國立交通大學
電機與控制工程學系

碩士論文

網路控制系統的穩定性分析

Stability Analysis of Networked Control Systems

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中華民國九十三年七月

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摘要

當控制系統由網路互相溝通聯繫形成一個迴路，此系統稱為網路控制系統。考慮網路控制系統中，狀態迴授需由網路傳送，所以有不能隨時更新的問題。因此，探討控制系統僅在某些時間點更新控制訊號對系統穩定性的影響是一門重要的課題。

本篇論文利用 Lyapunov Theorem，針對網路控制系統未及時更新迴授的控制推導一些數學不等式，這些式子包含用於取得使系統穩定的控制器其更新區間之最大容許時間範圍，以及有雜訊項的系統其在不即時更新迴授控制器，但仍然要使系統穩定其可接受之狀態誤差的最大值，最後亦導出網路控制系統的 H_∞ 控制器。利用這些推導結果可以得到在網路控制器中，系統狀態由網路傳送，其對於系統穩定度的影響以及所需的最小傳輸時間範圍。

Stability Analysis of Networked Control Systems

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Abstracts

Feedback control systems where the control loops are closed through a real-time network are called Networked Control Systems (NCSs). In the NCSs, state feedback must be transmitted by the network, but it may be jammed in the channel. So it is a matter in updated delay. Therefore, discussion and analysis of the stability of control systems with control signals being updated at some moment is an important topic.

In this thesis, we derive several conditions to guarantee stability of NCSs with control signals being updated in retard according to the Lyapunov Theorem. The maximum range of time interval to update the control signal to ensure the stability of NCSs is derived. The transmission error upper bound between two successive transmissions in NCSs with disturbances is also obtained. Moreover, the H_∞ control of NCSs is considered.

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謝謝我身邊的大家，謝謝你們的關照！

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Chapter1

Introduction

1.1 Motivation

There are already several network standards designed especially for control applications, including CAN (Controller area network) [24] for automotive application, BACNet (building automation and control networks) [25] [15] for building automation applications etc. The characteristic of a Networked Control System (NCS) is having one or more control loops closed by serial communication channel. NCSs have the merits of reducing the wires between components and easier to judgment, etc. However, its stability analysis problem is different from direct control loop controller. We are interested in the stability analysis of NCSs.

1.2 Survey on Related Work

The defining feature of an NCS is that signals (reference input, control input, plant output, etc.) are exchanged using a network that connects control system components (sensors, controllers, actuators, etc.). *Fig. 1-1* shows a sketch map of a generic NCS structure. A detailed view of NCSs can be found in [5].

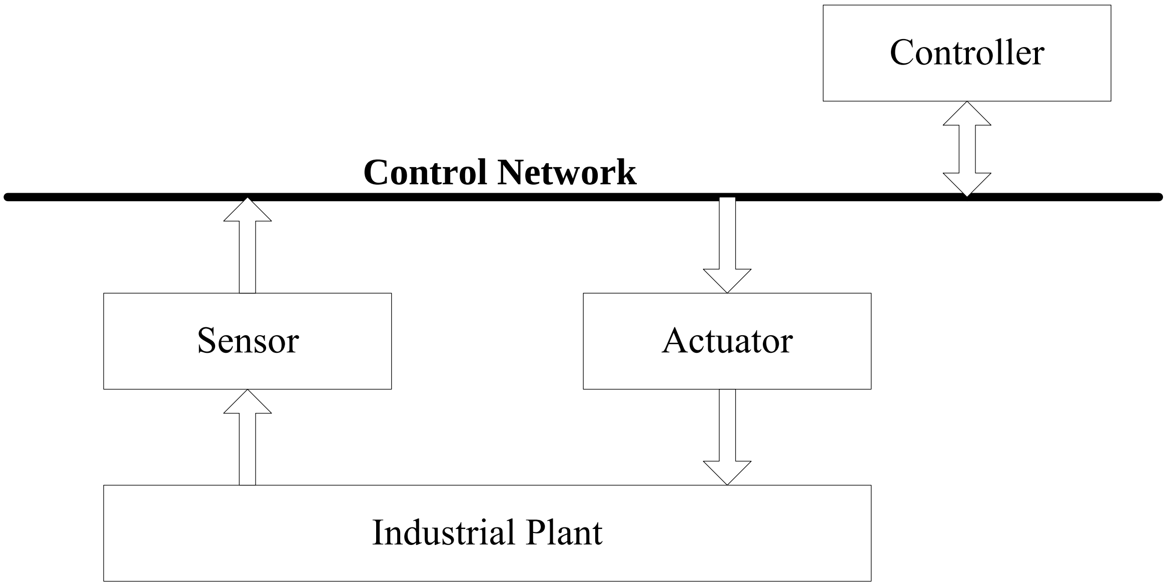

Figure 1-1 A networked control system setup.

There are a lot of issues to discuss in NCSs. A number of related works have explored the effects of communication constraints on control problems, including the relationship between practical stability of a dynamical system and the bit-rate available for feedback [21], and joint communication/control optimization problems [22] [23]. Besides these, in [11], the authors discuss about the network transmission delay deadlines that guarantee stability via Razumikhin-type stability theorem. In [9], Wei Zhang, Michael S. Branicky, and Stephen M. Philips analyze several topics in the network. The dropping network packets which happen on NCSs there are node failures or message collisions. Although most network protocols are equipped with transmission-retry mechanisms, they can only retransmit for a limited time. After this time has expired, the packets are dropped. Furthermore, for real-time feedback control data such as sensor measures and calculated control signals, it may be advantageous to discard the old, untransmitted message and transmit a new packet if it becomes available. In this way, the controller always receives fresh data for control. Normally, feedback-controlled plants can admit a certain amount of data loss, to compute acceptable lower bounds on the packet transmission rate. This is similar as our target in this thesis.

