A heuristic approach to multi-mode theory of superconductivity and the effect of mode-damping

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Abstract. Based on a clear physical picture, a theory of superconductivity capable of treating the simultaneous participation of multiple bosonic modes that mediate the pairing interaction is developed in terms of the dielectric function. The effect of mode damping is then incorporated in a simple manner that is free of the encumbrance of the strong-coupling, Green function formalism usually required for the retardation effect. Explicit formulae including such damping are derived for the critical temperature T_c and the energy gap Δ_0 showing, in particular, how the ratio $2\Delta_0/T_c$ may thus vary from sample to sample, a prediction consistent with the recent experimental findings on high-T_c superconductors.

The experimental discovery of the new crop of high-T_c superconductors has rekindled theoretical interests in superconductivity [1–4]. Although no definite physical mechanism has been identified unequivocally as the cause of the high critical temperature, it seems clear that whatever gives rise to it must be more complex than the conventional mechanisms. Indeed, there may be more than one simple mechanism responsible for the high T_c—phonons [1], plasmons [2], spin fluctuations [3], and magnetic couplings [4] may all play a role in contributing to the ultimate superconducting behaviour. In facing these complexities it is important to have a theoretical framework that can bring us to the heart of the physical mechanism as quickly as possible, without being heavily encumbered by impenetrable mathematical shells. For example, if some generalised BCS theory is still valid, we should at least have a formalism that can handle the pairing mechanism with the associated electron–electron attraction via exchange of many possible bosonic modes all at once. In fact, Kirzhnits et al [5] have developed such a theory based on a dielectric formulation of the strong-coupling system with the aid of Green functions [5]. Unfortunately the mathematical machinery is too cumbersome to yield immediate physical insights. A more intuitive, heuristic version is urgently needed. This is accentuated by any attempt to include the effect of boson decays on superconductivity. As is well known, boson modes such as phonons, plasmons, magnons, excitons, etc all suffer decay in one way or another. If any of these bosons are to mediate the

pairing interactions, their decay will, inevitably, affect the resultant superconducting properties. To incorporate this decay requires a treatment of the retardation effect that is beyond the simple version of the original BCS theory [6]. A Green function theory of the Kirzhnits type, however, is already so ponderous that any further complication from the decay effect would render it mathematically more unmanageable and physically more obscure. On the other hand, the high-T_c superconductors are so riddled with imperfections, grain boundaries, impurities, etc that it makes a proper inclusion of the decay of the elementary excitations caused by them almost mandatory. The observed wide range of the $2\Delta_0/T_c$ ratio [7] might be a tell-tale sign of how varied these imperfections are.

In the following we shall first develop an intuitive, heuristic approach to the pairing theory of superconductivity which enables us to take into account the multiple modes of bosons simultaneously. The resulting equations are then shown to be the same as those given by Kirzhnits et al [5]. Armed with such an approach, we can extend with ease on physical grounds to the calculation of the decay effects of the many boson modes on superconductivity. In particular, since each boson decay rate depends generally on temperature, being more pronounced at higher temperatures, the various boson modes contribute differently to the energy gap $\Delta(T)$ at different temperatures. Some modes that contribute to Δ_0 at $T = 0$ may have decayed and cease to contribute at temperatures close to T_c. This, in turn,

alters the ratio of $2\Delta_0/T_c$, rendering it variable from sample to sample with the decay-causing defects present in each sample. Explicit formulae relating $2\Delta_0/T_c$ to the boson decay rates will be presented.

