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THE MATHEMATICAL MODEL OF WILDHABER–NOVIKOV GEARS APPLICABLE TO FINITE ELEMENT STRESS ANALYSIS

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Abstract—The mathematical model and results of tooth contact analysis (TCA) of the Wildhaber–Novikov (W–N) gear are given in Ref. [1] [F. L. Litvin and C.-B. Tsay, *Trans. ASME, J. mech. Trans. Auto. Des.* 556–564 (December 1985)]. A finite element method (FEM), using the SAP IV package, is employed to the W–N gear stress analysis. An automatic nodal points and meshes generating (AMG) computer program has been developed. By utilizing the mathematical model of the W–N gear and the AMG computer program, nodal points and meshes for finite element analysis can be obtained automatically. By applying the results of TCA, the location and direction of applied loading on gear teeth can be defined. Root bending stresses of pinion and gear with different loading positions have been analyzed. Effects on root stresses due to variations of fillet radius and normal pressure angle of pinion have also been investigated.

INTRODUCTION

During the past years there have been a number of investigations into the nature properties of the Wildhaber–Novikov (helical gears with circular arc teeth) gear [1–8]. Two main differences between W–N gears and traditional involute helical gears are: (1) the profile of their normal sections, and (2) the nature of tooth contact. The generating line of normal section of rack cutter for involute helical gears is a straight line while the generating line for W–N gears is a circular arc. Meanwhile, the bearing contact of W–N gears is a point contact rather than a line contact for involute helical gears. The W–N gear possesses a favorable relative curvature in the contact area which decreases the contact stress and increases the loading capacity significantly. It also provides high rolling velocity and favorable orientation of bearing contact during meshing process which improve conditions of lubrication.

Recently, greater emphasis has been placed on the certain practical numerical methods such as computer-aided design (CAD), tooth contact analysis (TCA) technique and finite element method which may be employed to calculate transmission errors, stress analysis and to improve the dynamic performance of gears. In particular, computers have played an important role in the development of simulation models for both static and dynamic analysis for different types of gears. Tavakoli and Houser [9] presented an optimization algorithm which minimized the static transmission errors of spur gears by applying the profile modification. Ou and Seireg [10] employed an interactive computer graphics technique for the analysis of engagement and synthesis of circular arc tooth profiles for optimal contact where the axes misalignment and the manufacturing errors had been considered. Chang and Huston [11] employed finite element method to calculate the stresses of spur gear for a variety of loading conditions. Litvin and Tsay [1, 12] explored TCA to simulate the conditions of gear meshing and bearing contact for helical gears with circular arc teeth. In 1986, Oda *et al.* [13] studied the effects of pressure angle on tooth deflection and root stress. Ueno *et al.* [14] studied the effects of tooth form on stresses for involute helical gears.

Helical gears with screw involute tooth surfaces are widely used in industry for parallel axes power transmission. The bearing contact of involute helical gear is not localized because they contact each other at every instant at a line. The disadvantages of this type of gearing are their sensitivity to the axes misalignment and a high level of contact stress occurred when axes misalignment exists. However, the bearing contact of the W–N gear proposed here is localized because the gear tooth surfaces contact each other at every instant at a point instead of a line. Thus, the sensitivity of the gears to the misalignment is reduced. By choosing the curvature of circular arc teeth properly, the contact stress can be reduced to nearly 1/4 contact stress of traditional involute helical gears.

In this paper, the root bending stresses and contact stresses of the W–N gear will be investigated by applying the finite element method. Effects on root bending stresses with various shapes of tooth form which include the variations of fillet radius and normal pressure angle have also been studied.

MATHEMATICAL MODEL FOR W-N GEARS

Generation of the W–N gears with conjugate gear tooth surfaces can be performed by the method which had been proposed by Litvin [3] and Davidov [5]. Equations of pinion tooth surface and its unit normal for W–N gears are assumed to be given as follows [1]:

$$
\begin{aligned}
X_1^{(f)} &= (\rho_f \sin\theta_f - b_f + r_1)\cos\phi_1 + (\rho_f \cos\theta_f - b_f \cot\theta_f)\sin\phi_1 \sin\lambda_f \\
Y_1^{(f)} &= (\rho_f \sin\theta_f - b_f + r_1)\sin\phi_1 - (\rho_f \cos\theta_f - b_f \cot\theta_f)\cos\phi_1 \sin\lambda_f \\
Z_1^{(f)} &= \rho_f \cos\theta_f \cos\lambda_f - \frac{a_f}{\cos\lambda_f} + b_f \cot\theta_f \tan\lambda_f \sin\lambda_f + r_1\phi_f \tan\lambda_f
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
n_{1x}^{(f)} &= \sin\psi_n \cos\phi_1 + \cos\psi_n \sin\phi_1 \sin\lambda_f \\
n_{1y}^{(f)} &= \sin\psi_n \sin\phi_1 - \cos\psi_n \cos\phi_1 \sin\lambda_f \\
n_{1z}^{(f)} &= \cos\psi_n \cos\lambda_f
\end{aligned}
\tag{2}
$$

where ρ_f represents the radius of circular arc of rack cutter Σ_f; θ_f expresses surface coordinate of the pinion; λ_f and ψ_n indicate lead angle and nominal pressure angle of the pinion, respectively; r_1 and ϕ_1 represent pitch radius and rotation angle of the pinion, respectively; and a_f and b_f represent the tool setting of rack cutter Σ_f.

Equation (1) represents the working part of pinion tooth surface which given in coordinate system S_1. The pinion tooth surface of the fillet can also be obtained by changing the location $C_f^{(f)}$ of arc center and the radius of the arc $\rho_f^{(f)}$ as expressed in equation (1).

Equations of gear tooth surfaces and its unit normal for W–N gears are also assumed to be given as follows [1]:

$$
\begin{aligned}
X_2^{(p)} &= (\rho_p \sin\theta_p - b_p - r_2)\cos\phi_2 - (\rho_p \cos\theta_p - b_p \cot\theta_p)\sin\phi_2 \sin\lambda_p \\
Y_2^{(p)} &= -(\rho_p \sin\theta_p - b_p - r_2)\sin\phi_2 - (\rho_p \cos\theta_p - b_p \cot\theta_p)\cos\phi_2 \sin\lambda_p \\
Z_2^{(p)} &= \rho_p \cos\theta_p \cos\lambda_p - \frac{a_p}{\cos\lambda_p} + b_p \cot\theta_p \tan\lambda_p \sin\lambda_p + r_2\phi_2 \tan\lambda_p
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
n_{2x}^{(p)} &= \sin\psi_n \cos\phi_2 - \cos\psi_n \sin\phi_2 \sin\lambda_p \\
n_{2y}^{(p)} &= -\sin\psi_n \sin\phi_2 - \cos\psi_n \cos\phi_2 \sin\lambda_p \\
n_{2z}^{(p)} &= \cos\psi_n \cos\lambda_p
\end{aligned}
\tag{4}
$$

where ρ_p represents the radius of circular arc of rack cutter Σ_p; θ_p expresses surface coordinate of the gear; λ_p and ψ_n indicate lead angle and nominal pressure angle of the gear, respectively; r_2 and ϕ_2 repersent pitch radius and rotation angle of the gear, respectively; and a_p and b_p represent the tool seeting of rack cutter Σ_p.

Table 1. Parameters for pinion and gear

Parameters	Pinion	Gear
No. of teeth	12	94
Module (mm)	12.7	12.7
Profile shape	Convex	Concave
Pitch radius (mm)	78.74	617.22
Nominal pressure angle ψ_n (deg)	30	30
Helix angle β (deg)	15	15
Screw type	right hand	left hand
Fillet radius of rack cutter (mm)	3.56	8.93

STRESS ANALYSIS

In this section, the finite element method has been employed to analyze the bending stresses of the W–N gear. Stress analysis of the following topics have been studied: (a) sensitivity to the change of center distance, (b) effects due to different fillet radii, and (c) effects due to different pressure angles. Since the equations for tooth surface are given, an automatic meshes generation (AMG) computer program which generates nodal points and meshes has been developed for finite element stress analysis. With the AMG computer program and the FEM, the effects due to variations of fillet radius and pressure angle can be investigated by changing the input data of the computer program.

Analysis model

It is known that the W–N gear is in point contact, the bearing contact is thus localized. For simplicity, some assumptions are made for the stress analysis model: (a) three-dimensional finite element analysis may be replaced by two-dimensional finite element analysis where the computer time can be reduced substantially; (b) only one pair of teeth are in contact and the effect of overlapping is neglected during gear meshing; and (c) gear tooth under loading is considered as a plain strain loading.

