri net can be identified and corrected at most: k place faults, x amount transition faults and $k - x$ colour transition faults concurrently, where $0 \leq x \leq k$. The time complexity of the fault identification and correction method provided in this thesis is $O(k\gamma(\alpha+\beta))$, where α, β and γ are the number of transitions, places and colour types in a coloured Petri net, respectively.

Keywords: coloured Petri net, fault identification, fault detection, parity check.

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Chapter 1. Introduction

High-level applications such as *SOA* usually contain *workflow*, and workflow applications can be analyzed with coloured Petri nets (CPNs) and their associated techniques. One important issue in correcting the applications is how to deal with faulty states, and this is also an issue in CPNs. When faulty states occur, they may firstly be detected, after that the reasons which induce the system into faulty states may be determined, and finally these faulty states may be adjusted back to correct states.

The intuitive method on how to identify faulty states is firstly computing out the state which will be reached from initial states under recorded information of state transition and then check if these two states are identical. If these two states are different then the given state is a faulty state. In the case of CPNs, an occurrence graph [1] can firstly be constructed and then the markings are checked if they are identical with the markings in occurrence graph under the recorded firing sequences [1]. In this thesis, the proposed method can identify and correct the faulty markings in CPNs by algebraic operations without constructing an occurrence graph and recording firing sequences.

The CPNs considered in this thesis only have the marking information at a specific time point without firing sequence information, which is similar with the assumption in [3]. The faults considered in this thesis may occur in places or transitions [3] and fault types may be colour or amount. The purpose of this thesis is to identify and correct these based on the marking information.

The purposed method will firstly construct a *separate* [4] CPN which is bisimilar to the given CPN. While the state evolves, the marking on the redundant CPN will keep the encoding relation with the marking on the given CPN. Finally, the marking on the redundant CPN can be checked if it satisfies the encoding relation by algebraic operations on the marking matrix. If it violates the encoding relation, it can be identified where the faults occur and adjusted back to a correct state by algebraic operations on the marking matrix. However, since the purposed method is based on algebraic approach, matrices must be used to describe the markings and state evolution of CPNs before implementing the method. We will also describe how to construct a matrix for a CPN.

As the result, If there are k addition places in the redundant CPN compare with given one, it will have capability to simultaneously detect and correct k place faults, x amount transition faults and k - x colour transition faults concurrently, where $0 \leq x \leq k$. We will prove this capability in this thesis and also prove the separate CPN will be bisimilar to the given one.

The remainder of this thesis is organized as follows. Chapter 2 introduces CPNs , presents the notations and definitions of CPN, which will be used in the rest of this thesis, and show the related works. Chapter 3 describes how to use matrices to describe the markings of CPN and the state evolution of CPN. Chapter 4 proposes the methods on how to detect and correct these four kinds of fault and also give an example in each case. Chapter 5 gives a conclusion of this thesis and show future works.

Chapter 2. Notations and Background Overview

This chapter briefly introduces some background knowledge and gives the notations and definitions to be used in this thesis. In section 2.1, *multi-set* is introduced. Section 2.2 introduces CPN which uses multi-set widely. Section 2.3 describes the definition of bisimilation equivalence. Section 2.4 discusses the faults considered in the thesis and gives the corresponding definitions. And, the relate works of fault detection are presented in section 2.5.

2.1 Multi-sets

A multi-set is similar to a set, except that it can own one or more common elements. That a multi-set *A* is defined over a set *B* indicates all the elements in the *A* are the elements in *B*. The basic formal definitions of multi-set are described in [1] and some extended definitions are described in this section.

Definition 2.1: Let S be a non-empty set.

- A **multi-set** ms, over S, is a function $ms: S \to \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\forall s \in S$, the $ms(s)$ is the number of appearances of s in the multi-set ms, which is called the **coefficient** of s. A multi-set ms is usually represented by a formal sum: $\sum_{s \in S} ms(s)'s$.

- An **empty** multi-set is a multi-set where all the coefficients are zero, which can be denoted by $\varnothing$.
- An element $s \in S$ **belongs** to a multi-set ms iff $ms(s) \neq 0$, which can be written as $s \in ms$.
- S_{MS} is defined as the set of all multi-sets over S.

For example, if there is a multi-set $\{a, b, b, c, c, c\}$, it can be represented by a formal sum: $1'a + 2'b + 3'c$. Since the associated computations for an unlimited number is meaningless, the coefficients and the nonnegative integers in $\mathbb{N}$ applied in the thesis are assumed to be finite.

Definition 2.2: Let $ms_1, ms_2 \in S_{MS}$, and $n \in \mathbb{N}$.

- **Addition** of multi-sets is defined as

  $$ms_1 + ms_2 \equiv \sum_{s \in S} (ms_1(s) + ms_2(s))'s .$$

- **Comparison** between multi-sets is defined as

  $$ms_1 \neq ms_2 \equiv \exists s \in S : ms_1(s) \neq ms_2(s) ,$$

  $$ms_1 \leq ms_2 \equiv \forall s \in S : ms_1(s) \leq ms_2(s) ,$$

  $\geq$ and $=$ are defined analogously to $\leq$.

- **Subtraction** of multi-sets is defined as

  $$ms_1 - ms_2 \equiv \sum_{s \in S} (ms_1(s) - ms_2(s))'s .$$

  A subtraction is meaningless when $\exists s \in S$, $ms_1(s) \geq ms_2(s)$.

- **Inner product** of multi-sets is defined as

  $$ms_1 \cdot ms_2 \equiv \sum_{s \in S} (ms_1(s) * ms_2(s))'s .$$

- **Scalar multiplication** of multi-sets is defined as

  $$n * ms_1 \equiv \sum_{s \in S} (n * ms_1(s))'s .$$

Definition 2.2 defines the operations on multi-sets. And two additional properties which will be applied as the basis include: commutative[I] and associative[II].

2.2 Colored Petri Nets

The graphical and mathematical model of Petri nets are created by Carl Adam Petri in 1962 [6]. CPNs are one kind of extensions from Petri nets, which are completely backward compatible with original Petri nets [1, 2]. More specifically, an original Petri net can be treated as a CPN with single colour. CPNs can help design, specify, simulate, validate and implement systems (e.g., workflow systems, distributed systems, control systems). There are various types of definition of CPN (e.g., [1, 2, 7, 8]), and the contributions with different definitions may be different. In this thesis, the definition of CPN is based on [7, 8]. Besides, to make this discussion easier, we adopt some notions from [1, 2, 9].

[I] $ms_1 + ms_2 = ms_2 + ms_1$

[II] $ms_1 + (ms_2 + ms_3) = (ms_1 + ms_2) + ms_3$

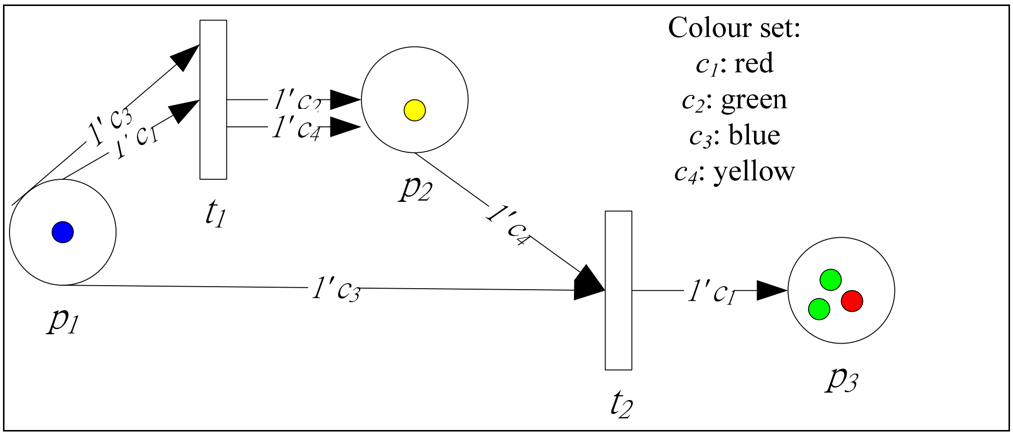

Figure 2.1 A CPN with a colour set which has four colours.

A CPN structure is a directed bipartite graph. There are two kinds of nodes in a CPN structure, called *place* and *transition*. The arcs in a CPN only connect between two different kinds of nodes. There are two functions, *input* and *output* assigning a colour and a weight to an arc. A place may be put *tokens* of which each has a colour. A *marking* describes the assignment of tokens to places. In general, transitions represent activities and markings represent states in a system. As in Figure 2.1 shows a CPN, where circles denote places, rectangles denote transitions and dots denote tokens. The marking of this figure is p_1 has one c_3 token, p_2 has one c_4 token and p_3 has one c_1 token and two c_2 tokens.

Definition 2.3: A CPN structure is a 6-tuple $N = (P, T, A, C, I, O)$

- P is a finite set of **places**.
- T is a finite set of **transitions** such that $P \cap T = \varnothing$.
- A is a finite set of **directed arcs**, $A \subseteq (P \cup T) \times (P \cup T)$, satisfying $A \cap (P \times P) = A \cap (T \times T) = \varnothing$.
- C is a finite set of **token colours**.
- I is an **input function** with domain $(P \times T)$, $I: P \times T \rightarrow C_{MS}$.
- O is an **output function** with domain $(P \times T)$, $O: P \times T \rightarrow C_{MS}$.

Definition 2.3 is a formal description of the CPN structure. Based on a transition, an input/output function describes the input/output with some connected place respectively. In Figure 2.1, the $I(p_1, t_1) = 1'c_1 + 0'c_2 + 1'c_3 + 0'c_4$ and $O(p_2, t_1) = 0'c_1 + 1'c_2 + 0'c_3 + 1'c_4$. To simplify the discussion, the P, T, and C are assumed to follow rules correspondingly from now on: (1) $P = \{p_1, p_2, \ldots, p_\beta\}$, $\beta > 0$, (2) $T = \{t_1, t_2, \ldots, t_\alpha\}$, $\alpha > 0$ and (3) $C = \{c_1, c_2, \ldots, c_\gamma\}$, $\gamma > 0$.

Definition 2.4:

- A **marking** of a set of places P is a mapping $m: P \rightarrow C_{MS}$.
- m_0 is the **initial marking**, $m_0: P \rightarrow C_{MS}$.
- A **CPN** $G = <N, m_0>$ is a CPN structure N with an initial marking m_0.

A place may have tokens of different colours. A marking expresses the colour elements and the token number of these colours in each place, which can be defined as a function from a set of places in a CPN to the set of pair sequence where a pair is a nonnegative integer and a colour element. For example, in Figure 2.1, the marking of p_3 is $1'c_1 + 2'c_2 + 0'c_3 + 0'c_4$. An initial marking represent the initial state of a system. A CPN should have both a CPN structure and an initial marking.

Definition 2.5: Let $N = (P, T, A, C, I, O)$ be a CPN structure.

- For an element $x \in P \cup T$, its **pre-set** $^{\bullet}x$ is defined as
  $^{\bullet}x = \{y \in P \cup T \mid (y, x) \in A\}$
- and its **post-set** $x^{\bullet}$ is defined as
  $x^{\bullet} = \{y \in P \cup T \mid (x, y) \in A\}$.

The pre-set or post-set of a place is a set of transitions. The pre-set or post-set of a transition is a set of places. If $t \in T$ and p $\notin {}^{\bullet}t$, the $I(p, t)$ must be an empty multi-set. If $t \in T$ and p $\notin t^{\bullet}$, the $O(p, t)$ must be an empty multi-set.

Definition 2.6: Let $G = <N, m_0>$ be a CPN and $N = (P, T, A, C, I, O)$, and a marking is m.

- A transition $t \in T$ is **enabled** iff $\forall p \in {}^{\bullet}t$: $m(p) \geq I(p, t)$.
- If a transition $t \in T$ is enabled by a marking m, it may **fire** and yield a new marking m', which can be denoted by $m\ [t> m'$, where

  $\forall p \in P$: $m'(p) = m(p) - I(p, t) + O(p, t)$.
- A **finite occurrence sequence** is a finite sequence of firings and its corresponding markings:

  $m\ [t_{s1}> m_1\ [t_{s2}> m_2 \ldots [t_{sq}> m'$, where $t_{si} \in T$ for all $1 \leq i \leq q$.

  This sequence can be denoted by $m\ [\sigma> m'$ and σ is called **firing vector**. The final marking m' can be calculated by following equation

  $\forall p \in P$: $m'(p) = m(p) - I(p, t_{s1}) + O(p, t_{s1}) - I(p, t_{s2}) + O(p, t_{s2}) - \ldots - I(p, t_{sq}) + O(p, t_{sq})$.

  The sequence of transitions in firing vector σ is called **firing transition sequence**. As a short hand, $\forall t \in T$, the $\sigma(t)$ is the number of appearances of t in the firing vector σ, where $\sigma(t) \in \mathbb{N}$. Hence, the above equation can be rewrite as

  $\forall p \in P$: $m'(p) = m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + \ldots + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha))$,

  where t_1 to t_α are the all elements in T.

- A marking m' is **reachable** from another marking m if there exists a firing vector σ such that $m\ [\sigma\!>\ m'$. The set of markings which are reachable from m is denoted by $[m\!>$. As a short hand, a set $M = m_0 \cup [m_0\!>$ is called **all markings** of CPN G.

Definition 2.6 defines the state transition of a Petri net, which is called firing. When a transition fires, it will take the tokens from its pre-set by the rule described in the input function and put the tokens to its post-set by the rule described in the output function. For example, the transition t_2 is the only transition which is enabled in Figure 2.1. When transition t_2 fires, the new marking will be p_1 and p_2 has no token and p_3 has two c_1 tokens and two c_2 tokens.

2.3 Related Works

C. N. Hadjicostis [3, 4, 11], G. C. Verghese [11], Y. Wu [3], L. Li and R. S. Sreenivas [4] proposed a series of algebraic methods to identify and correct the place and transition faults which may occur in Petri nets or Petri net controllers. The method in [11] can deal with place faults and transition faults individually. An extended method in [3] can deal with place and transition faults simultaneously. In [4], the authors extend the methods proposed in [3] and [11] to the Petri net controller. They also prove that the proposed methods could construct *redundant* Petri nets and analyze the complexity of proposed methods.

P. Jancar [5, 10] proves the decidability on bisimilarity of Petri nets. In [5], the author proves that the bisimilarity of free-labeled Petri nets is decidable and it can be mapped to language equivalence and reachability equivalence problems. In [10], the author proves that the bisimilarity of labeled Petri nets is undecidable.

V. K. Belikov [13, 14] and Y. F. Rutner [13] proposed methods to describe input and output functions in CPNs by algebraic matrices. In [14], the author proposes a method describing an input and an output function of a CPN by two three-dimensional matrices. In [8, 13], the authors propose a method using a four-dimensional matrix to describe an input and an output function of a CPN at the same time.

This thesis adopts the idea from [3, 4, 11] to CPNs. The methods in [3, 4, 11] should firstly use algebraic matrices to describe state transformation of Petri nets, so this thesis firstly proposes a method describing the state transformation of CPNs by algebraic matrices such matrices are different from that in [8, 13, 14] for the reason that the method proposed in this thesis can distinguish more faults. Second, this thesis use a method extended from [3, 4, 11] to code the matrices come from previous method into matrices that have fault detection and correction capabilities. Finally, these matrices with fault detection and correction capabilities can be mapped back to new CPNs with fault detection and correction capabilities. Since the CPNs discussed in this thesis is free-labeled, it can be proved that the CPNs constructed by the methods proposed in this thesis are bisimilar to the given ones, or more specifically the constructed CPNs are redundant CPNs.

Chapter 3. Matrix Approaches on CPNs

This chapter proposes a method describing the state transformation of CPNs in matrix approaches. In previous [3, 4, 7, 11], the algebraic matrix representations for input and output functions of CPNs are dimensions of three at least. The method proposed in this chapter using two-dimensional matrices to describe input and output functions of CPNs. By using this method, the pre-condition and post-condition colour transition faults can be distinguished. In section 3.1, the method to express CPNs by two-dimensional matrix approaches is introduced. Formal algorithms of this method are given in section 3.2. Section 3.3 gives an example and shows how this method works.

3.1 Two-dimensional Matrix Method

The proposed method is a concept of flattening. The information in matrices of three dimensions expressing input and output functions are flattened into the ones of two dimensions respectively. In Definition 2.6, it can be seen that the firing equation of a CPN is $\forall p \in P$: $m'(p) = m(p) - I(p, t) + O(p, t)$. Hence, there are four matrices that need to be designed for describing the states and state transitions of a CPN. They are *input matrix*, *output matrix*, *marking matrix* and *firing matrix*.

Definition 3.1: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, a marking is m, $|P| = \beta$ and $|C| = \gamma$.

- A **marking matrix** Q is a $\beta \times \gamma$ matrix describes the marking m. The entries in matrix Q are

  $q_{ij} = m(p_i)(c_j)$, where $1 \leq i \leq \beta$, $1 \leq j \leq \gamma$, $p_i \in P$, $c_j \in C$, and q_{ij} represents the entry in ith row and jth column of Q.

Definition 3.1 describes the format and content of a marking matrix of a CPN. Each row in Q represents the tokens of each colour in a place, and each column in Q represents the token distribution of a colour on each place. In Definition 3.1, $m(p_i)$ represents the tokens on place p_i, which is a multi-set over colour set C, hence $m(p_i)(c_j)$ is the coefficient of colour element c_j of $m(p_i)$, which means the number of tokens with colour c_j on place p_i.

Definition 3.2: Let $G = \langle N, m_0 \rangle$ be a CPN where $N = (P, T, A, C, I, O)$, $|P| = \beta$, $|T| = \alpha$ and $|C| = \gamma$.

- An **input matrix** B^- is a $\beta \times n$ matrix describes the input function I, where $n = \alpha\gamma$. The entries in matrix B^- are $b^-_{ij} = I(p_i, t_r)(c_s)$, where $r = \lceil j / \gamma \rceil$, $s = (j - 1) \bmod \gamma + 1$, $1 \leq i \leq \beta$, $1 \leq j \leq \alpha \times \gamma$, $1 \leq r \leq \alpha$, $1 \leq s \leq \gamma$, $p_i \in P$, $t_r \in T$, $c_s \in C$, and b^-_{ij} represents the entry in ith row and jth column of B^-.
- An **output matrix** B^+ is a $\beta \times n$ matrix describes the output function O, where $n = \alpha\gamma$. The entries in matrix B^+ are $b^+_{ij} = O(p_i, t_r)(c_s)$, where $r = \lceil j / \gamma \rceil$, $s = (j - 1) \bmod \gamma + 1$, $1 \leq i \leq \beta$, $1 \leq j \leq \alpha \times \gamma$, $1 \leq r \leq \alpha$, $1 \leq s \leq \gamma$, $p_i \in P$, $t_r \in T$, $c_s \in C$, and b^+_{ij} represents the entry in ith row and jth column of B^+.

In Definition 3.2, the input and output matrix of a CPN is defined. Each row in B^-/B^+ represents the token number of each colour that would be removed/deposited from/into a place while each transition fires. Each column in B^{-}/B^{+} represents the number of tokens with a colour that would be removed/deposited from/into each place while a transition fires. The operator “mod” in Definition 3.2 represents the modular arithmetic.

Definition 3.3: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $|T| = \alpha$, $|C| = \gamma$, $t_r \in T$, and two markings are m_1 and m_2 such that $m_1\ [t_r> m_2$.

- A **transition firing matrix** X_r is an $n \times \gamma$ matrix describes the firing transition t_r, where $n = \alpha\gamma$. The entries in matrix X_r are
$$x_{r_{ij}} = \begin{cases} 1 & if\ t_s = t_r\ and\ j = (i-1)\bmod\gamma + 1, where\ s = \lceil i/\gamma \rceil \\ 0 & otherwise \end{cases},$$
and $x_{r_{ij}}$ represents the entry in ith row and jth column of X_r.

Definition 3.3 describes how to indicate a firing transition of a CPN. On the other hand, a transition firing matrix X_r can be deemed as a column of α square sub-matrices, and each sub-matrix is a $\gamma \times \gamma$ matrix. The rth sub-matrix is an identity matrix, where t_r is the firing transition, and the entries of others entries are all zero.

Lemma 3.1: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $|P| = \beta$, $|T| = \alpha$, $|C| = \gamma$, $t_r \in T$, two markings are m_1 and m_2 such that $m_1\ [t_r> m_2$, Q_1 and Q_2 are the matrix representation of m_1 and m_2 respectively, B^{-} and B^{+} are the matrix representation of I and O respectively, and X_r is the matrix representation of firing transition t_r.

- The state transformation between Q_1 and Q_2 would satisfy
$$Q_2 = Q_1 - B^{-}X_r + B^{+}X_r.$$

In other words,

$Q_2 = Q_1 - B^-X_r + B^+X_r$ iff $\forall p_i \in P$: $m_2(p_i) = m_1(p_i) - I(p_i, t_r) + O(p_i, t_r)$.

Proof:

(→) Since, by Definition 3.3, $X_r =$

$$\begin{bmatrix} 0_{\gamma\times\gamma} \\ 0_{\gamma\times\gamma} \\ \vdots \\ I_{\gamma\times\gamma} \\ \vdots \\ 0_{\gamma\times\gamma} \\ 0_{\gamma\times\gamma} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} r-1\, square\, matrices \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} a-r\, square\, matrices \end{matrix}$$

, where $0_{\gamma\times\gamma}$ represents a $\gamma \times \gamma$ matrix whose entries are all zero, and $I_{\gamma\times\gamma}$ represents a $\gamma \times \gamma$ identity matrix, B^-X_r will be a $\beta \times \gamma$ matrix, where $(B^-X_r)_{ij} = b^-_{is}$, $s = (r - 1)\gamma + j$, and $(B^-X_r)_{ij}$ represents the entry in ith row and jth column of B^-X_r. Likewise, $(B^+X_r)_{ij} = b^+_{is}$. Since $Q_2 = Q_1 - B^-X_r + B^+X_r$, $1 \le i \le \beta$, $1 \le j \le \gamma$: $q_{2_{ij}} = q_{1_{ij}} - (B^-X_r)_{ij} + (B^+X_r)_{ij}$. Hence, $1 \le i \le \beta$, $1 \le j \le \gamma$: $q_{2_{ij}} = q_{1_{ij}} - b^-_{is} + b^+_{is}$, where $s = (r - 1)\gamma + j$. By Definition 3.1 and 3.2, $\forall p_i \in P$, $\forall c_j \in C$: $m_2(p_i)(c_j) = m_1(p_i)(c_j) - I(p_i, t_r)(c_j) + O(p_i, t_r)(c_j)$. Hence, $\forall p_i \in P : \sum_{j=1}^{\gamma} m_2(p_i)(c_j)'c_j = \sum_{j=1}^{\gamma} m_1(p_i)(c_j)'c_j - I(p_i,t_r)(c_j)'c_j + O(p_i,t_r)(c_j)'c_j$ .By Definition 2.2, $\forall p_i \in P$: $m_2(p_i) = m_1(p_i) - I(p_i, t_r) + O(p_i, t_r)$.

(←) Since m_1 $[t_r\!> m_2$, by Definition 2.6, they satisfy $\forall p_i \in P$: $m_2(p_i) = m_1(p_i) - I(p_i, t_r) + O(p_i, t_r)$. Hence, by Definition 2.2, $\forall p_i \in P$, $\forall c_j \in C$: $m_2(p_i)(c_j) = m_1(p_i)(c_j)$

- $I(p_i, t_r)(c_j) + O(p_i, t_r)(c_j)$. By Definition 3.1 and 3.2, $1 \le i \le \beta$, $1 \le j \le \gamma$: $q_{2_{ij}} = q_{1_{ij}} - b^-_{is} + b^+_{is}$, where $s = (r - 1) \times \gamma + j$[III]. By Definition 3.3, $1 \le i \le \beta$, $1 \le j \le \gamma$: $q_{2_{ij}} = q_{1_{ij}} - \sum_{u=1}^{\alpha\times\gamma} b^-_{iu} \times x_{r_{uj}} + \sum_{u=1}^{\alpha\times\gamma} b^+_{iu} \times x_{r_{uj}}$[IV], and therefore $Q_2 = Q_1 - B^-X_r + B^+X_r$.

The arithmetic in a state transformation equation, $Q_2 = Q_1 - B^-X_r + B^+X_r$, only contains the operations of matrix: addition, subtraction and multiplication. Lemma 3.1 proves that state transformation equation, $Q_2 = Q_1 - B^-X_r + B^+X_r$, conforms to the firing equation in Definition 2.6 by applying the matrix representations in Definition 3.1 to 3.3.

Definition 3.4: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $|T| = \alpha$, $|C| = \gamma$, two markings are m and m', and there exists a firing vector σ such that $m\ [\sigma> m'$.

- A **firing matrix** X is an $n \times \gamma$ matrix describes the firing vector σ, where $n = \alpha\gamma$.

  The entries in matrix X are

  $$x_{ij} = \begin{cases} \sigma(t_s) & if\ j = (i-1) \bmod \gamma + 1, where\ s = \lceil (i-1)/\gamma \rceil \\ 0 & otherwise \end{cases},$$

  and x_{ij} represents the entry in ith row and jth column of X.

Definition 3.4 describes a sequence of state transformations and can be deemed as a generalization of Definition 3.3. A firing matrix X can be deemed as a column of α square sub-matrices. The sth square sub-matrix is an identical matrix scalar multiplied by $\sigma(t_s)$.

[III] By Definition 3.2, it should satisfy $r = \lceil s/\gamma \rceil$ and $j = (s - 1) \bmod \gamma + 1$.

[IV] By Definition 3.3, $x_{r_{uj}}$ is 1 only when $u = (r - 1) \times \gamma + j$, otherwise $x_{r_{uj}}$ is 0.

Lemma 3.2: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $|T| = \alpha$, $|C| = \gamma$, two markings are m and m', and there exists a firing vector σ such that $m\ [\sigma> m'$, Q and Q' are matrix representation of m and m' respectively, B^- and B^+ are matrix representation of I and O respectively, and X is the matrix representation of firing vector σ.

- The state transformation between Q and Q' would satisfy

  $Q' = Q - B^-X + B^+X.$

Proof:

Assume the firing vector σ represents a finite occurrence sequence such that $m\ [t_{s1}> m_1\ [t_{s2}> m_2 \ldots [t_{sn}> m'$. Hence, $Q_1 = Q - B^-X_{s1} + B^+X_{s1}$, $Q_2 = Q_1 - B^-X_{s2} + B^+X_{s2}$, …, $Q' = Q_{n-1} - B^-X_{sn} + B^+X_{sn}$, where X_{si} represents a transition firing matrix describing t_{si}. Hence, $Q' = Q - B^-X_{s1} + B^+X_{s1} - B^-X_{s2} + B^+X_{s2} - \ldots - B^-X_{sn} + B^+X_{sn}$. Since all the operations in previous equation are matrix operations, it will obey the left distributive law[V] in matrix operations, and hence $Q' = Q - B^-(X_{s1} + X_{s2} + \ldots + X_{sn}) + B^+(X_{s1} + X_{s2} + \ldots + X_{sn})$. Since $X_{si} =$

$$\begin{bmatrix} 0_{\gamma\times\gamma} \\ 0_{\gamma\times\gamma} \\ \vdots \\ I_{\gamma\times\gamma} \\ \vdots \\ 0_{\gamma\times\gamma} \\ 0_{\gamma\times\gamma} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\\vdots\end{matrix}}\right\} si-1\,square\,matrices \\ \\ \left.\vphantom{\begin{matrix}\vdots\\0\\0\end{matrix}}\right\} \alpha - si\,square\,matrices \end{matrix}$$

, $(X_{s1} + X_{s2} + \ldots + X_{sn})$ is an $n \times \gamma$ matrix, where jth square matrix counted from the top of $(X_{s1} + X_{s2} + \ldots + X_{sn})$ is an identity matrix multiply by the number of appearance of X_j in $(X_{s1} + X_{s2} + \ldots + X_{sn})$. Since each transition firing matrix in $(X_{s1} + X_{s2} + \ldots + X_{sn})$ represents a transition, the number of appearance of X_j in $(X_{s1} + X_{s2} + \ldots + X_{sn})$

[V] A(B + C) = AB + AC, where A, B and C are matrices.

represents the number of appearances of t_j in the firing vector σ, i.e. $\sigma(t_j)$. Therefore, the firing matrix $X = X_{s1} + X_{s2} + \ldots + X_{sn}$ and $Q' = Q - B^{-}X + B^{+}X$.

