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# Approximation by Invertible and Noninvertible Operators

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## 1. INTRODUCTION

Let $T$ be a bounded linear operator and $\mathscr{A}$ a class of operators on a fixed complex separable Hilbert space $H$. The problem of operator approximation is to determine how closely $T$ can be approximated in the norm by operators in $\mathscr{A}$, or more precisely, to determine the distance from $T$ to $\mathscr{A}$: $\inf\{\|T-S\|: S \in \mathscr{A}\}$. Interest in problems of this type was aroused by P. R. Halmos (cf. [5, 6]). In the past decade, intensive investigations in this area have led to some deep results in operator theory (cf. [8, 1]).

The purpose of the present paper is more modest. We consider approximating $T$ by operators with unequal kernel dimension, and we determine $\inf\{\|T-S\|: \dim\ker S \neq \dim\ker T\}$. In one direction ("$\neq$" replaced by "$<$"), this problem is related to that of approximating $T$ by invertible operators; in the other direction ("$\neq$" replaced by "$>$"), it is related to the approximation of $T$ by noninvertible operators. These latter two approximations have been considered before: their distances were determined by Bouldin [2] and Franck [4], respectively. Thus by exploiting these relationships and using the known results, we are able to determine $\inf\{\|T-S\|: \dim\ker S \neq \dim\ker T\}$ completely.

In Section 2 below, we give the preliminary preparations and in Sections 3 and 4 we consider the approximations by invertible and non-invertible operators, respectively.

## 2. PRELIMINARIES

For an operator $T$ on $H$, let $\|T\|$ (resp. $\|T\|_e$) denote its operator norm (resp. essential norm) and let $\sigma(T)$ (resp. $\sigma_e(T)$) be its spectrum (resp.

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essential spectrum). Let $m(T)=\inf\{\lambda: \lambda\in\sigma((T^*T)^{1/2})\}$ (resp. $m_e(T)=\inf\{\lambda: \lambda\in\sigma_e((T^*T)^{1/2})\}$) be its *minimum modulus* (resp. *essential minimum modulus*). The following proposition contains their basic properties and can be found in [2].

PROPOSITION 2.1. (1) $m(T)=\inf\{\|Tx\|: \|x\|=1\}$.

(2) $m_e(T)=\inf\{\lambda: \dim E([\lambda, \lambda+\varepsilon))H=\infty\,\forall\varepsilon>0\}$, *where* $E(\cdot)$ *is the spectral measure of* $(T^*T)^{1/2}$.

(3) $m(T)>0$ (*resp.* $m_e(T)>0$) *if and only if* $T$ *is left invertible* (*resp. left Fredholm*).

(4) *If* $L$ *is a left inverse of* $T$ (*resp.* $L$ *is such that* $LT-1$ *is compact*), *then* $m(T)=1/\|L\|$ (*resp.* $m_e(T)=1/\|L\|_e$).

(5) $T$ *is invertible* (*resp. Fredholm*) *if and only if* $m(T), m(T^*)>0$ (*resp.* $m_e(T), m_e(T^*)>0$). *In this case,* $m(T)=m(T^*)$ (*resp.* $m_e(T)=m_e(T^*)$).

Recall that for an operator $T$, $\operatorname{ind} T=\dim\ker T-\dim\ker T^*$ if at least one of these numbers is finite and $\operatorname{ind} T=0$ otherwise. The next proposition will be used in establishing the lower bounds for the distances which we are interested in later.

PROPOSITION 2.2. *If* $T$ *and* $S$ *are operators on* $H$ *and* $\|T-S\|<m(T)$ (*resp.* $\|T-S\|_e<m_e(T)$), *then*

(1) $T$ *and* $S$ *are both left invertible* (*resp. left Fredholm*),

(2) $T$ *is invertible* (*resp. Fredholm*) *if and only if* $S$ *is, and*

(3) $\operatorname{ind} T=\operatorname{ind} S$.

*Proof.* We only prove $\|T-S\|<m(T)$. The proof of $\|T-S\|_e<m_e(T)$ appeared in [10, Theorem 1.1].

Since $m(T)>\|T-S\|\geqslant 0$, $T$ is left invertible by Proposition 2.1. Let $L$ be a left inverse of $T$. Then $\|1-LS\|\leqslant\|L\|\,\|T-S\|<1$ implies that $LS$ is invertible whence $S$ is left invertible. This proves (1). (2) follows immediately by noting that $T$ is invertible if and only if $L$ is. Since $\|T-S\|_e\leqslant\|T-S\|<m(T)\leqslant m_e(T)$, (3) follows as in [10, Theorem 1.1].

