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FACTORIZATION OF MATRICES INTO PARTIAL ISOMETRIES

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ABSTRACT. In this paper, we characterize complex square matrices which are expressible as products of partial isometries and orthogonal projections. More precisely, we show that a matrix T is the product of k partial isometries ($k \geq 1$) if and only if T is a contraction ($\|T\| \leq 1$) and rank $(1 - T^*T) \leq k \cdot \operatorname{nullity} T$. It follows, as a corollary, that any $n \times n$ singular contraction is the product of n partial isometries and n is the smallest such number. On the other hand, T is the product of finitely many orthogonal projections if and only if T is unitarily equivalent to $1 \oplus S$, where S is a singular strict contraction ($\|S\| < 1$). As contrasted to the previous case, the number of factors can be arbitrarily large.

1. INTRODUCTION

An $n \times n$ complex matrix T is a *partial isometry* if $\|Tx\| = \|x\|$ for any vector x in $\ker^{\perp} T$, the orthogonal complement of the kernel of T in $\mathbf{C}^n$, where $\|x\|$ denotes the 2-norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{1/2}$ of $x = [x_1 \cdots x_n]^t$ in $\mathbf{C}^n$. Examples of partial isometries are (*orthogonal*) *projections* ($T^2 = T = T^*$) and *unitary matrices* ($T^* = T^{-1}$). In this paper, we will characterize matrices which are expressible as products of partial isometries and projections.

As we will show below, the situations for these two types of products are quite different. For the former, we obtain that T is the product of k partial isometries ($k \geq 1$) if and only if T is a contraction ($\|T\| \leq 1$) and rank $(1 - T^*T) \leq k \cdot \operatorname{nullity} T$ (Theorem 2.2). This latter condition links our problem to that of factorization into idempotent matrices (cf. [1]). In particular, it follows that any $n \times n$ singular contraction is the product of n partial isometries and n is the smallest such number (Corollary 2.4). (Recall that a matrix is *singular* if it does not have an inverse.)

Products of partial isometries have also been considered before by Erdelyi [3]. However his concern is different from ours. He was interested in conditions under which a product of partial isometries is itself a partial isometry.

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As for products of projections, very few seem to be known in the literature. One exception is the characterization of products of two projections due to Crimmins (cf. [5, Theorem 8]) which is true even for bounded linear operators on infinite-dimensional Hilbert spaces: T is such a product if and only if $TT^*T = T^2$. In this paper, we characterize products of finitely many projections. More precisely, we will show that a matrix T is such a product if and only if T is unitarily equivalent to $1 \oplus S$, where S is a singular strict contraction ($\|S\| < 1$) (Theorem 3.1). Note the similarity of this result to that for partial isometries: T is the product of finitely many partial isometries if and only if T is unitarily equivalent to $U \oplus S$, where U is unitary and S is a singular contraction (Corollary 2.3). However, there is one big difference between these two types of products: unlike the partial isometry products, the number of projections in a product can be arbitrarily large.

2. Partial isometry

We start with the following simple observation.

Lemma 2.1. *If T is a partial isometry and U is unitary, then UT and TU are also partial isometries.*

Proof. This follows from the fact that a matrix S is a partial isometry if and only if $SS^*S = S$ (cf. [4, Corollary 3 to Problem 127]). It is also a consequence of [3, Theorem 1]. □

The preceding lemma reduces, via the singular-value decomposition, the partial isometry factorization of arbitrary matrices to that of positive semidefinite ones. In the following, nullity T denotes the dimension of $\ker T$. A matrix T is *idempotent* if $T^2 = T$.