Lemma 3.2 proves that the state transformation equation of a finite occurrence sequence is $Q' = Q - B^{-}X + B^{+}X$ by derived from Lemma 3.1. By the matrix representations proposed in Definition 3.1, Definition 3.2 and Definition 3.4 and the equation in Lemma 3.2, a CPN can be presented.

3.2 Matrix Constructing Algorithms for CPN

This section presents four constructing algorithms for above matrices: MARKING-MATRIX, FUNCTION-MATRIX, FIRING-MATRIX and NEXT-MARKING. Algorithm MARKING-MATRIX constructs matrix representations for markings of CPNs as in Definition 3.1. Algorithm FUNCTION-MATRIX constructs matrix representations for input functions or output functions of CPNs as in Definition 3.2. Algorithm FUNCTION-MATRIX constructs matrix representations for firing vectors of CPNs as in Definition 3.4. Since there can be only one firing transitions in a firing vector, hence the concept presented in Definition 3.3 is also contained in this algorithm. Algorithm NEXT-MARKING computes the marking after firing a certain transition under another marking by matrix operations.

Algorithm 3.1: **MARKING-MATRIX(*P*, *C*, *m*)**

```
$\beta \leftarrow |P|$
$\gamma \leftarrow |C|$
$Q \leftarrow$ a $\beta \times \gamma$ matrix initialized by 0
```

```
4   for $i \leftarrow 1$ to $\beta$
5       do for $j \leftarrow 1$ to $\gamma$
6           do $q_{ij} \leftarrow m(p_i)(c_j)$    $\triangleright$ $p_i \in P$, $c_j \in C$, and $q_{ij}$ is an entry in $Q$
7   return $Q$
```

The inputs of Algorithm 3.1 are a set of places, a set of colours and a marking function, and the output of Algorithm 3.1 is a marking matrix. The algorithm contains loops nested two deep. The outer **for** loop at line 4 iterates β times after initialization, and each time it constructs a row of marking matrix which represents the tokens on a place. The inner **for** loop at line 5 iterates γ times, and each time it assigns an entry of marking matrix by the number of tokens with a colour on a place. Hence, the assignment at line 6 within loops totally runs $\beta\gamma$ times. Besides initialization and return, Algorithm 3.1 contains the computation of these nested loops, and hence MARKING-MATRIX runs in time $\Theta(\beta\gamma)$.

```
Algorithm 3.2: FUNCTION-MATRIX(P, C, T, F)
$\alpha \leftarrow |T|$
$\beta \leftarrow |P|$
$\gamma \leftarrow |C|$
$B \leftarrow$ a $\beta \times \alpha\gamma$ matrix initialized by 0
for $i \leftarrow 1$ to $\beta$
    do for $j \leftarrow 1$ to $\alpha\gamma$
        do $r \leftarrow \lceil j / \gamma \rceil$
            $s \leftarrow (j - 1) \bmod \gamma + 1$
            $b_{ij} \leftarrow F(p_i, t_r)(c_s)$    $\triangleright$ $p_i \in P$, $t_r \in T$, $c_s \in C$, and $b_{ij}$ is an entry in $B$
```

```
10  return $B$
```

The inputs of Algorithm 3.2 are a set of places, a set of colours, a set of transitions and an input function (or an output function), and the output of Algorithm 3.2 is an input or output matrix. Similarly, Algorithm 3.2 contains loops nested two deep. The outer **for** loop at line 5 iterates *β* times after initialization, and each time it constructs a row of input or output matrix which represents the tokens to be removed from or deposited into a place while firings occur. The inner **for** loop at line 6 iterates *αγ* times when outer loop iterates once, and each time it assigns an entry of input or output matrix by the number of tokens with a colour to be removed from or deposited into a place during a transition fires. The assignments in lines 7-9 totally run *βαγ* times, and each time they firstly compute the transition and the colour that an entry of input or output matrix refers to, and then assign the value retrieved from an input or output function to this entry. Algorithm 3.2 has a similar structure as Algorithm 3.1, but there are only three assignments which run *βαγ* times in these nested loops. Thus, FUNCTION-MATRIX takes Θ(*βαγ*) time.

Algorithm 3.3: **FIRING-MATRIX(*C*, *T*, *σ*)**

```
$\alpha \leftarrow |T|$
$\gamma \leftarrow |C|$
$X \leftarrow$ a $\alpha\gamma \times \gamma$ matrix initialized by 0
for $s \leftarrow 1$ to $\alpha$
    do for $j \leftarrow 1$ to $\gamma$
           do $i \leftarrow (s-1)\gamma + j$
              $x_{ij} \leftarrow \sigma(t_s)$    ▷ $t_s \in T$, and $x_{ij}$ is an entry in $X$
```

```
8   return $X$
```

The inputs of Algorithm 3.3 are a set of colours, a set of transitions and a firing vector, and the output of Algorithm 3.3 is a firing matrix. The outer **for** loop at line 4 iterates α times, and each time it constructs the values of a $\gamma \times \gamma$ square sub-matrix in a firing matrix, which corresponds to a transition in firing vector. The inner **for** loop at line 5 iterates γ times when outer loop iterates once, and each time it assigns the values to diagonal entries on a $\gamma \times \gamma$ square matrix by the number of appearances of a transition in firing vector. Algorithm 3.3 also has a similar structure as Algorithm 3.2, except there are only two assignments which run $\alpha\gamma$ times in these nested loops. In general case, the initialization is considered to runs in constant time, and hence lines 1-3 takes constant time. Thus, FIRING-MATRIX takes $\Theta(\alpha\gamma)$ time.

```
Algorithm 3.4: NEXT-MARKING($Q, B^-, B^+, X$)
1   $U \leftarrow$ MATRIX-MULTIPLY($B^-, X$)
2   $V \leftarrow$ MATRIX-MULTIPLY($B^+, X$)
3   for $i \leftarrow 1$ to $rows[U]$                ▷ $rows[U]$ is the number of rows of U
4       do for $j \leftarrow 1$ to $columns[U]$     ▷ $columns[U]$ is the number of columns of U
5               do if $q_{ij} < u_{ij} - v_{ij}$
    ▷ $q_{ij}$, $u_{ij}$ and $v_{ij}$ is an entry in $Q$, $U$ and V respectively
6                   then return $Q$
7   return MATRIX-ADD(MATRIX-SUBTRACT($Q$, $U$), $V$)


8   MATRIX-MULTIPLY($A$, $B$)
```

```
9   $C \leftarrow$ a $rows[A] \times columns[B]$ matrix initialized by 0
10  for $i \leftarrow 1$ to $rows[A]$
11      do for $j \leftarrow 1$ to $columns[B]$
12            do for $k \leftarrow 1$ to $columns[A]$
13                  do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
14  return $C$


15  MATRIX-ADD($A$, $B$)
16  $C \leftarrow$ a $rows[A] \times columns[A]$ matrix initialized by 0
17  for $i \leftarrow 1$ to $rows[A]$
18      do for $j \leftarrow 1$ to $columns[A]$
19            do $c_{ij} \leftarrow a_{ij} + b_{ij}$
20  return $C$


21  MATRIX- SUBTRACT($A$, $B$)
22  $C \leftarrow$ a $rows[A] \times columns[A]$ matrix initialized by 0
23  for $i \leftarrow 1$ to $rows[A]$
24      do for $j \leftarrow 1$ to $columns[A]$
25            do $c_{ij} \leftarrow a_{ij}$ - $b_{ij}$
26  return $C$
```

The inputs of Algorithm 3.4 are a marking matrix, an input matrix, an output function and a firing matrix, and the output of Algorithm 3.4 is a marking matrix. In lines 3-6 it firstly checks if the sequence of firing is valid, i.e., if it would not cause a negative number of tokens in some places. Then, it computes the next marking by the state transformation equation in Lemma 3.2. In line 1-2, it contains two matrix multiplications defined in line 8-14, and each of them takes $\Theta(\beta\alpha\gamma^2)$ time obviously. In line 3-6, it contains loops nested two deep, and there has only a conditional return in these loops, hence they run in $O(\beta\gamma)$. The return at line 7 contains a matrix addition and a matrix subtraction, and each of them takes $\Theta(\beta\gamma)$ time. Thus, NEXT-MARKING takes $O(\beta\alpha\gamma^2)$ time.

3.3 Example for Matrix Representation of CPN

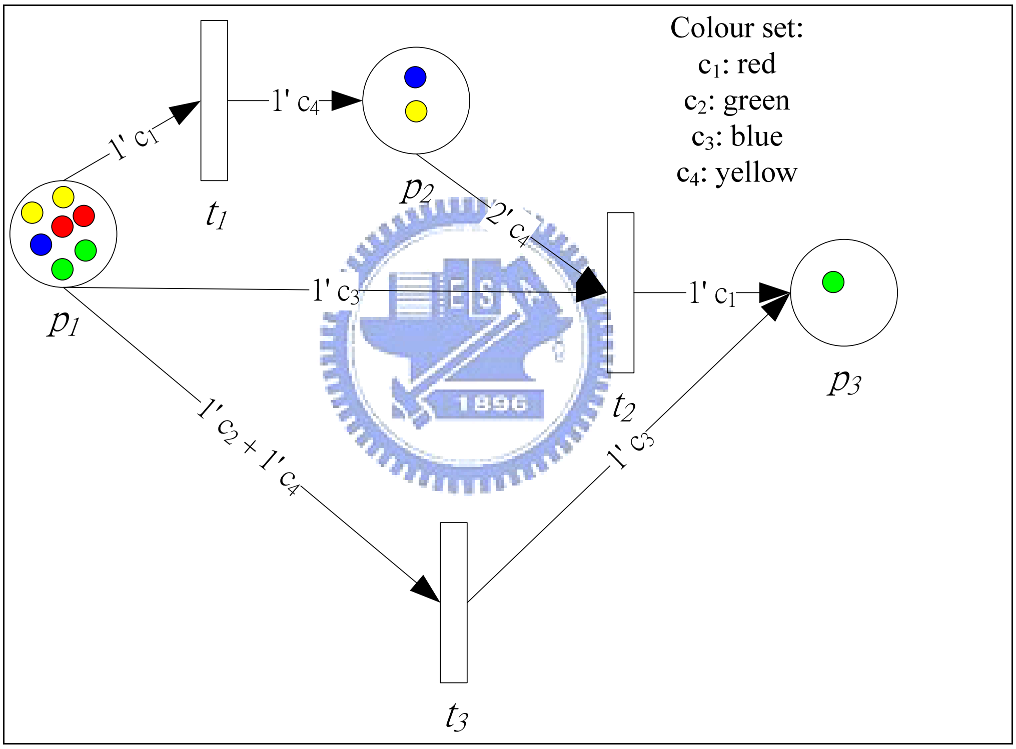

Figure 3.1 A CPN with enable transitions.

Consider the CPN in Figure 3.1, where the place set $P = \{p_1, p_2, p_3\}$, transition set $T = \{t_1, t_2, t_3\}$, and colour set $C = \{c_1, c_2, c_3, c_4\}$. The input function I satisfies $I(p_1, t_1) = 1'c_1$, $I(p_1, t_2) = 1'c_3$, $I(p_1, t_3) = 1'c_2 + 1'c_4$, and $I(p_2, t_2) = 2'c_4$. The output function O satisfies $O(p_2, t_1) = 1'c_4$, $O(p_3, t_2) = 1'c_1$, and $O(p_3, t_3) = 1'c_3$. The marking m in Figure 3.1 satisfies $m(p_1) = 2'c_1 + 2'c_2 + 1'c_3 + 2'c_4$, $m(p_2) = 1'c_3 + 1'c_4$, and $m(p_3) =$

1'c_2. There is an occurrence sequence m [t_3> m_1 [t_3> m_2 [t_1> m_3 [t_2> m', and corresponding firing vector is σ. The marking m' after the firing vector is σ satisfies $m'(p_1)$ = 1'c_1, $m'(p_2)$ = 1'c_3, and $m'(p_3)$ = 1'c_1 + 1'c_2 + 2'c_3. According to Definition 3.1, the marking matrix which represents marking m is

$$Q=\begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

According to Definition 3.2, the input matrix which represents input function I is

$$B^{-}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the output matrix which represents input function O is

$$B^{+}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

According to Definition 3.4, the firing matrix which represents firing vector σ is

$$X=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The marking matrix after the firing matrix X is

$$Q' = Q - B^{-}X + B^{+}X$$

$$= \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

, which satisfies the matrix representation of marking *m'*.

Chapter 4. Fault Detection and Correction Scheme

This chapter presents a scheme detecting and correcting the faults in CPNs. The presented scheme firstly encodes a CPN into a *redundant* CPN for detection and correction of faults later. The detection and correction in the redundant CPN is done via parity check. Since all the encoding and parity check need to operate upon linear algebra, the scheme works based on the matrix representations of CPNs in chapter 3 and shows the faults with matrix representations too. The definition of redundant CPNs is given in section 4.1. Section 4.2 discusses the faults considered in the thesis and gives the corresponding definitions. From section 4.3 to section 4.5, each section discusses the methodology dealing with a kind of faults defined in section 4.2. Section 4.6 proves the correctness of the proposed scheme.

4.1 Bisimilar and Redundant CPNs

Bisimulation equivalence is also called *bisimilarity* [10]. Bisimulations play an important role in the theory of parallelism and concurrency [5]. The techniques apply to language equivalence and reachability set equivalence can also apply to bisimulation equivalence of Petri nets. Several kinds of notations applied to bisimulation equivalence, such as based on markings [4, 5] and based on places [12]. The notations adopted in this thesis are the former with some modifications in order to suit for CPNs.

The *redundancy* [3, 4] is stricter than bisimulation equivalence, i.e., it contains all the conditions in bisimulation equivalence. If a Petri net is the redundant net of another Petri net, these two Petri nets can have a common firing sequence. The *redundant* CPNs discussed here are the notations based on the extensions of those in [4].

Definition 4.1: Let $G = <N, m_0>$ and $G' = <N', m_0'>$ be two CPNs where $N = (P, T, A, C, I, O)$ and $N' = (P', T', A', C', I', O')$, M and M' be the sets which are all markings of G and G' respectively, $R \subseteq M \times M'$ be a relation between marking m and m' where $m \in M$ and $m' \in M'$, and mRm' denote the relation.

- R is a **bisimulation** iff for all $m_1 \in M$ and $m_1' \in M'$ such that m_1Rm_1'.
  (1) For each enabled transition $t \in T$ and $m_2 \in M$ such that m_1 $[t> m_2$, there exists an enabled transition $t' \in T'$ and $m_2' \in M'$ such that m_1' $[t'> m_2'$ with m_2Rm_2'.
  (2) For each enabled transition $t' \in T'$ and $m_2' \in M'$ such that m_1' $[t'> m_2'$, there exists an enabled transition $t \in T$ and $m_2 \in M$ such that m_1 $[t> m_2$ with m_2Rm_2'.
- For a bisimulation R, CPNs $<N, m_0>$ and $<N', m_0'>$ are called **bisimilar**[VI] and are denoted by $<N, m_0> \sim <N', m_0'>$.

[VI] It can also call that there is bisimulation equivalence between $<N, m_0>$ and $<N', m_0'>$.

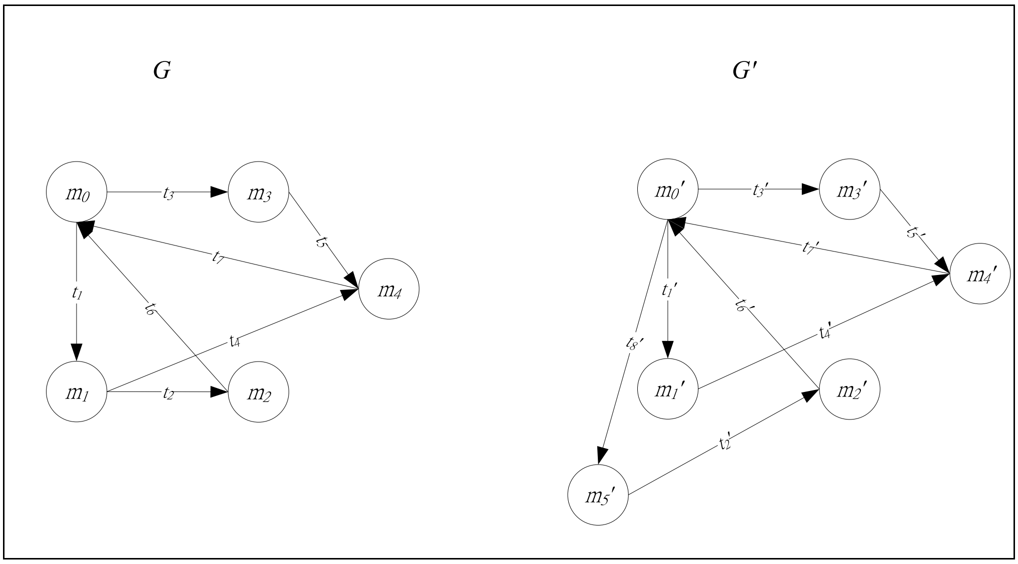

Figure 4.1 Occurrence graphs of two bisimilar CPNs.

The example in Figure 4.1 contains the Occurrence graphs of two CPNs which are $G = <N, m_0>$ and $G' = <N', m_0'>$. The CPN structures of these two CPNs are $N = (P, T, A, C, I, O)$ and $N' = (P', T', A', C', I', O')$. The markings m_1, m_2, m_3 and m_4 are reachable from initial marking m_0 in G, and the markings m_1', m_2', m_3', m_4' and m_5' are reachable from initial marking m_0' in G'. There is a relation $R = \{(m_0, m_0'), (m_1, m_1'), (m_1, m_5'), (m_2, m_2'), (m_2, m_4'), (m_3, m_3'), (m_4, m_2'), (m_4, m_4')\}$ which satisfies the conditions in definition 4.1, hence R is a bisimulation[VII] and $G \sim G'$.

Definition 4.2: Let m be a marking of a set of place P where $|P| = \beta$ and m' be a marking of a set of places P' where $|P'| = \beta'$.

- A **linear transformation function set** F of a marking m is defined by $F = \{f_i \mid i = 1\ldots q$, where $q > 0$; $f_i(m(p_1), m(p_2), \ldots, m(p_\beta)) = o_i$, where $p_1, p_2, \ldots, p_\beta \in P$, $o_i \in C_{MS}$, and f_i is a function only contains addition, subtraction, and inner product on input multi-sets$\}$.

[VII] It can be checked one by one in relation R.

- m' is **linearly related** to m if and only if there exists a linear transformation function set F of m which satisfies following two conditions.

  (1) $|F| = \beta'$.

  (2) $f_i(m(p_1), m(p_2), \ldots, m(p_\beta)) = m'(p_i')$.

  It is denoted by $F(m) = m'$.

The linear relation of two markings in Definition 4.2 is one kind of relation in Definition 4.1. The relation defined in Definition 4.2 is stricter than that in Definition 4.1 since the latter can be nonlinear.

Definition 4.3: Let $G = <N, m_0>$ and $G' = <N', m_0'>$ be two CPNs where $N = (P, T, A, C, I, O)$ and $N' = (P', T', A', C', I', O')$, M and M' be sets which are all markings of G and G' respectively, $m_1, m_2 \in M$, $m_1', m_2' \in M'$, $t \in T$, and F and H be linear transformation function sets.

- G and G' have **redundant** relation iff the following conditions are true.

  (1) $T = T'$.

  (2) $F(m_0) = m_0'$ and $H(m_0') = m_0$.

  (3) If $F(m_1) = m_1'$, $H(m_1') = m_1$ and m_1 $[t>$ m_2, then m_1' $[t>$ m_2' such that $F(m_2) = m_2'$ and $H(m_2') = m_2$.

  (4) If $F(m_1) = m_1'$, $H(m_1') = m_1$ and m_1' $[t>$ m_2', then m_1 $[t>$ m_2 such that $F(m_2) = m_2'$ and $H(m_2') = m_2$.

  It is denoted by $<N, m_0> \simeq <N', m_0'>$.

It is obvious that bisimilarity in Definition 4.1 doesn't constrain the relation of firing transitions between two bisimilar CPNs. They can have different sequences of firing transitions, as long as they satisfy the conditions in Definition 4.1. However, the definition of redundancy in Definition 4.3 gives stricter constrains that also define the sequences of firing transitions should be the same. If $<N, m_0> \simeq <N', m_0'>$, then $<N, m_0> \sim <N', m_0'>$ will also be true, since the condition 3 and 4 in Definition 4.3 are stricter than the condition 1 and 2 in Definition 4.1 respectively and the condition 1 and 2 in Definition 4.3 should be satisfied in addition. Redundancy has an additional property with respect to bisimilarity, which redundancy has a property of activity equivalence since sequences of firing transitions should be same.

Definition 4.4: Let $G = <N, m_0>$ and $G' = <N', m_0'>$ be two CPNs where $N = (P, T, A, C, I, O)$ and $N' = (P', T', A', C', I', O')$, and $[m_0>$ and $[m_0'>$ be sets of reachable markings from m_0 in G and m_0' in G' respectively.

- G' is a **separate** CPN of G iff the following conditions are true.
  (1) $G \simeq G'$.
  (2) $P \supseteq P'$.
  (3) $C = C'$.
  (4) $\forall p \in P$: $m_0(p) = m_0'(p)$.
  (5) $\forall p \in P$, $\forall m \in [m_0>$, $\forall m' \in [m_0'>$: if the sequences of firing transitions from m_0 to m is the same as that from m_0' to m', then $m(p) = m'(p)$.
- G' is a **nonseparate** CPN of G iff $G \simeq G'$ and $P \supseteq P'$ but G' is not a separate CPN of G.

The redundant relation can be classified into two classes, which are separate and nonseparate relations. If G' is a separate CPN of G, the markings of G can be identified from the markings on the subset of places in G' under the same sequences of firing transitions, however the markings of G can be identified by linear operations on the markings of G' in a nonseparate case. In addition, a separate CPN has a property of sub-marking equivalence which means the markings of a CPN is retained in the markings of another CPN on the subset of places. In this thesis, we mainly discuss separate CPNs since this case is more meaningful than a nonseparate case, though the method proposed in this thesis can be applied in a nonseparate case.

4.2 Fault Models

There are several kinds of faults that a CPN may suffer. In [11], two kinds of fault models in Petri nets, which are *place faults* and *transition faults*, are proposed. A place fault is caused by the corruption on a place, and a transition fault is caused by a firing problem. This thesis extends the fault models in CPNs, where fault model on transition faults is divided into two classes: *amount* and *colour*. The amount transition faults are similar to the transition faults in [11]. Since the tokens in CPNs are coloured, there is an additional fault on transition introduced: colour transition fault. The place faults in CPNs are extended the place faults in [11], where the place faults in CPNs would be caused by faulty colours or faulty amount.

Definition 4.5: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $\exists t \in T$, M be the all marking of G, and $m, m' \in M$ where there exists a firing vector σ such that m $[\sigma> m'$.

■ If m_f is a marking with a **pre-condition amount transition fault**,

$\forall p \in P$: $m_f(p) = m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)\ ) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + \ldots + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha)) + i * I(p, t) = m'(p) + i * I(p, t)$, where $i \in \mathbb{N} / \{0\}$ and $i \leq \sigma(t)$.

- If m_f is a marking with a **post-condition amount transition fault**,

  $\forall p \in P$: $m_f(p) = m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)\ ) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + \ldots + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha)) - i * O(p, t) = m'(p) - i * O(p, t)$, where $i \in \mathbb{N} / \{0\}$ and $i \leq \sigma(t)$.

There are two types of amount transition faults which are pre-condition and post-condition amount transition faults. A pre-condition amount transition fault occurs when the tokens are not removed from the pre-set of transition t being fired. A post-condition amount transition fault occurs when the tokens are not deposited into the post-set of transition t being fired. In Definition 4.5, the marking m' is reached by firing a sequence of transitions from marking m_1 under fault-free conditions, and its firing vector is σ, hence they satisfy $\forall p \in P$: $m'(p) = m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)\ ) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + \ldots + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha))$. Thus, if a pre-condition amount transition fault, which causes i times of not removing tokens from the pre-set of transition t in these $\sigma(t)$ times of transition t being fired, occurs during this firing sequence, the CPN will get into a faulty marking m_f and the difference between m_f and a fault-free marking is $i * I(p, t)$. If a post-condition amount transition fault, which causes i times of not depositing tokens into the post-set of transition t in these $\sigma(t)$ times of transition t being fired, occurs during this firing sequence, the CPN will get into a faulty marking m_f and the difference between m_f and a fault-free marking is - $i * O(p, t)$. An amount transition fault can be deemed as a fault cause by choking.

Definition 4.6: Let $G = \langle N, m_0 \rangle$ be a CPN where $N = (P, T, A, C, I, O)$, M be the all marking of G, $m \in M$ be a fault-free marking, and m_f be a marking with accumulated amount transition faults in respect of m, such that

$$\exists t_{rf_1} \cdots t_{rf_q} \in T, \exists t_{of_1} \cdots t_{of_s} \in T, \forall p \in P:$$
$$m_f(p) = m(p) + i_1 * I(p, t_{rf_1}) + i_2 * I(p, t_{rf_2}) + \cdots + i_q * I(p, t_{rf_q})$$
$$- j_1 * O(p, t_{of_1}) - j_2 * O(p, t_{of_2}) - \cdots - j_s * O(p, t_{of_s}).$$

- $|m_f|_{at}$ is the amount of amount transition faults in m_f, and $|m_f|_{at} = |T_f|$, where T_f is a set of transitions that have experienced amount transition faults, such that $\forall t_{rf_1}, \cdots, t_{rf_q}, t_{of_1}, \cdots t_{of_s} \in T_f$.

The amount of amount transition faults in a faulty marking is defined on the amount of transitions that have suffered amount transition faults, rather than the amount of firings with amount transition faults. Hence, if there is only one transition that has suffered several times of an amount transition fault in a firing sequence, it is still defined as an amount transition fault in this firing sequence.

Definition 4.7: Let $G = \langle N, m_0 \rangle$ be a CPN where $N = (P, T, A, C, I, O)$, $\exists t_j \in T$, $\exists c_r \in C$, M be the all marking of G, and $m, m' \in M$ where there exists a firing vector σ such that $m\ [\sigma \rangle\ m'$.

