Products of Positive Semidefinite Matrices

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ABSTRACT

We characterize the complex square matrices which are expressible as the product of finitely many positive semidefinite matrices; a matrix T can be expressed as such if and only if $\det T \geqslant 0$; moreover, the number of factors can always be limited to five. We also determine those matrices which can be expressed as the product of two or four positive semidefinite matrices. These results are analogous to the ones obtained before by C. S. Ballantine for products of positive definite matrices.

1. INTRODUCTION

An $n \times n$ complex matrix T is *positive semidefinite* [*positive definite*] if $(Tx, x) \geqslant 0$ for any $x \in \mathbb{C}^n$ [$(Tx, x) > 0$ for any $x \neq 0$ in $\mathbb{C}^n$], where $\mathbb{C}$ denotes the field of complex numbers. For convenience, we will abbreviate this to *nonnegative* [*positive*] and denote it by $T \geqslant 0$ [$T > 0$]. In this paper, we characterize matrices which are expressible as the product of finitely many nonnegative matrices. The corresponding question with "nonnegative" replaced by "positive" has been solved already by C. S. Ballantine [1]. He showed that a matrix T is the product of finitely many positive matrices if and only if the determinant of T is positive, and in this case five positive matrices suffice. We will prove that any singular square matrix is the product of at most four nonnegative matrices (Theorem 4.1). Combined with the

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above, this yields an analogous result for products of nonnegative matrices: T is such a product if and only if its determinant is nonnegative (Theorem 4.3).

In [1], Ballantine also completely determined those matrices which are expressible as products of two, three, or four positive matrices. We will show that similar results hold for the products of nonnegative matrices. In particular, it is shown that T is the product of two nonnegative matrices if and only if it is similar to a nonnegative one (Theorem 2.2). The proof for this is slightly more involved than the corresponding one for positive products. The main step is to show that no nilpotent matrix, except the zero one, can be the product of two nonnegative matrices (Lemma 2.1). In the case of three nonnegative matrices, our attempt for a complete characterization has been less successful. We are able to show that any nilpotent matrix is the product of three nonnegative matrices (Corollary 3.4) and obtain a slightly more general sufficient condition (Theorem 3.3), which is in terms of Ballantine's characterization of the products of three positive matrices.

In the following, $A^{1/2}$ always denotes the nonnegative square root of a nonnegative matrix A. For an arbitrary matrix A, $\sigma(A)$ denotes the set of its eigenvalues. For T_1 and T_2 on spaces H_1 and H_2, respectively,

$$T_1 \oplus T_2 = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

acts on $H_1 \oplus H_2$, the orthogonal direct sum of H_1 and H_2. Sections 2, 3, and 4 below are concerned with the products of two, three, and four nonnegative matrices, respectively.

2. TWO NONNEGATIVE MATRICES

We start by proving the following negative result for nilpotent matrices.

LEMMA 2.1. *No nilpotent matrix, except the zero matrix, is the product of two nonnegative matrices.*

Proof. Let $T = AB$ on the finite-dimensional space H, where T is nilpotent and $A, B \geqslant 0$. Since $\sigma((A^{1/2}B)A^{1/2}) = \sigma(A^{1/2}(A^{1/2}B))$ (cf. [4, p. 102, Exercise 13]) and $\sigma(T) = \{0\}$ by the nilpotency of T, we infer that $\sigma(A^{1/2}BA^{1/2}) = \{0\}$. This, together with the nonnegativity of $A^{1/2}BA^{1/2}$, implies that $A^{1/2}BA^{1/2} = 0$. For any x in H,

$$(BA^{1/2}x, A^{1/2}x) = (A^{1/2}BA^{1/2}x, x) = 0.$$

Since ran $A^{1/2}$ = ran A, this implies that

$$(BAy, Ay) = 0 \qquad \text{for any} \quad y \in H.$$

Hence

$$\|B^{1/2}Ay\|^2 = (B^{1/2}Ay, B^{1/2}Ay) = (BAy, Ay) = 0.$$

Thus $B^{1/2}A = 0$. It follows that $BA = 0$ or $T = 0$. ∎

A slight modification of the above arguments shows that no quasinilpotent operator on a (possibly infinite-dimensional) Hilbert space is the product of two nonnegative operators unless it is the zero operator. Now we are ready for the main result of this section.

THEOREM 2.2. *For an* $n \times n$ *complex matrix* T, *the following statements are equivalent*:

(1) $T = AB$, *where* $A, B \geqslant 0$;
(2) $T = AB$, *where* $A > 0$ *and* $B \geqslant 0$;
(3) T *is similar to a nonnegative matrix.*

Proof. The equivalence of (2) and (3) is easy to derive (and well known). Indeed, assuming (2), we infer that T is similar to $A^{-1/2}TA^{1/2} = A^{1/2}BA^{1/2}$, which is nonnegative, where the invertibility of $A^{1/2}$ follows from that of A. On the other hand, if (3) holds, say $T = X^{-1}CX$, where X is invertible and $C \geqslant 0$, then $T = (X^{-1}X^{-1*})(X^*CX)$ is the product of the positive $X^{-1}X^{-1*}$ and the nonnegative X^*CX. This proves (2).