The elements chosen for finite element analysis are 2-D four nodes plane strain elements. Three teeth instead of one or two teeth are selected for the consideration of boundary conditions. It is assumed that a loading of 1000 lb is applied at each contact position along the tooth surface normal. Material properties for steel with Young's modulus $= 30 \times 10^6$ lb/in^2 and Poisson's ratio $= 0.3$ are used for both pinion and gear. The gear data are given in Table 1.

Figures 1 and 2 show the meshes for finite element analysis of the pinion and gear, respectively. These meshes consist of 347 nodes and 288 elements for the pinion and of 471 nodes and 396 elements for the gear. The meshes are generated by an automatic meshes generation computer program. The number of elements and nodes may be changed from case to case with different input data. A fine mesh is used in fillet area and bottom land between two teeth while a coarse mesh covers the rest of the gear. It is noticed that nodes near the center of the blank (the gear–shaft interface) are fixed to simulate tight-fitting hubs. Nodes at the edge of both sides of gear model are assumed to displace along the radial direction only when loading is applied.

Stress analysis

Pinion tooth stress analysis. Loading of 1000 lb is applied at each node individually along the tooth surface of the pinion. The applied loading at each node is normal to the tooth surface and the loading positions are shown in Fig. 3. The results of finite element stress analysis are shown in Figs 4 and 5. Figure 4 shows the variation of maximum principal stress in each element along tooth surface for loadings applied at positions 1 and 5 of the pinion, respectively. Spikes in the tooth zone are local effects due to point loadings and corresponding curvature of nodal points. Figure 5 shows the maximum principal root bending stresses for loading of 1000 lb applied at each loading position. The maximum principal bending stress increases as the loading position moves from the tip (position 1) to the fillet of the tooth (position 9). This is opposite to what is predicted by customary beam theory. It can be explained by the fact that the applied loadings are normal to the tooth surface, the force component to produce bending stresses at loading position 1 is quite smaller than at loading position 9.

Due to the discrepancy, tooth shapes of W–N gears and standard involute gears are compared.

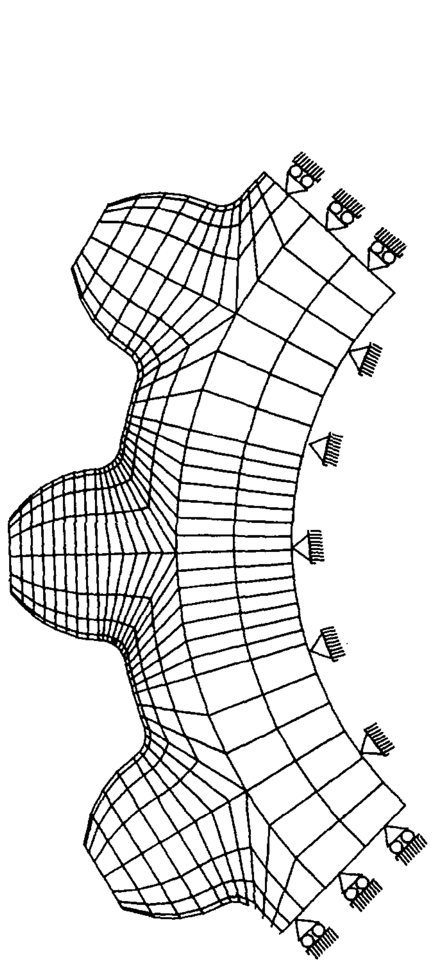

Fig. 1. Pinion meshes with $\rho_f = 0.14''$.

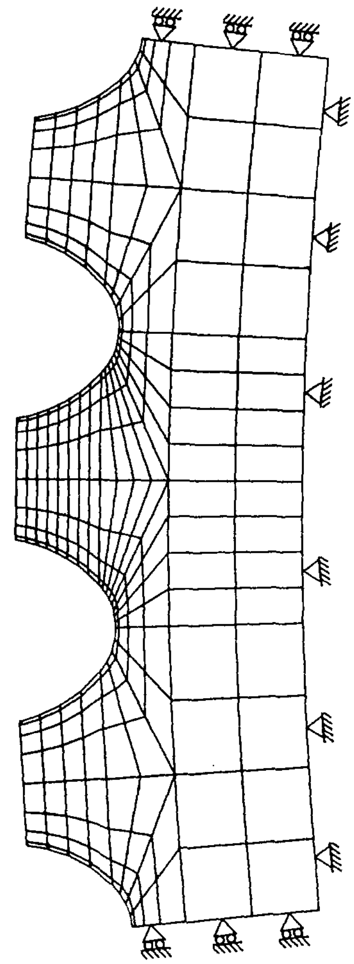

Fig. 2. Gear meshes with $\rho_p = 0.3515''$.