- If m_f is a marking with a **pre-condition colour transition fault**,

  $\forall p \in P$: $m_f(p) = m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)\ ) + \sigma(t_2) * (- I(p, t_2) + O(p,$

$t_2)) + ... + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha)) - I_f(p)= m'(p) - I_f(p)$,

where I_f is a input faulty function which satisfies $I_f(p_k) = -i * I(p_k, t_j)(c_r)'c_r + \sum_{h=1,h\neq r}^{\gamma} i_h * I(p_k, t_j)(c_r)'c_h$, $\sum_{h=1,h\neq r}^{\gamma} i_h = i$, $i \in \mathbb{N} / \{0\}$, $i \leq \sigma(t_j)$, and $i_h \in \mathbb{N}$. The pre-condition colour transition fault denoted by I_f is defined as the pre-condition colour transition fault occurring on the colour c_r of transition t_j.

- If m_f is a marking with a **post-condition colour transition fault**,

  $\forall p \in P$: $m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + ... + \sigma(t_\alpha) * (- I(p, t_\alpha) + O(p, t_\alpha)) - O_f(p)= m'(p) - O_f(p)$,

  where O_f is a output faulty function which satisfies $O_f(p_k) = i * O(p_k, t_j)(c_r)'c_r - \sum_{h=1,h\neq r}^{\gamma} i_h * O(p_k, t_j)(c_r)'c_h$, $\sum_{h=1,h\neq r}^{\gamma} i_h = i$, $i \in \mathbb{N} / \{0\}$, $i \leq \sigma(t_j)$, and $i_h \in \mathbb{N}$. The post-condition colour transition fault denoted by O_f is defined as the post-condition colour transition fault occurring on the colour c_r of transition t_j.

Colour transition faults also have two types, named as pre-condition and post-condition colour transition faults. A pre-condition colour transition fault occurs when the tokens removed are of wrong colours from the pre-set of transition t_j being fired. A post-condition colour transition fault occurs when the tokens deposited are of wrong colours into the post-set of transition t_j being fired. Similar to the explanation in Definition 4.5, $m(p) + \sigma(t_1) * (- I(p, t_1) + O(p, t_1)) + \sigma(t_2) * (- I(p, t_2) + O(p, t_2)) + ... + \sigma(t_a) * (- I(p, t_a) + O(p, t_a))$ in Definition 4.6 describes a fault-free process. In pre-condition case, I_f indicates that the faulty firing removes the tokens of wrong colour, here c_r is correct and $\sum_{h=1,h\neq r}^{\gamma} i_h * I(p_k,t_j)(c_r)'c_h$ describes the wrong colours, from the pre-set of transition t_j. It is similar in post-condition case where O_f indicates the faulty process deposits the wrong tokens, which should be with the color element c_r but with the color elements $\sum_{h=1,h\neq r}^{\gamma} i_h * O(p_k,t_j)(c_r)'c_h$, to the post-set of transition t_j. In both pre-condition and post-condition case, the multiplier i indicates that i times of colour transition faults occur on transition t_j in these $\sigma(t_j)$ times of transition t_j being fired. A colour transition fault can be deemed as a fault cause by noise during transmitting.

Definition 4.8: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, M be the all marking of G, $m \in M$ be a fault-free marking, and m_f be a marking with accumulated colour transition faults in respect of m, such that

$$\exists t_{rf_1}\cdots t_{rf_q} \in T, \exists c_{sf_1}\cdots c_{sf_q} \in C, \exists t_{of_1}\cdots t_{of_s} \in T, \exists c_{hf_1}\cdots c_{hf_s} \in C, \forall p \in P:$$

$$\begin{aligned} m_f(p) = m(p) &+ i_1 * I_{f_1}(p) + i_2 * I_{f_2}(p) + \cdots + i_q * I_{f_q}(p) \\ &- j_1 * O_{f_1}(p) - j_2 * O_{f_2}(p) - \cdots - j_s * O_{f_s}(p) \end{aligned} ,$$

where $I_{f_1},\cdots,I_{f_q}$ are input faulty functions denote the pre-condition colour faults on the colour c_{sf_1} of transition t_{rf_1}, …, the colour c_{sf_q} of transition t_{rf_q} respectively, and $O_{f_1},\cdots,O_{f_s}$ are output faulty functions denote the post-condition colour faults on the colour c_{hf_1} of transition t_{of_1}, …, the colour c_{hf_s} of transition t_{of_s} respectively.

- $|m_f|_{ct}$ is the amount of colour transition faults in m_f, and $|m_f|_{ct}$ = the amount of distinct pairs in $(t_{rf_1},c_{sf_1}),\cdots,(t_{rf_q},c_{sf_q}),(t_{of_1},c_{hf_1}),\cdots,(t_{of_s},c_{hf_s})$.

The amount of colour transition faults in a faulty marking is defined on the sum of colours with colour transition faults on each transition, rather than the amount of firings with colour transition faults. Hence, if the colour transition fault only occurs on one colour of transition in a firing sequence, it is still defined as a colour transition fault in this firing sequence.

Definition 4.9: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, $\exists p_f \in P$, $\exists\ c \in C$, M be the all marking of G, and $m \in M$.

- If m_f is a marking with a **place fault** on place p_f in respect of marking m, $m_f(p_f) = m(p_f) + a - s$ and $\forall\ p_c \in P/\{p_f\}: m_f(p_c) = m(p_c)$, where $a, s \in C_{MS}$, and $a \neq s$.

Token corruption in CPN may cause place faults which occur when the amounts of tokens on places are suddenly increased or decreased, or the colours of tokens on places are suddenly changed without firing. Formally, as in Definition 4.7, a and s are both multi-sets over C, and $a - s$ indicates the incorrect tokens on place p_f with respect to the fault-free marking m. Practically, an place fault can represent a fault caused by data missing, appearing of fake data or data errors. In [11], there is another fault, *additive faults*, indicating simultaneous occurrence(s) of place faults and transition faults. In this thesis, the additive faults indicate simultaneous occurrences of place faults, amount transition faults, and colour transition faults.

Definition 4.10: Let $G = <N, m_0>$ be a CPN where $N = (P, T, A, C, I, O)$, M be the all marking of G, $m \in M$ be a fault-free marking, and m_f be a marking with accumulated place faults in respect of m, such that $1 \le i \le \beta : m_f(p_i) = m(p_i) + a_i - s_i$, where a_i, s_i $\in C_{MS}$.

- $|m_f|_p$ is the amount of place faults in m_f, and $|m_f|_p$ = the amount of $1 \le i \le \beta$: $a_i \ne s_i$.

The amount of place faults in a faulty marking is defined on the amount of places that have suffered place faults, rather than the times of occurrence of place faults. Hence, if there is only one place that has suffered several times of a place fault, it is still defined as a place fault. In [11], there is another fault, *additive faults*, indicating simultaneous occurrence(s) of place faults and transition faults. In this thesis, the additive faults indicate simultaneous occurrences of place faults, amount transition faults, and colour transition faults.

4.3 Place Faults

According to Definition 4.9, this section firstly gives the problem formulation of place faults in matrix representations. Next, based on the problem formulation, this section presents a methodology encoding a CPN into a separate CPN with detection and correction capabilities on place faults. Then, this section gives the syndromes of place faults while they occur in CPNs. Finally, this section describes how to compute the correction markings via these syndromes.

4.3.1 Problem Formulation for Place Faults

Lemma 4.1: Let Q be a fault-free marking matrix of a CPN G.

- If Q_f is a marking matrix containing a place fault on place p_i corresponding to Q,

  $\exists F_p^i$, a $\beta \times \gamma$ matrix,

  $Q_f = Q + F_p^i$,

  and all the entries in F_p^i satisfy

  $\forall 1 \leq j \leq \gamma$: $f_{p_{ij}}^i \in \mathbb{Z}$, $\exists 1 \leq h \leq \gamma$: $f_{p_{ih}}^i \neq 0$,

  and $\forall 1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, $k \neq i$: $f_{p_{kl}}^i = 0$.

Proof:

Assume Q is the matrix representation of a fault-free marking m, and Q_f is the matrix representation of m_f which is a marking with a place fault on place p_i with respect to m. The entries in matrix Q and Q_f are $q_{yz} = m(p_y)(c_z)$ and $q_{f_{yz}} = m_f(p_y)(c_z)$, respectively. By Definition 4.9, $m_f(p_i) = m(p_i) + a - s$ and $\forall$ $p_k \in P/\{p_i\}$: $m_f(p_k) = m(p_k)$, where a, $s \in C_{MS}$, and $a \neq s$. Hence, by Definition 2.2, $\forall$ $p_k \in P/\{p_i\}$,$\forall$ $c_l \in C$: $m_f(p_i)(c_l) = m(p_i)(c_l) + a(c_l) - s(c_l)$ and $m_f(p_k)(c_l) = m(p_k)(c_l)$. Consider $m_f(p_k)(c_l) = m(p_k)(c_l)$ part, it would have

$$\forall 1 \leq l \leq \gamma,\ 1 \leq k \leq \beta,\ k \neq i:\ q_{f_{kl}} = q_{kl}. \qquad (1)$$

Since $a \neq s$, by Definition 2.2, $\exists c_h \in C$: $a(c_h) \neq s(c_h)$. By Definition 2.1, $\forall$ $c_l \in C$: $a(c_l) \in \mathbb{N}$ and $s(c_l) \in \mathbb{N}$. Hence, consider $m_f(p_i)(c_l) = m(p_i)(c_l) + a(c_l) - s(c_l)$ part, it would have

$\forall 1 \leq l \leq \gamma$: $q_{f_{il}} = q_{il} + e_l$, where $e_l \in \mathbb{Z}$, and (2)

$\exists 1 \leq h \leq \gamma$: $e_h \neq 0$. (3)

By combining (1), (2) and (3), $\forall 1 \leq l \leq \gamma$, $1 \leq k \leq \beta$, $k \neq i$: $q_{f_{kl}} = q_{kl} + 0$, $\forall 1 \leq l \leq \gamma$: $q_{f_{il}} = q_{il} + e_l$, and $\exists 1 \leq h \leq \gamma$: $e_h \neq 0$. Hence, there exist a $\beta \times \gamma$ matrix, F_p^i, the entries in F_p^i satisfy $\forall 1 \leq j \leq \gamma$: $f_{p_{ij}}^i \in \mathbb{Z}$, $\exists 1 \leq h \leq \gamma$: $f_{p_{ih}}^i \neq 0$, and $\forall 1 \leq l \leq \gamma$, $1 \leq k \leq \beta$, $k \neq i$: $f_{p_{kl}}^i = 0$, and the relation between Q and Q_f is $Q_f = Q + F_p^i$.

Lemma 4.1 formulates the problem of a place fault in linear algebra. The matrix F_p^i in the equation is an indicator matrix of a place fault, where it has only one row containing nonzero entries. If the nonzero entries exist in the ith row, it indicates that a place fault occurs on the place p_i. If the jth column in this row is a nonzero entry, the colour c_j tokens on place p_i are incorrect. The consistency between the description of place fault in Lemma 4.1 and the definition of place fault in Definition 4.9 is also proved.

Lemma 4.2: Let Q be a fault-free marking matrix of a CPN G.

- If Q_f is a marking matrix representing Q with n place faults, $\exists F_p$, a $\beta \times \gamma$ matrix,

  $Q_f = Q + F_p$,

  and all the entries in the **place fault indicator matrix** F_p satisfy

  $\exists 1 \leq i_1, i_2, \ldots, i_n \leq \beta$, $i_1 \neq i_2 \neq \ldots \neq i_n$, $\forall 1 \leq j \leq \gamma$: $f_{p_{i_1 j}}, f_{p_{i_2 j}}, \cdots, f_{p_{i_n j}} \in \mathbb{Z}$, $\forall 1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, $k \neq i_1 \neq i_2 \neq \ldots \neq i_n$: $f_{p_{kl}} = 0$, and $\exists 1 \leq h_1, h_2, \ldots, h_n \leq \gamma$:

$f_{p_{i_1h_1}}, f_{p_{i_2h_2}}, \cdots, f_{p_{i_nh_n}} \neq 0$.

Proof:

By Definition 4.10, *n* place faults denote that there are *n* places with place faults. Hence, by Lemma 4.1, there are *n* matrices, $F_p^{i_1}, F_p^{i_2}, \cdots, F_p^{i_n}$, indicate these place faults, where $i_1 \neq i_2 \neq \ldots \neq i_n$, and $Q_f = Q + F_p^{i_1} + F_p^{i_2} + \cdots + F_p^{i_n} = Q + F_p$. Therefore, the entries in F_p should satisfy $\exists 1 \leq i_1, i_2, \ldots, i_n \leq \beta$, $i_1 \neq i_2 \neq \ldots \neq i_n$, $\forall 1 \leq j \leq \gamma$: $f_{p_{i_1j}}, f_{p_{i_2j}}, \cdots, f_{p_{i_nj}} \in \mathbb{Z}$, $\forall 1 \leq l \leq \gamma$, $1 \leq k \leq \beta$, $k \neq i_1 \neq i_2 \neq \ldots \neq i_n$: $f_{p_{kl}} = 0$, and $\exists 1 \leq h_1, h_2, \ldots, h_n \leq \gamma$: $f_{p_{i_1h_1}}, f_{p_{i_2h_2}}, \cdots, f_{p_{i_nh_n}} \neq 0$.

A matrix F_p which indicates *n* place faults would have *n* rows with nonzero entries, and the other rows have only entries of zero. Each row with nonzero entries in F_p represents a place fault. The relation equation, $Q_f = Q + F_p$, could be proved by deriving from Lemma 4.1.

4.3.2 Separate CPNs with Place Faults Detection and Correction Capabilities

In order to detect and correct place faults, the strategy in this thesis encodes a CPN with additional tokens as a separate CPN firstly. Definitely, Let *G* be a CPN with no more than *k* place faults, the separate CPN with place faults detection and correction capabilities is constructed by adding 2*k* additional places to *G* and the colour sets of both CPNs are the same. Assume that G has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}. The separate CPN H with place faults detection and correction capabilities in respect to G would have α transitions, $\beta + 2k$ places, γ colours, input matrix B_h^-, output matrix B_h^+ and initial marking matrix Q_{0_h}. Besides, $B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-$, $B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$, where D is a $2k \times \beta$ matrix, and I_β denotes a $\beta \times \beta$ identity matrix. After H is constructed, the place faults occurring on H can be identified and corrected from the syndromes. These properties are proved in Lemma 4.3 and 4.4.

Lemma 4.3: Let G be a CPN which has α transitions, β places, γ colours, the input matrix B_g^-, the output matrix B_g^+ and the initial marking matrix Q_{0_g}.

If the CPN H, constructed by adding d additional places to G, has the same colour set with G, α transitions, $\beta + d$ places, γ colours, input matrix

$$B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-,$$

output matrix

$$B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+$$

and initial marking matrix

$$Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$$

, where D is a $d \times \beta$ matrix, all the entries in D are nonnegative, $d \in \mathbb{N}$, and I_β denotes a $\beta \times \beta$ identity matrix, H has following two properties.

- H is a separate CPN with respect to G.
- If a reachable marking matrix Q_g of G has the same firing transition sequence

with a reachable marking matrix Q_h of H, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$.

Proof:

First, Let P_g and P_h be the place sets of G and H respectively. Consider conditions 2 and 3 in Definition 4.4, CPN H is composed of CPN G and d additional places, and thus $P_g \supseteq P_h$. Since H and G have the same colour set, $C_g = C_h$ where C_g and C_h are the colour sets of G and H respectively.

Next, let i and j be two integers, where $1 \leq i \leq \beta$ and $1 \leq j \leq \gamma$. $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = \begin{bmatrix} I_\beta Q_{0_g} \\ DQ_{0_g} \end{bmatrix} = \begin{bmatrix} Q_{0_g} \\ DQ_{0_g} \end{bmatrix}$, thus $q_{0_{h_{ij}}} = q_{0_{g_{ij}}}$, for all possible i and j. By Definition 3.1, $q_{0_{h_{ij}}} = m_{0_h}(p_i)(c_j)$ and $q_{0_{g_{ij}}} = m_{0_g}(p_i)(c_j)$, and hence $\forall p_i \in P_g, \forall c_j \in C_g : m_{0_h}(p_i)(c_j) = m_{0_g}(p_i)(c_j)$. By Definition 2.2, since $C_g = C_h$, $\forall p_i \in P_g : m_{0_h}(p_i) = m_{0_g}(p_i)$. Therefore, condition 4 in Definition 4.4 is satisfied.

Next, since H has no additional transition compared to G,

$T_g = T_h$, (1)

where T_g and T_h are the transition sets of G and H respectively. Since $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = UQ_{0_g}$, there is a matrix $V = \begin{bmatrix} I_\beta & 0_{\beta\times d} \end{bmatrix}$ such that $VQ_{0_h} = VUQ_{0_g} = \begin{bmatrix} I_\beta & 0_{\beta\times d} \end{bmatrix} \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = Q_{0_g}$, where $0_{\beta\times d}$ is a $\beta\times d$ matrix with all entries of zero. By Definition 3.1, $m_{0_h}(p_i)(c_j) = q_{0_{h_{ij}}} = \sum_{l=1}^{\beta} u_{il} q_{0_{g_{lj}}} = \sum_{l=1}^{\beta} u_{il} m_{0_g}(p_l)(c_j)$. Hence, there is a linear transformation set F, such that

$$m_{0_h} = F(m_{0_g}), \tag{2}$$

and in the same way,

$$m_{0_g} = H(m_{0_h}). \tag{3}$$

Assume there are two markings m_{1_g} and m_{1_h} in G and H respectively. The marking matrices of m_{1_g} and m_{1_h} are Q_{1_g} and Q_{1_h} respectively. Assume that $Q_{1_h} = UQ_{1_g}$, and hence $Q_{1_g} = VQ_{1_h}$. Similar to the derivation in previous, $m_{1_h} = F(m_{1_g})$ and $m_{1_g} = H(m_{1_h})$. By Definition 3.1 and 2.6, a transition t_r is enabled by marking m_{1_g} if and only if $\forall 1 \le i \le \beta, 1 \le j \le \gamma : q_{1_{g_{ij}}} \ge b^-_{g_{is}}$, where s = (r − 1)γ + j. Hence, $\forall 1 \le i \le \beta + d, 1 \le j \le \gamma : q_{1_{h_{ij}}} = \sum_{l=1}^{\beta} u_{il} q_{1_{g_{lj}}} \ge \sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} = b^-_{h_{is}}$.

Therefore, transition t_r is also enabled by marking m_{1_h} in H. (4)

The marking after firing transition t_r by marking m_{1_g} is m_{2_g} , and the marking matrix of m_{2_g} is Q_{2_g} . The entries in Q_{2_g} satisfy $\forall 1 \le i \le \beta, 1 \le j \le \gamma : q_{2_{g_{ij}}} = q_{1_{g_{ij}}} - b^-_{g_{is}} + b^+_{g_{is}}$. The marking after firing transition t_r by marking m_{1_h} is m_{2_h} , and the marking matrix of m_{2_h} is Q_{2_h} . Hence, the entries in Q_{2_h} satisfy

$$\begin{aligned}
\forall 1 \le i \le \beta + d, 1 \le j \le \gamma : q_{2_{h_{ij}}} &= q_{1_{h_{ij}}} - b^-_{h_{is}} + b^+_{h_{is}} \\
&= \sum_{l=1}^{\beta} u_{il} q_{1_{g_{lj}}} - \sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} + \sum_{l=1}^{\beta} u_{il} b^+_{g_{ls}} \\
&= \sum_{l=1}^{\beta} u_{il} (q_{1_{g_{lj}}} - b^-_{g_{ls}} + b^+_{g_{ls}}) \\
&= \sum_{l=1}^{\beta} u_{il} q_{2_{g_{lj}}} .
\end{aligned}$$

Therefore, $m_{2_h} = F(m_{2_g})$ (5)

and $m_{2_g} = H(m_{2_h})$. (6)

In the same way, if $m_{1_h} = F(m_{1_g})$, $m_{1_g} = H(m_{1_h})$ and $m_{1_h}[t_r > m_{2_h}$, then $m_{1_g}[t_r > m_{2_g}$ such that $m_{2_h} = F(m_{2_g})$ and $m_{2_g} = H(m_{2_h})$. (7)

By (1), condition 1 in Definition 4.3, the definition of redundant relation, is satisfied. By (2) and (3), condition 2 in Definition 4.3 is satisfied. By (4), (5) and (6), condition 3 in Definition 4.3 is satisfied. By (7), condition 4 in Definition 4.3 is satisfied. Therefore, $G \simeq H$ which satisfies condition 1 in Definition 4.4.

Finally, since Q_g and Q_h are reachable marking matrices of G and H respectively, and Q_g and Q_h have the same firing transition sequence, $Q_g = Q_{0_g} - B_g^- X_g + B_g^+ X_g$ and $Q_h = Q_{0_h} - B_h^- X_h + B_h^+ X_h$, where $X_g = X_h$. Since $B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-$, $B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} - \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^- X_h + \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+ X_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} (Q_{0_g} - B_g^- X_g + B_g^+ X_g) = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$. Since $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$, it can be proved that $\forall p_i \in P_g : m_h(p_i) = m_g(p_i)$ by the same way proving $\forall p_i \in P_g : m_{0_h}(p_i) = m_{0_g}(p_i)$. Hence, condition 5 in Definition 4.4 is satisfied. Therefore two properties in Lemma 4.3 are proved.

In Lemma 4.3, A CPN H with input matrix $B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-$, output matrix $B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$ is proved to be a separate CPN of G. It is also derived that the marking matrix, Q_h, in H would be $\begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$ all along, where Q_g, the marking matrix in G, has the same firing transition sequence as

Q_h.

Lemma 4.4: Let G be a CPN which has α transitions, β places, γ colours, and H be a separate CPN with d additional places with respect to G. Besides, if a reachable marking matrix Q_g of G has the same firing transition sequence with a reachable marking matrix Q_h of H, $Q_h = UQ_g = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$.

- If there are place faults on H, it can be detected by a $d \times (\beta + d)$ check matrix W, such that $WU = \mathbf{0}_{d \times \beta}$, where $\mathbf{0}_{d \times \beta}$ is a $d \times \beta$ matrix with all entries of zero. The syndrome $S = WF_p$ iff the place fault indicator matrix is F_p.
- k place faults on H can be identified and corrected if any $2k$ columns of the check matrix W are linearly dependent.

Proof:

Assume Q_h is a fault-free marking matrix of H, then $WQ_h = WUQ_g = \mathbf{0}_{d \times \gamma}$. If there are place faults on H, by Lemma 4.2, the faulty marking matrix Q_f satisfies $Q_f = Q_h + F_p = UQ_g + F_p$. Hence, The syndrome $S = WQ_f = WUQ_g + WF_p = WF_p$. By the same way, if the syndrome is WF_p, it will be $S = WF_p = WF_p + \mathbf{0}_{d \times \gamma} = WF_p + WUQ_g = WF_p + WQ_h = W(F_p + Q_h)$, where Q_h is a fault-free marking matrix of H, and F_p is a place fault indicator matrix. Therefore, a marking matrix of H could be examined if it is a faulty marking matrix by multiplying the marking matrix with the check matrix W.

If Q_f is a faulty marking matrix of H, which states k place faults on H, by Lemma 4.2, $Q_f = Q_h + F_p$, where k rows of F_p have nonzero entries. In other words, each column in F_p has at most k nonzero entries, and hence each column in Q_f has at most k incorrect entries. Thus, the syndrome S = $\lfloor s_1 \quad s_2 \quad \cdots \quad s_\gamma \rfloor$ = WQ_f = $W\lfloor q_{f_1} \quad q_{f_2} \quad \cdots \quad q_{f_\gamma} \rfloor = \lfloor Wq_{f_1} \quad Wq_{f_2} \quad \cdots \quad Wq_{f_\gamma} \rfloor$ = WF_p = $W\lfloor f_{p_1} \quad f_{p_2} \quad \cdots \quad f_{p_\gamma} \rfloor = \lfloor Wf_{p_1} \quad Wf_{p_2} \quad \cdots \quad Wf_{p_\gamma} \rfloor$, where s_n, q_{f_n} and f_{p_n} represent the nth column in S, Q_f and F_p, respectively, and $1 \le n \le \gamma$. A column in Q_f, q_{f_n}, can be deemed as a linear code which is of length β. There are two theorems [15] in error control coding: (1) If a linear code with a check matrix, such that any $2k$ columns of the check matrix are linearly dependent, the linear code has minimum distance $2k + 1$. (2) A code with minimum distance $2k + 1$ can identify and correct k errors. Since each column in Q_f has at most k incorrect entries, the faults in Q_f can be identified and corrected by the check matrix W inside which any $2k$ columns are linearly dependent. After getting the syndrome by WQ_f, the place fault indicator matrix F_p can be found by solving equations $Wf_{p_n} = s_n$, where $1 \le n \le \gamma$.

Lemma 4.4 shows that if there are at most k place faults, it needs to find out a check matrix which has any $2k$ columns are linearly dependent, and the result of multiplying the check matrix with a fault-free marking matrix is a matrix with all entries of zero. By applying the method of *Reed-Solomon codes* [15], it would find a check matrix of $2k$ rows and any $2k$ columns are linearly dependent. Hence, $d = 2k$. In other word, if there are at most k place faults, $2k$ additional places is needed in the separate CPN by applying the method of Reed-Solomon codes in order to derive the detection and correction capabilities of place faults. Lemma 4.4 also shows that the place faults can be identified and corrected from the syndrome.

Let G be a CPN which has α transitions, β places, γ colours, the input matrix B_g^-, the output matrix B_g^+ and the initial marking matrix Q_{0_g}. From Lemmas 4.3 and 4.4, constructing a separate CPN H which can detect and correct at most k place faults is concluded as following steps: (1) First, constructing a $2k \times (\beta + 2k)$ check matrix W from the check matrix of Reed-Solomon codes. (2) Second, solving the equation $W\begin{bmatrix} I_\beta \\ D \end{bmatrix} = 0_{d\times\beta}$ and getting the entries of D. (3) Finally, deriving the separate CPN H containing input matrix $B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-$, output matrix $B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$. A marking Q_h in the separate CPN H can be examined if it is a correct marking by the check matrix W. If the marking Q_h is a faulty marking, it can be corrected by solving the place fault indicator matrix F_p from the equation $S = WF_p$, where S is the syndrome from WQ_h.