Theorem 2.2. *Let T be an $n \times n$ matrix and $k \geq 1$. Then the following statements are equivalent*:

(1) *T is the product of k partial isometries*;

(2) $\|T\| \leq 1$ *and* $\operatorname{rank}(1 - T^*T) \leq k \cdot \operatorname{nullity} T$;

(3) $\|T\| \leq 1$ *and* $(T^*T)^{1/2}$ *is the product of k idempotent matrices.*

Proof. (1) $\Rightarrow$ (2). Let $T = A_1 A_2 \cdots A_k$ be the product of k partial isometries. Since the norm of any nonzero partial isometry is one, we have $\|T\| \leq 1$. Next let $K = \{x \in \mathbf{C}^n : T^*Tx = x\}$. We claim that

$$K = \ker^{\perp} A_k \cap A_k^{-1}(\ker^{\perp} A_{k-1}) \cap \cdots \cap A_k^{-1}(A_{k-1}^{-1}(\cdots (A_2^{-1}(\ker^{\perp} A_1))\ldots)). \tag{$*$}$$

Indeed, if $x \in K$, then $T^*Tx = x$ whence $\|Tx\|^2 = (T^*Tx, x) = (x, x) = \|x\|^2$. For each $j = 1, 2, \ldots, k$, let $A_{j+1} \cdots A_k x = y_1 + y_2$, where $y_1 \in \ker A_j$

and $y_2 \in \ker^{\perp} A_j$. Since

$$\begin{aligned}\|y_1\|^2 + \|y_2\|^2 &= \|A_{j+1} \cdots A_k x\|^2 \leq \|x\|^2 \\ &= \|Tx\|^2 = \|A_1 A_2 \cdots A_k x\|^2 \\ &= \|A_1 \cdots A_j y_2\|^2 \leq \|y_2\|^2,\end{aligned}$$

we infer that $y_1 = 0$ or, equivalently, $x \in A_k^{-1}(A_{k-1}^{-1}(\cdots(A_{j+1}^{-1}(\ker^{\perp} A_j))\cdots))$. This shows that x belongs to the right hand side of $(*)$. Conversely, if x belongs to this subspace, then $A_{j+1} \cdots A_k x \in \ker^{\perp} A_j$ for each $j = 1, 2, \ldots, k$. Hence

$$\|A_j A_{j+1} \cdots A_k x\| = \|A_{j+1} \cdots A_k x\| \quad \text{for each } j.$$

Therefore, $\|Tx\| = \|x\|$. This implies that $(T^*Tx, x) = (x, x)$ or $((1 - T^*T)x, x) = 0$. Since $0 \leq T^*T \leq 1$, we may consider the positive square root of $1 - T^*T$ and obtain

$$\|(1 - T^*T)^{1/2} x\|^2 = ((1 - T^*T)x, x) = 0.$$

Thus $(1 - T^*T)^{1/2} x = 0$ which implies that $(1 - T^*T)x = 0$ or $T^*Tx = x$. This proves $(*)$.

To conclude the proof of $(1) \Rightarrow (2)$, let $m = \operatorname{nullity} T$. Then $\operatorname{rank} A_j \geq n - m$ for each j. It is easily seen that

$$\begin{aligned}A_k K = \operatorname{ran} A_k \cap \ker^{\perp} A_{k-1} \cap A_{k-1}^{-1}(\ker^{\perp} A_{k-2}) \\ \cap \cdots \cap A_{k-1}^{-1}(\cdots(A_2^{-1}(\ker^{\perp} A_1))\cdots).\end{aligned}$$

Hence

$$\begin{aligned}\dim K &\geq \dim A_k K \\ &\geq \operatorname{rank} A_k + \dim(\ker^{\perp} A_{k-1} \cap A_{k-1}^{-1}(\ker^{\perp} A_{k-2}) \cap \cdots \\ &\qquad \cap A_{k-1}^{-1}(\cdots(A_2^{-1}(\ker^{\perp} A_1))\cdots)) - n \\ &\geq \cdots \\ &\geq \sum_{j=1}^{k} \operatorname{rank} A_j - (k-1)n \\ &\geq k(n-m) - (k-1)n = n - km.\end{aligned}$$

On the other hand, we also have

$$\dim K = \operatorname{nullity}(1 - T^*T) = n - \operatorname{rank}(1 - T^*T).$$

Hence $\operatorname{rank}(1 - T^*T) \leq km$ as asserted.