It is observed that the tooth form is short and stubby. It is much less like a long and slender beam than the involute tooth. The W–N gear tooth also possesses a higher pressure angle than that of the involute gear tooth, particularly at the tip of pinion. It results in a larger radial loading component. However, the "bending effect" will be significant when the loading position moves from the tip to the fillet because the loading direction is progressively horizontal. Figures 4 and

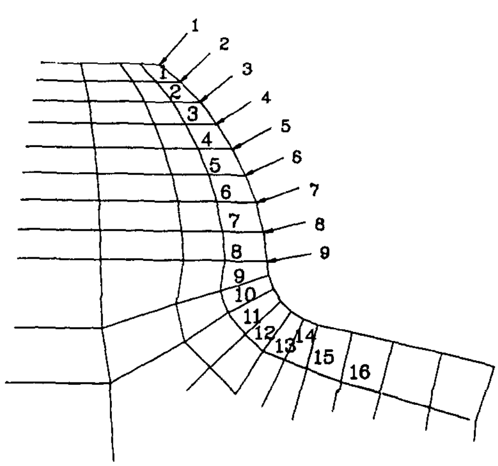

Fig. 3. Loading positions of the pinion.

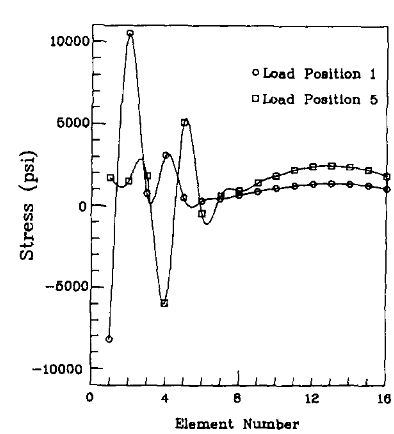

Fig. 4. Maximum principal stress vs element number for loading applied at positions 1 and 5 of the pinion.

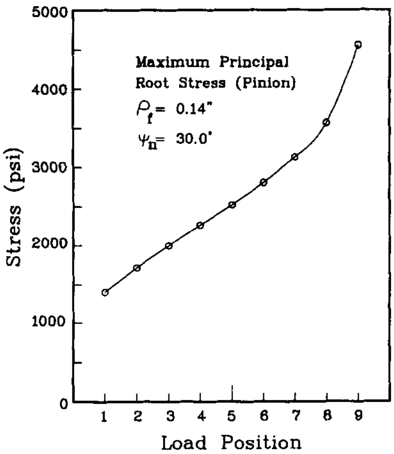

Fig. 5. Bending stress vs load position for the pinion.

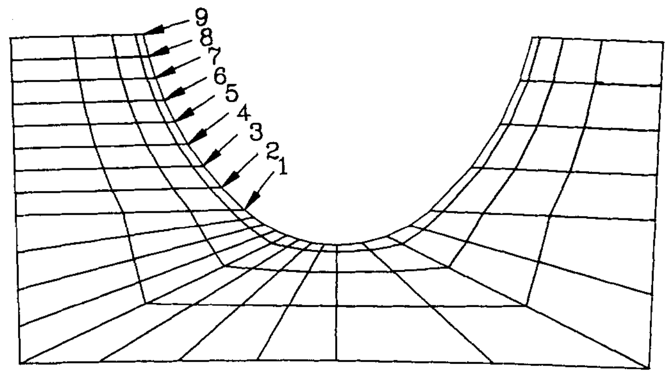

Fig. 6. Loading positions of the gear.

5 also show that the maximum principal stresses occur on the fillet surface just below the center of the fillet profile. This coincides with the stress concentration for the small radius of the curve.

Gear tooth stress analysis. An automatic nodal points and meshes generation program which generates a 2-D finite element model for a gear normal section has been developed. This model is similar to that created for the pinion. The nodal points and meshes of the gear have been shown in Fig. 2. Loading positions of the gear are shown in Fig. 6 which correspond to those of the pinion.