4.3.3 An Example of Identifying and Correcting Place Faults

This section uses the example in Figure 3.1 as the given CPN G and sets the marking in this figure as the initial marking. Let B_g^-, B_g^+ and Q_{0_g} represent the input matrix, the output matrix and the initial marking matrix of G respectively. Hence, the given CPN G has $\alpha = 3$ transitions, $\beta = 3$ places, $\gamma = 4$ colours,

$$B_g^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_g^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{and}$$

$$Q_{0_g} = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Assume there are at most $k = 3$ place faults, a 6×9 check matrix W can be constructed by the method of Reed-Solomon codes. The followings are the steps of constructing check matrix W: (1) Choosing a prime number a which satisfies $a \geq \beta + 2k = 9$, and hence it can take $a = 11$. (2) Second, finding a nature number r which is a primitive root mod a[VIII], it can choose $r = 2$. (3)Third, giving a polynomial of degree $a - 1 - 2k = 10 - 6 = 4$ used to construct check matrix, such that $w(x) = (x - r^{a-2} \bmod a)(x - r^{a-3} \bmod a)(x - r^{a-4} \bmod a)(x - 1) = (x - 2^9 \bmod 11)(x - 2^8 \bmod 11)(x - 2^7 \bmod 11)(x - 1) = x^4 + 5x^3 + 9x^2 + 2x + 5 = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. (4) Finally, constructing the check matrix W from b_4, b_3, b_2, b_1 and b_0, such that

$$W = \begin{bmatrix} b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 & 0 \\ b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 \\ 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 \end{bmatrix}.$$

Next, the matrix $\begin{bmatrix} I_\beta \\ D \end{bmatrix}$ is obtained by solving the equation

$$0_{d \times \beta} = W \begin{bmatrix} I_\beta \\ D \end{bmatrix} \bmod 11$$

[VIII] If r is a primitive root mod a, it would satisfy $\{r^1 \bmod a, r^2 \bmod a, \ldots, r^{a-1} \bmod a\} = \{1, 2, \ldots, a-1\}$.

$$= \begin{bmatrix} 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \\ d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} \\ d_{16} & d_{17} & d_{18} \end{bmatrix} \text{ mod 11, and hence}$$

$d_1 = 10$, $d_2 = 7$, $d_3 = 4$, $d_4 = 9$, $d_5 = 5$, $d_6 = 1$, $d_7 = 7$, $d_8 = 5$, $d_9 = 9$, $d_{10} = 4$, $d_{11} = 4$, $d_{12} = 8$, $d_{13} = 10$, $d_{14} = 5$, $d_{15} = 3$, $d_{16} = 2$, $d_{17} = 9$, $d_{18} = 6$,

$$\begin{bmatrix} I_\beta \\ D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 7 & 4 \\ 9 & 5 & 1 \\ 7 & 5 & 9 \\ 4 & 4 & 8 \\ 10 & 5 & 3 \\ 2 & 9 & 6 \end{bmatrix}.$$

Finally, the separate CPN H is constructed, where

$$B_h^- = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^-$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 7 & 4 \\ 9 & 5 & 1 \\ 7 & 5 & 9 \\ 4 & 4 & 8 \\ 10 & 5 & 3 \\ 2 & 9 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 10 & 14 & 0 & 10 & 0 & 10 \\ 9 & 0 & 0 & 0 & 0 & 0 & 9 & 10 & 0 & 9 & 0 & 9 \\ 7 & 0 & 0 & 0 & 0 & 0 & 7 & 10 & 0 & 7 & 0 & 7 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & 0 & 4 & 0 & 4 \\ 10 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 0 & 10 & 0 & 10 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 18 & 0 & 2 & 0 & 2 \end{bmatrix},
$$

$$
B_h^+ = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 7 & 4 \\ 9 & 5 & 1 \\ 7 & 5 & 9 \\ 4 & 4 & 8 \\ 10 & 5 & 3 \\ 2 & 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 9 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 9 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}, \text{ and}
$$

$$
Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}
$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 7 & 4 \\ 9 & 5 & 1 \\ 7 & 5 & 9 \\ 4 & 4 & 8 \\ 10 & 5 & 3 \\ 2 & 9 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 20 & 24 & 17 & 27 \\ 18 & 19 & 14 & 23 \\ 14 & 23 & 12 & 19 \\ 8 & 16 & 8 & 12 \\ 20 & 23 & 15 & 25 \\ 4 & 10 & 11 & 13 \end{bmatrix}.$$

The CPN *H* with its initial marking is illustrated in Figure 4.2, where, in each place, the number above each token denotes the amount of that token.

Since *H* is the separate CPN of *G*, *G* and *H* can have the same firing transition sequence. As in section 3.3, *G* has a firing transition sequence: t_3, t_3, t_1, t_2. Hence, consider the same firing transition sequence in *H* and assume the corresponding fault-free firing sequence is m_{0_h} [t_3> m_{1_h} [t_3> m_{2_h} [t_1> m_{3_h} [t_2> m'_h. The markings m_{0_h}, m_{1_h}, m_{2_h}, m_{3_h} and m'_h are the fault-free marking in *H*, and it is assumed Q_{0_h}, Q_{1_h}, Q_{2_h}, Q_{3_h} and Q'_h are the marking matrices of m_{0_h}, m_{1_h}, m_{2_h}, m_{3_h} and m'_h respectively. Assume there are amount transition faults inside the firing sequence, such that m_{0_h} $\xrightarrow{F_{p1}}$ m_{1_f} [t_3> m_{2_f} [t_3> m_{3_f} $\xrightarrow{F_{p2}}$ m_{4_f} [t_1> m_{5_f} $\xrightarrow{F_{p3}}$ m_{6_f} [t_2> m_{7_f} $\xrightarrow{F_{p4}}$ m'_f which is an informal representation and means that the place faults occur (1) before the first firing, (2) after the second firing and before the third firing, (3) after the third firing and before the last firing, and (4) after the last firing. F_{p_1}, F_{p_2}, F_{p_3} and F_{p_4} are place fault indicator matrices, where

$$F_{p_1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_{p_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_{p_3} = \begin{bmatrix} 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$F_{p_4} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. From Lemma 4.2, F_{p_1} represents that there is lack of one red token in p_1, F_{p_2} represents that there are five extra yellow tokens in p_3, four extra green tokens in p_7, and lack of four red tokens in p_7, F_{p_3} represents that there are nine extra green tokens in p_1, and lack of one blue token in p_3, and F_{p_4} represents that there are two extra red tokens and three extra blue tokens in p_1, and lack of one green tokens in p_7. Hence, the marking matrices represent m_{1_f}, m_{2_f}, m_{3_f}, m_{4_f}, m_{5_f}, m_{6_f}, m_{7_f} and m'_f are

$$Q_{1_f} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 20 & 24 & 17 & 27 \\ 18 & 19 & 14 & 23 \\ 14 & 23 & 12 & 19 \\ 8 & 16 & 8 & 12 \\ 20 & 23 & 15 & 25 \\ 4 & 10 & 11 & 13 \end{bmatrix}, \quad Q_{2_f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 20 & 14 & 21 & 17 \\ 18 & 10 & 15 & 14 \\ 14 & 16 & 21 & 12 \\ 8 & 12 & 16 & 8 \\ 20 & 13 & 18 & 15 \\ 4 & 8 & 17 & 11 \end{bmatrix}, \quad Q_{3_f} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 20 & 4 & 25 & 7 \\ 18 & 1 & 16 & 5 \\ 14 & 9 & 30 & 5 \\ 8 & 8 & 24 & 4 \\ 20 & 3 & 21 & 5 \\ 4 & 6 & 23 & 9 \end{bmatrix},$$

$$Q_{4_f} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 20 & 4 & 25 & 7 \\ 18 & 1 & 16 & 5 \\ 14 & 9 & 30 & 5 \\ 1 & 15 & 24 & 4 \\ 20 & 3 & 21 & 5 \\ 4 & 6 & 23 & 9 \end{bmatrix}, \quad Q_{5_f} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 5 \\ 10 & 4 & 25 & 14 \\ 9 & 1 & 16 & 10 \\ 7 & 9 & 30 & 10 \\ 0 & 12 & 24 & 8 \\ 10 & 3 & 21 & 10 \\ 2 & 6 & 23 & 18 \end{bmatrix}, \quad Q_{6_f} = \begin{bmatrix} 0 & 9 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 10 & 4 & 25 & 14 \\ 9 & 1 & 16 & 10 \\ 7 & 9 & 30 & 10 \\ 0 & 12 & 24 & 8 \\ 10 & 3 & 21 & 10 \\ 2 & 6 & 23 & 18 \end{bmatrix},$$

$$Q_{7_f} = \begin{bmatrix} 0 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 5 \\ 14 & 4 & 15 & 0 \\ 10 & 1 & 7 & 0 \\ 16 & 9 & 23 & 0 \\ 8 & 12 & 20 & 0 \\ 13 & 3 & 11 & 0 \\ 8 & 6 & 21 & 0 \end{bmatrix} \text{ and } Q'_f = \begin{bmatrix} 2 & 9 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 5 \\ 14 & 4 & 15 & 0 \\ 10 & 1 & 7 & 0 \\ 16 & 9 & 23 & 0 \\ 8 & 11 & 20 & 0 \\ 13 & 3 & 11 & 0 \\ 8 & 6 & 21 & 0 \end{bmatrix}.$$

The following steps identify and correct these three place faults from the marking matrix Q'_f and the check matrix W. First, since

$$WQ'_f \mod 11 = \begin{bmatrix} 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 9 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 9 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 5 \\ 14 & 4 & 15 & 0 \\ 10 & 1 & 7 & 0 \\ 16 & 9 & 23 & 0 \\ 8 & 11 & 20 & 0 \\ 13 & 3 & 11 & 0 \\ 8 & 6 & 21 & 0 \end{bmatrix} \mod 11$$

$$= \begin{bmatrix} 5 & 1 & 2 & 10 \\ 1 & 9 & 5 & 1 \\ 0 & 0 & 6 & 3 \\ 2 & 4 & 10 & 5 \\ 3 & 6 & 0 & 0 \\ 8 & 5 & 0 & 0 \end{bmatrix} \neq \mathbf{0}_{6\times 4},$$

there are place faults in Q'_f. Second, since

$$WF_p \mod 11 = WQ'_f \mod 11 = \begin{bmatrix} 5 & 1 & 2 & 10 \\ 1 & 9 & 5 & 1 \\ 0 & 0 & 6 & 3 \\ 2 & 4 & 10 & 5 \\ 3 & 6 & 0 & 0 \\ 8 & 5 & 0 & 0 \end{bmatrix},$$

the following sets of equations are figured out:

$$\begin{cases} (5f_{p_{11}} + 9f_{p_{21}} + 2f_{p_{31}} + 5f_{p_{41}}) \mathrm{mod} 11 = 5 \\ (f_{p_{11}} + 5f_{p_{21}} + 9f_{p_{31}} + 2f_{p_{41}} + 5f_{p_{51}}) \mathrm{mod} 11 = 1 \\ (f_{p_{21}} + 5f_{p_{31}} + 9f_{p_{41}} + 2f_{p_{51}} + 5f_{p_{61}}) \mathrm{mod} 11 = 0 \\ (f_{p_{31}} + 5f_{p_{41}} + 9f_{p_{51}} + 2f_{p_{61}} + 5f_{p_{71}}) \mathrm{mod} 11 = 2 \\ (f_{p_{41}} + 5f_{p_{51}} + 9f_{p_{61}} + 2f_{p_{71}} + 5f_{p_{81}}) \mathrm{mod} 11 = 3 \\ (f_{p_{51}} + 5f_{p_{61}} + 9f_{p_{71}} + 2f_{p_{81}} + 5f_{p_{91}}) \mathrm{mod} 11 = 8 \end{cases} ,$$

$$\begin{cases} (5f_{p_{12}} + 9f_{p_{22}} + 2f_{p_{32}} + 5f_{p_{42}}) \mathrm{mod} 11 = 5 \\ (f_{p_{12}} + 5f_{p_{22}} + 9f_{p_{32}} + 2f_{p_{42}} + 5f_{p_{52}}) \mathrm{mod} 11 = 1 \\ (f_{p_{22}} + 5f_{p_{32}} + 9f_{p_{42}} + 2f_{p_{52}} + 5f_{p_{62}}) \mathrm{mod} 11 = 0 \\ (f_{p_{32}} + 5f_{p_{42}} + 9f_{p_{52}} + 2f_{p_{62}} + 5f_{p_{72}}) \mathrm{mod} 11 = 2 \\ (f_{p_{42}} + 5f_{p_{52}} + 9f_{p_{62}} + 2f_{p_{72}} + 5f_{p_{82}}) \mathrm{mod} 11 = 3 \\ (f_{p_{52}} + 5f_{p_{62}} + 9f_{p_{72}} + 2f_{p_{82}} + 5f_{p_{92}}) \mathrm{mod} 11 = 8 \end{cases} \quad \text{and}$$

$$\begin{cases} (5f_{p_{13}} + 9f_{p_{23}} + 2f_{p_{33}} + 5f_{p_{43}})\bmod 11 = 5 \\ (f_{p_{13}} + 5f_{p_{23}} + 9f_{p_{33}} + 2f_{p_{43}} + 5f_{p_{53}})\bmod 11 = 1 \\ (f_{p_{23}} + 5f_{p_{33}} + 9f_{p_{43}} + 2f_{p_{53}} + 5f_{p_{63}})\bmod 11 = 0 \\ (f_{p_{33}} + 5f_{p_{43}} + 9f_{p_{53}} + 2f_{p_{63}} + 5f_{p_{73}})\bmod 11 = 2' \\ (f_{p_{43}} + 5f_{p_{53}} + 9f_{p_{63}} + 2f_{p_{73}} + 5f_{p_{83}})\bmod 11 = 3 \\ (f_{p_{53}} + 5f_{p_{63}} + 9f_{p_{73}} + 2f_{p_{83}} + 5f_{p_{93}})\bmod 11 = 8 \end{cases}$$

and there are two more restrictions: (1) each set of equations has at most three nonzero variables and (2) $\forall 1 \le i \le 9,\ 1 \le j \le 3$: $-\frac{11}{2} < f_{p_{ij}} < -\frac{11}{2}$. Therefore, $f_{p_{11}} = 1$, $f_{p_{71}} = -4$, $f_{p_{12}} = 9$, $f_{p_{72}} = 3$, $f_{p_{13}} = 3$, $f_{p_{33}} = -1$, $f_{p_{34}} = 5$, and the other entries in F_p are zero. It can be inferred from F_p that three place faults occur in p_1, p_3 and p_7, and the colours of fault for these places are: red, green and blue in p_1, blue and yellow in p_3, red and green in p_7. The correct marking matrix with respect to Q'_f is

$$Q'_f - F_p = \begin{bmatrix} 2 & 9 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 5 \\ 14 & 4 & 15 & 0 \\ 10 & 1 & 7 & 0 \\ 16 & 9 & 23 & 0 \\ 8 & 11 & 20 & 0 \\ 13 & 3 & 11 & 0 \\ 8 & 6 & 21 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 9 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 14 & 4 & 15 & 0 \\ 10 & 1 & 7 & 0 \\ 16 & 9 & 23 & 0 \\ 12 & 8 & 20 & 0 \\ 13 & 3 & 11 & 0 \\ 8 & 6 & 21 & 0 \end{bmatrix},$$ which is equal to Q'_h.

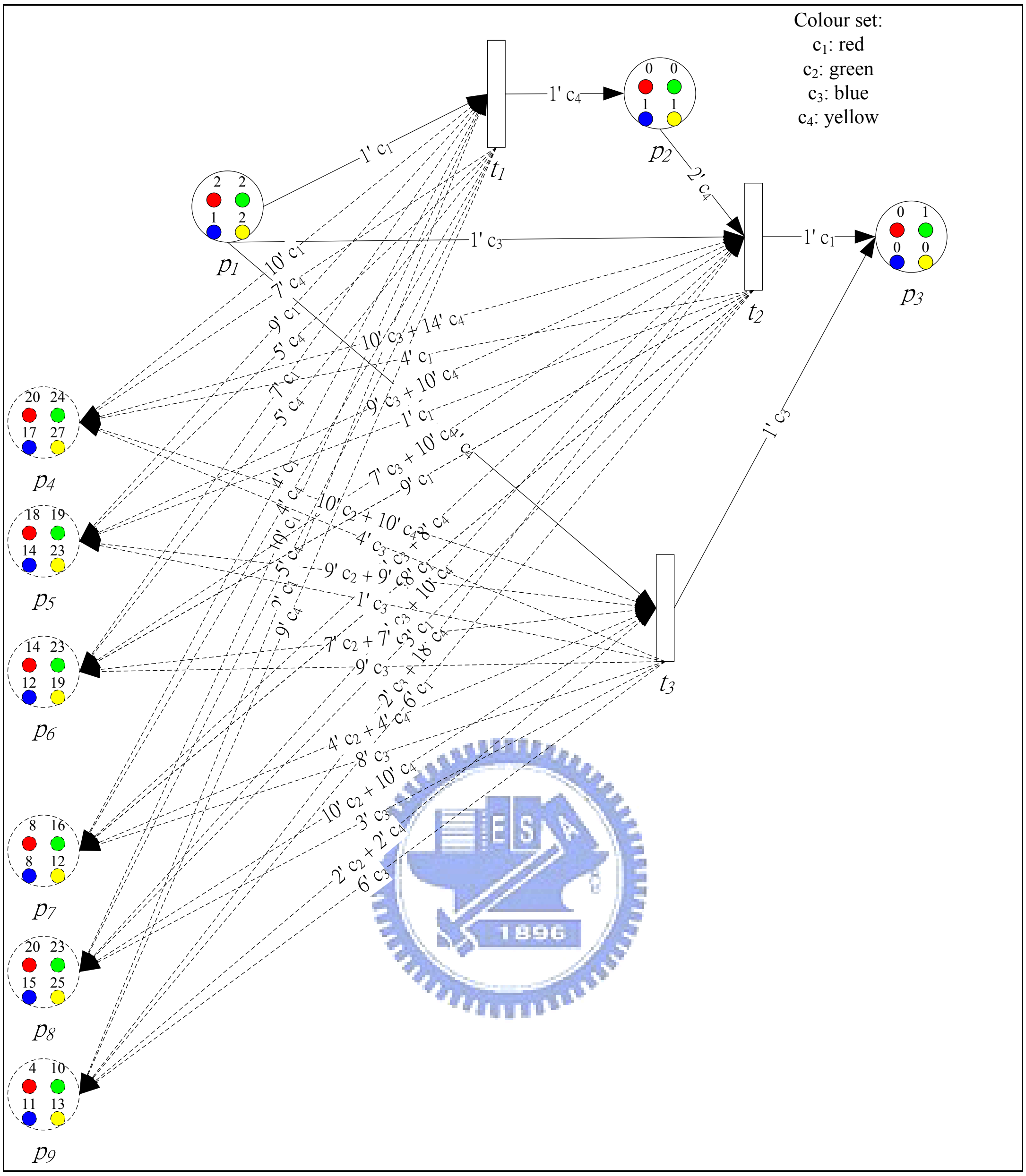

Figure 4.2 A CPN with place faults detection and correction capabilities.

4.4 Amount Transition Faults

The organization of this section is similar to the one of previous section. This section firstly gives the problem formulation of amount transition faults in matrix representations according to Definition 4.5. Based on the problem formulation, a methodology encoding a CPN into a separeate CPN with detection and correction capabilities on amount transition faults is presented, but the encoding matrix adopted is different from the one in previous section. Then, this section gives the syndromes of amount transition faults while they occur in CPNs, where a new check matrix is applied. Finally, it describes how to compute the correction markings via these syndromes.

4.4.1 Problem Formulation for Amount Transition Faults

Lemma 4.5: Let Q be a fault-free marking matrix of a CPN G.

- If Q_f is a marking matrix containing a pre-condition amount transition fault on transition t_i corresponding to Q, $\exists F_a^{i-}$, a $n \times \gamma$ matrix, where $n = \alpha\gamma$,

  $Q_f = Q + B^- F_a^{i-}$,

  and all the entries in F_a^{i-} satisfy

  $\forall (i\text{-}1)\gamma + 1 \le h \le i\gamma, j = (h\text{-}1) \bmod \gamma + 1 : f_{a_{hj}}^{i-} \in \mathbb{N}/\{0\}$

  and $\forall 1 \le k \le n,\ 1 \le l \le \gamma,\ k \ne h \ or\ l \ne j :\ f_{a_{kl}}^{i-} = 0$.

- If Q_f is a marking matrix containing a post-condition amount transition fault on transition t_i corresponding to Q, $\exists F_a^{i+}$, a $n \times \gamma$ matrix, where $n = \alpha\gamma$,

  $Q_f = Q - B^+ F_a^{i+}$,

  and all the entries in F_a^{i+} satisfy

  $\forall (i\text{-}1)\gamma + 1 \le h \le i\gamma, j = (h\text{-}1) \bmod \gamma + 1 : f_{a_{hj}}^{i+} \in \mathbb{N}/\{0\}$

  and $\forall 1 \le k \le n,\ 1 \le l \le \gamma,\ k \ne h$ or $l \ne j :\ f_{a_{kl}}^{i+} = 0$.

Proof:

Assume Q is the matrix representation of a fault-free marking m, and Q_f is the matrix representation of m_f which is a marking with a pre-condition amount transition fault on transition t_i with respect to m. By Definition 4.5, $\forall p_g \in P$: $m_f(p_g) = m(p_g) + z * I(p_g, t_i)$, where $z \in \mathbb{N} / \{0\}$ and $z \leq \sigma(t_i)$. Hence, by Definition 2.2, $\forall p_g \in P$, $\forall\ c_r \in C$: $m_f(p_g)(c_r) = m\,(p_g)(c_r) + z * I(p_g, t_i)(c_r)$. By Definition 3.1 and 3.2, $\forall\, 1 \leq g \leq \beta$, $1 \leq r \leq \gamma$: $q_{f_{gr}} = q_{gr} + z * b^-_{gs} = q_{gr} + z * b^-_{gs} + \sum_{k=1,k\neq s}^{n} 0 * b^-_{gk}$, where $s = (i - 1)\gamma + r$.

Hence, $\forall\, 1 \leq g \leq \beta$, $1 \leq r \leq \gamma$: $q_{f_{gr}} = q_{gr} + \sum_{k=1}^{n} f^{i-}_{a_{kr}} * b^-_{gk}$,

where $f^{i-}_{a_{xy}} = \begin{cases} z & if\ x = (i-1)\gamma + y \\ 0 & otherwise \end{cases}$. Since $1 \leq r \leq \gamma$, the previous equation is equivalent to $\forall\, 1 \leq g \leq \beta$, $1 \leq r \leq \gamma$: $q_{f_{gr}} = q_{gr} + \sum_{k=1}^{n} f^{i-}_{a_{kr}} * b^-_{gk}$,

where $f^{i-}_{a_{xy}} = \begin{cases} z & if\ (i-1)\gamma + 1 \leq x \leq i\gamma\ and\ y = (x-1) \bmod \gamma + 1 \\ 0 & otherwise \end{cases}$. Therefore, there is an $n \times \gamma$ matrix and all the entries in F^{i-}_a satisfy $\forall (i\text{-}1)\gamma + 1 \leq h \leq i\gamma$, $j = (h\text{-}1) \bmod \gamma + 1$: $f^{i-}_{a_{hj}} \in \mathbb{N}/\{0\}$ and $\forall\, 1 \leq k \leq n$, $1 \leq l \leq \gamma$, $k \neq h\ or\ l \neq j$: $f^{i-}_{a_{kl}} = 0$, such that $Q_f = Q - B^+ F^{i+}_a$. The case of post-condition amount transition fault can be proved by the same way.

Lemma 4.5 formulates the problem of an amount transition fault in linear algebra. The matrix F^{i-}_a and F^{i+}_a in the equation indicate the transition suffering a pre-condition and post-condition amount transition fault respectively. Both F^{i-}_a and

F_a^{i+} have similar structures to transition firing matrix, where all of them can be deemed as a column of *α* square sub-matrices, and each sub-matrix is a $\gamma \times \gamma$ matrix. There is only one sub-matrix in F_a^{i-} and F_a^{i+} with non-zero entries, which is an identity matrix scalar multiplied by a positive integer, and all the other sub-matrices are matrices with all entries of zero. If the *i*th sub-matrix in F_a^{i-} (F_a^{i+}) is an identity matrix scalar multiplied by *z*, it denotes that transition t_i suffers a pre-condition (post-condition) amount transition fault *z* times. In this thesis, it is assumed that a transition wouldn't suffer both a pre-condition and a post-condition amount transition fault, since it would seem like the transition without firings in this case, and the effects are cancelled [3]. The consistency between the description of amount transition fault in Lemma 4.5 and the definition of amount transition fault in Definition 4.5 is also proved.

Lemma 4.6: Let *Q* be a fault-free marking matrix of a CPN *G*.