$(2) \Rightarrow (1)$. Let $T = UPV$ be the singular-value decomposition of T, where U and V are unitary and $P = \operatorname{diag}(a_1, \ldots, a_n)$ is the diagonal matrix with the singular values $1 \geq a_1 \geq \cdots \geq a_n \geq 0$ of T on its diagonal. By Lemma 2.1, it suffices to factor P into k partial isometries. Let $l = \operatorname{rank}(1 - T^*T)$ and

$m = \text{nullity}\, T$. In terms of the singular values, this says that $a_1 = \cdots = a_{n-l} = 1$, $0 < a_{n-l+1}, \ldots, a_{n-m} < 1$ and $a_{n-m+1} = \cdots = a_n = 0$. We only need factor the $l \times l$ matrix $P' = \text{diag}(a_{n-l+1}, \ldots, a_{n-m}, 0, \ldots, 0)$. Let $l - m = 2ms + t$, where $0 \le t < 2m$ and let $r = s$ or $s + 1$ depending on whether $t = 0$ or $t > 0$. Then we have $P' = P_1 P_2 \ldots P_r$, where P_j, $j = 1, 2, \ldots, r$, is the diagonal matrix obtained from P' by retaining $a_{n-l+2(j-1)m+1}, \ldots, a_{n-l+2jm}$ and replacing the remaining nonzero diagonal entries by 1's. Note that, other than P_r, each P_j can be written as

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \oplus \begin{pmatrix} x_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & x_m & & & & & & \\ & & & y_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & y_m & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

whose second summand is the product of three partial isometries

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} x_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & x_m & & & & & & \\ & & & y_1 & & & (1-y_1^2)^{1/2} & & \\ & & & & \ddots & & & \ddots & \\ & & & & & y_m & & & (1-y_m^2)^{1/2} \\ (1-x_1^2)^{1/2} & & & & & & 0 & & \\ & \ddots & & & & & & \ddots & \\ & & (1-x_m^2)^{1/2} & & & & & & 0 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}.$$

Hence the same is true for each P_j, $j = 1, \dots, r-1$, say, $P_j = JQ_jJ$, where Q_j is a partial isometry and J denotes the matrix

$$\operatorname{diag}(\underbrace{1, \dots, 1}_{l-m}, \underbrace{0, \dots, 0}_{m}).$$

Similar arguments applied to P_r yields that $P_r = JQ_rJ$ or JQ_r depending on whether $t > m$ or $\leq m$. In the former case, we have

$$P' = (JQ_1J)\cdots(JQ_{r-1}J)(JQ_rJ) = JQ_1JQ_2\cdots JQ_rJ$$

with $2r+1$ factors. Since $l \leq km$, we have $(l-m)/2m \leq 1/2(k-1)$. If k is odd, say, $k = 2q+1$, then $s + t/2m \leq q$ whence $r = s+1 \leq q$ and we have $2r+1 \leq 2q+1 = k$ as required. If k is even, say $k = 2q$, then $s + t/2m \leq q - 1/2$ and, since $t > m$, we have $s + 2 \leq q$ which implies that $2r+1 = 2s+3 \leq 2q-1 \leq k-1$ as required. Analogously, for $t \leq m$ we can prove that P' is the product of $2r$ partial isometries and $2r \leq k$.

The equivalence of (2) and (3) follows from the main theorem in [1]. This completes the proof. □

Here are some immediate corollaries.

Corollary 2.3. *A complex square matrix is the product of finitely many partial isometries if and only if it is either unitary or a singular contraction.*

Corollary 2.4. *Any $n \times n$ singular contraction is the product of n partial isometries and there are such matrices which are not the product of $n-1$ partial isometries.*

Proof. The assertions follow from Theorem 2.2 and from considering matrices of the form $\operatorname{diag}(a_1, \dots, a_{n-1}, 0)$, where $|a_i| < 1$ for $i = 1, \dots, n-1$. □

We remark in passing that on an infinite-dimensional Hilbert space, every contraction is the product of two partial isometries. More precisely, a contraction T can be factored as $S_1^*S_2$, where S_1 and S_2 are unilateral shifts with infinite multiplicity (cf. [2]).