Consider the usual [8] quantum mechanical electron–electron interaction matrix element effecting a transition from the pair of states k, $-k$ to k', $-k'$ via the exchange of an undamped boson of momentum q and frequency $\omega_j(q)$ of the jth branch, given by

$$\langle k \,|\, V_j \,|\, k' \rangle = \frac{\hbar\omega_j(q)\,|\,D_q^{(j)}\,|^2}{(\varepsilon_k - \varepsilon_{k'})^2 - \hbar^2\omega_j^2(q)} \qquad q = k - k' \quad (1)$$

where $D_q^{(j)}$ is the electron–boson interaction matrix element. For any type of deformation potential, since the deformation is generally proportional to $(1/\omega_q)^{1/2}$ from normalisation, Δ_q itself also has the same $(1/\omega_q)^{1/2}$ dependence. Consequently, when the energy transfer $\Delta\varepsilon = \varepsilon_k - \varepsilon_{k'}$ is negligible compared with $\hbar\omega_j(q)$ an attractive e–e interaction proportional to $1/\omega_j^2(q)$ results from (1). At this point, we remind ourselves that no retardation effect arising from the propagation of the boson from one electron to the other has been taken into account in (1), because of the inadequacy of the Hamiltonian formulation of the electron system alone. Only an Eliashberg-type Green function treatment [9] of the fully interacting electron–boson system would be able to handle the retardation. We shall, instead, start from a simple physical picture adapted from Weisskopf [10].

An electron e moving with velocity v_e ($v_e \sim v_F$, the Fermi velocity) will scatter the e' electrons (and ions) in the surrounding medium, thereby transferring a momentum of magnitude p, causing a displacement of the latter from their equilibrium configuration. The amount of displacement δ_j depends also on the type of harmonic coupling of stiffness $m\omega_j^2$ that tends to restore the perturbed charges e' to equilibrium. The perturbed, polarised region tailing the moving electron e is positively charged, providing an attractive potential of strength U_j for other electrons. It is proportional, of course, to the displacement δ_j

$$U_j \sim \left(\frac{\mathrm{d}}{\mathrm{d}r}\frac{e^2}{r}\right)_{r=d}\delta_j = -\frac{e^2}{d^2}\delta_j \qquad (2)$$

where d is the distance between e and the neighbouring charges. The length of the attractive tail is $l_j = v_e/\omega_j$ since, after a time $\tau_j \sim 1/\omega_j$, the displaced charges in the tail will have bounced back as a result of the harmonic coupling, erasing the positive bias. The displacement amplitude δ_j is obviously proportional to $1/\omega_j$ from elementary considerations of momentum transfer and energy balance. It follows that the electron–electron attractive potential associated with the harmonic mode ω_j of the medium is given approximately by

$$V_j(r) = \begin{cases} U_j & r \leqslant l_j \\ 0 & r > l_j \end{cases} \qquad (3)$$

where r is the distance between the two electrons. The corresponding matrix element of this interaction is

approximated by [10]

$$\langle k \,|\, V_j \,|\, k' \rangle = \int \phi_k(r)V_j(r)\phi_{k'}(r)\,\mathrm{d}r$$

$$\simeq \begin{cases} U_j l_j/R & \text{for } (k - k')l_j < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

where R is the radius of the spherical box in which the relative S-wave motion of the pair of electrons is confined. This simple Weisskopf picture thus yields correctly the matrix element in the phase space region of $(k - k')l_j < 1$, or equivalently

$$\varepsilon_k - \varepsilon_{k'} = \frac{\hbar^2}{2m}(k^2 - k'^2) \simeq v_F\hbar(k - k') < \frac{v_F\hbar}{l_j} = \hbar\omega_j \quad (5)$$

is given by $U_j l_j \propto \delta_j l_j \propto 1/\omega_j^2$, in agreement with and providing a physical basis for (1), as well as for the original version of the BCS theory [10].