If the network speed is high and the traffic sparse, the effect of inserting such a network into the feedback loop is that of creating a small, randomly varying time delay between the records and their images. This approach has many merits. First, the network may be treated abstractly, and hence the interface between the control system and the network can take place at a high level of the open systems interconnect (OSI) [16] model, with the associated benefits of robustness and flexibility. In addition, because the impact on control design methodology is minor, standard techniques may be applied without considering the network. This highly desirable approach is supported by several analytic results [17] [18] [19].

Another topics of NCSs is the stability with network -induced delay [9]. There are two sources of delays from the network: sensor-to-controller τ_{sc} and controller-to-actuator τ_{ca}. Any controller computational delay can be absorbed into either τ_{sc} or τ_{ca} without loss of generality [20]. For fixed control law (time-invariant controllers), the sensor-to-controller delay and controller-to-actuator delay can be lumped together as $\tau = \tau_{sc} + \tau_{ca}$ for analysis purposes.

Finally, scheduling for networked control systems are discussed in [5] [6] [9].

1.3 Contribution

- Aim at the influence of the un-real time feedback in NCSs.
- Find out the maximum range of time interval to update the control signal to ensure the stability of NCSs
- Find out the upper bound of transmission error in NCSs with disturbances, validate it, and suggest the condition good for using.
- Provide a method to design H_∞ controller for NCSs.

1.4 Organization of the Thesis

The thesis is organized as follows. In Chap. 1, the related work in Networked Control Systems (NCSs), and motivations of research are given. In Chap. 2, the definition of NCSs and the problem we focus on is given. In Chap. 3, several important results for NCSs with no disturbance are provided. Afterward, they are validated by simulation and analyzed in detail. In Chap. 4, the error bound of NCSs with disturbance and a method to design H_∞ controller for NCSs are presented. Finally, conclusions and future works are given in Chap. 5.

Chapter2

Definitions of Networked Control Systems

2.1 Introduction to Networked Control Systems

Computer-Control systems started to emerge in the 1950s. At the beginning stage, since the cubage of the computer was too big, and it required much power, the competency of using digital computers as control system components was misdoubt. This situation changed when the Direct Digital Control (DDC) system was developed. DDC placed stress on the computer, which controlled the process directly. *Fig. 2-1* shows a generic DDC system architecture. In DDC systems, the analog-control instrumentation for the process control was replaced by a computer. Sensors with analog outputs and actuators with analog inputs were point-to-point connected with the digital computer. Sensing, control signal calculation, and actuation were all handled by the computer itself.

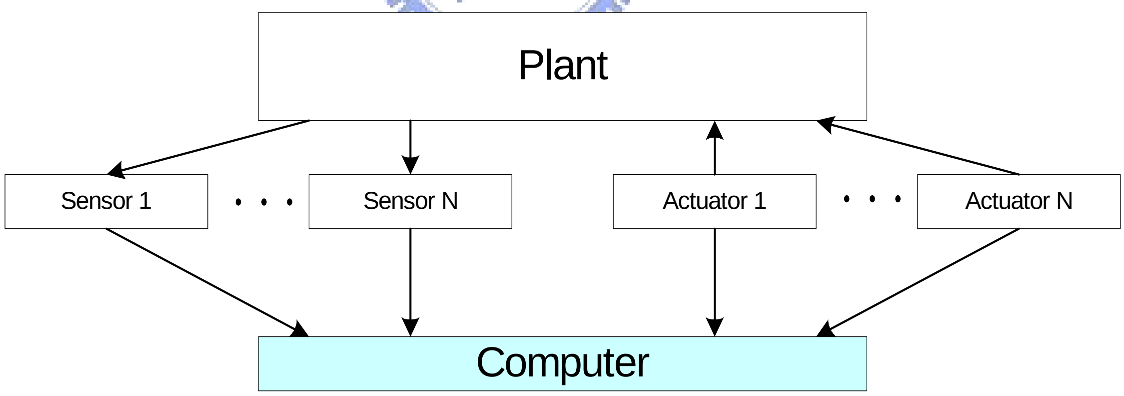

Figure 2-1 Direct digital control system.

Because of the evolution of the control system and the fast development of computer technology, distribution of computing load was required and had become possible. This system was called the Distributed Control System (DCS), which is illustrated in *Fig. 2-2*. In a DCS, several computers connected to a serial network shared the workload. The processing order of every computer in the system is monitored by operators, and is various aided stations for data logging and processing optimization.

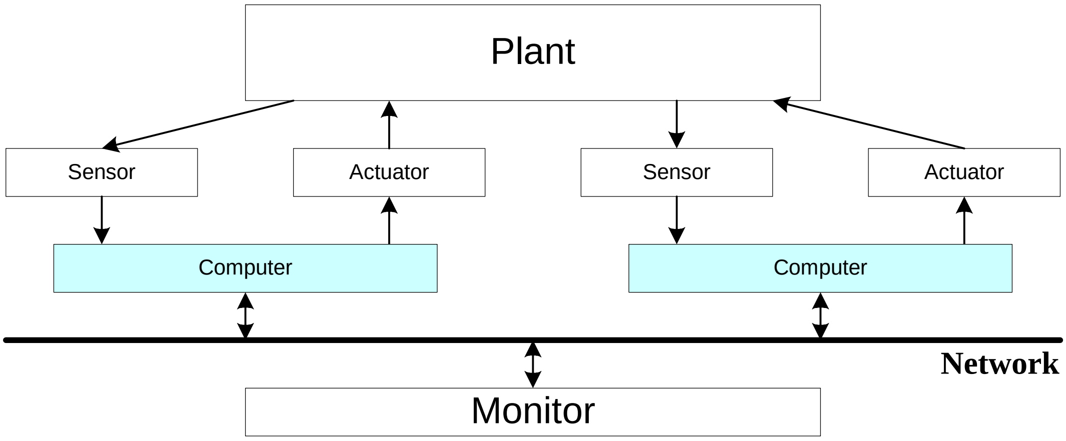

Figure 2-2 Distributed control system.