3. Projection

The main result of this section is the following characterization of products of projections.

Theorem 3.1. *An $n \times n$ matrix T is the product of finitely many projections if and only if T is unitarily equivalent to $1 \oplus S$, where S is singular with $\|S\| < 1$. Moreover, for each $n \geq 2$, the number of projections in such a factorization can be arbitrarily large.*

Proof of necessity. Assume that $T = P_1 P_2 \ldots P_m$, where P_j's are projections. Then T is a contraction and the subspace $K = \{x\colon Tx = x\}$ reduces T (cf. [6, p. 8]). Let $T = 1 \oplus S$ with respect to the decomposition $K \oplus K^{\perp}$. Note that if x is any vector satisfying $\|Tx\| = \|x\|$, then x must be in K. Indeed, from

$$\|x\| = \|Tx\| = \|P_1 P_2 \cdots P_m x\| \leq \|P_2 \cdots P_m x\| \leq \cdots \leq \|P_m x\| \leq \|x\|$$

we infer that

$$\|P_1 P_2 \cdots P_m x\| = \|P_2 \cdots P_m x\| = \cdots = \|P_m x\| = \|x\|.$$

The last equality $\|P_m x\| = \|x\|$ implies that $P_m x = x$. Then from $\|P_{m-1} P_m x\| = \|x\|$ we have $\|P_{m-1} x\| = \|x\|$ which implies that $P_{m-1} x = x$. Arguing successively, we obtain that $P_j x = x$ for all j whence $Tx = P_1 P_2 \cdots P_m x = x$ as asserted. Note that if $\|S\| = 1$, then there exists a unit vector x in $K^{\perp}$ such that $\|Sx\| = \|Tx\| = 1$. From above we have $x \in K$. This together with $x \in K^{\perp}$ implies that $x = 0$, a contradiction. Thus we must have $\|S\| < 1$. That S is singular is trivial. □

To prove the sufficiency, we start with the following two elementary lemmas whose proofs we omit.

Lemma 3.2. (1) *For any real θ and α,*

$$P(\theta, \alpha) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta e^{i\alpha} \\ \sin\theta\cos\theta e^{-i\alpha} & \sin^2\theta \end{pmatrix}$$

is the projection onto the subspace of $\mathbf{C}^2$ generated by

$$\begin{pmatrix} \cos\theta e^{i\alpha} \\ \sin\theta \end{pmatrix}.$$

(2) *Any 2×2 projection with rank 1 is of the form $P(\theta, \alpha)$ for some θ and α.*

Lemma 3.3. (1) *For any $0 < \theta \leq \frac{1}{2}\pi$, $(\cos(\theta/n))^n$ is strictly increasing with limit 1 as n approaches infinity.*

(2) $\prod_{j=1}^n |\cos\theta_j| \leq (\cos(\pi/2n))^n$ *for any real $\theta_1, \ldots, \theta_n$ satisfying $\sum_{j=1}^n \theta_j = \frac{1}{2}\pi$.*

Our next lemma is an easy observation. It holds even for operators on infinite-dimensional spaces.

Lemma 3.4. *If* $T = P_1P_2 \ldots P_m$ *is the product of* m (≥ 2) *projections, then* $T = QP_2 \cdots P_{m-1}R$*, where* Q *and* R *are the projections onto the subspaces* $\operatorname{ran} T$ *and* $\ker^{\perp} T$*, respectively.*

Proof. Since $T = P_1P_2 \ldots P_m$ implies that $\operatorname{ran} T \subseteq \operatorname{ran} P_1$ and $\ker^{\perp} T \subseteq \operatorname{ran} P_m$, we have $QP_1 = Q$ and $P_mR = R$. Thus $T = QTR = QP_1P_2 \cdots P_mR = QP_2 \cdots P_{m-1}R$ as asserted. □

The next result is, in nature, a two-dimensional one. It is the main step toward our sufficiency proof and may have some independent interest.