The results of the finite element stress analysis are shown in Fig. 7. It shows the maximum principal root stress vs loading positions. Unlike the pinion, the root stresses go down as the loading position is moved from the tip to the bottom of the tooth.

Finally, comparison has been made for the results of root bending stresses of the pinion and gear under various meshing conditions. The results are shown in Fig. 8 as a plot of the maximum root stresses vs loading positions for both pinion and gear. This figure shows an interesting result that the maximum root stresses on the fillet of pinion are always higher than those on the fillet of gear.

To minimize root stresses in both pinion and gear, the tip of pinion should contact the lowest point of the gear tooth surface. Conditions get progressively worse as the loading position moves to the fillet of the pinion and the tip of the gear tooth. The results suggest a preferable contact zone ranging from the tip of the pinion to just below the center of the tooth profile (pinion loading positions 1–6) and correspondingly from bottom to just above the center zone of the gear (gear loading positions 1–6). This indicates that a center distance variation could lead to bending stress failure because the contact point may be out of the suggested contact zone. From the TCA results

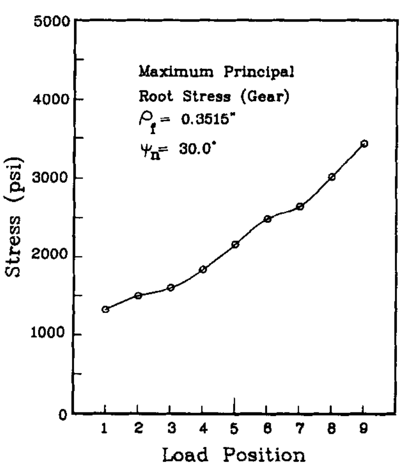

Fig. 7. Bending stresses vs load positions for the gear.

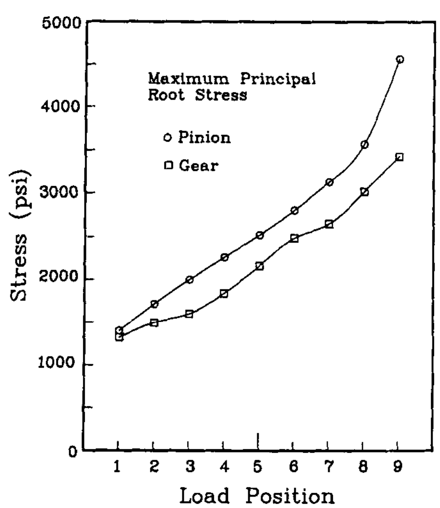

Fig. 8. Comparison of bending stress for pinion and gear.

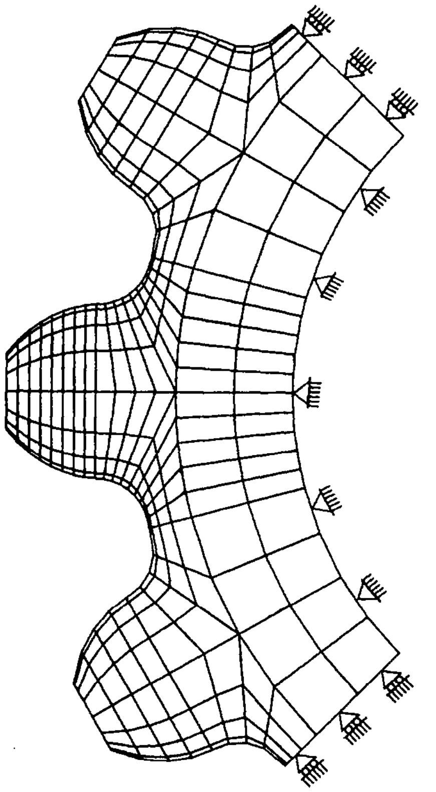

Fig. 9. Meshes of the pinion with $\rho_p = 0.28''$.