However, the DCS is loosely connected because most of the real-time control data (sensing, calculation, and actuation) are carried out within their individual process stations. For only on/off signals, alarm information, and the like were more suitable to be transmitted on the serial network.

In the 1990s, the development of the microprocessor had a serious impact on the way which computers ordinary are applied to control entire plants. Furthermore, the advancement in the technology of ASIC and the cheaper price in silicon so sensors and actuators can be equipped with network interfaces, and thus Computer-Control Systems become independent nodes on a real-time control network. The status creates a nice situation for Networked Control System (NCS), as presented in *Fig. 2-3*.

In NCS, real-time sensing, actuating and control data are transmitted on the network. In the other words, the network is the key for sensing, actuating and control data to co-operation.

Since the limitation of network, the data (sensing, controlled, actuator, etc.) in NCS can be transmitted once at a time only. Therefore, how to decrease the use for transmitting information on the network is very important.

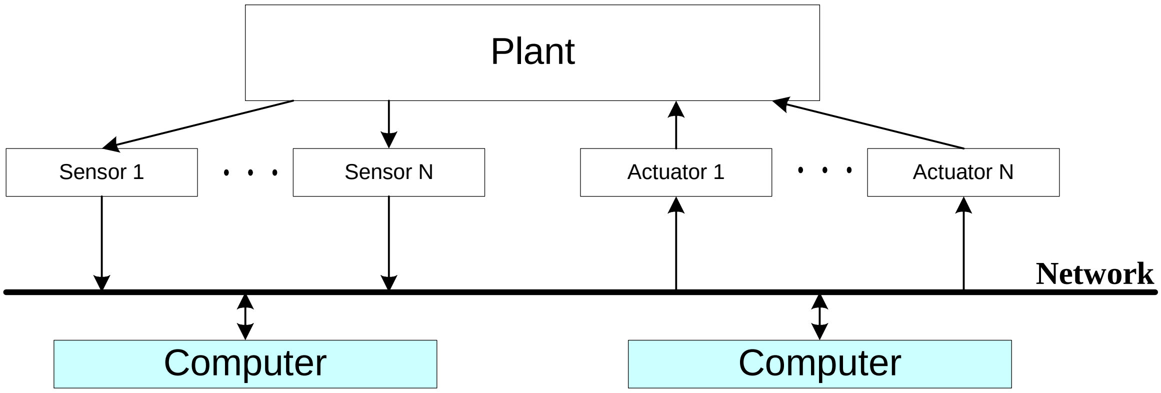

Figure 2-3 Networked control systems.

2.2 Problem Description

Networked control systems (NCSs) are being adopted in many application areas for a number of reasons [16] including their low cost, reduced weight, and power requirements, simple installation and maintenance, and higher reliability. However, using a network presents some new analytical challenges because the network imposes a communication constraint: only one data could be transmitted at a time. For the limitation of network, if we decrease the time of each node to be connected to network, it is efficacy to keep channel of network from jam-packed.

In general, the data we want to transfer must tie to network in NCSs, so the defect is the data does not be updated on real time. Ordinarily, the data includes three parts sensing data, control data, and actuator data. In this thesis, we aim at the "un-real time" sensing data. The sensing data can be transmitted at switch 'ON' only as shown in *Fig. 2-4*. It means when $S(t)$ is '1', then the states for now could be transferred by the network to update the feedback controller. Otherwise, the control signal could not be changed.

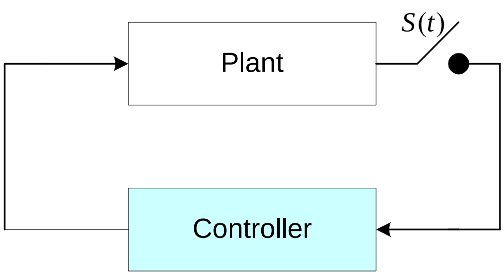

Figure 2-4 Un-real time feedback.

In our thesis, we assume there is a zero-order-hold in the controller, so when the switch is 'OFF', then the transmitted state will be hold as last state shown in *Fig. 2-5*. It means that, $x(t) = x(t_0)$ as $t_0 \le t < t_1$, $x(t) = x(t_1)$ as $t_1 \le t < t_2$, $x(t) = x(t_2)$ as $t_2 \le t < t_3$, etc. The figure also indicates the signification of the norm of the state error $\|e(t)\| = \|x(t) - x(t_k)\|$, $t_k \le t < t_{k+1}$.