Lemma 3.5. *Let* x *and* y *be vectors in* $\mathbf{C}^n$. *Then a necessary and sufficient condition that* $x = P_1 \cdots P_m y$ *for some projections* $P_1, \ldots, P_m$ *is that either* $x = y$ *or* $\|x\| < \|y\|$. *Moreover, in this case, the* P_j *'s may be chosen to fix all the vectors which are orthogonal to a fixed two-dimensional subspace containing* x *and* y.

Proof. If $x = P_1 \cdots P_m y$ and $\|x\| = \|y\|$, then, as proved in the necessity part of Theorem 3.1, $x = y$. To prove the converse, let x and y be such that $\|x\| < \|y\|$. By restricting to a fixed two-dimensional subspace containing x and y, changing the scale and rotating this subspace appropriately, we may assume that $x = \binom{a}{b}$ with $0 < |a|^2 + |b|^2 < 1$ and $y = \binom{0}{1}$. We consider the following four cases successively:

(1) $|a|^2 + |b|^2 = b$. In this case, $x = Py$, where

$$P = \begin{pmatrix} |a|^2/b & a \\ \bar{a} & b \end{pmatrix}.$$

(2) $|a|^2 + |b|^2 < b$. Let P be the projection from $\mathbf{C}^2$ onto the subspace generated by x, let s and t be a pair of positive solutions of the equations $s^2 + t^2 = t$ and $(s - |a|)|a| + (t - b)b = 0$, and let $c = sa/|a|$ if $a \neq 0$ and s if $a = 0$, and $d = t$. Then it is easily seen that $x = P\binom{c}{d}$. Since $|c|^2 + |d|^2 = d$, (1) yields that $\binom{c}{d} = P'y$ for some projection P'. Hence $x = PP'y$ as required.

(3) $|a|^2 + |b|^2 > b \geq 0$. Let $r = (|a|^2 + |b|^2)^{1/2}$ and $\theta = \tan^{-1} \frac{b}{|a|}$. By Lemma 3.3(1), there exists an integer N such that $r(\sec(1/N)(\frac{1}{2}\pi - \theta))^N < 1$. Let $\eta = (1/N)(\frac{1}{2}\pi - \theta)$ and $\theta_j = \theta + (j-1)\eta$ for $j = 1, 2, \ldots, N$. Let $a_0 = a$, $b_0 = b$, and, for $j = 1, 2, \ldots, N$, let

$$a_j = r(\sec\eta)^{j-1} \cos\theta_j \frac{a_{j-1}}{|a_{j-1}|}$$

and

$$b_j = r(\sec\eta)^{j-1} \sin\theta_j.$$

Note that $a_1 = a$ and $b_1 = b$. Let $P_j = P(\theta_j, \arg a_{j-1})$ be the projection onto the subspace generated by $(\cos\theta_j \frac{a_{j-1}}{|a_{j-1}|} \ \sin\theta_j)^t$, $j = 1, 2, \ldots, N-1$, or, equivalently, by $\binom{a_j}{b_j}$. It is easily seen that $\binom{a_j}{b_j} = P_j\binom{a_{j+1}}{b_{j+1}}$ for $j = 1, 2, \ldots, N-1$. Hence we have $x = P_1P_2\cdots P_{N-1}\binom{a_N}{b_N}$. Since

$$(|a_N|^2 + |b_N|^2)^{1/2} = r(\sec\eta)^{N-1} < \cos\eta = \sin\theta_N = b_N/(|a_N|^2 + |b_N|^2)^{1/2},$$

that is, $|a_N|^2 + |b_N|^2 < b_N$, it follows from (2) that there exist projections P and P' such that

$$\begin{pmatrix} a_N \\ b_N \end{pmatrix} = PP'y$$

whence $x = P_1 \ldots P_{N-1}PP'y$.