- If Q_f is a marking matrix representing *Q* with *z* amount transition faults, where *x* of *z* are pre-condition amount transition faults, and *y* of *z* are post-condition amount transition faults, $\exists\, F_a^-$ and F_a^+, $n \times \gamma$ matrices, where $n = \alpha\gamma$,

  $Q_f = Q + B^- F_a^- - B^+ F_a^+$,

  , all the entries in the **pre-condition amount transition fault indicator matrix** F_a^- satisfy

  $\exists 1 \le i_1^-, i_2^-, \cdots, i_x^- \le \beta,\ i_1^- \ne i_2^- \ne \cdots \ne i_x^-$,

  $\forall\ (i_1^- - 1)\gamma + 1 \le h_1^- \le i_1^-\gamma, (i_2^- - 1)\gamma + 1 \le h_2^- \le i_2^-\gamma, \cdots, (i_x^- - 1)\gamma + 1 \le h_x^- \le i_x^-\gamma$,

$$j_1^- = (h_1^- - 1) \bmod \gamma + 1, j_2^- = (h_2^- - 1) \bmod \gamma + 1, \cdots, j_x^- = (h_x^- - 1) \bmod \gamma + 1 :$$

$$f_{a_{h_1^- j_1^-}}^-, f_{a_{h_2^- j_2^-}}^-, \cdots, f_{a_{h_x^- j_x^-}}^- \in \mathbb{N}/\{0\}, \text{ and}$$

$$\forall 1 \le k^- \le n, 1 \le l^- \le \gamma,$$

$$k^- \ne h_1^- \ne h_2^- \ne \cdots \ne h_x^- \text{ or } l^- \ne j_1^- \ne j_2^- \ne \cdots \ne j_x^- :$$

$$f_{a_{k^- l^-}}^- = 0,$$

and all the entries in the **post-condition amount transition fault indicator matrix** F_a^+ satisfy

$$\exists 1 \le i_1^+, i_2^+, \cdots, i_y^+ \le \beta, \; i_1^+ \ne i_2^+ \ne \cdots \ne i_y^+ \ne i_1^- \ne i_2^- \ne \cdots \ne i_x^-,$$

$$\forall (i_1^+ - 1)\gamma + 1 \le h_1^+ \le i_1^+\gamma, (i_2^+ - 1)\gamma + 1 \le h_2^+ \le i_2^+\gamma, \cdots, (i_y^+ - 1)\gamma + 1 \le h_y^+ \le i_y^+\gamma,$$

$$j_1^+ = (h_1^+ - 1) \bmod \gamma + 1, j_2^+ = (h_2^+ - 1) \bmod \gamma + 1, \cdots, j_y^+ = (h_y^+ - 1) \bmod \gamma + 1 :$$

$$f_{a_{h_1^+ j_1^+}}^+, f_{a_{h_2^+ j_2^+}}^+, \cdots, f_{a_{h_y^+ j_y^+}}^+ \in \mathbb{N}/\{0\}, \text{ and}$$

$$\forall 1 \le k^+ \le n, 1 \le l^+ \le \gamma,$$

$$k^+ \ne h_1^+ \ne h_2^+ \ne \cdots \ne h_y^+ \text{ or } l^+ \ne j_1^+ \ne j_2^+ \ne \cdots \ne j_y^+ :$$

$$f_{a_{k^+ l^+}}^+ = 0.$$

Proof:

By Definition 4.6, *z* amount transition faults denote that there are *z* transitions have suffered amount transition faults, and *x* of these *z* transitions have suffered pre-condition amount transition faults, *y* of these *z* transitions have suffered post-condition amount transition faults. In previous assumption, a transition wouldn't suffer both a pre-condition amount transition fault and a post-condition amount transition fault, thus $z = x + y$. Hence, By Lemma 4.6, there are x matrices, $F_a^{i_1^-\,-}, F_a^{i_2^-\,-}, \cdots, F_a^{i_x^-\,-}$, indicate these pre-condition amount transition faults and y matrices, $F_a^{i_1^+\,+}, F_a^{i_2^+\,+}, \cdots, F_a^{i_y^+\,+}$, indicate these post-condition amount transition faults, where $i_1^+ \neq i_2^+ \neq \cdots \neq i_y^+ \neq i_1^- \neq i_2^- \neq \cdots \neq i_x^-$, and the marking Q_f would be

$$\begin{aligned} Q_f &= Q + B^- F_a^{i_1^-\,-} + B^- F_a^{i_2^-\,-} + \cdots + B^- F_a^{i_x^-\,-} - B^+ F_a^{i_1^+\,+} - B^+ F_a^{i_2^+\,+} - \cdots - B^+ F_a^{i_y^+\,+} \\ &= Q + B^-(F_a^{i_1^-\,-} + F_a^{i_2^-\,-} + \cdots + F_a^{i_x^-\,-}) - B^+(F_a^{i_1^+\,+} + F_a^{i_2^+\,+} + \cdots + F_a^{i_y^+\,+}) \\ &= Q + B^- F_a^- - B^+ F_a^+ \end{aligned}.$$

From the previous equation, it would have the relations that $F_a^- = F_a^{i_1^-\,-} + F_a^{i_2^-\,-} + \cdots + F_a^{i_x^-\,-}$ and $F_a^+ = F_a^{i_1^+\,+} + F_a^{i_2^+\,+} + \cdots + F_a^{i_y^+\,+}$. Therefore, the entries in F_a^- should satisfy

$\forall\, (i_1^- - 1)\gamma + 1 \leq h_1^- \leq i_1^-\gamma, (i_2^- - 1)\gamma + 1 \leq h_2^- \leq i_2^-\gamma, \cdots, (i_x^- - 1)\gamma + 1 \leq h_x^- \leq i_x^-\gamma$,

$j_1^- = (h_1^- - 1) \bmod \gamma + 1, j_2^- = (h_2^- - 1) \bmod \gamma + 1, \cdots, j_x^- = (h_x^- - 1) \bmod \gamma + 1$:

$f^-_{a_{h_1^- j_1^-}}, f^-_{a_{h_2^- j_2^-}}, \cdots, f^-_{a_{h_x^- j_x^-}} \in \mathbb{N}/\{0\}$, and

$\forall\, 1 \leq k^- \leq n,\ 1 \leq l^- \leq \gamma$,

$k^- \neq h_1^- \neq h_2^- \neq \cdots \neq h_x^-$ or $l^- \neq j_1^- \neq j_2^- \neq \cdots \neq j_x^-$:

$f^-_{a_{k^- l^-}} = 0$,

and the entries in F_a^+ should satisfy

$\forall\, (i_1^+ - 1)\gamma + 1 \leq h_1^+ \leq i_1^+\gamma, (i_2^+ - 1)\gamma + 1 \leq h_2^+ \leq i_2^+\gamma, \cdots, (i_y^+ - 1)\gamma + 1 \leq h_y^+ \leq i_y^+\gamma$,

$j_1^+ = (h_1^+ - 1) \bmod \gamma + 1, j_2^+ = (h_2^+ - 1) \bmod \gamma + 1, \cdots, j_y^+ = (h_y^+ - 1) \bmod \gamma + 1$:

$f^+_{a_{h_1^+ j_1^+}}, f^+_{a_{h_2^+ j_2^+}}, \cdots, f^+_{a_{h_y^+ j_y^+}} \in \mathbb{N}/\{0\}$, and

$\forall 1 \le k^+ \le n,\ 1 \le l^+ \le \gamma,$

$k^+ \ne h_1^+ \ne h_2^+ \ne \cdots \ne h_y^+$ or $l^+ \ne j_1^+ \ne j_2^+ \ne \cdots \ne j_y^+$:

$f_{a_{k^+l^+}}^+ = 0$.

Both pre-condition and post-condition amount transition fault indicator matrices, F_a^- and F_a^+, have similar structures to firing matrices. There are x (y) sub-matrices in F_a^- (F_a^+), which indicates x (y) pre-condition (post-condition) amount transition faults, have nonzero entries, each of x (y) sub-matrices is an identity matrix scalar multiplied by a positive integer, and all the other sub-matrices in F_a^- (F_a^+) are matrices with all entries of zero. Since the assumption in previous that pre-condition and post-condition amount transition fault can't both appear in a transition, there is a constrain between F_a^- and F_a^+, such that the rth sub-matrix in F_a^- is a matrix with all entries of zero if the rth sub-matrix in F_a^+ is a matrix with nonzero entries. The poof at here is similar to the one in Lemma 4.2. It proves the consistency between Lemma 4.6 and Definition 4.6 by deriving from Lemma 4.5.

4.4.2 Separate CPNs with Amount Transition Faults Detection and Correction Capabilities

The strategy in this section is used to detect and correct amount transition faults, which has the same steps with the strategy in section 4.3.2 but with different encoding matrices on input and output matrices. Let G be a CPN with no more than k amount transition faults, the separate CPN with amount transition faults detection and correction capabilities is constructed by adding $2k$ additional places to G and the colour sets of both CPNs are the same. Assume that G has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}. The separate CPN H with amount transition faults detection and correction capabilities in respect to G would have α transitions, $\beta + 2k$ places, γ colours, input matrix B_h^-, output matrix B_h^+ and initial marking matrix Q_{0_h}. Besides, $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$, where D is a $2k \times \beta$ matrix, E is a $2k \times \alpha\gamma$ matrix, and I_β denotes a $\beta \times \beta$ identity matrix. After H is constructed, the amount transition faults occurring on H can be identified and corrected from the syndromes. These properties are proved in Lemma 4.7 and 4.8.

Lemma 4.7: Let G be a CPN which has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}.

If the CPN H, constructed by adding d additional places to G, has the same colour set with G, α transitions, $\beta + d$ places, γ colours, input matrix

$$B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix},$$

output matrix

$$B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$$

and initial marking matrix

$$Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$$

, where D is a $d \times \beta$ matrix, E is a $d \times \alpha\gamma$ matrix, all the entries in D and E are nonnegative, $d \in \mathbb{N}$, I_β denotes a $\beta \times \beta$ identity matrix, and all the entries in $DB_g^- - E$ and $DB_g^+ - E$ are nonnegative, *H* has following two properties.

- *H* is a separate CPN with respect to *G*.
- If a reachable marking matrix Q_g of *G* has the same firing transition sequence with a reachable marking matrix Q_h of *H*, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$.

Proof:

The satisfaction of conditions 2, 3 and 4 in Definition 4.4 can be proved as the proof in Lemma 4.3, and thus it is omitted here. Next, consider condition 1 in Definition 4.4, the satisfaction of conditions 1 and 2 in Definition 4.3 can also be proved as the proof in Lemma 4.3, thus it only needs to be proved the satisfaction of conditions 3 and 4 in Definition 4.3 at here. Assume there are two markings m_{1_g} and m_{1_h} in *G* and *H* respectively. The marking matrices of m_{1_g} and m_{1_h} are Q_{1_g} and Q_{1_h} respectively. Assume that $Q_{1_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{1_g} = UQ_{1_g}$, and hence there is a matrix $V = \begin{bmatrix} I_\beta & 0_{\beta\times d} \end{bmatrix}$ such that $VQ_{1_h} = VUQ_{1_g} = \begin{bmatrix} I_\beta & 0_{\beta\times d} \end{bmatrix} \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{1_g} = Q_{1_g}$, where $0_{\beta\times d}$ is a $\beta\times d$ matrix with all entries of zero. By Definition 3.1, $\forall 1 \le i \le \beta$, $1 \le j \le \gamma$: $m_{1_h}(p_i)(c_j) = q_{1_{hij}} = \sum_{l=1}^{\beta} u_{il} q_{1_{glj}} = \sum_{l=1}^{\beta} u_{il} m_{1_g}(p_l)(c_j)$. Hence, there is a linear transformation set *F*, such that $m_{0_h} = F(m_{0_g})$, and in the same way, $m_{0_g} = H(m_{0_h})$.

Assume that $R = \begin{bmatrix} 0_{\beta\times\alpha\gamma} \\ E \end{bmatrix}$, thus input matrix of H would be $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^- - \begin{bmatrix} 0_{\beta\times\alpha\gamma} \\ E \end{bmatrix} = UB_g^- - R$ and output matrix of H would be $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} B_g^+ - \begin{bmatrix} 0_{\beta\times\alpha\gamma} \\ E \end{bmatrix} = UB_g^+ - R$. By Definition 3.1 and 2.6, a transition t_r is enabled by marking m_{1_g} if and only if $\forall 1 \le i \le \beta, 1 \le j \le \gamma : q_{1_{g_{ij}}} \ge b^-_{g_{is}}$, where $s = (r - 1)\gamma + j$. Since all the entries in E are nonnegative, all the entries in R have constrains which are $\forall 1 \le i \le \beta + d$, $1 \le j \le \alpha\gamma$: $r_{ij} \ge 0$. Hence, $\forall 1 \le i \le \beta + d, 1 \le j \le \gamma : q_{1_{h_{ij}}} = \sum_{l=1}^{\beta} u_{il} q_{1_{g_{lj}}} \ge \sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} \ge \sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} - r_{is} = b^-_{h_{is}}$.

Therefore, transition t_r is also enabled by marking m_{1_h} in H. (1)

The marking after firing transition t_r by marking m_{1_g} is m_{2_g}, and the marking matrix of m_{2_g} is Q_{2_g}. The entries in Q_{2_g} satisfy $\forall 1 \le i \le \beta, 1 \le j \le \gamma : q_{2_{g_{ij}}} = q_{1_{g_{ij}}} - b^-_{g_{is}} + b^+_{g_{is}}$. The marking after firing transition t_r by marking m_{1_h} is m_{2_h}, and the marking matrix of m_{2_h} is Q_{2_h}. Hence, The entries in Q_{2_h} satisfy

$$\begin{aligned}
\forall 1 \le i \le \beta + d, 1 \le j \le \gamma : q_{2_{h_{ij}}} &= q_{1_{h_{ij}}} - b^-_{h_{is}} + b^+_{h_{is}} \\
&= \sum_{l=1}^{\beta} u_{il} q_{1_{g_{lj}}} - (\sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} - r_{is}) + (\sum_{l=1}^{\beta} u_{il} b^+_{g_{ls}} - r_{is}) \\
&= \sum_{l=1}^{\beta} u_{il} q_{1_{g_{lj}}} - \sum_{l=1}^{\beta} u_{il} b^-_{g_{ls}} + \sum_{l=1}^{\beta} u_{il} b^+_{g_{ls}} \\
&= \sum_{l=1}^{\beta} u_{il} (q_{1_{g_{lj}}} - b^-_{g_{ls}} + b^+_{g_{ls}}) \\
&= \sum_{l=1}^{\beta} u_{il} q_{2_{g_{lj}}}.
\end{aligned}$$

Therefore, $m_{2_h} = F(m_{2_g})$ (2)

and $m_{2_g} = H(m_{2_h})$. (3)

In the same way, if $m_{1_h} = F(m_{1_g})$, $m_{1_g} = H(m_{1_h})$ and $m_{1_h}[t_r > m_{2_h}$, then $m_{1_g}[t_r > m_{2_g}$ such that $m_{2_h} = F(m_{2_g})$ and $m_{2_g} = H(m_{2_h})$. (4)

By (1), (2) and (3), condition 3 in Definition 4.3 is satisfied. By (4), condition 4 in Definition 4.3 is satisfied. Therefore, $G \simeq H$ which satisfies condition 1 in Definition 4.4.

Finally, Since Q_g and Q_h are reachable marking matrices of G and H respectively, and Q_g and Q_h have the same firing transition sequence, $Q_g = Q_{0_g} - B_g^- X_g + B_g^+ X_g$ and $Q_h = Q_{0_h} - B_h^- X_h + B_h^+ X_h$, where $X_g = X_h$. Since $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$,

$$\begin{aligned} Q_h &= \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} - \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix} X_h + \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix} X_h \\ &= \begin{bmatrix} Q_{0_g} - B_g^- X_h + B_g^+ X_h \\ DQ_{0_g} - DB_g^- X_h + EX_h + DB_g^+ X_h - EX_h \end{bmatrix} Q_{0_g} \\ &= \begin{bmatrix} Q_{0_g} - B_g^- X_g + B_g^+ X_g \\ DQ_{0_g} - DB_g^- X_g + DB_g^+ X_g \end{bmatrix} Q_{0_g} \\ &= \begin{bmatrix} I_\beta \\ D \end{bmatrix} (Q_{0_g} - B_g^- X_g + B_g^+ X_g) = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g. \end{aligned}$$

Since $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$, it can be proved that $\forall p_i \in P_g : m_h(p_i) = m_g(p_i)$ as in Lemma 4.3. Hence, condition 5 in Definition 4.4 is satisfied. Therefore two properties in Lemma 4.7 are proved.

In Lemma 4.7, A CPN H with input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$ also has two properties in Lemma 4.3. Most parts of the proof of Lemma 4.7 are the same as the proof of Lemma 4.3, thus these parts are omitted here.

Lemma 4.8: Let G be a CPN which has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}, and H be a separate CPN with d additional places with respect to G, input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = UQ_{0_g}$.

- If there are amount transition faults on H, it can be detected by a $d \times (\beta + d)$ check matrix W, such that $WU = \mathbf{0}_{d \times \beta}$, where $\mathbf{0}_{d \times \beta}$ is a $d \times \beta$ matrix with all entries of zero. The syndrome $S = WB_h^- F_a^-$ - $WB_h^+ F_a^+$, iff the pre-condition and post-condition amount transition fault indicator matrices are F_a^- and F_a^+ respectively.
- k amount transition faults on H can be identified and corrected if $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, and any $2k$ columns of the matrix E are linearly dependent.

Proof:

Assume Q_g and Q_h are fault-free marking matrices of G and H respectively, and they have the same firing transition sequence. Hence, by Lemma 4.7, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$, and Q_h multiplied by W would be $WQ_h = WUQ_g = \mathbf{0}_{d\times\gamma}$. If there are amount transition faults on H, by Lemma 4.6, the faulty marking matrix Q_f satisfies $Q_f = Q_h + B_h^- F_a^- - B_h^+ F_a^+ = UQ_g + B_h^- F_a^- - B_h^+ F_a^+$. Hence, The syndrome $S = WQ_f = WUQ_g + WB_h^- F_a^- - WB_h^+ F_a^+ = WB_h^- F_a^- - WB_h^+ F_a^+$. By the same way, if the syndrome is $WB_h^- F_a^- - WB_h^+ F_a^+$, it will be $S = WB_h^- F_a^- - WB_h^+ F_a^+ = WB_h^- F_a^- - WB_h^+ F_a^+ + \mathbf{0}_{d\times\gamma} = WB_h^- F_a^- - WB_h^+ F_a^+ + WUQ_g = WB_h^- F_a^- - WB_h^+ F_a^+ + WQ_h = W(B_h^- F_a^- - B_h^+ F_a^+ + Q_h)$, where Q_h is a fault-free marking matrix of H, and F_a^- and F_a^+ are pre-condition and post-condition amount transition fault indicator matrix respectively. Therefore, a marking matrix of H could be examined if it is a faulty marking matrix by multiplying the marking matrix with the check matrix W.

Assume Q_f is a faulty marking matrix of H, which states k amount transition faults on H. Besides, x of k are pre-condition amount transition faults, and $(k-x)$ of k are post-condition amount transition faults. By Lemma 4.6, $Q_f = Q_h + B_h^- F_a^- - B_h^+ F_a^+$, where x sub-matrices in F_a^- are identity matrices scalar multiplied by positive integers, and $(k - x)$ sub-matrices in F_a^+ are identity matrices scalar multiplied by positive integers. Hence, each column in F_a^- has x nonzero entries, and each column in F_a^+ has $(k - x)$ nonzero entries. If the check matrix $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, the syndrome will be

$S = \begin{bmatrix} s_1 & s_2 & \cdots & s_\gamma \end{bmatrix}$

$$= WQ_f$$

$$= WUQ_g + WB_h^- F_a^- \text{ - } WB_h^+ F_a^+$$

$$= \begin{bmatrix} -D & I_d \end{bmatrix} \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix} F_a^- - \begin{bmatrix} -D & I_d \end{bmatrix} \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix} F_a^+$$

$$= (-DB_g^- + DB_g^- - E)F_a^- - (-DB_g^+ + DB_g^+ - E)F_a^+$$

$$= E(F_a^+ - F_a^-)$$

$$= EF_a$$

$$= E\begin{bmatrix} f_{a_1} & f_{a_2} & \cdots & f_{a_\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} Ef_{a_1} & Ef_{a_2} & \cdots & Ef_{a_\gamma} \end{bmatrix},$$

where s_n and f_{a_n} represent the nth column in S and F_a, respectively, and $1 \le n \le \gamma$. Hence, it can be deem as the problem of correcting the faults, $f_{a_1}, f_{a_2}, \cdots, f_{a_\gamma}$, from linear codes with length $\alpha\gamma$ by multiplied with the matrix E. Since $F_a = F_a^+ - F_a^-$, each column in F_a, f_{a_n}, has k nonzero entries. From the theorems in error control coding, each code word with a fault indicator f_{a_n} can be corrected by E if any $2k$ columns of E are linearly dependent. After getting the syndrome by WQ_f, the amount transition fault indicator matrix F_a can be found by solving equations $Ef_{a_n} = s_n$, where $1 \le n \le \gamma$.

Lemma 4.8 shows that if there are at most k amount transition faults, it needs to design the matrix E with any $2k$ columns are linearly dependent, and the result of multiplying the check matrix W with a fault-free marking matrix is a matrix with all entries of zero. Same as previous section, the matrix E can have $2k$ rows and any $2k$

columns of E are linearly dependent if it is designed by the method of Reed-Solomon codes. Since E is a $d \times \alpha\gamma$ matrix, it would have $d = 2k$. In other word, if there are at most k amount transition faults, $2k$ additional places is needed in the separate CPN by applying the method of Reed-Solomon codes in order to derive the detection and correction capabilities of amount transition faults. Lemma 4.8 also shows that the amount transition faults can be identified and corrected from the syndrome.

Let G be a CPN which has α transitions, β places, γ colours, the input matrix B_g^-, the output matrix B_g^+ and the initial marking matrix Q_{0_g}. From Lemmas 4.7 and 4.8, constructing a separate CPN H which can detect and correct at most k amount transition faults is concluded as following steps: (1) First, designing a $d \times \alpha\gamma$ matrix E from the check matrix of Reed-Solomon codes. (2) Second, choosing a $d \times \beta$ matrix D which satisfies all the entries in $DB_g^- - E$ and $DB_g^+ - E$ are nonnegative. (3) Third, constructing the check matrix W from $W = \begin{bmatrix} -D & I_\beta \end{bmatrix}$. (4) Finally, deriving the separate CPN H containing input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$. A marking Q_h in the separate CPN H can be examined if it is a correct marking by the check matrix W. If the marking Q_h is a faulty marking, it can be corrected by solving the pre-condition and post-condition amount transition fault indicator matrix, F_a^- and F_a^+, from the equation $S = EF_a = E(F_a^+ - F_a^-)$, where S is the syndrome from WQ_h.

4.4.3 An Example of Identifying and Correcting Amount Transition Faults

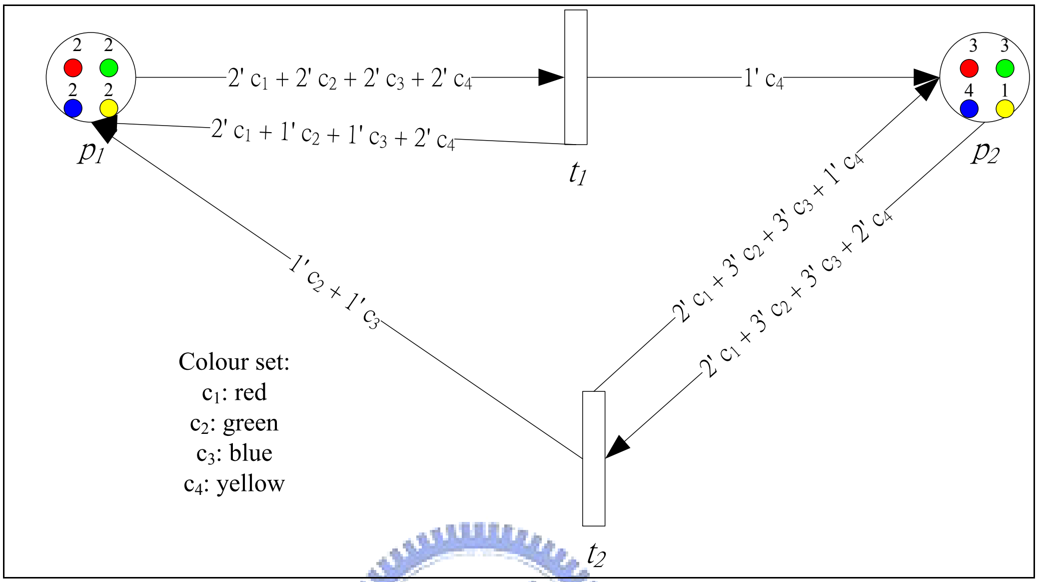

Figure 4.3 A CPN with nested loops.

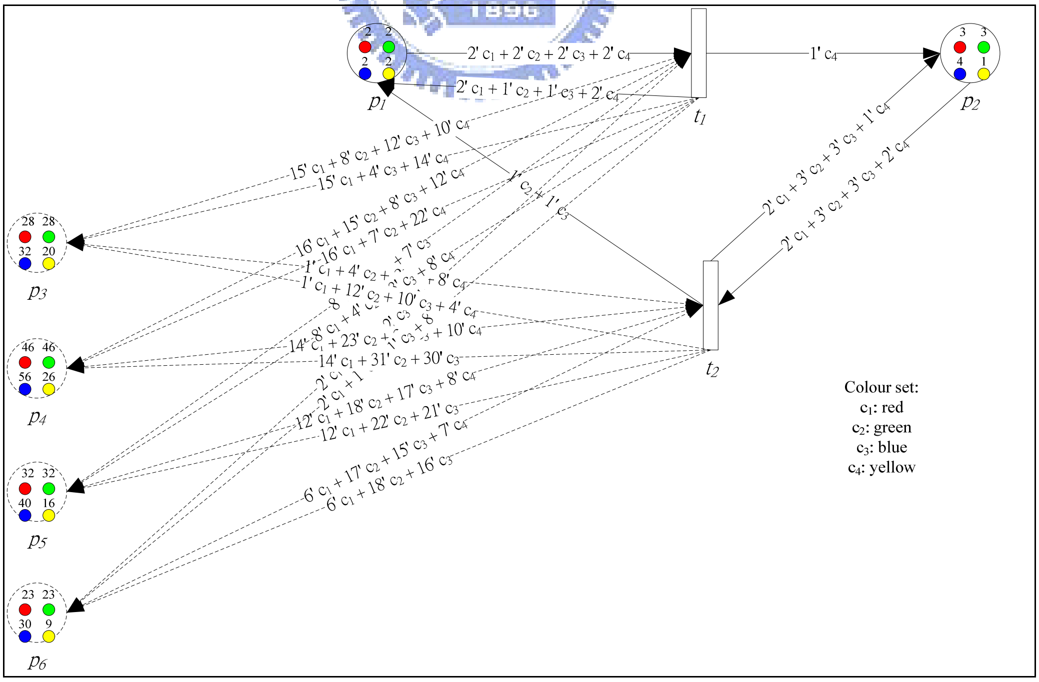

Figure 4.4 A separate CPN of the CPN in Figure 4.3.

This section uses the example in Figure 4.3 as the given CPN *G* and sets the marking in this figure as the initial marking. Let B_g^-, B_g^+ and Q_{0_g} represent the input matrix, the output matrix and the initial marking matrix of G respectively. Hence, the given CPN G has $\alpha = 2$ transitions, $\beta = 2$ places, $\gamma = 4$ colours,

$$B_g^- = \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \end{bmatrix},$$

$$B_g^+ = \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \end{bmatrix}, \text{ and}$$

$$Q_{0_g} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}.$$

Assume there are at most $k = 2$ amount transition faults, a 4×8 matrix E can be constructed by the method of Reed-Solomon codes as following steps: (1) Choosing a prime number a which satisfies $a \geq \alpha\gamma = 8$, and hence it can take $a = 11$. (2) Second, finding a nature number r which is a primitive root mod a, it can choose $r = 2$. (3)Third, giving a polynomial of degree $a - 1 - 2k = 10 - 4 = 6$ used to construct check matrix, such that $e(x) = (x - r^{a-2} \bmod a)(x - r^{a-3} \bmod a)(x - r^{a-4} \bmod a)(x - r^{a-5} \bmod a)(x - r^{a-6} \bmod a)(x - 1) = (x - 2^9 \bmod 11)(x - 2^8 \bmod 11)(x - 2^7 \bmod 11)(x - 2^6 \bmod 11)(x - 2^5 \bmod 11)(x - 1) = x^6 + 8x^5 + 4x^4 + 6x^3 + 7x^2 + 8x + 10 = b_6x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. (4) Finally, constructing the matrix E from b_6, b_5, b_4, b_3, b_2, b_1 and b_0, such that

$$E = \begin{bmatrix} b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_6 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \\ 0 & 0 & b_6 & b_5 & b_4 & b_3 & b_2 & b_1 \\ 0 & 0 & 0 & b_6 & b_5 & b_4 & b_3 & b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}.$$

Next, the matrix D can be chose as

$$D = \begin{bmatrix} 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix}$$ which satisfies all the entries in

$$DB_g^- - E = \begin{bmatrix} 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 16 & 16 & 16 & 8 & 12 & 12 & 8 \\ 16 & 16 & 16 & 16 & 20 & 30 & 30 & 20 \\ 8 & 8 & 8 & 8 & 16 & 24 & 24 & 16 \\ 2 & 2 & 2 & 2 & 14 & 21 & 21 & 14 \end{bmatrix} - \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix}$$ and

$$DB_g^+ - E = \begin{bmatrix} 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 8 & 8 & 20 & 8 & 20 & 20 & 4 \\ 16 & 8 & 8 & 26 & 20 & 38 & 38 & 10 \\ 8 & 4 & 4 & 16 & 16 & 28 & 28 & 8 \\ 2 & 1 & 1 & 9 & 14 & 22 & 22 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix}$$

are nonnegative. Finally, the check matrix W is obtained from

$$W = \begin{bmatrix} -D & I_d \end{bmatrix} = \begin{bmatrix} -8 & -4 & 1 & 0 & 0 & 0 \\ -8 & -10 & 0 & 1 & 0 & 0 \\ -4 & -8 & 0 & 0 & 1 & 0 \\ -1 & -7 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the separate CPN H is constructed, where

$$B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix},$$

$$B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix}, \text{ and}$$

$$Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 1 \\ 28 & 28 & 32 & 20 \\ 46 & 46 & 56 & 26 \\ 32 & 32 & 40 & 16 \\ 23 & 23 & 30 & 9 \end{bmatrix}.$$

The CPN *H* with its initial marking is illustrated in Figure 4.4.