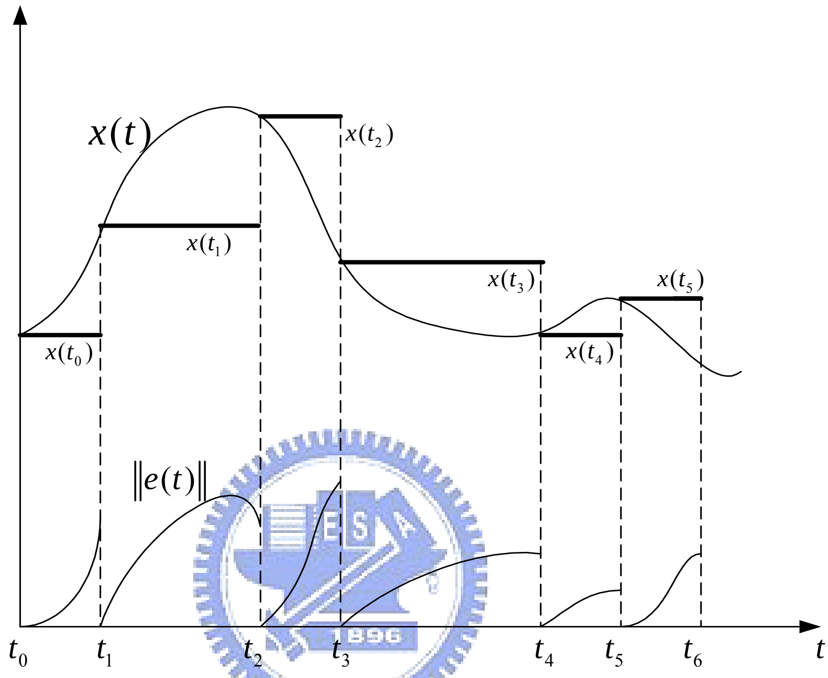

Figure 2-5 Signification of $x(t_k)$ and $\|e(t)\|$.

2.3 Preliminary

Before accessing the topic, we make some preliminary. First, we introduce the *Lyapunov theorem* which is an important theorem to verify whether a system is stable.

Theorem 1. **(Lyapunov theorem)** [8]

Consider the system $\dot{x}(t) = Ax(t)$, all eigenvalues of A have negative real parts if and only if for any given positive definite symmetric matrix Q, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique symmetric solution P and P is positive definite.

Theorem 2. **(Lyapunov theorem for non-autonomous systems)** [14]

Stability: If, in a ball B_{R_0} around the equilibrium point 0, there exists a scalar function $V(x,t)$ with continuous partial derivatives such that

1. V is positive definite
2. $\dot{V}$ is negative semi-definite

then the equilibrium point 0 is stable in the sense of Lyapunov.

Next, we represent the Bellman-Gronwall lemma for deriving Lemma 4, and Lemma 5.

Lemma 1. **(Bellman-Gronwall lemma 1 for fixed-inintial-time)**

Given $\lambda(t)$ and $k(t)$ non-negative piecewise continuous and differentiable functions of time t. If a function $y(t)$ satisfies $y(t) \le \lambda(t) + \int_{t_i}^{t} k(w)y(w)dw, \ \forall t \ge t_i \ge 0$, then

$$y(t) \le \lambda(t_i)e^{\int_{t_i}^{t} k(w)dw} + \int_{t_i}^{t} \dot{\lambda}(w)e^{\int_{w}^{t} k(\tau)d\tau} dw, \ \forall t \ge t_i \ge 0.$$

Lemma 2. **(Bellman-Gronwall lemma 2 for fixed-final-time)**

Given $\lambda(t)$ and $k(t)$ non-negative piecewise continuous functions of time t, with $\lambda(t)$ differentiable. If a function $y(t)$ satisfies $y(t) \le \lambda(t) + \int_{t}^{t_f} k(w)y(w)dw, \ \forall t_f \ge t \ge 0$, then

$$y(t) \le \lambda(t_f)e^{\int_{t}^{t_f} k(w)dw} - \int_{t}^{t_f} \dot{\lambda}(w)e^{\int_{t}^{w} k(\tau)d\tau} dw, \ \forall t_f \ge t \ge 0.$$

The proofs in detail are given in the *Appendix*. The general form of Bellman-Gronwall lemma has been detailed in [7] and [2].

Consider the feedback control system as shown in *Fig. 2-6:*

$$\dot{x}(t) = Ax(t) + Bu(t), \ u(t) = -Kx(t_k), \ t \in [t_k, t_{k+1}) \tag{2-1}$$

where $x(t) \in \Re^n$ is the state of the system, $u(t) \in \Re^m$ is the control input, A and B are known matrices with proper dimensions, and t_i is the time of switch 'ON'.

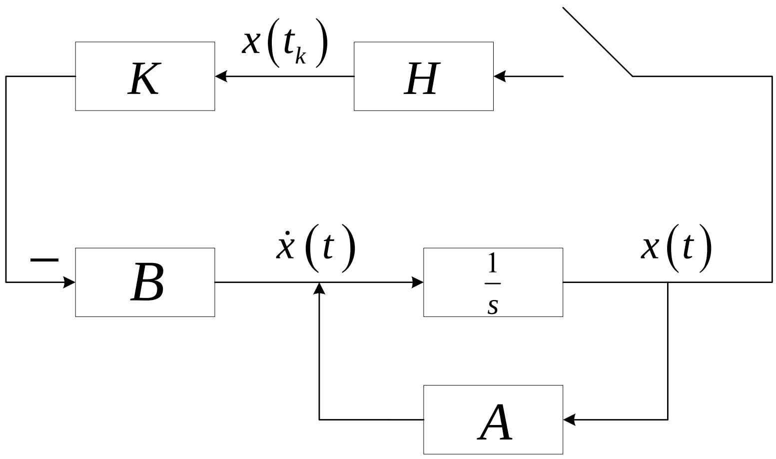

Figure 2-6　System $\dot{x}(t) = Ax(t) + Bu(t)$, where $u(t) = -Kx(t_k)$, H denotes a zero-order-hold stage.

Define $e(t) = x(t) - x(t_k)$, where $t_k \le t < t_{k+1}$. Then we call $\|e(t)\|$ as "transmission error". In the following, the Lemma 3 gives a bound on the transmission error. The proof in detail is given in the *Appendix*.

Lemma 3. **(Transmission Error Upper Bound)** [3]

The transmission error $e(t)$ *defined as* $[x(t) - x(t_k)]$ *is bounded by*

$$\|e(t)\| \le \frac{\|A - BK\|}{\|A\|}(e^{\|A-BK\|(t-t_k)} - 1)\|x(t_k)\|, \qquad t \in [t_k, t_{k+1})$$

between two successive transmissions at t_k and t_{k+1}.