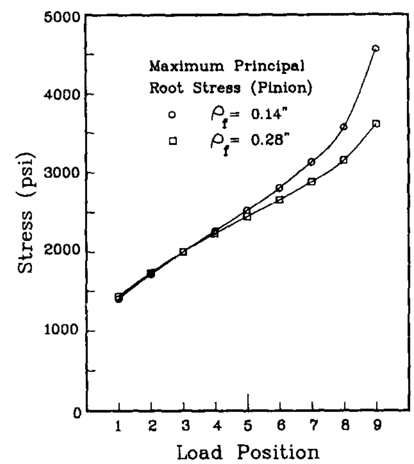

Fig. 10. Effect of fillet radius on bending stress.

of Litvin and Tsay [1], it is demonstrated that a positive change in center distance $\Delta C > 0$, causes the contact position moving from the original position (load position 5 for pinion or $\theta_f = \theta_p = 30°$) to the fillet of the tooth profile and this increases the root bending stress. Therefore, this type of errors should be prevented.

The effects on bending stress due to change of pinion tooth profile. In this section, the effects on the pinion root bending stress due to fillet radius and nominal pressure angle will be examined.

Figure 9 shows a pinion with meshes having the fillet radius of $\rho_f = 7.1$ mm. The fillet radius of the original meshes as shown in Fig. 1 was $\rho_f = 3.56$ mm. Figure 10 shows the maximum principal root stresses for both pinions with different fillet radii. Interestingly enough, the stress values for the larger fillet radius [$\rho_f = 7.1$ mm] were slightly higher than those for the original radius [$\rho_f = 3.56$ mm] at loading positions 1–3. This can be explained by increasing the loading component for the bending effect. Meanwhile, the stress values for the larger fillet radius reduced at loading positions 4–9. This is due to a lower stress concentration at fillet part for larger fillet radius.

In the final case, the effects of the normal pressure angle on the root bending stress are examined. The normal pressure angles of the pinion used for analysis are chosen as $\psi_n = 30$ and $20°$. Figure 11 shows a comparison of the normal section of pinion with the selected pressure angles. The results of the finite element stress analysis are shown in Fig. 12. It shows the maximum principal root

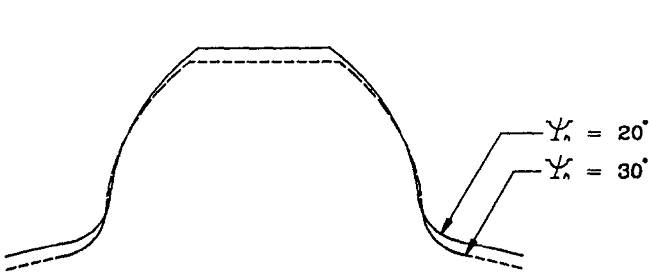

Fig. 11. Tooth profile variations due to different nominal pressure angles.

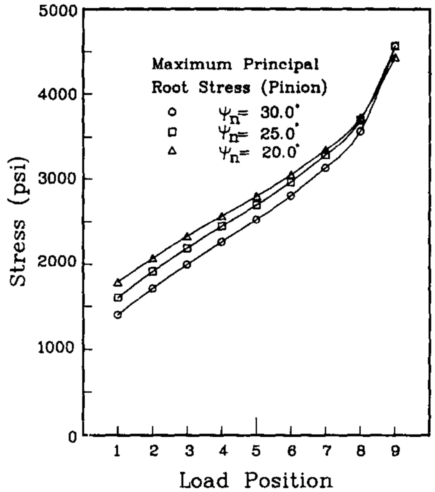

Fig. 12. Effect of nominal pressure angle on bending stress.

stresses are smaller for a larger pressure angle of pinion tooth with identical loading position and magnitude along the tooth surface. This is due to a smaller transverse loading component for a larger pressure angle. Since pinion tooth with a larger pressure angle results in a lower bending stress, a higher normal pressure angle is recommended for circular arc helical gear under the consideration of the root bending stress.

CONCLUSIONS

This paper applies the finite element method, tooth contact analysis and the mathematical model for Wildhaber–Novikov gears to study the stress behavior of the gears. According to the results of investigation, the following conclusions can be made:

(1) Compared with traditional helical gears, W–N gears are short and stubby.
(2) The bending stresses are sensitive to center distance variation. Gear housing of the W–N gear should be carefully designed to prevent center distance variation.
(3) Larger pressure angles are suggested for the pinion tooth of the W–N gear due to lower bending stresses occurring at its fillet.
(4) Pinion tooth of the W–N gear with a larger fillet radius results in a lower bending stress at the fillet.
(5) Applying the mathematical model of W–N gears to generate finite element meshes for stress analysis is very convenient and powerful.

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