There is a firing sequence in *G*, which is m_{0_g} [t_1> m_{1_g} [t_2> m_{2_g} [t_1> m_{3_g} [t_2> m_{4_g} [t_1> m'_g, and m_{0_g}, m_{1_g}, m_{2_g}, m_{3_g}, m_{4_g} and m'_g are the fault-free marking in *G*. Since *H* is the separate CPN of *G*, *G* and *H* can have the same firing transition sequence. Hence, consider the same firing transition sequence in *H* and assume the corresponding fault-free firing sequence is m_{0_h} [t_1> m_{1_h} [t_2> m_{2_h} [t_1> m_{3_h} [t_2> m_{4_h} [t_1> m'_h. The markings m_{0_h}, m_{1_h}, m_{2_h}, m_{3_h}, m_{4_h} and m'_h are the fault-free marking in CPN *H*, and it is assumed Q_{0_h}, Q_{1_h}, Q_{2_h}, Q_{3_h}, Q_{4_h} and Q'_h are the marking matrices of m_{0_h}, m_{1_h}, m_{2_h}, m_{3_h}, m_{4_h} and m'_h respectively. Assume there are place faults inside the firing sequence, such that m_{0_h} [t_1> m_{1_h} [t_2, $F^-_{a_1}$ > m_{2_f} [t_1> m_{3_f} [t_2, $F^-_{a_2}$ > m_{4_f} [t_1, $F^+_{a_3}$ > m'_f which is an informal representation and means that a post-condition amount transition fault occur when the last time firing t_1, and a pre-condition amount transition fault occur when firing t_2. $F^-_{a_1}$ and $F^-_{a_2}$ are pre-condition amount transition fault indicator matrices, and $F^+_{a_3}$ is a post-condition amount transition fault indicator matrix, where

$$F^-_{a_1} = F^-_{a_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$F^+_{a_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the marking matrices represent m_{1_h}, m_{2_f}, m_{3_f}, m_{4_f} and m'_f are

$$Q_{1_h} = Q_{0_h} - B^-_h X_1 + B^+_h X_1$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 1 \\ 28 & 28 & 32 & 20 \\ 46 & 46 & 56 & 26 \\ 32 & 32 & 40 & 16 \\ 23 & 23 & 30 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 3 & 4 & 2 \\ 28 & 20 & 24 & 24 \\ 46 & 38 & 48 & 36 \\ 32 & 28 & 36 & 24 \\ 23 & 22 & 29 & 16 \end{bmatrix},$$

$$Q_{2_f} = Q_{1_h} - B_h^- \mathbf{0}_{8\times 4} + B_h^+ X_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 5 & 6 & 7 & 3 \\ 29 & 32 & 34 & 28 \\ 60 & 69 & 78 & 36 \\ 44 & 50 & 57 & 24 \\ 29 & 40 & 45 & 16 \end{bmatrix},$$

$$Q_{3_f} = Q_{2_f} - B_h^- X_1 + B_h^+ X_1 = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 5 & 6 & 7 & 4 \\ 29 & 24 & 26 & 32 \\ 60 & 61 & 70 & 46 \\ 44 & 46 & 53 & 32 \\ 29 & 39 & 44 & 23 \end{bmatrix},$$

$$Q_{4_f} = Q_{3_f} - B_h^- \mathbf{0}_{8\times 4} + B_h^+ X_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 7 & 9 & 10 & 5 \\ 30 & 36 & 36 & 36 \\ 74 & 92 & 100 & 46 \\ 56 & 68 & 74 & 32 \\ 35 & 57 & 60 & 23 \end{bmatrix}, \text{ and}$$

$$Q'_f = Q_{4_f} - B_h^- X_1 + B_h^+ \mathbf{0}_{8\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 7 & 9 & 10 & 5 \\ 15 & 28 & 24 & 26 \\ 58 & 77 & 92 & 34 \\ 48 & 60 & 67 & 32 \\ 33 & 55 & 58 & 22 \end{bmatrix}.$$

Following steps are identifying and correcting these two amount transition faults from the marking matrix Q'_f and the check matrix W. First, Since

$$WQ'_f \bmod 11 = \begin{bmatrix} -8 & -4 & 1 & 0 & 0 & 0 \\ -8 & -10 & 0 & 1 & 0 & 0 \\ -4 & -8 & 0 & 0 & 1 & 0 \\ -1 & -7 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 7 & 9 & 10 & 5 \\ 15 & 28 & 24 & 26 \\ 58 & 77 & 92 & 34 \\ 48 & 60 & 67 & 32 \\ 33 & 55 & 58 & 22 \end{bmatrix} \bmod 11$$

$$= \begin{bmatrix} 9 & 3 & 6 & 6 \\ 10 & 9 & 3 & 6 \\ 3 & 10 & 9 & 3 \\ 6 & 3 & 10 & 9 \end{bmatrix} \neq \mathbf{0}_{4\times 4},$$

there are amount transition faults in Q'_f. Second, since

$EF_a \bmod 11$

$$= \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix} \begin{bmatrix} f_{a_{11}} & 0 & 0 & 0 \\ 0 & f_{a_{22}} & 0 & 0 \\ 0 & 0 & f_{a_{33}} & 0 \\ 0 & 0 & 0 & f_{a_{44}} \\ f_{a_{51}} & 0 & 0 & 0 \\ 0 & f_{a_{62}} & 0 & 0 \\ 0 & 0 & f_{a_{73}} & 0 \\ 0 & 0 & 0 & f_{a_{84}} \end{bmatrix} \bmod 11$$

$$= WQ'_f \bmod 11 = \begin{bmatrix} 9 & 3 & 6 & 6 \\ 10 & 9 & 3 & 6 \\ 3 & 10 & 9 & 3 \\ 6 & 3 & 10 & 9 \end{bmatrix},$$

the following sets of equations are figured out:

$$\begin{cases}(f_{a_{11}}+7f_{a_{51}})\bmod 11=9\\(6f_{p_{51}})\bmod 11=10\\(4f_{p_{51}})\bmod 11=3\\(8f_{p_{51}})\bmod 11=6\end{cases},\quad\begin{cases}(8f_{a_{22}}+8f_{a_{62}})\bmod 11=3\\(f_{a_{22}}+7f_{a_{62}})\bmod 11=9\\(6f_{a_{62}})\bmod 11=10\\(4f_{a_{62}})\bmod 11=3\end{cases},$$

$$\begin{cases}(4f_{a_{33}}+10f_{a_{73}})\bmod 11=6\\(8f_{a_{33}}+8f_{a_{73}})\bmod 11=3\\(f_{a_{33}}+7f_{a_{73}})\bmod 11=9\\(6f_{a_{73}})\bmod 11=10\end{cases}\text{ and }\begin{cases}(6f_{a_{44}})\bmod 11=6\\(4f_{a_{44}}+10f_{a_{84}})\bmod 11=6\\(8f_{a_{44}}+8f_{a_{84}})\bmod 11=3\\(f_{a_{44}}+7f_{a_{84}})\bmod 11=9\end{cases},$$

and there are three more restrictions: (1) $f_{a_{11}}$ = $f_{a_{22}}$ = $f_{a_{33}}$ = $f_{a_{44}}$ and $f_{a_{51}}$ = $f_{a_{62}}$ = $f_{a_{73}}$ = $f_{a_{84}}$, (2) each set of equations has at most two nonzero variables and (3) $\forall 1 \leq i \leq 8,\ 1 \leq j \leq 4$: $-\frac{11}{2} < f_{a_{ij}} < \frac{11}{2}$. Therefore, $f_{a_{11}}$ = $f_{a_{22}}$ = $f_{a_{33}}$ = $f_{a_{44}}$ = 1 and $f_{a_{51}}$ = $f_{a_{62}}$ = $f_{a_{73}}$ = $f_{a_{84}}$ = -2. F_a^- and F_a^+ can be derived from F_a that $F_a^- = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$ and $F_a^+ = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. By Lemma 4.6, it can be inferred from F_a^- and F_a^+ that a pre-condition amount transition fault occurs in t_2 two times, and a post-condition amount transition fault occurs in t_1 one time. The correct marking matrix with respect to Q'_f is

Q'_f - B_h^- F_a^- + B_h^+ F_a^+

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 7 & 9 & 10 & 5 \\ 15 & 28 & 24 & 26 \\ 58 & 77 & 92 & 34 \\ 48 & 60 & 67 & 32 \\ 33 & 55 & 58 & 22 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 3 & 4 & 2 \\ 28 & 20 & 24 & 24 \\ 46 & 38 & 48 & 36 \\ 32 & 28 & 36 & 24 \\ 23 & 22 & 29 & 16 \end{bmatrix}, \text{ which is equal to } Q'_h.$$

4.5 Colour Transition Faults

The organization of this section is similar to the one of previous section. This section firstly gives the problem formulation of colour transition faults in matrix representations according to Definition 4.7. Next, based on the problem formulation, a methodology encoding a CPN into a separeate CPN with detection and correction capabilities on amount transition faults is presented, and the encoding matrix is the same as it in previous section. Then, this section gives the syndromes of colour transition faults while they occur in CPNs, where the check matrix is also the same as it in previous section. Finally, this section describes how to compute the correction markings via these syndromes.

4.5.1 Problem Formulation for Colour Transition Faults

Lemma 4.9: Let Q be a fault-free marking matrix of a CPN G.

- If Q_f is a marking matrix containing a pre-condition colour transition fault on transition t_i and colour c_j corresponding to Q, $\exists F_c^{ij-}$, a $n \times \gamma$ matrix, where $n = \alpha\gamma$,

  $Q_f = Q + B^- F_c^{ij-}$,

  and all the entries in F_c^{ij-} satisfy

  $\exists h = (i - 1)\gamma + j : f_{c_{hj}}^{ij-} \in \mathbb{N} / \{0\}$,

  $\forall 1 \leq g \leq \gamma, g \neq j$: $f_{c_{hg}}^{ij-} \in \mathbb{Z} / \mathbb{N} \cup \{0\}$, $\sum_{o=1}^{\gamma} f_{c_{ho}}^{ij-} = 0$, and

  $\forall 1 \leq k \leq n, 1 \leq l \leq \gamma, k \neq h$: $f_{c_{kl}}^{ij-} = 0$.

- If Q_f is a marking matrix containing a post-condition colour transition fault on transition t_i and colour c_j corresponding to Q, $\exists F_c^{ij+}$, a $n \times \gamma$ matrix, where $n = \alpha\gamma$,

  $Q_f = Q - B^+ F_c^{ij+}$,

  and all the entries in F_c^{ij+} satisfy

  $\exists h = (i - 1)\gamma + j : f_{c_{hj}}^{ij+} \in \mathbb{N} / \{0\}$,

$\forall 1 \le g \le \gamma, g \neq j$: $f^{ij+}_{c_{hg}} \in \mathbb{Z} / \mathbb{N} \cup \{0\}$, $\sum_{o=1}^{\gamma} f^{ij+}_{c_{ho}} = 0$, and

$\forall 1 \le k \le n, 1 \le l \le \gamma, k \neq h$: $f^{ij+}_{c_{kl}} = 0$.

Proof:

Assume Q is the matrix representation of a fault-free marking m, and Q_f is the matrix representation of m_f which is a marking with a pre-condition colour transition on transition t_i and colour c_j with respect to m. By Definition 4.7,

$\forall p_x \in P$: $m_f(p_x) = m(p_x) - I_f(p_x)$, where

$I_f(p_y) = -z * I(p_y, t_i)(c_j)'c_j + \sum_{g=1, g\neq j}^{\gamma} i_g * I(p_y, t_i)(c_j)'c_g$,

$\sum_{g=1, g\neq r}^{\gamma} i_g = z$, (1)

$z \in \mathbb{N} / \{0\}$, $z \le \sigma(t_i)$, and $i_g \in \mathbb{N}$. (2)

Hence, by Definition 2.2, $\forall p_x \in P$, $\forall c_r \in C$: $m_f(p_x)(c_r) = m(p_x)(c_r) + I_f(p_x)(c_r)$. First, consider the colour c_j, it would have $\forall p_x \in P$: $m_f(p_x)(c_j) = m(p_x)(c_j) - I_f(p_x)(c_j) = m(p_x)(c_j) + z * I(p_x, t_i)(c_j)$. By Definition 3.1 and 3.2,

$\forall 1 \le x \le \beta$: $q_{f_{xj}} = q_{xj} + z * b^-_{xh} = q_{xj} + z * b^-_{xh} + \sum_{s=1, s\neq h}^{n} 0 * b^-_{xs}$, (3)

where $h = (i - 1)\gamma + j$. Next, consider all the colour $c_g \neq c_j$, it would have $\forall p_x \in P$: $m_f(p_x)(c_g) = m(p_x)(c_g) - I_f(p_x)(c_g) = m(p_x)(c_g) - i_g * I(p_x, t_i)(c_j)$. By Definition 3.1 and 3.2,

$\forall 1 \le x \le \beta, 1 \le g \le \gamma, g \neq j$: $q_{f_{xg}} = q_{xg} - i_g * b^-_{xh} = q_{xg} - i_g * b^-_{xh} + \sum_{s=1, s\neq h}^{n} 0 * b^-_{xs}$, (4)

where $h = (i - 1)\gamma + j$. By combining (3) and (4), it could have a $n \times \gamma$ matrix, F_c^{ij-}, the entries in F_c^{ij-} are

$$f_{c_{\kappa\lambda}}^{ij+} = \begin{cases} z & if\ \kappa = h\, and\ \lambda = j \\ i_\lambda & if\ \kappa = h\, and\ \lambda \neq j\ , \\ 0 & otherwise \end{cases}$$

and $Q_f = Q + B^{-} F_c^{ij-}$. From (1) and (2), it would have

$f_{c_{hj}}^{ij-} \in \mathbb{N} / \{0\}$,

$\forall 1 \leq g \leq \gamma, g \neq j$: $f_{c_{hg}}^{ij-} \in \mathbb{Z} / \mathbb{N} \cup \{0\}$, $\sum_{o=1}^{\gamma} f_{c_{ho}}^{ij-} = 0$, and

$\forall 1 \leq k \leq n,\ 1 \leq l \leq \gamma,\ k \neq h$: $f_{c_{kl}}^{ij-} = 0$. The case of post-condition colour transition fault can be proved by the same way.

Lemma 4.9 formulates the problem of a colour transition fault in linear algebra. The matrix F_c^{ij-} and F_c^{ij+} in the equation indicate the jth colour in the ith transition suffering a pre-condition and post-condition colour transition fault respectively. There is only one row in F_c^{ij-} (F_c^{ij+}) containing nonzero entries, which is $((i - 1)\gamma + j)$th row, and the jth column in this row is positive (negative), and all the other entries in this row are negative (positive) or zero. The sum of all the entries in $((i - 1)\gamma + j)$th row of F_c^{ij-} and F_c^{ij+} are zero. In this thesis, it is assumed that a transition wouldn't suffer both a pre-condition and a post-condition colour transition fault on the same colour. The consistency between the description of amount transition fault in Lemma 4.9 and the definition of amount transition fault in Definition 4.7 is also proved.

Lemma 4.10: Let Q be a fault-free marking matrix of a CPN G.

- If Q_f is a marking matrix representing Q with z colour transition faults, where x

of z are pre-condition colour transition faults, and y of z are post-condition colour transition faults, $\exists\, F_c^-$ and F_c^+, $n \times \gamma$ matrices, where $n = \alpha\gamma$,

$Q_f = Q + B^- F_c^- - B^+ F_c^+$,

, all the entries in the **pre-condition colour transition fault indicator matrix** F_c^- satisfy

$\exists 1 \le h_1^-, h_2^-, \cdots, h_x^- \le n,\ h_1^- \ne h_2^- \ne \cdots \ne h_x^-$,

$j_1^- = (h_1^- - 1) \bmod \gamma + 1, j_2^- = (h_2^- - 1) \bmod \gamma + 1, \cdots, j_x^- = (h_x^- - 1) \bmod \gamma + 1:$

$f^-_{c_{h_1^- j_1^-}}, f^-_{c_{h_2^- j_2^-}}, \cdots, f^-_{c_{h_x^- j_x^-}} \in \mathbb{N} / \{0\}$,

$\forall 1 \le k^- \le n,\ 1 \le l^- \le \gamma$,

$k^- \ne h_1^- \ne h_2^- \ne \cdots \ne h_x^-$ or $l^- \ne j_1^- \ne j_2^- \ne \cdots \ne j_x^-:$

$f^-_{c_{k^- l^-}} \in \mathbb{Z} / \mathbb{N} \cup \{0\}$, and

$\forall 1 \le g^- \le n$: $\sum_{o=1}^{\gamma} f^-_{c_{g^- o}} = 0$,

and all the entries in the **post-condition colour transition fault indicator matrix** F_c^+ satisfy

$\exists 1 \le h_1^+, h_2^+, \cdots, h_y^+ \le n,\ h_1^+ \ne h_2^+ \ne \cdots \ne h_y^+ \ne h_1^- \ne h_2^- \ne \cdots \ne h_x^-$,

$j_1^+ = (h_1^+ - 1) \bmod \gamma + 1, j_2^+ = (h_2^+ - 1) \bmod \gamma + 1, \cdots, j_y^+ = (h_y^+ - 1) \bmod \gamma + 1:$

$f^+_{c_{h_1^+ j_1^+}}, f^+_{c_{h_2^+ j_2^+}}, \cdots, f^+_{c_{h_y^+ j_y^+}} \in \mathbb{N} / \{0\}$,

$\forall 1 \le k^+ \le n,\ 1 \le l^+ \le \gamma$,

$k^+ \ne h_1^+ \ne h_2^+ \ne \cdots \ne h_y^+$ or $l^+ \ne j_1^+ \ne j_2^+ \ne \cdots \ne j_y^+:$

$f^+_{c_{k^+ l^+}} \in \mathbb{Z} / \mathbb{N} \cup \{0\}$, and

$\forall 1 \leq g^{+} \leq n$: $\sum_{o=1}^{\gamma} f^{+}_{c_{g^{+}o}} = 0$.

Proof:

By Definition 4.8, z colour transition faults denote that the faults occur on z colour and transition pairs, and x of these z pairs suffer pre-condition colour transition faults, y of these z pairs suffer post-condition colour transition faults. In previous assumption, a transition wouldn't suffer both a pre-condition and a post-condition colour transition fault on the same colour, thus $z = x + y$. Hence, By Lemma 4.9, there are x matrices, $F_c^{i_1^- j_1^- -}, F_c^{i_2^- j_2^- -}, \cdots, F_c^{i_x^- j_x^- -}$, indicate these pre-condition colour transition faults and y matrices, $F_c^{i_1^+ j_1^+ +}, F_c^{i_2^+ j_2^+ +}, \cdots, F_c^{i_y^+ j_y^+ +}$, indicate these post-condition colour transition faults, where $i_1^+ \neq i_2^+ \neq \cdots \neq i_y^+ \neq i_1^- \neq i_2^- \neq \cdots \neq i_x^-$ or $j_1^+ \neq j_2^+ \neq \cdots \neq j_y^+ \neq j_1^- \neq j_2^- \neq \cdots \neq j_x^-$, and the marking Q_f would be

$$\begin{aligned} Q_f &= Q + B^- F_c^{i_1^- j_1^- -} + B^- F_c^{i_2^- j_2^- -} + \cdots + B^- F_c^{i_x^- j_x^- -} - B^+ F_c^{i_1^+ j_1^+ +} - B^+ F_c^{i_2^+ j_2^+ +} - \cdots - B^+ F_c^{i_y^+ j_y^+ +} \\ &= Q + B^- (F_c^{i_1^- j_1^- -} + F_c^{i_2^- j_2^- -} + \cdots + F_c^{i_x^- j_x^- -}) - B^+ (F_c^{i_1^+ j_1^+ +} + F_c^{i_2^+ j_2^+ +} + \cdots + F_c^{i_y^+ j_y^+ +}) \\ &= Q + B^- F_c^- - B^+ F_c^+ \end{aligned} .$$

From the previous equation, it would have the relations that $F_c^- = F_c^{i_1^- j_1^- -} + F_c^{i_2^- j_2^- -} + \cdots + F_c^{i_x^- j_x^- -}$ and $F_c^+ = F_c^{i_1^+ j_1^+ +} + F_c^{i_2^+ j_2^+ +} + \cdots + F_c^{i_y^+ j_y^+ +}$. By Lemma 4.9, the rows with nonzero entries of $F_c^{i_1^- j_1^- -}, F_c^{i_2^- j_2^- -}, \cdots, F_c^{i_x^- j_x^- -}, F_c^{i_1^+ j_1^+ +}, F_c^{i_2^+ j_2^+ +}, \cdots, F_c^{i_y^+ j_y^+ +}$ are $h_1^- = (i_1^- - 1)\gamma + j_1^-$ th, $h_2^- = (i_2^- - 1)\gamma + j_2^-$ th, …, $h_x^- = (i_x^- - 1)\gamma + j_x^-$ th, $h_1^+ = (i_1^+ - 1)\gamma + j_1^+$ th, $h_2^+ = (i_2^+ - 1)\gamma + j_2^+$ th, …, $h_y^+ = (i_y^+ - 1)\gamma + j_y^+$ th rows respectively. Since $i_1^+ \neq i_2^+ \neq \cdots \neq i_y^+ \neq i_1^- \neq i_2^- \neq \cdots \neq i_x^-$ or

$j_1^+ \neq j_2^+ \neq \cdots \neq j_y^+ \neq j_1^- \neq j_2^- \neq \cdots \neq j_x^-$, $h_1^+ \neq h_2^+ \neq \cdots \neq h_y^+ \neq h_1^- \neq h_2^- \neq \cdots \neq h_x^-$.

Therefore, the entries in F_c^- should satisfy

$f^-_{c_{h_1^- j_1^-}}, f^-_{c_{h_2^- j_2^-}}, \cdots, f^-_{c_{h_x^- j_x^-}} \in \mathbb{N} / \{0\},$

$\forall 1 \leq k^- \leq n,\ 1 \leq l^- \leq \gamma,$

$k^- \neq h_1^- \neq h_2^- \neq \cdots \neq h_x^-$ or $l^- \neq j_1^- \neq j_2^- \neq \cdots \neq j_x^-$:

$f^-_{c_{k^- l^-}} \in \mathbb{Z} / \mathbb{N} \cup \{0\},$

$\forall 1 \leq g^- \leq n$: $\sum_{o=1}^{\gamma} f^-_{c_{g^- o}} = 0$,

and the entries in F_c^+ should satisfy

$f^+_{c_{h_1^+ j_1^+}}, f^+_{c_{h_2^+ j_2^+}}, \cdots, f^+_{c_{h_y^+ j_y^+}} \in \mathbb{N} / \{0\},$

$\forall 1 \leq k^+ \leq n,\ 1 \leq l^+ \leq \gamma,$

$k^+ \neq h_1^+ \neq h_2^+ \neq \cdots \neq h_y^+$ or $l^+ \neq j_1^+ \neq j_2^+ \neq \cdots \neq j_y^+$:

$f^+_{c_{k^+ l^+}} \in \mathbb{Z} / \mathbb{N} \cup \{0\},$

$\forall 1 \leq g^+ \leq n$: $\sum_{o=1}^{\gamma} f^+_{c_{g^+ o}} = 0$.

There are x (y) rows in F_c^- (F_c^+) have nonzero entries, each of which indicates a pre-condition (post-condition) amount transition fault. The summation of the entries in each row of F_c^- (F_c^+) is zero. The poof at here is similar to the one in Lemma 4.6. It proves the consistency between Lemma 4.10 and Definition 4.8 by deriving from Lemma 4.9.

4.5.2 Separate CPNs with Colour Transition Faults Detection and Correction Capabilities

The strategy in this section is used to detect and correct colour transition faults. As it is mentioned in the beginning of this section, the encoding method in section 4.4.2 could also applied to encode a CPN into a separate CPN with colour transition faults detection and correction capabilities. Therefore, let G be a CPN with no more than k colour transition faults, the separate CPN H with colour transition faults detection and correction capabilities is constructed by adding $2k$ additional places to G and the colour sets of both CPNs are the same. The input matrix, output matrix, initial marking matrix of G are $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$ respectively. After H is constructed, the colour transition faults occurring on H can be identified and corrected from the syndromes which are different from the syndromes in section 4.4.2. These properties are proved in Lemma 4.11.

Lemma 4.11: Let G be a CPN which has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}, and H be a separate CPN with d additional places with respect to G, input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = UQ_{0_g}$.

- If there are colour transition faults on H, it can be detected by a $d \times (\beta + d)$ check matrix W, such that $WU = \mathbf{0}_{d \times \beta}$, where $\mathbf{0}_{d \times \beta}$ is a $d \times \beta$ matrix with all entries of zero. The syndrome $S = W B_h^- F_c^- - W B_h^+ F_c^+$, iff the pre-condition and post-condition colour transition fault indicator matrices are F_c^- and F_c^+ respectively.
- k colour transition faults on H can be identified and corrected if $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, and any $2k$ columns of the matrix E are linearly dependent.

Proof:

Assume Q_g and Q_h are fault-free marking matrices of G and H respectively, and they have the same firing transition sequence. Hence, by Lemma 4.7, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$, and Q_h multiplied by W would be $WQ_h = WUQ_g = \mathbf{0}_{d \times \gamma}$. If there are colour transition faults on H, by Lemma 4.10, the faulty marking matrix Q_f satisfies $Q_f = Q_h + B_h^- F_c^- - B_h^+ F_c^+ = UQ_g + B_h^- F_c^- - B_h^+ F_c^+$. Hence, The syndrome $S = WQ_f = WUQ_g + WB_h^- F_c^- - WB_h^+ F_c^+ = WB_h^- F_c^- - WB_h^+ F_c^+$. By the same way, if the syndrome is $WB_h^- F_c^- - WB_h^+ F_c^+$, it will be $S = WB_h^- F_c^- - WB_h^+ F_c^+ = WB_h^- F_c^- - WB_h^+ F_c^+ + \mathbf{0}_{d \times \gamma} = WB_h^- F_c^- - WB_h^+ F_c^+ + WUQ_g = WB_h^- F_c^- - WB_h^+ F_c^+ + WQ_h = W(B_h^- F_c^- - B_h^+ F_c^+ + Q_h)$, where Q_h is a fault-free marking matrix of H, and F_c^- and F_c^+ are pre-condition and post-condition colour transition fault indicator matrices respectively. Therefore, a marking matrix of H could be examined if it is a faulty marking matrix by multiplying the marking matrix with the check matrix W.