Chapter3

Networked Control Systems with No Disturbance

3.1 System Model

3.1.1 Normal Control Systems

Now we model the NCSs for system with no disturbance first. If there is no network, the sketch map of this system can be shown as in *Fig. 3-1-1*.

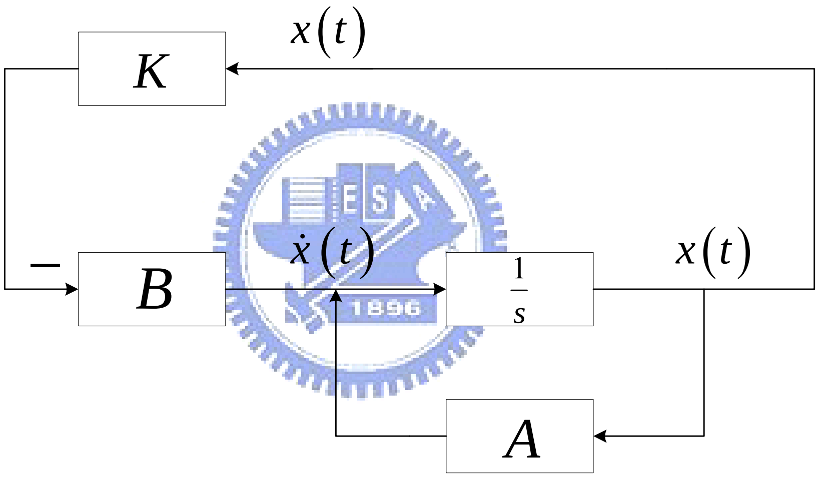

Figure 3-1-1 System $\dot{x}(t) = Ax(t) + Bu(t)$, where $u(t) = -Kx(t)$.

The dynamic equation of the system is as:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3-1}$$

where $x(t) \in \Re^n$ is the state of the system, $u(t) \in \Re^m$ is the control, $u(t) = -Kx(t)$, and A and B are known matrices with proper dimensions. Suppose the original system ($\dot{x}(t) = Ax(t)$) has eigenvalues in the right-half-plane, so it is an unstable system. We must design a state feedback controller such that the system to be stable. For convenience, we define $\bar{A} = A - BK$, and the controlled system can be rewritten as $\dot{x}(t) = (A - BK)x(t)$ $= \bar{A}x(t)$.

From Theorem 1 and 2, if $x = 0$ is a globally exponentially stable equilibrium point of the non-networked system $\dot{x}(t) = \bar{A}x(t)$, there exists a unique symmetric positive definite matrix P to the Lyapunov equation:

$$\bar{A}^T P + P\bar{A} = -I \tag{3-2}$$

Let $V(x(t)) = x^T(t)Px(t)$ be a Lyapunov function of the non-networked system, then the closed-loop system satisfies the following inequalities:

$$\sigma_{\min} \|x(t)\|^2 \le V(x(t)) \le \sigma_{\max} \|x(t)\|^2$$

where $\sigma_{\min}$ is the minimum eigenvalue of P, and $\sigma_{\max}$ is the maximum eigenvalue of P. We have

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &= x^T(t)(\bar{A}^T P + P\bar{A})x(t) \\ &= -x^T(t)x(t) \\ &= -\|x(t)\|^2 < 0 \end{aligned} \tag{3-3}$$

Therefore, the system (3-1) is stable.

3.1.2 Networked Control Systems

When we connect the feedback channel of sensors to the network, the system (3-1) can be given as in *Fig. 3-1-2*. In this case, $u(t) = -Kx(t_k),\ \ as\ t \in [t_k, t_{k+1})$. Thus, it becomes

$$\dot{x}(t) = Ax(t) - BKx(t_k),\ t \in [t_k, t_{k+1}). \tag{3-4}$$

Now, we drive a useful form between $x(t)$ and $x(t_k)$.

$$\begin{aligned} x(t) &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-s)}Bu(s)ds \\ &= e^{A(t-t_k)}x(t_k) - \int_{t_k}^{t} e^{A(t-s)}BKx(t_k)ds \\ &= e^{A(t-t_k)}x(t_k) + A^{-1}\left(I - e^{A(t-t_k)}\right)BKx(t_k) \end{aligned}$$

Define $\tau = t - t_k$, then

$$x(t) = \left[e^{A\tau} + A^{-1}\left(I - e^{A\tau}\right)BK\right]x(t_k) \tag{3-5}$$

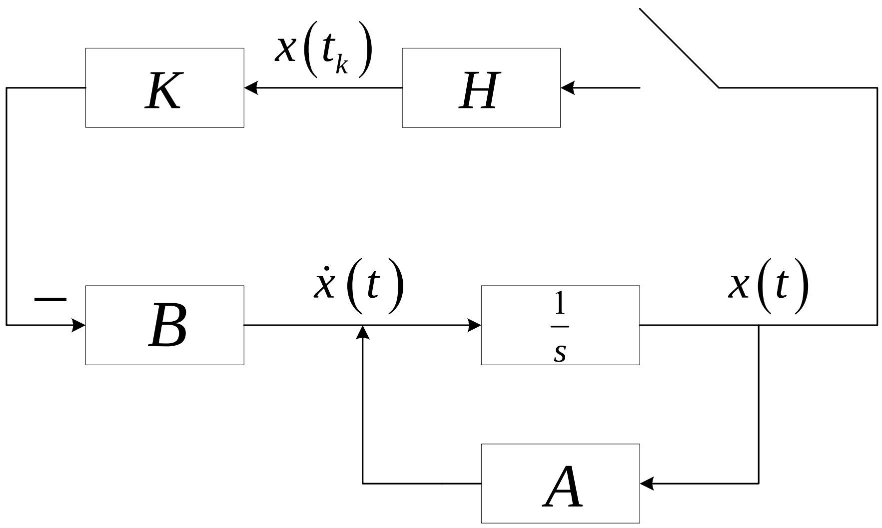

Figure 3-1-2 System $\dot{x}(t) = Ax(t) + Bu(t)$, where $u(t) = -Kx(t_k)$, H denotes a zero-order-hold stage.