So far, the harmonic mode ω_j has been assumed to be undamped. The displaced charge would thus oscillate as $\delta_j \mathrm{e}^{-\mathrm{i}\omega_j t}$. On the other hand, when damping is taken into account, the oscillatory movement becomes $\delta_j \mathrm{e}^{-\mathrm{i}\omega_j t}\mathrm{e}^{-\Gamma_j t/2}$, where Γ_j is the corresponding damping rate. As in the determination of l_j above, the effective time interval for the attractive potential is $\tau_j = 1/\omega_j$, the bounce-back time. This means that the original δ_j in (2) should now be replaced by $\delta_j \mathrm{e}^{-\Gamma_j/2\omega_j}$ in the presence of damping. In turn, the interaction $V_j(r)$ of (3) and its matrix element in (4) should be modified by a multiplicative factor $\mathrm{e}^{-\Gamma_j/2\omega_j}$. In view of the relation between (4) and (1) discussed above we may simply adapt this factor from the *generalised* Weisskopf picture to rewrite our model interaction† of (1) still in a non-retarded form

$$V_{k,k'}^{(j)} = \begin{cases} -\dfrac{|\,D_q^{(j)}\,|^2}{\hbar\omega_j(q)}\,\mathrm{e}^{-\Gamma_j(q)/2\omega_j(q)} & \text{for } |\,\varepsilon_k - \varepsilon_{k'}\,| < \hbar\tilde{\omega}_j \\ 0 & \text{otherwise} \end{cases}$$

$$(6)$$

where we have introduced $\tilde{\omega}_j$ as an effective frequency for the jth mode which will be discussed later.

Next we turn our attention to the electron–electron attraction that is mediated by the exchange of multiple bosons. Since each boson mode is like a harmonic oscillator of definite frequency we recall first the simplest case of an atomic electron identified as $e'(=e)$, of mass m, connected to a classical harmonic spring of frequency

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† A more rigorous way to derive this interaction would entail first modifying the non-retarded interaction (1) into a properly retarded and hence frequency-dependent form (replacing $\Delta\varepsilon \equiv \varepsilon_k - \varepsilon_{k'}$ by $\hbar\omega$)

$$\langle k \,|\, V_j(\omega) \,|\, k' \rangle = \frac{\hbar\omega_j(q)\,|\,D_q^{(j)}\,|^2}{(\hbar\omega)^2 - \hbar^2\omega_j^2(q)}.$$

The inverse Fourier transform $\langle k \,|\, V_j(t) \,|\, k' \rangle = \int \langle k \,|\, V_j(\omega) \,|\, k' \rangle \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$ would then depend explicitly on the retarded time interval t. Factors like $\mathrm{e}^{\mathrm{i}\omega_j|t|}$ there would probably have to be changed to $\mathrm{e}^{\mathrm{i}(\omega_j + \mathrm{i}\Gamma_j/2)|t|}$ to account for damping. Then another Fourier transform back to the ω-variable, with $\omega \ll \omega_j$, might yield (6). But even here the complex details are not entirely clear, not mentioning a full-fledged Green function theory.
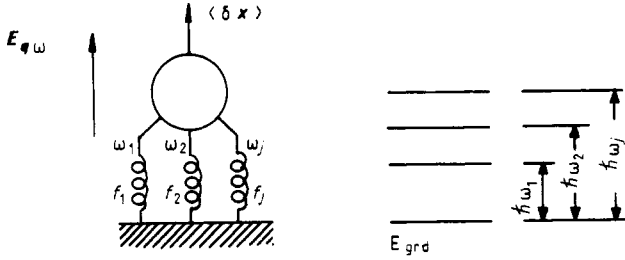
Figure 1. Displacement of an 'atom' connected to a quantum spring symbolising the strength of excitation via various modes ω_j with probability f_j when perturbed by $E_{q\omega}$. $E_{q\omega}$.

ω_j. Under an external electric field $E_\omega e^{-i\omega t}$ the displacement amplitude of the classical atom is

$$\langle \delta x \rangle_j \equiv \delta_j = -\frac{e'}{m} \frac{1}{\omega^2 - \omega_j^2} E_\omega.$$