Assume Q_f is a faulty marking matrix of H, which states k colour transition faults on H. Besides, x of k are pre-condition colour transition faults, and $(k - x)$ of k are post-condition colour transition faults. By Lemma 4.6, $Q_f = Q_h + B_h^- F_c^-$ - $B_h^+ F_c^+$, where x rows in F_c^- with nonzero entries, and $(k - x)$ rows in F_c^+ with nonzero entries. If the check matrix $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, the syndrome will be

$$\begin{aligned} S &= \begin{bmatrix} s_1 & s_2 & \cdots & s_\gamma \end{bmatrix} \\ &= WQ_f \\ &= \begin{bmatrix} -D & I_d \end{bmatrix} \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix} F_c^- - \begin{bmatrix} -D & I_d \end{bmatrix} \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix} F_c^+ \\ &= E(F_c^+ - F_c^-) \\ &= EF_c \\ &= \begin{bmatrix} Ef_{c_1} & Ef_{c_2} & \cdots & Ef_{c_\gamma} \end{bmatrix}, \end{aligned}$$

where s_n and f_{c_n} represent the nth column in S and F_c, respectively, and $1 \le n \le \gamma$. Hence, it can be deem as the problem of correcting the faults, $f_{c_1}, f_{c_2}, \cdots, f_{c_\gamma}$, from linear codes with length $\alpha\gamma$ by multiplied with the matrix E. From Lemma 4.10, F_c^- and F_c^+ wouldn't have nonzero entries on the same row, and hence there are k rows in F_c with nonzero entries. In other words, each column in F_c, f_{c_n}, has at most k nonzero entries. From the theorems in error control coding, each code word with a fault indicator f_{c_n} can be corrected by E if any $2k$ columns of E are linearly dependent. After getting the syndrome by WQ_f, the colour transition fault indicator matrix F_c can be found by solving equations $Ef_{c_n} = s_n$, where $1 \le n \le \gamma$.

Lemma 4.11 shows that if there are at most k amount transition faults, it needs to design the matrix E with any $2k$ columns are linearly dependent. Thus, the separate CPN designed for detecting and correcting amount transition faults can also applied for detecting and correcting colour transition faults. The pre-condition and post-condition colour transition fault indicator matrices can also be solved out by the method in 4.4.2. The only difference at here is how to analyze the colours and the transitions which have suffered colour transition faults from the pre-condition and post-condition colour transition fault indicator matrices. From Lemma 4.10, the rows with nonzero entries denote the colour transition faults. If ith row in a pre-condition (post-condition) colour transition fault indicator matrix has nonzero entries, a pre-condition (post-condition) colour transition fault is on transition t_l and colour c_s, where $l = \lceil i/\gamma \rceil$ and $s = (i\ \text{-}1) \bmod \gamma + 1$. If jth column in this ith row is a negative entry, it denotes the faulty phenomenon is the colour c_s changing into colour c_j.

4.5.3 An Example of Identifying and Correcting Colour Transition Faults

This section also uses the example in Figure 4.3 as the given CPN G but the initial marking matrix is changed into $Q_{0_g} = \begin{bmatrix} 4 & 2 & 5 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$ which adds two red tokens and two blue tokens in to place p_1. As it proved in section 4.5.2, the separate CPN designed for detecting and correcting amount transition faults can also be applied for detecting and correcting colour transition faults. Thus, the CPN in Figure 4.4 can be used for detecting and correcting colour transition faults, but the initial marking matrix is changed into

$$
\begin{aligned}
Q_{0_h} &= \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 2 & 5 & 2 \\ 3 & 3 & 4 & 1 \\ 44 & 28 & 56 & 20 \\ 62 & 46 & 80 & 26 \\ 40 & 32 & 52 & 16 \\ 25 & 23 & 33 & 9 \end{bmatrix}.
\end{aligned}
$$

Unless otherwise specified, the assumptions and variables used at here are the same as they in section 4.4.3. Assume there are colour transition faults inside the firing sequence m_{0_h} [t_1> m_{1_h} [t_2> m_{2_h} [t_1> m'_h, which are

$$
F_{c_1}^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
F_{c_2}^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}
$$

$$F_{c_3}^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$F_{c_1}^-$ is a pre-condition colour transition fault which occurs when the first time firing t_1, $F_{c_2}^+$ is a post-condition colour transition fault which occurs when firing t_2, $F_{c_3}^-$ is a pre-condition colour transition fault which occurs when the last time firing t_1, and they can be represented as m_{0_h} $[t_1, F_{c_1}^- >$ m_{1_f} $[t_2, F_{c_2}^+ >$ m_{2_f} $[t_1, F_{c_3}^- >$ m'_f. Hence, the marking matrices represent m_{1_f}, m_{2_f} and m'_f are

$$Q_{1_f} = Q_{0_h} - B_h^-(X_1 - F_{c_1}^-) + B_h^+ X_1$$

$$= \begin{bmatrix} 4 & 2 & 5 & 2 \\ 3 & 3 & 4 & 1 \\ 44 & 28 & 56 & 20 \\ 62 & 46 & 80 & 26 \\ 40 & 32 & 52 & 16 \\ 25 & 23 & 33 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 23 & 0 & 12 & 10 & 1 & 4 & 2 & 8 \\ 31 & 0 & 8 & 12 & 14 & 23 & 22 & 10 \\ 16 & 0 & 7 & 0 & 12 & 18 & 17 & 8 \\ 4 & 0 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 2 \\ 3 & 3 & 4 & 2 \\ 36 & 28 & 48 & 24 \\ 47 & 53 & 72 & 36 \\ 32 & 36 & 48 & 24 \\ 23 & 24 & 32 & 16 \end{bmatrix},$$

$$Q_{2_f} = Q_{1_f} - B_h^- X_2 + B_h^+(X_2 - F_{c_2}^+)$$

$$= \begin{bmatrix} 2 & 3 & 4 & 2 \\ 3 & 3 & 4 & 2 \\ 36 & 28 & 48 & 24 \\ 47 & 53 & 72 & 36 \\ 32 & 36 & 48 & 24 \\ 23 & 24 & 32 & 16 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 6 & 0 & 1 \\ 15 & 0 & 4 & 14 & 1 & 22 & 0 & 4 \\ 16 & 7 & 0 & 22 & 14 & 61 & 0 & 0 \\ 8 & 4 & 3 & 8 & 12 & 43 & 0 & 0 \\ 2 & 1 & 1 & 8 & 6 & 34 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 4 & 2 \\ 3 & 6 & 1 & 1 \\ 36 & 46 & 46 & 20 \\ 47 & 91 & 50 & 26 \\ 32 & 61 & 31 & 16 \\ 23 & 41 & 17 & 9 \end{bmatrix}, \text{ and}$$

$$Q'_f = Q_{2_f} - B_h^-(X_1 - F_{c_3}^-) + B_h^+ X_1$$

$$= \begin{bmatrix} 2 & 5 & 4 & 2 \\ 3 & 6 & 1 & 1 \\ 36 & 46 & 46 & 20 \\ 47 & 91 & 50 & 26 \\ 32 & 61 & 31 & 16 \\ 23 & 41 & 17 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 0 & 20 & 10 & 1 & 4 & 2 & 8 \\ 16 & 0 & 23 & 12 & 14 & 23 & 22 & 10 \\ 8 & 0 & 15 & 0 & 12 & 18 & 17 & 8 \\ 2 & 0 & 4 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 1 & 2 \\ 3 & 6 & 1 & 2 \\ 36 & 46 & 30 & 24 \\ 47 & 98 & 27 & 36 \\ 32 & 65 & 19 & 24 \\ 23 & 42 & 14 & 16 \end{bmatrix}.$$

Following steps are identifying and correcting these two colour transition faults from the marking matrix Q'_f and the check matrix *W*. First, Since

$$WQ'_f \mod 11 = \begin{bmatrix} -8 & -4 & 1 & 0 & 0 & 0 \\ -8 & -10 & 0 & 1 & 0 & 0 \\ -4 & -8 & 0 & 0 & 1 & 0 \\ -1 & -7 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 1 & 2 \\ 3 & 6 & 1 & 2 \\ 36 & 46 & 30 & 24 \\ 47 & 98 & 27 & 36 \\ 32 & 65 & 19 & 24 \\ 23 & 42 & 14 & 16 \end{bmatrix} \mod 11$$

$$= \begin{bmatrix} 8 & 7 & 7 & 0 \\ 1 & 1 & 9 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 5 & 6 & 0 \end{bmatrix} \neq \mathbf{0}_{4\times 4},$$

there are colour transition faults in Q'_f. Second, since

$EF_c \mod 11$

$$= \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix} \begin{bmatrix} f_{c_{11}} & f_{c_{12}} & f_{c_{13}} & f_{c_{14}} \\ f_{c_{21}} & f_{c_{22}} & f_{c_{23}} & f_{c_{24}} \\ f_{c_{31}} & f_{c_{32}} & f_{c_{33}} & f_{c_{34}} \\ f_{c_{41}} & f_{c_{42}} & f_{c_{43}} & f_{c_{44}} \\ f_{c_{51}} & f_{c_{52}} & f_{c_{53}} & f_{c_{54}} \\ f_{c_{61}} & f_{c_{62}} & f_{c_{63}} & f_{c_{64}} \\ f_{c_{71}} & f_{c_{72}} & f_{c_{73}} & f_{c_{74}} \\ f_{c_{81}} & f_{c_{82}} & f_{c_{83}} & f_{c_{84}} \end{bmatrix} \mod 11$$

$$= WQ'_f \mod 11 = \begin{bmatrix} 8 & 7 & 7 & 0 \\ 1 & 1 & 9 & 0 \\ 4 & 4 & 7 & 0 \\ 0 & 5 & 6 & 0 \end{bmatrix},$$

the following sets of equations are figured out:

$$\begin{cases} (f_{c_{11}} + 8f_{c_{21}} + 4f_{c_{31}} + 6f_{c_{41}} + 7f_{c_{51}} + 8f_{c_{61}} + 10f_{c_{71}}) \bmod 11 = 8 \\ (f_{c_{21}} + 8f_{c_{31}} + 4f_{c_{41}} + 6f_{c_{51}} + 7f_{c_{61}} + 8f_{c_{71}} + 10f_{c_{81}}) \bmod 11 = 1 \\ (f_{c_{31}} + 8f_{c_{41}} + 4f_{c_{51}} + 6f_{c_{61}} + 7f_{c_{71}} + 8f_{c_{81}}) \bmod 11 = 0 \\ (f_{c_{41}} + 8f_{c_{51}} + 4f_{c_{61}} + 6f_{c_{71}} + 7f_{c_{81}}) \bmod 11 = 0 \end{cases},$$

$$\begin{cases} (f_{c_{12}} + 8f_{c_{22}} + 4f_{c_{32}} + 6f_{c_{42}} + 7f_{c_{52}} + 8f_{c_{62}} + 10f_{c_{72}}) \bmod 11 = 7 \\ (f_{c_{22}} + 8f_{c_{32}} + 4f_{c_{42}} + 6f_{c_{52}} + 7f_{c_{62}} + 8f_{c_{72}} + 10f_{c_{82}}) \bmod 11 = 1 \\ (f_{c_{32}} + 8f_{c_{42}} + 4f_{c_{52}} + 6f_{c_{62}} + 7f_{c_{72}} + 8f_{c_{82}}) \bmod 11 = 4 \\ (f_{c_{42}} + 8f_{c_{52}} + 4f_{c_{62}} + 6f_{c_{72}} + 7f_{c_{82}}) \bmod 11 = 5 \end{cases},$$

$$\begin{cases} (f_{c_{13}} + 8f_{c_{23}} + 4f_{c_{33}} + 6f_{c_{43}} + 7f_{c_{53}} + 8f_{c_{63}} + 10f_{c_{73}}) \bmod 11 = 7 \\ (f_{c_{23}} + 8f_{c_{33}} + 4f_{c_{43}} + 6f_{c_{53}} + 7f_{c_{63}} + 8f_{c_{73}} + 10f_{c_{83}}) \bmod 11 = 9 \\ \qquad (f_{c_{33}} + 8f_{c_{43}} + 4f_{c_{53}} + 6f_{c_{63}} + 7f_{c_{73}} + 8f_{c_{83}}) \bmod 11 = 7 \\ \qquad\qquad (f_{c_{43}} + 8f_{c_{53}} + 4f_{c_{63}} + 6f_{c_{73}} + 7f_{c_{83}}) \bmod 11 = 6 \end{cases} \quad \text{and}$$

$$\begin{cases} (f_{c_{14}} + 8f_{c_{24}} + 4f_{c_{34}} + 6f_{c_{44}} + 7f_{c_{54}} + 8f_{c_{64}} + 10f_{c_{74}}) \bmod 11 = 0 \\ (f_{c_{24}} + 8f_{c_{34}} + 4f_{c_{44}} + 6f_{c_{54}} + 7f_{c_{64}} + 8f_{c_{74}} + 10f_{c_{84}}) \bmod 11 = 0 \\ \qquad (f_{c_{34}} + 8f_{c_{44}} + 4f_{c_{54}} + 6f_{c_{64}} + 7f_{c_{74}} + 8f_{c_{84}}) \bmod 11 = 0 \\ \qquad\qquad (f_{c_{44}} + 8f_{c_{54}} + 4f_{c_{64}} + 6f_{c_{74}} + 7f_{c_{84}}) \bmod 11 = 0 \end{cases},$$

and there are three more restrictions: (1) each set of equations has at most two nonzero variables and (2) $\forall 1 \le i \le 8$, $1 \le j \le 4$: $-\frac{11}{2} < f_{c_{ij}} < \frac{11}{2}$. Therefore, $f_{c_{21}} = f_{c_{73}} = f_{c_{23}} = 1$, $f_{c_{22}} = -2$, $f_{c_{72}} = -1$ and all the other entries in F_c are zero. By Lemma 4.11, $F_c = F_c^+ - F_c^-$, and hence it can have $F_c^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $F_c^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By Lemma 4.10, it can be inferred from F_c^- that a pre-condition colour transition fault occurs on t_1 two times, where one time is c_2 changing into c_1, and one time is c_2 changing into c_3, and it can be inferred from F_c^+ that a post-condition colour transition fault occurs on t_2 one times, where it is c_3 changing into c_2. The correct marking matrix with respect to Q'_f is

Q'_f - $B_h^- F_c^-$ + $B_h^+ F_c^+$

$$= \begin{bmatrix} 2 & 6 & 1 & 2 \\ 3 & 6 & 1 & 2 \\ 36 & 46 & 30 & 24 \\ 47 & 98 & 27 & 36 \\ 32 & 65 & 19 & 24 \\ 23 & 42 & 14 & 16 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 15 & 8 & 12 & 10 & 1 & 4 & 2 & 8 \\ 16 & 15 & 8 & 12 & 14 & 23 & 22 & 10 \\ 8 & 8 & 7 & 0 & 12 & 18 & 17 & 8 \\ 2 & 2 & 2 & 1 & 6 & 17 & 15 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 15 & 0 & 4 & 14 & 1 & 12 & 10 & 4 \\ 16 & 7 & 0 & 22 & 14 & 31 & 30 & 0 \\ 8 & 4 & 3 & 8 & 12 & 22 & 21 & 0 \\ 2 & 1 & 1 & 8 & 6 & 18 & 16 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 4 & 2 \\ 3 & 3 & 4 & 2 \\ 44 & 20 & 48 & 24 \\ 62 & 38 & 72 & 36 \\ 40 & 28 & 48 & 24 \\ 25 & 22 & 32 & 16 \end{bmatrix}$$, which is equal to Q'_h.

4.6 Additive Faults

From Lemma 4.1, 4.5 and 4.9, the problem formulation of additive faults in matrix representations can be concluded. Thus, based on the problem formulation, this section gives the methodology encoding a CPN into a separeate CPN, and the place faults, amount treansition faults and colour transition faults in the separeate CPN can be extracted from the syndromes. This section also describes how to compute the correction markings via these syndromes and gives an example which shows how the methodology works.

4.6.1 Separate CPNs with Additive Faults Detection and Correction Capabilities

The strategy in this section is used to detect and correct additive faults. The encoded input matrix, encoded output matrix and encoded initial marking matrix here would have the same forms as those in the previous section, but with stricter restrictions. Therefore, let G be a CPN with no more than k place faults, x amount transition faults and k - x colour transition faults, where $0 \leq x \leq k$, the separate CPN H with additive faults detection and correction capabilities is constructed by adding $2k$ additional places to G and the colour sets of both CPNs are the same. The input matrix, output matrix and initial marking matrix of G are $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$ respectively, where all the entries in D are coprime with a prime number, and all the entries in E are multiples of the same prime number. After H is constructed, the place faults, amount treansition faults and colour transition faults occurring on H can be extracted from the syndromes, and the correct marking can be obtained. These properties are proved in Lemma 4.12.

Lemma 4.12: Let G be a CPN which has α transitions, β places, γ colours, input matrix B_g^-, output matrix B_g^+ and initial marking matrix Q_{0_g}, and H be a separate CPN with d additional places with respect to G, input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g} = UQ_{0_g}$.

- If there are additive faults on H, it can be detected by a $d \times (\beta + d)$ check matrix W, such that $WU = \mathbf{0}_{d \times \beta}$, where $\mathbf{0}_{d \times \beta}$ is a $d \times \beta$ matrix with all entries of zero. The syndrome $S = WF_p + WB_h^-(F_a^- + F_c^-)$ - $WB_h^+(F_a^+ + F_c^+)$ $= WF_p + WB_h^-F_t^-$ - $WB_h^+F_t^+$, iff the place fault, pre-condition and post-condition amount transition fault, pre-condition and post-condition colour transition fault indicator matrices are F_p, F_a^-, F_a^+, F_c^- and F_c^+ respectively. F_t^- and F_t^+ are named as **pre-condition transition fault indicator matrix** and **post-condition transition fault indicator matrix** respectively. $F = F_p + F_t^-$ - F_t^+ is named as **fault indicator matrix**.
- k place faults, x amount transition faults and k - x colour transition faults on H can be identified and corrected if $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, where $0 \le x \le k$, $E = j * E'$, $D = j * i * \mathbf{1}_{d \times \beta}$ - D', j is a prime number lager than all the entries in E' and D', i is a positive integer lager than all the entries in D', $\mathbf{1}_{d \times \beta}$ is a $d \times \beta$ matrix with all entries of one, any $2k$ columns of the $d \times \alpha\gamma$ matrix E' are linearly dependent, and any $2k$ columns of the $d \times 2\beta$ matrix $\begin{bmatrix} D' & I_d \end{bmatrix}$ are linearly dependent.

Proof:

Assume Q_g and Q_h are fault-free marking matrices of G and H respectively, and they have the same firing transition sequence. Hence, by Lemma 4.7, $Q_h = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_g$, and Q_h multiplied by W would be $WQ_h = WUQ_g = \mathbf{0}_{d \times \gamma}$. By Lemma 4.1, 4.5 and 4.9, if there are additive faults on H, the faulty marking matrix Q_f satisfies $Q_f = Q_h + F_p + B_h^- F_c^- - B_h^+ F_c^+ + B^- F_a^- - B^+ F_a^+ = UQ_g + F_p + B_h^- F_c^- - B_h^+ F_c^+ + B^- F_a^- - B^+ F_a^+$. Hence, The syndrome $S = WQ_f = WUQ_g + WF_p + WB_h^- F_c^- - WB_h^+ F_c^+ + WB^- F_a^- - WB^+ F_a^+ = WF_p + WB_h^-(F_a^- + F_c^-) - WB_h^+(F_a^+ + F_c^+)$. By the same way, if the syndrome is $WF_p + WB_h^-(F_a^- + F_c^-) - WB_h^+(F_a^+ + F_c^+)$, it will be $S = WF_p + WB_h^-(F_a^- + F_c^-) - WB_h^+(F_a^+ + F_c^+) = WF_p + WB_h^-(F_a^- + F_c^-) - WB_h^+(F_a^+ + F_c^+) + WUQ_g = WF_p + WB_h^- F_c^- - WB_h^+ F_c^+ + WB^- F_a^- - WB^+ F_a^+ + WQ_h = W(F_p + B_h^- F_c^- - B_h^+ F_c^+ + B^- F_a^- - B^+ F_a^+ + Q_h)$, where Q_h is a fault-free marking matrix of H, and F_p, F_a^-, F_a^+, F_c^- and F_c^+ are place fault, pre-condition and post-condition amount transition fault, pre-condition and post-condition colour transition fault indicator matrices respectively. Therefore, a marking matrix of H could be examined if it is a faulty marking matrix by multiplying the marking matrix with the check matrix W.

Assume Q_f is a faulty marking matrix of H, which states k place faults, k amount transition faults and k colour transition faults on H. By Lemma 4.1, 4.5 and 4.9, $Q_f = Q_h + F_p + B_h^- F_c^- - B_h^+ F_c^+ + B^- F_a^- - B^+ F_a^+$. If the check matrix $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, the syndrome will be

$$S = \begin{bmatrix} s_1 & s_2 & \cdots & s_\gamma \end{bmatrix}$$

$$= WQ_f$$

$$= \begin{bmatrix}-D & I_d\end{bmatrix}F_p + \begin{bmatrix}-D & I_d\end{bmatrix}\begin{bmatrix}B_g^- \\ DB_g^- - E\end{bmatrix}(F_a^- + F_c^-) - \begin{bmatrix}-D & I_d\end{bmatrix}\begin{bmatrix}B_g^+ \\ DB_g^+ - E\end{bmatrix}(F_a^+ + F_c^+)$$

$$= \begin{bmatrix}-D & I_d\end{bmatrix}F_p + E(F_a^+ + F_c^+ - F_a^- - F_c^-)$$

$$= \begin{bmatrix}D' - j*i*1_{d\times\beta} & I_d\end{bmatrix}F_p + j*E'(F_a^+ + F_c^+ - F_a^- - F_c^-)$$

$$= \begin{bmatrix}D' - j*i*1_{d\times\beta} & I_d\end{bmatrix}F_p + j*E'(F_t^+ - F_t^-)$$

$$= \lfloor D' - j*i*1_{d\times\beta} \quad I_d \rfloor F_p + j*E'F_t .$$

From S mod j,

$$S_p = \lfloor s_{p_1} \quad s_{p_2} \quad \cdots \quad s_{p_\gamma} \rfloor$$

$$= S \bmod j$$

$$= \begin{bmatrix}D' - j*i*1_{d\times\beta} & I_d\end{bmatrix}F_p \bmod j + j*E'(F_a^+ + F_c^+ - F_a^- - F_c^-) \bmod j$$

$$= \begin{bmatrix}D' & I_d\end{bmatrix}F_p$$

$$= \begin{bmatrix}D' & I_d\end{bmatrix}\lfloor f_{p_1} \quad f_{p_2} \quad \cdots \quad f_{p_\gamma} \rfloor$$

is obtained, which contains only place fault part. Since any $2k$ columns of the matrix $\begin{bmatrix}D' & I_d\end{bmatrix}$ are linearly dependent and each column of F_p has at most k nonzero entries, the place fault indicator matrix F_p can be be found by solving equations $\begin{bmatrix}D' & I_d\end{bmatrix}f_{p_n} = s_{p_n}$, where $1 \leq n \leq \gamma$. After F_p is solved out, it can have

$$S_t = \lfloor s_{t_1} \quad s_{t_2} \quad \cdots \quad s_{t_\gamma} \rfloor$$

$$= S - \begin{bmatrix}-D & I_d\end{bmatrix}F_p$$

$$= EF_t$$

$$= E\lfloor f_{t_1} \quad f_{t_2} \quad \cdots \quad f_{t_\gamma} \rfloor$$

which contains only amount and colour transition fault part. Since any $2k$ columns of the matrix E' are linearly dependent, any $2k$ columns of the matrix $E = j * E'$ are linearly dependent. Since each column of F_t has at most k nonzero entries, the transition fault indicator matrix F_t can be be found by solving equations $Ef_{t_n} = s_{t_n}$, where $1 \leq n \leq \gamma$. By Lemma 4.6 and 4.10, the F_a would not have nonzero entries on $j \neq (i\text{-}1) \bmod \gamma + 1$: $f_{a_{ij}}$, but F_c would have nonzero entries on $j \neq (i\text{-}1) \bmod \gamma + 1$: $f_{c_{ij}}$. Hence, if an entry on $j \neq (i\text{-}1) \bmod \gamma + 1$: $f_{t_{ij}}$ is an nonzero entry, this nonzero value is belong to F_c. By Lemma 4.10, the sum of all the entries in each row is zero, all the entries on $j = (i\text{-}1) \bmod \gamma + 1$: $f_{c_{ij}}$ can be solved out from this conditions, and all the remining entries in F_c are zero. After F_c is obtained, F_a can be obtained from $F_a = F_t - F_c$.

Let G be a CPN which has α transitions, β places, γ colours, the input matrix B_g^-, the output matrix B_g^+ and the initial marking matrix Q_{0_g} . From Lemma 4.12, constructing a separate CPN H which can detect and correct at most k place faults, k amount transition faults and k colour transition faults is concluded as following steps: (1) designing a $d \times \alpha\gamma$ matrix E' from the check matrix of Reed-Solomon codes, (2) choosing a $d \times \beta$ matrix D' which satisfies any $2k$ columns of $\begin{bmatrix} D' & I_d \end{bmatrix}$ are linearly dependent, (3) choosing a prime number j lager than all the entries in E' and D', (4) choosing a positive integer i lager than all the entries in D', (5) constructing matrices $E = j * E'$ and $D = j * i * \mathbf{1}_{d \times \beta} - D'$, (6) constructing the check matrix W from $W = \begin{bmatrix} -D & I_d \end{bmatrix}$, and finally (7) deriving H containing input matrix $B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix}$, output matrix $B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix}$ and initial marking matrix $Q_{0_h} = \begin{bmatrix} I_\beta \\ D \end{bmatrix} Q_{0_g}$. By applying the method of Reed-Solomon codes, it would have $d = 2k$. Whether a marking Q_h in H is correct can be examined with the check matrix W, and the faults can be distinguished from the following steps: (1) obtaining the syndrome S from $S = WQ_h$, (2) obtaining part of syndrome S_p containing only place fault part from $S_p = S \bmod j$, (3) solving the equation $S_p = \begin{bmatrix} D' & I_d \end{bmatrix} F_p$ to obtain the place fault indicator matrix F_p, and the place faults can be interpreted from F_p, (4) obtaining part of syndrome S_t containing only amount and colour transition fault part from $S_t = S - \begin{bmatrix} -D & I_d \end{bmatrix} F_p$, (5) solving the equation $S_t = EF_t$ to obtain the transition fault indicator matrix F_t, (6) obtaining the entries $\forall j \neq (i-1) \bmod \gamma + 1$: $f_{c_{ij}}$ in the colour transition fault indicator matrix F_c from $f_{c_{ij}} = f_{t_{ij}}$, (7) obtaining the entries $\forall j = (i-1) \bmod \gamma + 1$: $f_{c_{ij}}$ in the colour transition fault indicator matrix F_c from $f_{c_{ij}} = -\sum_{h=1, h \neq j}^{\gamma} f_{c_{ih}}$, (8) obtaining the pre-condition and post-condition colour transition fault indicator matrices, F_c^- and F_c^+, from F_c as the steps in section 4.5.2, and the pre-condition and post-condition colour transition faults can be interpreted from F_c^- and F_c^+ respectively, (9) obtaining the amount transition fault indicator matrix F_a from $F_a = F_t - F_c$, and (10) obtaining the pre-condition and post-condition colour transition fault indicator matrices, F_a^- and F_a^+, from F_a as the steps in section 4.4.2, and the pre-condition and post-condition amount transition faults can be interpreted from F_a^- and F_a^+ respectively.