(4) b is not a nonnegative real number. As in (3), let N be such that $r(\sec(\theta/N))^N < 1$, where $r = (|a|^2 + |b|^2)^{1/2}$ and $\theta = \tan^{-1}\frac{|b|}{|a|}$. Let $\eta = \theta/N$ and $\theta_j = \theta - (j-1)\eta$ for $j = 1, 2, \ldots, N+1$. Let $a_0 = a, b_0 = b$, and, for $j = 1, 2, \ldots, N+1$, let

$$a_j = r(\sec\eta)^{j-1}\cos\theta_j\frac{a_{j-1}}{|a_{j-1}|}$$

and

$$b_j = r(\sec\eta)^{j-1}\sin\theta_j\frac{b_{j-1}}{|b_{j-1}|}.$$

As before, let $P_j = P(\theta_j, \arg a_{j-1} - \arg b_{j-1})$ be the projection onto the subspace generated by $(\cos\theta_j\frac{a_{j-1}}{|a_{j-1}|}\frac{|b_{j-1}|}{b_{j-1}}\ \sin\theta_j)^t$, $j = 1, 2, \ldots, N$, or, equivalently, by $\binom{a_j}{b_j}$. It is easily seen that $a_1 = a$, $b_1 = b$, $a_{N+1} = r(\sec\eta)^N\frac{a_N}{|a_N|}$, $b_{N+1} = 0$ and $\binom{a_j}{b_j} = P_j\binom{a_{j+1}}{b_{j+1}}$ for $j = 1, 2, \ldots, N$. Hence we have $x = P_1P_2\ldots P_N\binom{a_{N+1}}{b_{N+1}}$. Since $|a_{N+1}|^2+|b_{N+1}|^2 = r^2(\sec\eta)^{2N} > 0$, $\binom{a_{N+1}}{b_{N+1}} = Q_1\cdots Q_m y$ for some projections $Q_1, \ldots, Q_m$ by (3). We conclude that $x = P_1\cdots P_NQ_1\cdots Q_my$ as asserted. □

The next lemma says that in the factorization of 2×2 matrices, the number of projection factors may be arbitrarily large.

Lemma 3.6. *For any $m \geq 2$, let*

$$S_m = \begin{pmatrix} 0 & (\cos\frac{\pi}{2m})^m \\ 0 & 0 \end{pmatrix}.$$

Then S_m is the product of at least $m+1$ projections.

Proof. For $j = 1, 2, \ldots, m+1$, let $\theta_j = (j-1)\pi/2m$, and $P_j = P(\theta_j, 0)$. Since $P_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $P_{m+1} = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, a little computation yields that

$$P_1P_2\ldots P_{m+1} = \begin{pmatrix} 0 & \prod_{j=1}^m\cos(\theta_{j+1}-\theta_j) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\cos\frac{\pi}{2m})^m \\ 0 & 0 \end{pmatrix} = S_m.$$

To prove the minimality of $m+1$, assume that $S_m = Q_1Q_2\cdots Q_{k+1}$, where $k < m$ and the Q_j's are projections $\neq 1$. By Lemma 3.4, we may take

$Q_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $Q_{k+1} = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Since each Q_j, $j = 2, \dots, k$, is of the form $P(\theta_j, \alpha_j)$ for some $-\frac{1}{2}\pi \le \theta_j < \frac{1}{2}\pi$ and α_j, carrying out the multiplications in $S_m = Q_1 Q_2 \cdots Q_{k+1}$ and taking the absolute values of the resulting quantities we obtain $(\cos(\pi/2m))^m \le \prod_{j=1}^{k} |\cos\eta_j|$, where $\eta_j = \theta_{j+1} - \theta_j$ if θ_j and θ_{j+1} are in the same quadrant and $\theta_{j+1} + \theta_j$ otherwise, and $\theta_1 = 0$, $\theta_{k+1} = \frac{1}{2}\pi$. Since $\cos x$ is an even function of x, we may suitably add a "+" or " − " sign in front of each η_j such that their algebraic sum equals $\frac{\pi}{2}$. Thus Lemma 3.3(2) is applicable and we infer that the right hand side of the above inequality is no greater than $(\cos(\pi/2k))^k$. It follows that $(\cos(\pi/2m))^m \le (\cos(\pi/2k))^k$. This contradicts Lemma 3.3(1) since $k < m$. The proof is complete. □