In the case of a quantum mechanical atom in the ground state, the same external field can cause transitions of energy $\hbar\omega_j$, $j = 1, 2, 3, \ldots$ to the many possible excited states. It is as if such an atom were connected to a quantum mechanical spring which acted like a collection of classical springs of various frequencies ω_j, but each with a distinct probability f_j, $j = 1, 2, 3, \ldots$ (see figure 1). This picture leads immediately to the well known result

$$\langle \delta x \rangle = \sum_j f_j \delta_j = -\frac{e'}{m} \sum_j \frac{f_j}{\omega^2 - \omega_j^2} E_\omega \qquad (7)$$

with $\sum_j f_j = 1$. The so-called 'oscillator strength' f_j is given by the familiar expression

$$f_j = \frac{2m}{\hbar} \omega_j |x_{0j}|^2 \qquad (8)$$

which is proportional to the absolute square of the atomic dipole transition matrix element. For a collection of N such atoms in a volume Ω, the induced charge density by an external potential $\phi_{ex}(q, \omega)$ is

$$\rho_{ind}(q, \omega) = -iq \cdot P(q, \omega) = -\frac{Ne}{\Omega} ig \cdot \langle \delta x \rangle$$

$$= \frac{q^2 e^2}{m} \frac{N}{\Omega} \sum_j \frac{f_j}{\omega^2 - \omega_j^2} \phi_{ex}(q, \omega) \qquad (9)$$

where (7) has been used.

Analogous to (2), what we need now in the Weisskopf picture is the displacement $\langle \delta x \rangle$ of the perturbed charge e' upon the passing of the electron e with velocity v_e. Although the field exerted by e on e' is unlike the monochromatic E_ω in (7), we note that the probability f_j given by (8) is independent of ω, meaning that f_j is an intrinsic property of the atom system and is hence divorced from the time dependence of any external field. Accordingly, the displacement of e' is given, like the first equality in (7), by

$$\langle \delta x \rangle = \sum_j f_j \delta_j \qquad (10)$$

where δ_j is, unlike the second equality in (7), just that used in calculating U_j of (2) and $V_j(r)$ of (3). Hence,

when e' is coupled not just to one harmonic spring but to a medium symbolised by the quantum spring, the resulting electron–electron interaction that generalises $V_j(r)$ of (3) is

$$V(r) = \sum_j f_j V_j(r). \qquad (11)$$

This is due to the linear relation between U_j (and hence $V_j(r)$) and the displacement δ_j as expressed in (2).

At this point one might object that the e' electron is not an atomic electron but one that is embedded in a medium consisting of other electrons, ions, etc, while the concept of f_j was introduced in (7), (8) in the context of the displacement of an electron in an atom. Indeed, (8) is not appropriate for our purpose. However, we may now analogously look upon our N-electron system as a 'giant atom' being connected to a giant quantum spring. As in the case of single atoms, this quantum spring represents collectively the relative 'stiffness' or difficulty with which each possible excited state of the system can be attained when perturbed from equilibrium. While it is no longer possible to introduce the probabilities f_j for this quantum spring through the calculation of the displacement of any one electron e' which is intimately correlated with other charges, we can nonetheless calculate [11], as in (9), the induced charge density $\rho_{ind}(q, \omega)$ caused by an external potential $\phi_{ex}(q, \omega)$:

$$\rho_{ind}(q, \omega) = \frac{e^2}{\Omega\hbar} \sum_j |(\rho_q)_{0j}|^2 \frac{2\omega_j}{\omega^2 - \omega_j^2} \phi_{ex}(q, \omega) \qquad (12)$$

where $\hbar\omega_j = E_j - E_{grd}$ and Ω is the volume of the system. Here $(\rho_q)_{0j}$ is the transition matrix element of the density operator ρ_q between the exact eigenstate $|j\rangle$ and the ground state $|0\rangle$. It plays the same role as the dipole matrix element x_{0j} in (8) or (9). Upon comparing (12) with (9) it is natural to identify the 'oscillator strength' f_j in the present case as

$$f_j = \frac{2m\omega_j}{N\hbar q^2} |(\rho_q)_{0j}|^2. \qquad (13)$$

The interpretation of the above f_j as a probability is indeed confirmed by the familiar dipole sum rule [11]

$$\sum_j f_j = 1.$$

Accordingly we conclude that it is this f_j of (13) that should be used in (11). This conclusion is thus reached by invoking merely one general result (12) for the electron gas. As before, this f_j is again independent of the perturbing frequency ω. The multi-mode electron–electron interaction matrix element is then generalised from (6) as

$$V_{k, k'} = \sum_j f_j V_{k, k'}^{(j)} \qquad (14)$$

which retains the simple unretarded form.