To complete the proof, we need only show that (1) $\Rightarrow$ (3). Assume that (1) is true. Since the property of being the product of two nonnegative matrices is preserved under similarity, we may, in view of the Jordan canonical form for matrices, assume that $T = T_1 \oplus T_2$ on the space $H = H_1 \oplus H_2$, where T_1 is invertible and T_2 is nilpotent. It is easily seen that $H_1 = \operatorname{ran} T^n$ and $H_2 = \ker T^n$. Let $S = A^{1/2}BA^{1/2}$, $K_1 = \operatorname{ran} S^n$, and $K_2 = \ker S^n$. Since S is nonnegative, K_1 and K_2 are orthogonal complements to each other. From $A^{1/2}S = TA^{1/2}$, we deduce that $A^{1/2}S^n = T^nA^{1/2}$, whence $A^{1/2}K_1 \subseteq H_1$ and $A^{1/2}K_2 \subseteq H_2$. It follows that $A^{1/2*}H_1^{\perp} \subseteq K_1^{\perp}$ and $A^{1/2*}H_2^{\perp} \subseteq K_2^{\perp}$, or, equivalently, $A^{1/2}H_2 \subseteq K_2$ and $A^{1/2}H_1 \subseteq K_1$. Thus $AH_1 = A^{1/2}A^{1/2}H_1 \subseteq A^{1/2}K_1 \subseteq H_1$ and, similarly, $AH_2 \subseteq H_2$. Therefore, $A = A_1 \oplus A_2$ on $H = H_1 \oplus H_2$, where $A_j = A \mid H_j$, $j = 1, 2$. In a similar fashion, $B = B_1 \oplus B_2$ on $H = H_1 \oplus H_2$. From $T = AB$, we have $T_1 = A_1 B_1$ and $T_2 = A_2 B_2$. Since T_1 is invertible, A_1 and B_1 are both positive. Thus T_1 is similar to a positive matrix by the proof of (2) $\Rightarrow$ (3). On the other hand, since the nilpotent T_2 is the product of the nonnegative matrices A_2 and B_2, we infer from Lemma 2.1 that $T_2 = 0$. Therefore $T = T_1 \oplus 0$ is similar to a nonnegative matrix. ■

It is interesting to compare the preceding theorem with the corresponding results for the product of two positive matrices and the product of two Hermitian matrices: a matrix is the product of two positive matrices if and only if it is similar to a positive one (cf. [1, Theorem 2]); it is the product of two Hermitian matrices if and only if it is similar to a matrix with real entries, and in this case one of the Hermitian matrices may be taken to be invertible (cf. [2] or [6, Theorem 1]). Other related results are [7, Propositions 2.1 and 2.3] concerning the products of a (real) symmetric matrix and a nonnegative matrix.

The next corollary will be needed in Section 3 (in the proof of Proposition 3.5).

COROLLARY 2.3. *If*

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$$

is the product of two nonnegative matrices, so are T_1 and T_2.

Proof. By Theorem 2.2, T is similar to a nonnegative diagonal matrix. It is easily seen that T_1 and T_2 are also similar to diagonal matrices and that their eigenvalues are nonnegative. Hence, by Theorem 2.2 again, they are products of two nonnegative matrices. ■

3. THREE NONNEGATIVE MATRICES

In this section, we will give a sufficient condition, in terms of Ballantine's characterization of the products of three positive matrices, for a matrix to be factored as the product of three nonnegative matrices. We start with the following

LEMMA 3.1. *Let A be an $n \times n$ positive matrix and a be an arbitrary $n \times 1$ matrix. Then there exists a positive number x such that the $(n+1) \times (n+1)$ matrix*

$$T = \begin{bmatrix} A & a \\ \bar{a}^t & x \end{bmatrix}$$

is positive.

Proof. Since a matrix is positive if and only if all its leading principal minors are positive (cf. [3, Theorem X.3]), to prove the positivity of T it suffices to show that for an appropriate choice of x, the determinant of T is positive. This can be seen by expanding the determinant along the last row, noting that $\det A > 0$, and letting x be sufficiently large. ■

LEMMA 3.2. *Any $n \times n$ matrix T is unitarily equivalent to a matrix of the form*

$$\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix},$$

where T_1 is invertible and T_2 is nilpotent. Moreover, T_1 and T_2 are uniquely determined, up to unitary equivalence, by T.

Proof. The first assertion follows from the Schur triangulation of T. Since the spaces on which T_1 and T_2 act are $\operatorname{ran} T^n$ and $\ker T^{*n}$, respectively, the uniqueness of T_1 and T_2 follows easily. ■

The next theorem gives the promised sufficient condition for the product of three nonnegative matrices. It is in terms of the "invertible part" of a matrix as given in Lemma 3.2.

THEOREM 3.3. *Let*

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix},$$

where T_1 *is invertible and* T_2 *is nilpotent. If* T_1 *is the product of three positive matrices, then* T *is the product of three nonnegative matrices.*

Proof. We may assume that T_2 is upper triangular. Let m and n be the sizes of T_1 and T, respectively. We prove our assertion by induction on n starting with $n = m$.