Now we derive the $\dot{V}(x(t))$ of the system (3-4) with $V(x(t)) = x^T(t)Px(t)$, where P is the solution of (3-2).

$$
\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= (Ax(t) - BKx(t_k))^T Px(t) + x^T(t)P(Ax(t) - BKx(t_k)) \\
&= x^T(t)(\bar{A}^T P + P\bar{A})x(t) + x^T(t)(K^T B^T P + PBK)x(t) \\
&\qquad - x^T(t_k)K^T B^T Px(t) - x^T(t)PBKx(t_k) \\
&= -\|x(t)\|^2 + [x(t) - x(t_k)]^T K^T B^T Px(t) + x^T(t)PBK[x(t) - x(t_k)] \\
&= -\|x(t)\|^2 + e^T(t)K^T B^T Px(t) + x^T(t)PBKe(t) \\
&\le -\|x(t)\|^2 + 2\|PBK\|\|x(t)\|\|e(t)\|.
\end{aligned}
$$

If we can get

$$-\|x(t)\|^2 + 2\|PBK\|\|e(t)\|\|x(t)\| < 0, \text{ for all } x \ne 0 \qquad (3\text{-}7)$$

then the system (3-4) is stable according to the Lyapunov theorem.

3.2 Transmission Stability

Combining the goal of control to make system to be stable and reduce the network usage, to find out the maximum allowable un-updated interval, τ_m, to ensure the stability of NCS is our work. The NCSs can be guaranteed to be stable if the control signal is updated in a period of τ_m. We call $(t_{k+1} - t_k)$ as 'transmission period', and the stability of the system under this transmission period as 'transmission stability'.

Lemma 4. **(Transmission Stability of Networked Control Systems)**

Let $x = 0$ be the globally exponentially stable equilibrium point of the non-networked system $\dot{x}(t) = (A - BK)x(t)$ with transmission period τ =0. If the transmission period, τ (>0), satisfies the following conditions (a) and (b), then the origin in period τ is also the globally exponentially stable equilibrium point of the NCS ($\dot{x}(t) = Ax(t) - BKx(t - \tau)$).

(a) $\tau = \frac{1}{\|A\|}\ln W$, where W satisfies the inequality

$$\|\bar{A}\|W^2 + \left(\frac{1}{2\|PBK\|} - \|\bar{A}\|\right)W + \left(-\frac{1}{2\|PBK\|}\|A\|\|BK\| - \frac{1}{2\|PBK\|}\right) < 0,$$

where $\bar{A} = A - BK$.

(b) $\tau < \frac{1}{\|A\|}\ln\left(\frac{\|BK\| + \|A\|}{\|BK\|}\right)$.

The maximum accepted value of updating period τ is the minimum of maximum of (a) and (b).

Proof.

In order to derive the relation between $x(t)$ and $x(t_k)$, we fix the final time, t_f , and let the initial time, t , be changeable, i.e., $t_k \le t \le t_f \le t_{k+1}$.

From the system (3-4), we have

$$x(t) = x(t_f) + \int_{t_f}^{t} \left[Ax(w) - BKx(t_k)\right]dw$$

Let $G = -BKx(t_k)$. Then $\|G\| \le \|BK\|\|x(t_k)\|$.

We can obtain

$$\|x(t)\| \le \underbrace{\|x(t_f)\| + \|G\|(t_f - t)}_{\lambda(t)} + \int_{t_f}^{t} \underbrace{\|A\|}_{k(w)}\|x(w)\|dw$$

Set $\lambda(t) = \|x(t_f)\| + \|G\|(t_f - t)$ and $k(w) = \|A\|$. Using Lemma 2, we get

$$\|x(t)\| \le \lambda(t_f)e^{\int_t^{t_f}\|A\|dw} - \int_t^{t_f}\left(-\|G\|\right)e^{\int_t^{w}\|A\|ds}dw$$

$$= \|x(t_f)\|e^{\|A\|(t_f - t)} + \|G\|\int_t^{t_f} e^{\|A\|(w-t)}dw$$

$$= \|x(t_f)\| e^{\|A\|(t_f - t)} + \|G\| \|A\|^{-1} (e^{\|A\|(t_f - t)} - 1) .$$

Let $t = t_k$, $t_f = t$, $t = t_k + \tau$, then we have

$$\|x(t_k)\| \le \|x(t)\| e^{\|A\|(t - t_k)} + \|G\| \|A\|^{-1} (e^{\|A\|(t - t_k)} - 1) .$$

It becomes

$$\|x(t_k)\| \le \|x(t)\| e^{\|A\|(t - t_k)} + \|BK\| \|x(t_k)\| \|A\|^{-1} (e^{\|A\|(t - t_k)} - 1) .$$

That is,

$$\left(1 - \|BK\| \|A\|^{-1} (e^{\|A\|(t - t_k)} - 1)\right) \|x(t_k)\| \le \|x(t)\| e^{\|A\|(t - t_k)} \tag{3-8}$$

Let $\tau = t - t_k$. If $1 - \|BK\| \|A\|^{-1} (e^{\|A\|\tau} - 1) > 0$, then (3-8) becomes

$$\|x(t_k)\| \le \frac{e^{\|A\|\tau} \|x(t)\|}{1 - \|BK\| \|A\|^{-1} (e^{\|A\|\tau} - 1)} . \tag{3-9}$$