Another related parameter of an operator $T$ is the *reduced minimum modulus*: $\gamma(T)=\inf\{\|Tx\|: \|x\|=1 \text{ and } x\perp\ker T\}$. The proof of the next proposition is in [3, Proposition XI.3.16].

PROPOSITION 2.3. (1) $\gamma(T)>0$ *if and only if* ran $T$ *is closed.*

(2) $\gamma(T)=\gamma(T^*)$.

The next result is also used in establishing the lower bounds for the distances. Its proof can be found in [3, Propositions XI.3.20 and XI.3.24].

PROPOSITION 2.4. *If $T$ and $S$ are operators on $H$ and* $\|T-S\| < \gamma(T)$, *then* $\dim\ker S \leqslant \dim\ker T$ *and* $\dim\ker S^* \leqslant \dim\ker T^*$. *If, in addition, $T$ or $T^*$ is injective, then* $\dim\ker S = \dim\ker T$ *and* $\dim\ker S^* = \dim\ker T^*$.

## 3. INVERTIBILITY

What is the distance from an arbitrary operator $T$ to the class of invertible ones? It has been shown by Bouldin [2] that the distance is expressible in terms of $m_e(T)$ and $m_e(T^*)$. An elaboration of his arguments can yield the following sharpening form. From now on, $n$ will denote an integer, positive, negative, or zero, or $\pm\infty$.

THEOREM 3.1. *For any operator $T$ on $H$ and* $-\infty \leqslant n < 0$,

$$\inf\{\|T-S\| : S \text{ left invertible and } \operatorname{ind} S = n\}$$
$$= \begin{cases} \max\{m_e(T), m_e(T^*)\} & \text{if } \operatorname{ind} T \neq n \\ 0 & \text{otherwise.} \end{cases}$$

*Corresponding assertions hold for* $0 < n \leqslant \infty$ *when "$S$ left invertible" is replaced by "$S$ right invertible" and for $n = 0$ when it is replaced by "$S$ invertible".*

This theorem was essentially proved in [1, Theorem 12.2]. When $T$ is not semi-Fredholm, the proof given there depends on the Apostol–Morrel simple models (cf. [8, Theorem 6.1]). In the following, we give a simpler proof for this case which is more in line with Bouldin's arguments.

*Proof of Theorem* 3.1. Let $\alpha(T)$ be the distance above. We first show that $\alpha(T) = 0$ for non-semi-Fredholm $T$. Let $T = VR$ be the polar decomposition of $T$, where $V$ is the partial isometry with $\ker V = \ker T$ and $R = (T^*T)^{1/2}$ (cf. [7, Problem 134]), and let $E(\cdot)$ denote the spectral measure of $R$. For any $m \geqslant 1$, let $H_m = E[0, 1/m)H$. Then $\dim H_m = \infty$ by Proposition 2.1(2). Next we show that $\dim(TH_m^\perp)^\perp = \infty$.

Indeed, since $T$ is not semi-Fredholm, either $\dim\ker T = \dim\ker T^* = \infty$ or $\operatorname{ran} T$ is not closed. In the former case, $TH_m^\perp \subseteq \operatorname{ran} T$ yields that $\ker T^* = (\operatorname{ran} T)^\perp \subseteq (TH_m^\perp)^\perp$. Therefore $\dim(TH_m^\perp)^\perp = \infty$ as asserted. Now assume that $\operatorname{ran} T$ is not closed. Let $K_m = E(0, 1/m)H$. If $\dim K_m < \infty$, then $R \mid K_m$, being injective, is surjective. Hence $RK_m = K_m \subseteq (\ker R)^\perp = (\ker T)^\perp$. Since $V$ is isometric on $(\ker T)^\perp$, we infer that $T \mid K_m$ is invertible from $K_m$ onto $TK_m$. It follows that $\dim TK_m = \dim K_m < \infty$.

Thus ran $T = TK_m + TH_m^{\perp}$, being the sum of a finite-dimensional subspace and a closed one, is closed, contradicting our assumption. Hence we must have $\dim K_m = \infty$ and so $\dim TK_m = \infty$. It follows from $TK_m \subseteq (TH_m^{\perp})^{\perp}$ that $\dim (TH_m^{\perp})^{\perp} = \infty$.