Since the oscillator strength f_j is involved in the relation between $\rho_{ind}(q, \omega)$ and $\phi_{ex}(q, \omega)$ (see (9)) and so is the usual longitudinal dielectric function $\epsilon(q, \omega)$, it is clear that these quantities must be related. A straightforward algebraic manipulation of the defining equation

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of $\epsilon(q, \omega)$ shows that

$$f_j(q) = \frac{1}{\omega_p^2} \left[\frac{\partial \epsilon(q, \omega)}{\partial(\omega^2)} \bigg|_{\omega^2 = \omega_j^2(q)} \right]^{-1} \quad (15)$$

which links our $V_{k, k'}$ of (14) with the dielectric function and eventually with the dielectric formulation of the theory of superconductivity [5]. In (15), ω_p is the plasma frequency, and $\omega_j(q)$ are the longitudinal oscillation modes of the medium satisfying

$$\epsilon(q, \omega_j(q)) = 0. \quad (16)$$

With the multi-modes and their damping incorporated into (14) and (6) that retain the simple non-retarded form of the electron pairing interaction, we can now follow pretty much the usual BCS theory [14]. We include also the Coulomb repulsion among electrons in the style of Bogoliubov [12]

$$V_q^{(c)} = \begin{cases} \dfrac{4\pi e^2}{q^2 \epsilon(q, 0)} & \text{for } |\varepsilon_k|, |\varepsilon_{k'}| < \varepsilon_F \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

where $\epsilon(q, 0)$ is the static electronic screening and ε_F the Fermi energy. The total interaction is then given by combining (14) with (17)

$$V_{k, k'}^{(total)} = \sum_j f_j(q) V_q^{(j)} + V_q^{(c)} \quad (18)$$

where $q = k - k'$. This interaction is now used to solve the usual BCS gap equation [6]

$$\Delta_k(T) = -\frac{1}{2} \sum_k V_{k, k'}^{tot} \frac{\Delta_{k'}(T)}{\sqrt{\xi_{k'}^2 + \Delta_{k'}^2(T)}}$$
$$\times \tanh\left(\frac{\sqrt{\xi_{k'}^2 + \Delta_{k'}^2(T)}}{2T} \right) \quad (19)$$

where

$$\xi_k = \frac{\hbar^2 k^2}{2m^*} - \varepsilon_F$$

is the single-particle Bloch energy measured from the Fermi level, and m^* is the electron effective mass.

Recognising that $\hbar\tilde{\omega}_j$, the energy-range of the interaction $V_{k, k'}^{(j)}$ of (6), is small compared with ε_F we may as usual convert $\sum_{k'}$ into a product of an energy integral over $\xi_{k'}$ and a solid-angle integral over the magnitude of the momentum transfer q which amounts to an average over the Fermi surface after the extraction of the density-of-states factor at the Fermi surface, N_F. Equation (19) can then be rewritten as

$$\Delta(\varepsilon, T) = \sum_j N_F \int_0^{2k_F} f_j(q) \frac{|D_q^{(j)}|^2}{\hbar\omega_j(q)} e^{-\Gamma_j(q, T)/2\omega_j(q)} \frac{q \, dq}{2k_F^2}$$
$$\times \int_0^{\hbar\tilde{\omega}_j} d\varepsilon' \frac{\Delta(\varepsilon', T)}{\sqrt{\varepsilon'^2 + \Delta^2(\varepsilon', T)}} \tanh \frac{\sqrt{\varepsilon'^2 + \Delta^2(\varepsilon', T)}}{2T}$$
$$- N_F \int_0^{2k_F} \frac{4\pi e^2}{q^2 \epsilon(q, 0)} \frac{q \, dq}{2k_F^2}$$
$$\times \int_0^{\varepsilon_F} d\varepsilon' \frac{\Delta(\varepsilon', T)}{\sqrt{\varepsilon'^2 + \Delta^2(\varepsilon', T)}} \tanh \frac{\sqrt{\varepsilon'^2 + \Delta^2(\varepsilon', T)}}{2T}$$
$$\quad (20)$$