With inequalities (3-9) and Lemma 3, we can drive

$$\|e(t)\| \le \frac{\|\bar{A}\|}{\|A\|} (e^{\|A\|\tau} - 1) \|x(t_k)\| \le \frac{\|\bar{A}\|}{\|A\|} (e^{\|A\|\tau} - 1) \left[\frac{e^{\|A\|\tau}}{1 - \|BK\| \|A\|^{-1} (e^{\|A\|\tau} - 1)} \|x(t)\| \right] .$$

Therefore,

$$\|e(t)\| \le \frac{\|\bar{A}\|}{\|A\|} (e^{\|A\|\tau} - 1) \frac{e^{\|A\|\tau}}{1 - \|BK\| \|A\|^{-1} (e^{\|A\|\tau} - 1)} \|x(t)\| . \tag{3-10}$$

Select $\gamma = (e^{\|A\|\tau} - 1) \dfrac{\|\bar{A}\| e^{\|A\|\tau}}{\|A\| \|BK\| - (e^{\|A\|\tau} - 1)}$, then inequality becomes $\|e(t)\| \le \gamma \|x(t)\|$.

From (3-7) and (3-10), we have

$$(-1 + 2\gamma \|PBK\|) \|x(t)\|^2 < 0 . \tag{3-11}$$

That is, $\gamma < \dfrac{1}{2\|PBK\|}$ must hold.

If we summarize the results, the interval τ must satisfy:

(a) $(e^{\|A\|\tau} - 1) \dfrac{\|\bar{A}\| e^{\|A\|\tau}}{\|A\| \|BK\| - (e^{\|A\|\tau} - 1)} < \dfrac{1}{2\|PBK\|}$

(b) $1 - \|BK\| \|A\|^{-1} (e^{\|A\|\tau} - 1) > 0$

Our goal is to fetch out τ**,** so rewrite (a)

$$\Rightarrow \|\bar{A}\|(e^{\|A\|\tau}-1)e^{\|A\|\tau} < \frac{1}{2\|PBK\|}\left[\|A\|\|BK\|-(e^{\|A\|\tau}-1)\right]$$

$$\Rightarrow \|\bar{A}\|(e^{\|A\|\tau})^2-\|\bar{A}\|e^{\|A\|\tau} < \frac{1}{2\|PBK\|}\|A\|\|BK\|-\frac{1}{2\|PBK\|}e^{\|A\|\tau}+\frac{1}{2\|PBK\|}$$

$$\Rightarrow \|\bar{A}\|(e^{\|A\|\tau})^2+(\frac{1}{2\|PBK\|}-\|\bar{A}\|)e^{\|A\|\tau}+(-\frac{1}{2\|PBK\|}\|A\|\|BK\|-\frac{1}{2\|PBK\|})<0$$

let $W=e^{\|A\|\tau}$, then we get

$$\|\bar{A}\|W^2+(\frac{1}{2\|PBK\|}-\|\bar{A}\|)W+(-\frac{1}{2\|PBK\|}\|A\|\|BK\|-\frac{1}{2\|PBK\|})<0\,.$$

We can solve it, and get the value of W. After that we can take $\tau=\frac{1}{\|A\|}\ln W$.

Rewrite (b), it can be $e^{\|A\|\tau}<\frac{\|BK\|\|A\|^{-1}+1}{\|BK\|\|A\|^{-1}}$.

Then we can gain the expectance

$$\tau<\frac{1}{\|A\|}\ln\left(\frac{\|BK\|+\|A\|}{\|BK\|}\right).$$

3.3 Simulation

3.3.1 Validating Lemma 4

In this section, Lemma 4 is verified by simulatinons. The target of this lemma is getting the maximum interval time between two updated points. We denote it as τ_m.

The step of verifying is as following. First, we consider some unstable systems. Next we find the feedback gain, K, by using the Pole-assignment method in order to make all of the eigenvalues of $A-BK$ lie in the left-half-plane. Finally, we take τ_m found by Lemma 4 to be the interval time of two updated points in these system.

System Ⅰ.

The system is $\dot{x}(t)=Ax(t)+Bu(t)$, where $u(t)=-Kx(t)$, $x(t)\in\Re^{2\times1}$ is the state of the system, $u(t)\in R$ is the control signal, and $A=\begin{bmatrix}0 & 1.5\\ -1.5 & 0\end{bmatrix}$, $B=\begin{bmatrix}3\\ 2\end{bmatrix}$, $x(0)=\begin{bmatrix}10\\ 20\end{bmatrix}$.

The poles of matrix A are $\pm 1.5i$, so it is an oscillating system. When we choose $K = \begin{bmatrix} 0.6667 & 0.5 \end{bmatrix}$, then eigenvalues of $\bar{A} = A - BK$ are -2 and -1. The closed-loop system is stable.

From Lemma 4, we can obtain τ_m is the smallest one of 0.8475s and 0.27s, so we select $\tau_m = 0.27s$ for this system.

The feedback state is updated in a period of 0.27 sec. *Fig. 3-3-1(a)* shows the states of original system Ⅰ. The meaning of OFF and ON is shown in *Fig.3-3(a)* and *Fig.3-3(b). Fig. 3-3-1(b)* shows two kinds of controlled states in system Ⅰ. One of them is real-time control. The other one is as controller updated in a period of 0.27s. *Fig. 3-3-1(c)* shows the above control inputs $u(t) = -Kx(t)$ and $u(t) = -Kx(t_k)$.