4.6.2 An Example of Identifying and Correcting Additive Faults

This section adopts the same given CPN G in section 4.5.3 to show how to identify and correct additive faults. Assume there are at most $k = 2$ place faults, $x = 1$ amount transition faults and $k - x = 1$ colour transition faults. As described in section 4.6.1, it need design a 4 × 8 matrix E' from the check matrix of Reed-Solomon codes in order to design the separate CPN H first. The matrix E in section 4.4.3 is a 4 × 8 matrix designed from the check matrix of Reed-Solomon codes, thus the matrix E in section 4.4.3 can adopted as the matrix below

$$E' = \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix}.$$

Next, choosing a 4 × 2 matrix D' which satisfies any 4 columns of $\begin{bmatrix} D' & I_4 \end{bmatrix}$ are linearly dependent. D' can be declared as

$$D' = \begin{bmatrix} 8 & 4 \\ 8 & 10 \\ 4 & 8 \\ 1 & 7 \end{bmatrix}$$

which is the same as D in section 4.4.3, and any 4 columns of $\begin{bmatrix} D' & I_4 \end{bmatrix}$ are linearly dependent. Then, a prime number j lager than all the entries in E' and D' is chosen, e.g.,

$j = 11.$

Then, a positive integer i lager than all the entries in D' is chosen, e.g.,

$i = 11.$

Thus,

$$E = j * E' = \begin{bmatrix} 11 & 88 & 44 & 66 & 77 & 88 & 110 & 0 \\ 0 & 11 & 88 & 44 & 66 & 77 & 88 & 110 \\ 0 & 0 & 11 & 88 & 44 & 66 & 77 & 88 \\ 0 & 0 & 0 & 11 & 88 & 44 & 66 & 77 \end{bmatrix} \text{ and}$$

$$D = j * i * \mathbf{1}_{4\times 2} - D' = \begin{bmatrix} 113 & 117 \\ 113 & 111 \\ 117 & 113 \\ 120 & 114 \end{bmatrix}$$

are obtained. Next, computing out

$$DB_g^- - E$$

$$= \begin{bmatrix} 113 & 117 \\ 113 & 111 \\ 117 & 113 \\ 120 & 114 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 11 & 88 & 44 & 66 & 77 & 88 & 110 & 0 \\ 0 & 11 & 88 & 44 & 66 & 77 & 88 & 110 \\ 0 & 0 & 11 & 88 & 44 & 66 & 77 & 88 \\ 0 & 0 & 0 & 11 & 88 & 44 & 66 & 77 \end{bmatrix}$$

$$= \begin{bmatrix} 215 & 138 & 182 & 160 & 157 & 263 & 241 & 234 \\ 226 & 215 & 138 & 182 & 156 & 256 & 245 & 112 \\ 234 & 234 & 223 & 146 & 182 & 273 & 262 & 138 \\ 240 & 240 & 240 & 229 & 140 & 298 & 276 & 151 \end{bmatrix} \text{ and}$$

$$DB_g^+ - E$$

$$= \begin{bmatrix} 113 & 117 \\ 113 & 111 \\ 117 & 113 \\ 120 & 114 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 11 & 88 & 44 & 66 & 77 & 88 & 110 & 0 \\ 0 & 11 & 88 & 44 & 66 & 77 & 88 & 110 \\ 0 & 0 & 11 & 88 & 44 & 66 & 77 & 88 \\ 0 & 0 & 0 & 11 & 88 & 44 & 66 & 77 \end{bmatrix}$$

$$= \begin{bmatrix} 215 & 25 & 69 & 277 & 157 & 376 & 354 & 117 \\ 226 & 102 & 25 & 293 & 156 & 369 & 358 & 1 \\ 234 & 117 & 106 & 259 & 182 & 390 & 379 & 25 \\ 240 & 120 & 120 & 343 & 140 & 418 & 396 & 37 \end{bmatrix}.$$

Therefore, input matrix

$$B_h^- = \begin{bmatrix} B_g^- \\ DB_g^- - E \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 215 & 138 & 182 & 160 & 157 & 263 & 241 & 234 \\ 226 & 215 & 138 & 182 & 156 & 256 & 245 & 112 \\ 234 & 234 & 223 & 146 & 182 & 273 & 262 & 138 \\ 240 & 240 & 240 & 229 & 140 & 298 & 276 & 151 \end{bmatrix},$$

output matrix

$$B_h^+ = \begin{bmatrix} B_g^+ \\ DB_g^+ - E \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 215 & 25 & 69 & 277 & 157 & 376 & 354 & 117 \\ 226 & 102 & 25 & 293 & 156 & 369 & 358 & 1 \\ 234 & 117 & 106 & 259 & 182 & 390 & 379 & 25 \\ 240 & 120 & 120 & 343 & 140 & 418 & 396 & 37 \end{bmatrix}$$

and initial marking

$$Q_{0_h} = \begin{bmatrix} I_2 \\ D \end{bmatrix} Q_{0_g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 113 & 117 \\ 113 & 111 \\ 117 & 113 \\ 120 & 114 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 & 2 \\ 3 & 3 & 4 & 1 \\ 803 & 577 & 920 & 343 \\ 785 & 559 & 896 & 337 \\ 807 & 573 & 920 & 347 \\ 822 & 582 & 936 & 354 \end{bmatrix}$$

of separate CPN H are obtained, and the check matrix is

$$W = \begin{bmatrix} -D & I_4 \end{bmatrix} = \begin{bmatrix} -113 & -117 & 1 & 0 & 0 & 0 \\ -113 & -111 & 0 & 1 & 0 & 0 \\ -117 & -113 & 0 & 0 & 1 & 0 \\ -120 & -114 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The H can be construct from B_h^-, B_h^+ and Q_{0_h}, which is illustrated in Figure 4.5.

Assume there are place faults, amount transition faults and colour transition faults inside the firing sequence, which is represented as $m_{0_h} \xrightarrow{F_{p1}} m_{1_f}$ [t_1, $F_{a_1}^-$> $m_{2_f} \xrightarrow{F_{p2}} m_{3_f}$ [t_2, $F_{c_1}^+$> $m_{4_f} \xrightarrow{F_{p3}} m'_f$, where

$$F_{p_1} = \begin{bmatrix} -2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_{p_2} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_{p_3} = \begin{bmatrix} 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_{a_1}^{-} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } F_{c_1}^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the matrix representations of m_{1_f}, m_{2_f}, m_{3_f}, m_{4_f} and m'_f are

$$Q_{1_f} = Q_{0_h} + F_{p_1} = \begin{bmatrix} 2 & 2 & 4 & 5 \\ 3 & 3 & 4 & 1 \\ 803 & 577 & 920 & 343 \\ 785 & 559 & 896 & 337 \\ 807 & 577 & 920 & 347 \\ 822 & 582 & 936 & 354 \end{bmatrix},$$

$$Q_{2_f} = Q_{1_f} + B_h^{+} X_1 = \begin{bmatrix} 4 & 3 & 5 & 7 \\ 3 & 3 & 4 & 2 \\ 1018 & 602 & 989 & 620 \\ 1011 & 661 & 921 & 630 \\ 1041 & 694 & 1026 & 606 \\ 1062 & 702 & 1056 & 697 \end{bmatrix},$$

$$Q_{3_f} = Q_{2_f} + F_{p_2} = \begin{bmatrix} 4 & 2 & 5 & 7 \\ 3 & 3 & 4 & 2 \\ 1018 & 602 & 989 & 620 \\ 1011 & 661 & 921 & 630 \\ 1041 & 695 & 1030 & 606 \\ 1062 & 702 & 1056 & 697 \end{bmatrix},$$

$$Q_{4_f} = Q_{3_f} - B_h^- X_2 + B_h^+ (X_2 - F_{c_1}^+) = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 6 & 3 & 1 & 1 \\ 1372 & 715 & 748 & 503 \\ 1369 & 774 & 676 & 519 \\ 1420 & 812 & 768 & 493 \\ 1458 & 822 & 780 & 583 \end{bmatrix} \text{ and}$$

$$Q'_f = Q_{4_f} + F_{p_3} = \begin{bmatrix} 5 & 3 & 2 & 8 \\ 6 & 3 & 1 & 1 \\ 1372 & 715 & 748 & 503 \\ 1369 & 774 & 676 & 519 \\ 1423 & 812 & 768 & 495 \\ 1458 & 822 & 780 & 583 \end{bmatrix} \text{ respectively.}$$

Following steps are identifying and correcting these faults from the marking matrix Q'_f and the check matrix *W*. First,

$$S = WQ'_f = \begin{bmatrix} -113 & -117 & 1 & 0 & 0 & 0 \\ -113 & -111 & 0 & 1 & 0 & 0 \\ -117 & -113 & 0 & 0 & 1 & 0 \\ -120 & -114 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 & 2 & 8 \\ 6 & 3 & 1 & 1 \\ 1372 & 715 & 748 & 503 \\ 1369 & 774 & 676 & 519 \\ 1423 & 812 & 768 & 495 \\ 1458 & 822 & 780 & 583 \end{bmatrix}$$

$$= \begin{bmatrix} 105 & 25 & 405 & -518 \\ 138 & 102 & 339 & -496 \\ 160 & 122 & 421 & -554 \\ 174 & 120 & 426 & -491 \end{bmatrix}$$

is obtained. Second, since

$$S_p = S \bmod 11 = \begin{bmatrix} 105 & 25 & 405 & -518 \\ 138 & 102 & 339 & -496 \\ 160 & 122 & 421 & -554 \\ 174 & 120 & 426 & -491 \end{bmatrix} \bmod 11$$

$$= \begin{bmatrix} 6 & 3 & 9 & 10 \\ 6 & 3 & 9 & 10 \\ 6 & 1 & 3 & 7 \\ 9 & 10 & 8 & 4 \end{bmatrix} \neq \mathbf{0}_{4\times 4},$$

there are place faults in Q'_f. Third, since

$$[D' \quad I_d]F_p \bmod 11 = \begin{bmatrix} 8 & 4 & 1 & 0 & 0 & 0 \\ 8 & 10 & 0 & 1 & 0 & 0 \\ 4 & 8 & 0 & 0 & 1 & 0 \\ 1 & 7 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{p_{11}} & f_{p_{12}} & f_{p_{13}} & f_{p_{14}} \\ f_{p_{21}} & f_{p_{22}} & f_{p_{23}} & f_{p_{24}} \\ f_{p_{31}} & f_{p_{32}} & f_{p_{33}} & f_{p_{34}} \\ f_{p_{41}} & f_{p_{42}} & f_{p_{43}} & f_{p_{44}} \\ f_{p_{51}} & f_{p_{52}} & f_{p_{53}} & f_{p_{54}} \\ f_{p_{61}} & f_{p_{62}} & f_{p_{63}} & f_{p_{64}} \end{bmatrix} \bmod 11$$

$$= S_p = \begin{bmatrix} 6 & 3 & 9 & 10 \\ 6 & 3 & 9 & 10 \\ 6 & 1 & 3 & 7 \\ 9 & 10 & 8 & 4 \end{bmatrix},$$

the following sets of equations are figured out:

$$\begin{cases} (-8f_{p_{11}} - 4f_{p_{21}} + f_{p_{31}}) \bmod 11 = 6 \\ (-8f_{p_{11}} - 10f_{p_{21}} + f_{p_{41}}) \bmod 11 = 6 \\ (-4f_{p_{11}} - 8f_{p_{21}} + f_{p_{51}}) \bmod 11 = 6 \\ (-f_{p_{11}} - 7f_{p_{21}} + f_{p_{61}}) \bmod 11 = 9 \end{cases}, \quad \begin{cases} (-8f_{p_{12}} - 4f_{p_{22}} + f_{p_{32}}) \bmod 11 = 3 \\ (-8f_{p_{12}} - 10f_{p_{22}} + f_{p_{42}}) \bmod 11 = 3 \\ (-4f_{p_{12}} - 8f_{p_{22}} + f_{p_{52}}) \bmod 11 = 1 \\ (-f_{p_{12}} - 7f_{p_{22}} + f_{p_{62}}) \bmod 11 = 10 \end{cases},$$

$$\begin{cases} (-8f_{p_{13}} - 4f_{p_{23}} + f_{p_{33}}) \bmod 11 = 9 \\ (-8f_{p_{13}} - 10f_{p_{23}} + f_{p_{43}}) \bmod 11 = 9 \\ (-4f_{p_{13}} - 8f_{p_{23}} + f_{p_{53}}) \bmod 11 = 3 \\ (-f_{p_{13}} - 7f_{p_{23}} + f_{p_{63}}) \bmod 11 = 8 \end{cases} \text{and} \begin{cases} (-8f_{p_{14}} - 4f_{p_{24}} + f_{p_{34}}) \bmod 11 = 10 \\ (-8f_{p_{14}} - 10f_{p_{24}} + f_{p_{44}}) \bmod 11 = 10 \\ (-4f_{p_{14}} - 8f_{p_{24}} + f_{p_{54}}) \bmod 11 = 7 \\ (-f_{p_{14}} - 7f_{p_{24}} + f_{p_{64}}) \bmod 11 = 4 \end{cases},$$

and there are two more restrictions: (1) each set of equations has at most two nonzero variables and (2) $\forall 1 \le i \le 6,\ 1 \le j \le 4$: $-\frac{11}{2} < f_{p_{ij}} < \frac{11}{2}$. Therefore, $f_{p_{11}}$ = -2, $f_{p_{12}}$ = -1, $f_{p_{13}}$ = -3, $f_{p_{14}}$ = $f_{p_{53}}$ = 4, $f_{p_{51}}$ = 3, $f_{p_{52}}$ = 5 , $f_{p_{54}}$ = 2 and all the other entries in F_p are zero. Fourth, S_t can be obtained from

$$S_t = S - [-D \quad I_4]F_p$$

$$= \begin{bmatrix} 105 & 25 & 405 & -518 \\ 138 & 102 & 339 & -496 \\ 160 & 122 & 421 & -554 \\ 174 & 120 & 426 & -491 \end{bmatrix} - \begin{bmatrix} -113 & -117 & 1 & 0 & 0 & 0 \\ -113 & -111 & 0 & 1 & 0 & 0 \\ -117 & -113 & 0 & 0 & 1 & 0 \\ -120 & -114 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 5 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -121 & -88 & 66 & -66 \\ -88 & -11 & 0 & -44 \\ -77 & 0 & 66 & -88 \\ -66 & 0 & 66 & -11 \end{bmatrix}.$$

By Lemma 4.2, it can be interpreted from F_p that there are two place faults, where one is on p_1, which denotes lack of two red tokens, one green token and three blue tokens, and appearing four extra yellow tokens, and the other place fault is on p_5, which denotes appearing three extra red tokens, five extra green tokens, four extra blue tokens and two extra yellow tokens. Fifth, since

$$EF_t / 11 \bmod 11 = \begin{bmatrix} 1 & 8 & 4 & 6 & 7 & 8 & 10 & 0 \\ 0 & 1 & 8 & 4 & 6 & 7 & 8 & 10 \\ 0 & 0 & 1 & 8 & 4 & 6 & 7 & 8 \\ 0 & 0 & 0 & 1 & 8 & 4 & 6 & 7 \end{bmatrix} \begin{bmatrix} f_{t_{11}} & f_{t_{12}} & f_{t_{13}} & f_{t_{14}} \\ f_{t_{21}} & f_{t_{22}} & f_{t_{23}} & f_{t_{24}} \\ f_{t_{31}} & f_{t_{32}} & f_{t_{33}} & f_{t_{34}} \\ f_{t_{41}} & f_{t_{42}} & f_{t_{43}} & f_{t_{44}} \\ f_{t_{51}} & f_{t_{52}} & f_{t_{53}} & f_{t_{54}} \\ f_{t_{61}} & f_{t_{62}} & f_{t_{63}} & f_{t_{64}} \\ f_{t_{71}} & f_{t_{72}} & f_{t_{73}} & f_{t_{74}} \\ f_{t_{81}} & f_{t_{82}} & f_{t_{83}} & f_{t_{84}} \end{bmatrix} \bmod 11$$

$$= S_t / 11 \bmod 11 = \begin{bmatrix} 0 & 3 & 6 & 5 \\ 3 & 10 & 0 & 7 \\ 4 & 0 & 6 & 3 \\ 5 & 0 & 6 & 10 \end{bmatrix},$$

the following sets of equations are figured out:

$$\begin{cases} (f_{t_{11}} + 8f_{t_{21}} + 4f_{t_{31}} + 6f_{t_{41}} + 7f_{t_{51}} + 8f_{t_{61}} + 10f_{t_{71}}) \bmod 11 = 0 \\ (f_{t_{21}} + 8f_{t_{31}} + 4f_{t_{41}} + 6f_{t_{51}} + 7f_{t_{61}} + 8f_{t_{71}} + 10f_{t_{81}}) \bmod 11 = 3 \\ (f_{t_{31}} + 8f_{t_{41}} + 4f_{t_{51}} + 6f_{t_{61}} + 7f_{t_{71}} + 8f_{t_{81}}) \bmod 11 = 4 \\ (f_{t_{41}} + 8f_{t_{51}} + 4f_{t_{61}} + 6f_{t_{71}} + 7f_{t_{81}}) \bmod 11 = 5 \end{cases},$$

$$\begin{cases} (f_{t_{12}} + 8f_{t_{22}} + 4f_{t_{32}} + 6f_{t_{42}} + 7f_{t_{52}} + 8f_{t_{62}} + 10f_{t_{72}}) \bmod 11 = 3 \\ (f_{t_{22}} + 8f_{t_{32}} + 4f_{t_{42}} + 6f_{t_{52}} + 7f_{t_{62}} + 8f_{t_{72}} + 10f_{t_{82}}) \bmod 11 = 10 \\ (f_{t_{32}} + 8f_{t_{42}} + 4f_{t_{52}} + 6f_{t_{62}} + 7f_{t_{72}} + 8f_{t_{82}}) \bmod 11 = 0 \\ (f_{t_{42}} + 8f_{t_{52}} + 4f_{t_{62}} + 6f_{t_{72}} + 7f_{t_{82}}) \bmod 11 = 0 \end{cases},$$

$$\begin{cases}(f_{t_{13}}+8f_{t_{23}}+4f_{t_{33}}+6f_{t_{43}}+7f_{t_{53}}+8f_{t_{63}}+10f_{t_{73}})\bmod 11=6\\(f_{t_{23}}+8f_{t_{33}}+4f_{t_{43}}+6f_{t_{53}}+7f_{t_{63}}+8f_{t_{73}}+10f_{t_{83}})\bmod 11=0\\(f_{t_{33}}+8f_{t_{43}}+4f_{t_{53}}+6f_{t_{63}}+7f_{t_{73}}+8f_{t_{83}})\bmod 11=6\\(f_{t_{43}}+8f_{t_{53}}+4f_{t_{63}}+6f_{t_{73}}+7f_{t_{83}})\bmod 11=6\end{cases}\text{ and}$$

$$\begin{cases}(f_{t_{14}}+8f_{t_{24}}+4f_{t_{34}}+6f_{t_{44}}+7f_{t_{54}}+8f_{t_{64}}+10f_{t_{74}})\bmod 11=5\\(f_{t_{24}}+8f_{t_{34}}+4f_{t_{44}}+6f_{t_{54}}+7f_{t_{64}}+8f_{t_{74}}+10f_{t_{84}})\bmod 11=7\\(f_{t_{34}}+8f_{t_{44}}+4f_{t_{54}}+6f_{t_{64}}+7f_{t_{74}}+8f_{t_{84}})\bmod 11=3\\(f_{t_{44}}+8f_{t_{54}}+4f_{t_{64}}+6f_{t_{74}}+7f_{t_{84}})\bmod 11=10\end{cases},$$

and there are two more restrictions: (1) each set of equations has at most two nonzero variables and (2) $\forall 1 \le i \le 8$, $1 \le j \le 4$: $-\frac{11}{2} < f_{t_{ij}} < \frac{11}{2}$. Therefore, $f_{t_{11}} = f_{t_{22}} = f_{t_{33}} = f_{t_{44}} = -1$, $f_{t_{71}} = -1$, $f_{t_{73}} = 1$ and all the other entries in F_t are zero. Sixth, from $\forall 1 \le i \le 8$, $1 \le j \le 4$, $j \ne (i-1) \bmod 5 + 1$: $f_{c_{ij}} = f_{t_{ij}}$, $f_{c_{71}} = -1$ and all the other $\forall 1 \le i \le 8$, $1 \le j \le 4$, $j \ne (i-1) \bmod 5 + 1$: $f_{c_{ij}}$ are zero. Seventh, from $\forall 1 \le i \le 8$, $1 \le j \le 4$, $j = (i-1) \bmod 5 + 1$: $f_{c_{ij}} = -\sum_{h=1,h\ne j}^{4} f_{c_{ih}}$, $f_{c_{73}} = 1$ and $f_{c_{11}} = f_{c_{22}} = f_{c_{33}} = f_{c_{44}} = f_{c_{51}} = f_{c_{62}} = f_{c_{84}} = 0$. Since $F_c = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix} \ne \mathbf{0}_{8\times 4}$, there is a colour transition fault in Q'_f. Eighth, Since $F_c = F_c^+ - F_c^-$, $F_c^+ = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}$

and F_c^- = $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are obtained. By Lemma 4.10, F_c^+ and F_c^- can be interpreted as: There is a post-condition colour transition fault on t_2, which denotes c_3 changing into c_1. Ninth, F_a = $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is obtained from $F_a = F_t - F_c$.

Since $F_a \neq \mathbf{0}_{8 \times 4}$, there is a amount transition fault in Q'_f. Tenth, Since F_a = $F_a^+ - F_a^-$, F_a^+ = $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and F_a^- = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By Lemma 4.6, F_a^+ and F_a^- can be interpreted as: There is a pre-condition amount transition fault on t_1, and it occurs one time. The correct marking matrix with respect to Q'_f is

Q'_f - F_p - $B_h^-(F_a^- + F_c^-) + B_h^+(F_a^+ + F_c^+)$

$$= \begin{bmatrix} 5 & 3 & 2 & 8 \\ 6 & 3 & 1 & 1 \\ 1372 & 715 & 748 & 503 \\ 1369 & 774 & 676 & 519 \\ 1423 & 812 & 768 & 495 \\ 1458 & 822 & 780 & 583 \end{bmatrix} - \begin{bmatrix} -2 & -1 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 5 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 3 & 2 \\ 215 & 138 & 182 & 160 & 157 & 263 & 241 & 234 \\ 226 & 215 & 138 & 182 & 156 & 256 & 245 & 112 \\ 234 & 234 & 223 & 146 & 182 & 273 & 262 & 138 \\ 240 & 240 & 240 & 229 & 140 & 298 & 276 & 151 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 \\ 215 & 25 & 69 & 277 & 157 & 376 & 354 & 117 \\ 226 & 102 & 25 & 293 & 156 & 369 & 358 & 1 \\ 234 & 117 & 106 & 259 & 182 & 390 & 379 & 25 \\ 240 & 120 & 120 & 343 & 140 & 418 & 396 & 37 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 4 & 2 \\ 3 & 3 & 4 & 1 \\ 803 & 577 & 920 & 343 \\ 785 & 559 & 896 & 337 \\ 807 & 573 & 920 & 347 \\ 822 & 582 & 936 & 354 \end{bmatrix},$$

which is equal to the matrix representation of m'_h in m_{0_h} $[t_1>$ m_{1_h} $[t_2>$ m'_h.

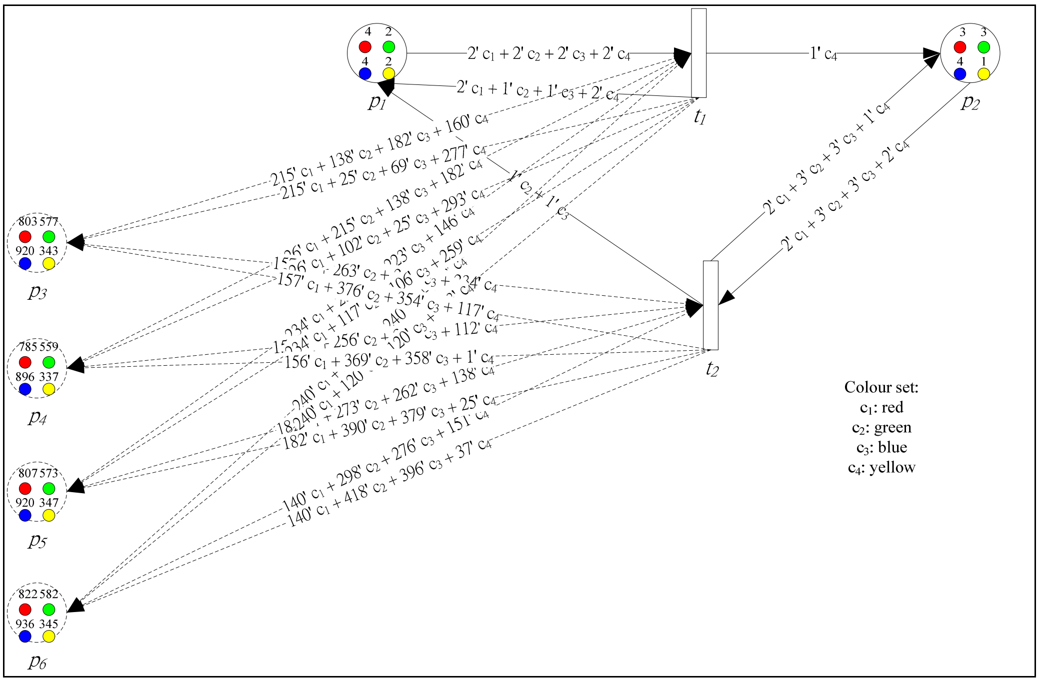

Figure 4.5 A CPN with Additive Faults detection and correction capabilities.

Chapter 5. Conclusion and Future Works

This thesis proposed a methodology to determine whether a marking in a coloured Petri net is a faulty marking which can be mapped to a faulty state in a system modeled by a coloured Petri net. The main idea of the methodology is applying the methods of error control coding on coloured Petri nets. In [3], the authors present the methods detecting the faults on Petri nets only. There are more issues to be studied on coloured Petri nets. Since a Petri net can be deem as a coloured Petri net with only single colour, the method for this special case would be the same as [3] in this thesis. Thus, the methods presented in this thesis are more general than those in [3]. If the applied error control coding is Reed-Solomon code, the methodology in this thesis can simultaneously detect and correct k place faults, x amount transition faults and k - x colour transition faults after adding $2k$ places, where $0 \le x \le k$. There is a corresponding code correction algorithm in Reed-Solomon code, which is *Berlekamp-Massey* algorithm [16]. By appling Berlekamp-Massey algorithm on the syndrome of a faulty marking, the equation sets obtained from the syndrome can be solved out in time complexity $O(k\gamma(\alpha+\beta))$, and hence the marking can be corrected in time complexity $O(k\gamma(\alpha+\beta))$, where α, β and γ are the number of transitions, places and colour types in a coloured Petri net, respectively.

There are two further research topics which can be extended from this thesis. First, from the marking, input and output matrices of the CPN with fault detection and correction capability, it can be seen that the values in these matrices are large. The reason is that the Reed-Solomon code has only minimized the length of a code but hasn’t minimized the value of a code word. Thus, one of the future works is to come out the encoding matrices which can also minimize the values in these matrices. Second, there are several kinds of high level CPNs extended from basic CPN discussed in this thesis. Thus, the other future work is to extend the methodology in this thesis to these high level CPNs.

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