where the familiar approximation appropriate for $\Delta\xi \ll \varepsilon_F, \Delta\xi' \ll \varepsilon_F$,

$$\sum_{k'} = \int_{-\varepsilon_F}^{\infty} d\xi' N(\xi') \int_{k(\xi) - k(\xi')}^{k(\xi) + k(\xi')} \frac{q \, dq}{2k(\xi)k(\xi')}$$
$$\simeq N_F \int_{-\varepsilon_F}^{\infty} d\xi' \int_0^{2k_F} \frac{q \, dq}{2k_F^2} \quad \text{with } \frac{N_F}{\Omega} = \frac{m^* k_F}{2\pi^2 \hbar^2}$$

and the symmetry of $\Delta(\varepsilon', T)$ about the Fermi surface have been used.

We introduce here the effective electron–boson–electron attractive coupling strength as the average of the model interaction (6) over the Fermi surface multiplied by the Fermi surface density of states

$$\lambda_j(T) = N_F \int_0^{2k_F} f_j(q) \frac{|D_q^{(j)}|^2}{\hbar\omega_j(q)} e^{-\Gamma_j(q, T)/2\omega_j(q)} \frac{q \, dq}{2k_F^2}. \quad (21)$$

Note that through the dependence of the damping rate $\Gamma_j(q, T)$ on the temperature T, each mode j makes its own characteristic contribution to e–e coupling, changing with T. Also, we denote as usual [13] the pseudo-Coulomb effective repulsion averaged over the Fermi surface by the parameter μ

$$\mu = N_F \int_0^{2k_F} \frac{4\pi e^2}{q^2 \epsilon(q, 0)} \frac{q \, dq}{2k_F^2}. \quad (22)$$

Equation (19) can then be solved for the critical temperature T_c by using the three-step approximation

$$\Delta(\varepsilon') = \begin{cases} \Delta_1 & |\varepsilon'| < \hbar\omega_N(T) \\ \Delta_2 & \hbar\omega_N < |\varepsilon'| < \varepsilon_F \\ 0 & |\varepsilon'| > \varepsilon_F \end{cases} \quad (23)$$

where $\omega_N(T) \equiv (\tilde{\omega}_j(T))_{max}$ and $\Delta_1 < \hbar\omega_N < \varepsilon_F$ is assumed. Clearly, Δ_1 arises from all the bosonic modes and Δ_2 from the direct repulsive Coulomb force. After some familiar algebra [14] for disentangling the coupled Δ_1 and Δ_2, the resulting equation for T_c is found to be

$$k_B T_c = 1.13\hbar\bar{\omega}(T_c) \exp\left(-\frac{1 + \mu^* \ln(\omega_N(T_c)/\bar{\omega}(T_c))}{\lambda(T_c) - \mu^*} \right)$$
$$\quad (24)$$

where

$$\mu^* = \frac{\mu}{1 + \mu \ln(\epsilon_F/\hbar\omega_N)}$$

is the well-known [13] renormalised pseudo-Coulomb repulsion parameter. The total coupling strength from all bosonic modes, denoted by λ in (24), is just a sum of the contributions from the individual ones

$$\lambda(T) = \sum_j \lambda_j(T) \quad (25)$$

where the proper weighting factor f_j for each individual mode has already been put in λ_j of (21).