Now let $S_m = (1/m) W \oplus (T \mid H_m^{\perp})$, where $W: H_m \to (TH_m^{\perp})^{\perp}$ is a unilateral shift with index $-n$ and $T \mid H_m^{\perp}: H_m^{\perp} \to TH_m^{\perp}$. Since $T \mid H_m^{\perp}$ is invertible, $S_m$ is left invertible with index $n$. Moreover, $\|T - S_m\| = \|T \mid H_m - (1/m) W\| \leqslant \|T \mid H_m\| + \|(1/m) W\| \leqslant 2/m$. This shows that $\alpha(T) = 0$ as asserted.

If $T$ is left Fredholm and ind $T \neq n$, then consider $H_m = E[0, m_e(T) + 1/m) H$ instead and show that $\dim H_m = \dim (TH_m^{\perp})^{\perp} = \infty$ whence $\alpha(T) \leqslant m_e(T) = \max\{m_e(T), m_e(T^*)\}$ as before. The reverse inequality follows by Proposition 2.2(3). The omitted details resemble the arguments in the preceding paragraphs (also cf. [1, pp. 145–146]). If ind $T = n < 0$, then $T$ can be decomposed as $VR$, where $V$ is an isometry with ind $V =$ ind $T$ and $R = (T^*T)^{1/2}$ as before. Let $E(\cdot)$ be the spectral measure of $R$. For any $m \geqslant 1$, let $H_m = E[0, 1/m) H$, $Q_m = (1/m) I \oplus (R \mid H_m^{\perp})$, where $I$ denotes the identity operator on $H_m$, and $S_m = VQ_m$. The invertibility of $Q_m$ implies that $S_m$ is left invertible and that ind $S_m =$ ind $T$. Moreover, $\|T - S_m\| = \|VR - VQ_m\| \leqslant \|R - Q_m\| \leqslant \|R \mid H_m\| + 1/m \leqslant 2/m$. This proves that $\alpha(T) = 0$. Other cases can be handled in a similar fashion.

The following corollaries follow easily from the preceding theorem (or its proof) and Proposition 2.2.

COROLLARY 3.2. *For any operator $T$ and* $-\infty \leqslant n \leqslant \infty$,

$$\inf\{\|T - S\| : \text{ind } S = n\} = \begin{cases} \max\{m_e(T), m_e(T^*)\} & \textit{if } \text{ind } T \neq n \\ 0 & \textit{otherwise.} \end{cases}$$

COROLLARY 3.3. $\text{Inf}\{\|T - S\| : \text{ind } S \neq \text{ind } T\} = \max\{m_e(T), m_e(T^*)\}$.

COROLLARY 3.4.

$$\text{Inf}\{\|T - S\| : S \textit{ left invertible}\} = \begin{cases} m_e(T^*) & \textit{if } \text{ind } T > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

COROLLARY 3.5.

$$\text{Inf}\{\|T - S\| : S \textit{ left invertible but not invertible}\} = \begin{cases} m_e(T^*) & \textit{if } \text{ind } T \geqslant 0 \\ 0 & \textit{otherwise.} \end{cases}$$

From the last two corollaries and Proposition 2.2, we can further deduce the following.

COROLLARY 3.6.

$$\operatorname{Inf}\{\|T-S\|: S \text{ injective}\} = \begin{cases} m_e(T^*) & \text{if } \operatorname{ind} T > 0 \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY 3.7.

$$\operatorname{Inf}\{\|T-S\|: S \text{ surjective}\} = \begin{cases} m_e(T) & \text{if } \operatorname{ind} T < 0 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Since an operator is surjective if and only if it is right invertible, the assertion follows by replacing $T$ in Corollary 3.4 by $T^*$.

COROLLARY 3.8. $\operatorname{Inf}\{\|T-S\|: S \text{ one-sided invertible}\} = 0$.

COROLLARY 3.9.

$$\operatorname{Inf}\{T-S\|: S \text{ one-sided invertible but not both}\}$$
$$= \begin{cases} m_e(T) & \text{if } \operatorname{ind} T = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that Corollary 3.8 appears in [7, Problem 140]. In preparation for determining the distance $\inf\{\|T-S\|: \dim\ker S \neq \dim\ker T\}$ in Section 4, we first consider the distance from $T$ to those operators with smaller kernel dimension.