Proof of Sufficiency in Theorem 3.1. Assume that S is a singular strict contraction. Let $S = AB$ be its polar decomposition, where A is a partial isometry and $B = (S^*S)^{1/2}$ is positive semidefinite with $\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} S$ (cf. [4, Problem 134]), and let α be a positive number satisfying $\|S\| < \alpha < 1$. Since $S = (\alpha A)(\frac{1}{\alpha}B)$ and both αA and $\frac{1}{\alpha}B$ are singular strict contractions, to complete the proof we need only decompose these two factors into projections.

We first consider αA. Let $A = \left(\begin{smallmatrix}A_1&0\\A_2&0\end{smallmatrix}\right)$ with respect to the decomposition $\operatorname{ran} A^* \oplus \ker A$. We may assume that A_1 is lower triangular. Next express αA in column vectors as $\alpha A = (a_1 \cdots a_k\ 0 \cdots 0)$, where $k = \operatorname{rank} A^*$. Since A is a partial isometry, a_j's are mutually orthogonal with norms less than 1. For $j = 1, 2, \dots, k$, let

$$e_j = (0 \cdots 0 \underset{j\text{th}}{1} 0 \cdots 0)^t.$$

Since e_1 and a_1 are both orthogonal to $a_2, \dots, a_k$ and $\|a_1\| < 1 = \|e_1\|$, by Lemma 3.5 we may transform e_1 to a_1 by a sequence of projections $P_1, \dots, P_{n_1}$ while preserving $a_2, \dots, a_k$, that is, $\alpha A = P_1 \cdots P_{n_1}(e_1\ a_2 \cdots a_k\ 0 \cdots 0)$. Repeating the argument, since e_2 and a_2 are orthogonal to e_1, $a_3, \dots, a_k$, there are projections $P_{n_1+1}, \dots, P_{n_2}$ such that $(e_1\ a_2 \cdots a_k\ 0 \cdots 0) = P_{n_1+1} \dots P_{n_2}$ $(e_1\, e_2\, a_3 \cdots a_k\, 0 \cdots 0)$. In k steps, we obtain that $\alpha A = P_1 \cdots P_{n_k}(e_1 \cdots e_k\ 0 \cdots 0)$ as a product of $n_k + 1$ projections.

The factorization of $(1/\alpha)B$ is even easier. Assuming that $\frac{1}{\alpha}B$ is diagonal, we may proceed as before since the column vectors of $\frac{1}{\alpha}B$ are mutually orthogonal. This proves the factorization of S.

To prove the assertion for the number of factors, let $T_m = I_{n-2} \oplus S_m$, where I_{n-2} denotes the identity matrix of size $n-2$ and S_m $(m \ge 2)$ is the 2×2 matrix as in Lemma 3.6. If $T_m = P_1 P_2 \cdots P_{k+1}$ is the product of $k+1$ projections, then, by Lemma 3.4, we may assume that $P_{k+1} = I_{n-2} \oplus \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Let $e_i = (0 \cdots 0\ 1\ 0 \cdots 0)^t$ be the ith column of T_m and also of P_{k+1}, $i = 1, 2, \dots, n-2$. From $T_m = P_1 P_2 \cdots P_{k+1}$, we have $e_i = P_1 P_2 \cdots P_k e_i$. An argument as in the proof of the necessity part yields that $P_j e_i = e_i$ for all i and j. Hence $P_j = I_{n-2} \oplus P_j'$ for some 2×2 projection P_j', and we have

$S_m = P_1' P_2' \cdots P_{k+1}'$. It follows from Lemma 3.6 that $k \geq m$ completing the proof. □

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