In (24) an effective average frequency $\bar{\omega}$ of the multi-boson modes that mediate the e–e attraction also emerges. It is just the logarithmic average

$$\ln \bar{\omega}(T) = \sum_j \frac{\lambda_j(T)}{\lambda(T)} \ln \tilde{\omega}_j(T) \quad (26)$$

in which each mode is weighted by a fraction that represents its relative contribution to the total e–e coupling. These weighting factors in (21), (25) and (26) associated with the various modes satisfying (16) are entirely consistent with those of Kirzhnits et al [5], with the logarithmic average arising from the integral of ε' to the upper limit $\hbar\tilde{\omega}_j$ in (20) as $\Delta(\varepsilon', T_c) \to 0$.

Unlike the usual BCS-type equation for T_c, our equation (24) must be solved self-consistently since T_c is also involved implicitly on the right side through $\lambda(T_c)$, $\omega_N(T_c)$ and $\tilde{\omega}(T_c)$.

We can similarly solve (20) for the energy gap $\Delta_0 \equiv \Delta_1(T = 0)$ at zero temperature:

$$\Delta_0 = 2\hbar\tilde{\omega}(T = 0) \exp\left(-\frac{1 + \mu^* \ln(\omega_N/\tilde{\omega}(T = 0))}{\lambda(T = 0) - \mu^*}\right)$$

(27)

where $\tilde{\omega}$, ω_N and λ are evaluated at $T = 0$.

Depending on the type of mechanism of damping, such as Landau damping, impurity scattering, etc, the amount of damping generally increases with temperature and is therefore minimised at zero temperature. It follows then from (21) and (25) that

$$\lambda(T = 0) > \lambda(T = T_c).$$

(28)

Similarly, the average boson frequency $\tilde{\omega}$ is larger at $T = 0$ since the higher frequencies in (26) can be shown [15] to be more susceptible to Landau damping at $T > 0$ and hence carries a diminished weight compared with the case $T = 0$. As a result of (24), (27) and (28) we see that the ratio of $2\Delta_0$ to $k_B T_c$ generally exceeds the BCS value of 3.5:

$$2\Delta_0/k_B T_c > 3.5$$

(29)

and this value should increase as T_c increases. This is qualitatively consistent with the recent experimental findings [7] about high-T_c superconductors. Explicit numerical results on T_c and Δ_0 as well as $\lambda(T)$ and $\tilde{\omega}(T)$ will be given elsewhere [16].

So far, the effective frequency $\tilde{\omega}_j(T)$ of the jth mode that appears in (6), (20) and (26) has not been specified. Since the jth mode frequency $\omega_j(q)$ depends generally on the wavenumber q, we have to take an appropriate average over q. In the presence of decay mechanisms such as Landau damping, a given mode j may have suffered serious damping if q is too large or the temperature is too high. It should not then be weighted as much in its contribution not only to the averaging q-integral over the Fermi surface, but also to the ε'-integral in (20), where the double integral is approximated† as a product of two separate integrals. This approximation is in the spirit of a separable poten-

† Strictly speaking, the upper limit of the ε'-integral should have been $\omega_j(q)$ which depends on q, rendering the two integrals linked with each other.

tial as assumed in the original BCS theory. In conformity with this approximation we propose an effective frequency for the jth mode

$$\tilde{\omega}_j(T) = \int_0^{2k_F} \omega_j(q)e^{-\Gamma_j(q, T)/2\omega_j(q)} \, dq$$

$$\times \left(\int_0^{2k_F} e^{-\Gamma_j(q, T)/2\omega_j(q)} \, dq\right)^{-1}$$

(30)

which, though not unique, does adequately account for the relevant physics involved. If we take the limit of no damping and evaluate the matrix elements at some average momentum transfer q, our present theory reduces to the dielectric formulation of Kirzhnits et al [5] based on the Eliashberg Green function theory [9]. However, our much simpler heuristic approach leads to a clearer physical picture and, with relative ease, has enabled us to incorporate the boson damping effect into this picture, culminating in the new results (24), (27), (29) on the critical temperature T_c, the energy gap Δ_0 and the ratio $2\Delta_0/k_B T_c$.

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