PROPOSITION 3.10. *Let T be a noninjective operator. Then*

$$\inf\{\|T-S\|: \dim\ker S < \dim\ker T\}$$
$$= \begin{cases} m_e(T^*) & \text{if } \dim\ker T < \infty \quad \text{and} \\ & \quad\ \ \dim\ker T^* = 0 \quad \text{or} \\ & \quad\ \ \dim\ker T = \infty \quad \text{and} \\ & \quad\ \ \dim\ker T^* < \infty \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let $\beta(T)$ be the distance above. If $\dim\ker T < \infty$ and $\dim\ker T^* = 0$ or $\dim\ker T = \infty$ and $\dim\ker T^* < \infty$, then $\operatorname{ind} T = \dim\ker T > 0$. Hence $\beta(T) \leqslant \inf\{T-S\|: S \text{ injective}\} = m_e(T^*)$ by Corollary 3.6. For the reverse inequality, let $S$ be any operator with dim ker $S < \dim \ker T$. If $\|T-S\| < m_e(T^*)$, then, since $\|T-S\|_e \leqslant \|T-S\|$, we infer from Proposition 2.2 that ind $T = \operatorname{ind} S$. Thus dim ker $T =$ dim ker $S -$ dim ker $S^* \leqslant$ dim ker $S$, a contradiction. We conclude that in this case $\beta(T) \geqslant m_e(T^*)$. Therefore $\beta(T) = m_e(T^*)$.

For the remaining case, we have ind $T <$ dim ker $T$. If ind $T \leqslant 0$, then, again, $\beta(T) \leqslant \inf\{\|T-S\|: S \text{ injective}\} = 0$ by Corollary 3.6. If ind $T > 0$, Theorem 3.1 implies that $\inf\{\|T-S\|: S$ right invertible and ind $S =$ ind $T\} = 0$. For any right invertible $S$ with ind $S =$ ind $T$, we have dim ker $S =$ ind $S =$ ind $T <$ dim ker $T$. It follows that $\beta(T) = 0$ in this case.

As a side result, we also have the following.

PROPOSITION 3.11. *Let $T$ be an operator such that both $T$ and $T^*$ are noninjective. Then*

$$\inf\{\|T-S\|: \dim\ker S < \dim\ker T \textit{ and } \dim\ker S^* < \dim\ker T^*\}$$
$$= \begin{cases} \max\{m_e(T), m_e(T^*)\} & \textit{if } \operatorname{ind} T = \pm\infty \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* Let $\delta(T)$ be the distance above. If ind $T = \pm\infty$, then $\delta(T) \leqslant \inf\{\|T-S\|: S \text{ invertible}\} = \max\{m_e(T), m_e(T^*)\}$ by Theorem 3.1. On the other hand, Proposition 3.10 implies that $\delta(T) \geqslant \beta(T) = m_e(T^*) = \max\{m_e(T), m_e(T^*)\}$ if ind $T = \infty$. Thus $\delta(T) = \max\{m_e(T), m_e(T^*)\}$ for ind $T = \infty$. If ind $T = -\infty$, the same conclusion follows by considering $T^*$ instead.

For ind $T$ finite, the assertion is a consequence of Theorem 3.1. If ind $T = 0$, then $\delta(T) \leqslant \inf\{\|T-S\|: S \text{ invertible}\} = 0$. If ind $T < 0$, then $\delta(T) \leqslant \inf\{\|T-S\|: S$ left invertible and ind $S =$ ind $T\} = 0$ since for such an $S$, dim ker $S = 0 <$ dim ker $T$ and dim ker $S^* = -$ind $S = -$ind $T <$ dim ker $T^*$. Similar arguments apply in the case ind $T > 0$.

Part of the approximation in the preceding proposition appears in [3, p. 374, Ex. 1]. Note that there is an error there: $\delta(T)$ may be strictly greater than 0 for semi-Fredholm $T$. As an example, let $T = T_1 \oplus 0$, where $T_1$ is a unilateral shift with dim ker $T_1^* = \infty$ and 0 denotes the zero operator on a finite-dimensional space. Then $T$ is left Fredholm but $\delta(T) = m_e(T) > 0$.

## 4. NONINVERTIBILITY

We start by first determining the distance from an operator to the class of non-left-invertible operators using the polar decomposition of operators. The next proposition appeared in [9, Proposition 2]; its proof made use of the Hahn–Banach theorem and is thus applicable to operators on any Banach space. Our approach, though valid only in the context of Hilbert space, is useful in other approximation problems later.

PROPOSITION 4.1. *For any operator* $T$, $\inf\{\|T-S\|: S$ *non-left-invertible*$\} = m(T)$.