If $n = m$, that is, $T = T_1$, then this is trivially true. Assume that the assertion holds for $n - 1$ ($\geqslant m$). Let T' be the matrix obtained from T by deleting its last row and column. We have

$$T = \begin{bmatrix} T' & a \\ 0 & 0 \end{bmatrix}$$

and, by the induction hypothesis, $T' = A'_1 A'_2 A'_3$, where $A'_i \geqslant 0$ for $i = 1,2,3$. By Theorem 2.2, we may assume that A'_1 and A'_2 are positive. Let $b = A'^{-1}_1 a$ and let x be such that

$$A_2 \equiv \begin{bmatrix} A'_2 & b \\ \bar{b}^t & x \end{bmatrix}$$

is positive (by Lemma 3.1). Moreover, let

$$A_1 = \begin{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} A'_3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $T = A_1 A_2 A_3$ is a product of three nonnegative matrices. ■

COROLLARY 3.4. *Any nilpotent matrix is the product of three nonnegative matrices.*

It seems plausible that the converse of Theorem 3.3 is also true. In this respect, we have only the following special case as a supporting evidence.

PROPOSITION 3.5. *Let* $T = T_1 \oplus 0$. *Then* T *is the product of three nonnegative matrices if and only if* T_1 *is.*

Proof. Assume that $T = A_1 A_2 A_3$, where $A_i \geqslant 0$ for $i = 1,2,3$. By Theorem 2.2, we may assume that A_3 is invertible. Let

$$A_3^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$TA_3^{-1}=\begin{bmatrix} T_1B_1 & T_1B_2 \\ 0 & 0 \end{bmatrix}=A_1A_2$$

is the product of two nonnegative matrices. It follows from Corollary 2.3 that the same is true for T_1B_1. Since B_1 is positive, we conclude that T_1 is the product of three nonnegative matrices. ■

4. FOUR NONNEGATIVE MATRICES

THEOREM 4.1. *Any singular square matrix is the product of four nonnegative matrices, and four is the smallest such number. Moreover, any three of these matrices may be taken to be positive.*

Proof. Let T be an $n\times n$ singular matrix. By the Jordan canonical form, T is similar to a matrix T' of the form $R\oplus S$, where

$$S=\begin{bmatrix} 0 & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & 0 \end{bmatrix}$$

is of size m. Let $R=UP$, where U is unitary and $P=(R^*R)^{1/2}\geqslant 0$, be the polar decomposition of R (cf. [4, p. 169]), and assume that $U=[a_{ij}]$. Let b satisfy $(-1)^{m+1}b\det U>0$, and c be such that 0 belongs to the interior of the numerical range of the 2×2 matrix

$$\begin{bmatrix} a_{n-m\,n-m} & c \\ 0 & b \end{bmatrix}\quad \text{if}\quad m=1$$

or

$$\begin{bmatrix} a_{n-m\,n-m} & c \\ 0 & 0 \end{bmatrix}\quad \text{if}\quad m\geqslant 2.$$

This is possible because the numerical range of a 2×2 matrix $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ is an elliptical disk with foci at a and b and minor axis of length $|c|$ (cf. [5]). Let

$$A = \left[\begin{array}{c|cccc} & 0 & \cdots & & 0 \\ U & \vdots & & & \vdots \\ & c & \cdots & & 0 \\ \hline & 0 & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \\ & & & & 1 \\ & b & & & 0 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|ccc} P & & 0 & \\ \hline & 0 & & 0 \\ 0 & & 1 & \\ & & & \ddots \\ & 0 & & 1 \end{array}\right].$$

Then $T' = AB$, $\det A = (-1)^{m+1} b \det U > 0$, and 0 belongs to the interior of the numerical range of A. The latter two conditions imply that A is the product of three positive matrices (cf. [1, Theorem 3]). Since $B \geqslant 0$, we have that T' is the product of four nonnegative matrices. Say $T' = A_1 A_2 A_3 A_4$, where $A_i \geqslant 0$ for all i. Since $T = X^{-1} T' X$ for some invertible X, we have

$$T = (X^{-1} A_1 X^{-1*})(X^* A_2 X)(X^{-1} A_3 X^{-1*})(X^* A_4 X)$$

expresses T as a product of four nonnegative matrices. That any three of these matrices may be taken to be positive follows from Theorem 2.2. Since

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not the product of three nonnegative matrices by Proposition 3.5, the minimality of four follows. ∎

Using the preceding theorem and Ballantine's results [1, Theorems 4 and 5], we can characterize the products of four or more nonnegative matrices.

THEOREM 4.2. *An* $n \times n$ *matrix* T *is the product of four nonnegative matrices if and only if* $\det T \geqslant 0$ *and* T *is not a scalar matrix* cI_n *with* c *in* $\mathbb{C} \setminus \{z : z \geqslant 0\}$.

THEOREM 4.3. *A matrix* T *is the product of finitely many nonnegative matrices if and only if* $\det T \geqslant 0$. *In this case, five such matrices suffice.*

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