*Proof.* The distance above is not less than $m(T)$ by Proposition 2.2(1). For the other direction, we may assume that $T$ is left invertible. Let $T = VR$ be the polar decomposition of $T$, where $V$ is a partial isometry with $\ker V = \ker R$ and $R = (T^*T)^{1/2}$ (cf. [7, Problem 134]), and let $S = T - m(T)V$. If $S$ is left invertible, then, since $S = V(R - m(T))$, we deduce that $R - m(T)$ is left invertible. For Hermitian operators, this is the same as invertibility, which contradicts the fact that $m(T) \in \sigma(R)$. Thus $S$ is non-left-invertible and $\|T-S\| = \|m(T)V\| = m(T)$. Therefore the distance in question is equal to $m(T)$ as asserted.

COROLLARY 4.2. *For any invertible operator* $T$, $\inf\{\|T-S\|: S$ *non-invertible*$\} = m(T)$.

*Proof.* This follows easily from the preceding proposition and Proposition 2.2(2).

Corollary 4.2 is first proved by Franck [4] using the Hahn–Banach theorem. Next we consider approximation by operators which are neither left invertible nor right invertible.

THEOREM 4.3. *For any operator* $T$, $\inf\{\|T-S\|: S$ *non-left-invertible and non-right-invertible*$\} = \max\{m(T), m(T^*)\}$.

*Proof.* Let $\mu(T)$ be the distance above. From Proposition 4.1, we deduce easily that $\mu(T) \geq \max\{m(T), m(T^*)\}$. For the reverse inequality, assume that $T$ is left invertible. Let $T = VR$ and $S = T - m(T)V$ as in the proof of Proposition 4.1. As before, $S$ is not left invertible and $\|T-S\| = m(T)$. Now we show that $S$ is not right invertible. Indeed, if it is, let $W$, say, be a right inverse of $S$. Since $SW = V(R - m(T))W = 1$, $V$ is right invertible. However, the left invertibility of $T$ implies that $V$ is an isometry. Thus, it is in fact a unitary operator. From $V(R - m(T))W = 1$, we infer that $(R - m(T))WV = 1$, that is, $R - m(T)$ is right invertible. This leads to the invertibility of $R - m(T)$, a contradiction. We conclude that $\mu(T) = m(T) = \max\{m(T), m(T^*)\}$ as asserted. If $T$ is right invertible, the assertion follows by symmetry.

Analogous assertions can be made with non-Fredholm operators replacing noninvertible operators in the preceding propositions. The next result appears in [11].

PROPOSITION 4.4. *For any operator $T$, $\inf\{\|T-S\|: S$ non-left-Fredholm$\} = m_e(T)$.*

COROLLARY 4.5. *For any Fredholm operator $T$, $\inf\{\|T-S\|: S$ non-Fredholm$\} = m_e(T)$.*

THEOREM 4.6. *For any operator $T$, $\inf\{\|T-S\|: S$ non-left-Fredholm and non-right-Fredholm$\} = \max\{m_e(T), m_e(T^*)\}$.*

*Proof.* Let $v(T)$ denote the distance above. We only prove that $v(T) = m_e(T)$ for left Fredholm $T$. The arguments are parallel to those in the proof of Theorem 4.3. Let $T = VR$ be as before and let $E(\cdot)$ denote the spectral measure of $R$. For any $n \geqslant 1$, let $P_n = E[m_e(T), m_e(T) + 1/n)$, $H_n = P_n H$, and $S_n = T(1 - P_n)$. Since $S_n H_n = T(1 - P_n) P_n H = \{0\}$ and $\dim H_n = \infty$ by Proposition 2.1(2), we have $\dim \ker S_n = \infty$. Thus $S_n$ is not left Fredholm. Next we show that $S_n$ is not right Fredholm. If it is, let $W$ be such that $S_n W - 1$ is compact. It is easily seen that $V$ is also right Fredholm. On the other hand, since $T$ is left Fredholm, $\dim \ker V = \dim \ker T < \infty$. Thus $V$ is Fredholm, and therefore $R(1 - P_n) WV - 1$, together with $VR(1 - P_n) W - 1$, is compact. This shows that $R(1 - P_n)$ is right Fredholm. For a normal operator, this is equivalent to $R(1 - P_n)$ being Fredholm. However, $H_n \subseteq \ker R(1 - P_n)$ implies that $\dim \ker R(1 - P_n) = \infty$, a contradiction. We conclude that $S_n$ is not right Fredholm. Moreover, $\|T - S_n\| = \|TP_n\| \leqslant \|RP_n\| \leqslant m_e(T) + 1/n$. Therefore, $v(T) = m_e(T)$ as asserted.

The last problem we address is the determination of the distance from an operator to the class of those with unequal kernel dimension. We start with the following proposition.

PROPOSITION 4.7. *Let $T$ be an operator on $H$ with* $\dim \ker T < \infty$. *Then* $\inf\{\|T - S\|: \dim \ker S > \dim \ker T\} = \gamma(T)$.

*Proof.* The distance above is not less than $\gamma(T)$ by Proposition 2.4. For the other direction, let $\{x_n\}$ be a sequence of unit vectors in $(\ker T)^\perp$ such that $\|Tx_n\| \to \gamma(T)$ as $n \to \infty$. Let $P_n$ denote the orthogonal projection from $H$ onto the one-dimensional subspace $H_n$ generated by $x_n$, and let $S_n = T(1 - P_n)$. It is easily seen that $\ker S_n = \ker T \oplus H_n$, and so $\dim \ker S_n = \dim \ker T + 1$. For any vector $y$, if $P_n y = a x_n$ where $a$ is some scalar, then

$$\|TP_n y\| = |a| \, \|Tx_n\| = \|a x_n\| \, \|Tx_n\| = \|P_n y\| \, \|Tx_n\| \leqslant \|y\| \, \|Tx_n\|.$$

This shows that $\|T - S_n\| = \|TP_n\| = \|Tx_n\| \to \gamma(T)$ as $n \to \infty$ whence our assertation.

In the proof above, we actually showed that $\inf\{\|T-S\|: \dim\ker S=\dim\ker T+1\}=\gamma(T)$ if $\dim\ker T<\infty$. Hence, in particular, if $T$ is injective, then $\inf\{\|T-S\|: \dim\ker S=1\}=m(T)$. This latter result generalizes Proposition 4.1.

THEOREM 4.8. *For any operator $T$,*

$$\inf\{\|T-S\|: \dim\ker S\neq\dim\ker T\} = \begin{cases} \gamma(T) & \text{if } \dim\ker T=0 \quad \text{or} \\ & \quad \dim\ker T<\infty \quad \text{and} \\ & \quad \dim\ker T^*=0 \\ m_e(T^*) & \text{if } \dim\ker T=\infty \quad \text{and} \\ & \quad \dim\ker T^*<\infty \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let $\omega(T)$ be the distance above. If $\dim\ker T=0$ or $\dim\ker T<\infty$ and $\dim\ker T^*=0$, then $\omega(T)\leqslant\gamma(T)$ by Proposition 4.7. The reverse inequality follows from Proposition 2.4. Hence $\omega(T)=\gamma(T)$ in this case.

If $\dim\ker T=\infty$ and $\dim\ker T^*<\infty$, then $\omega(T)\leqslant m_e(T^*)$ by Proposition 3.10. On the other hand, if $\|T-S\|<m_e(T^*)$ for some $S$ with $\dim\ker S\neq\dim\ker T$, then $T^*$ and $S^*$ are both left Fredholm and $\operatorname{ind} T^*=\operatorname{ind} S^*$ by Proposition 2.2. It follows that $\dim\ker T=\operatorname{ind} T=\operatorname{ind} S=\dim\ker S$, a contradiction. Thus $\omega(T)=m_e(T^*)$. The remaining case follows by Proposition 3.10.

We conclude this paper with a corollary. It is of a similar nature as [7, Problem 130]. The verification of this corollary, and that of the next lemma, are left to the reader.

LEMMA 4.9. *If $T$ is a partial isometry, then*

$$m(T)=\begin{cases}1 & \text{if } \dim\ker T=0\\ 0 & \text{otherwise,}\end{cases} \qquad m_e(T)=\begin{cases}1 & \text{if } \dim\ker T<\infty\\ 0 & \text{otherwise,}\end{cases}$$

*and $\gamma(T)=1$.*

COROLLARY 4.10. *Let $T$ be a partial isometry with* $\operatorname{ind} T=\infty$, *an isometry, or a coisometry. If* $\|T-S\|<1$, *then* $\dim\ker T=\dim\ker S$.

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*Note added in proof.* Recently, R. Bouldin [14] determined the distance $\inf\{\|T-S\|: \dim\ker S = n\}$ for any nonnegative integer $n$. This is more refined than the distances considered here.

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