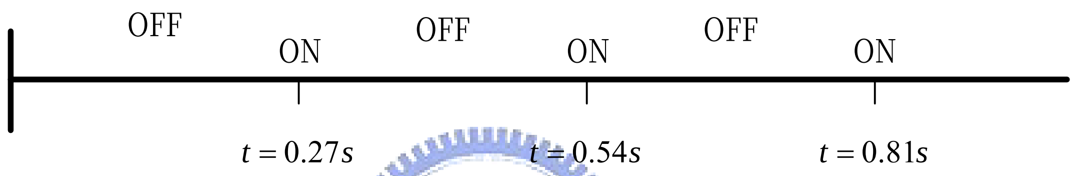

Figure 3-3-1 The switch assignment in system Ⅰ.

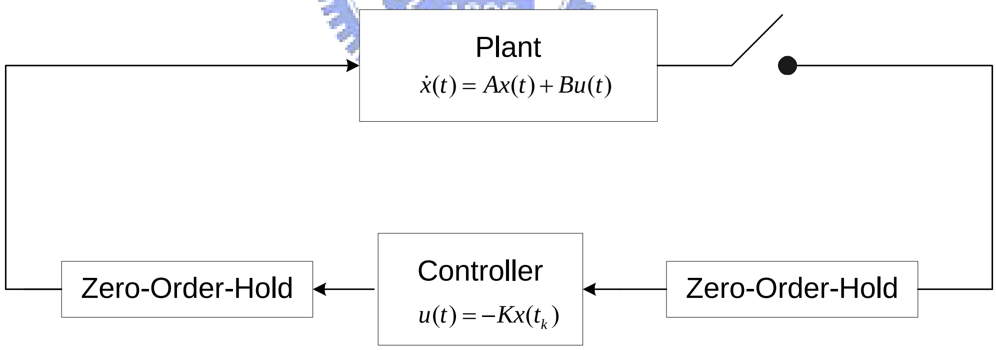

Figure 3-3(a) Switch OFF.

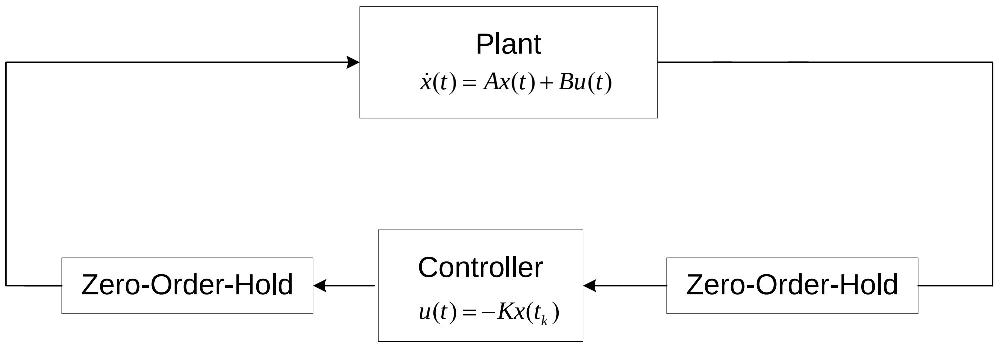

Figure 3-3(b) Switch ON.

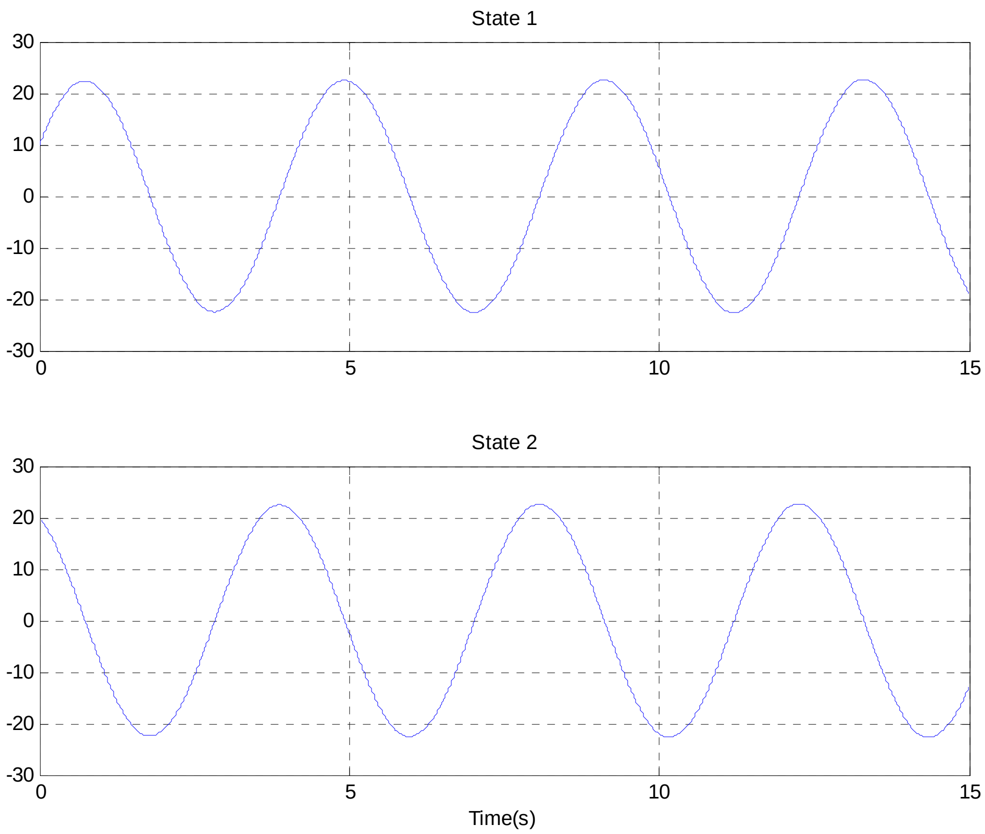

Figure 3-3-1(a) Original states in system Ⅰ.

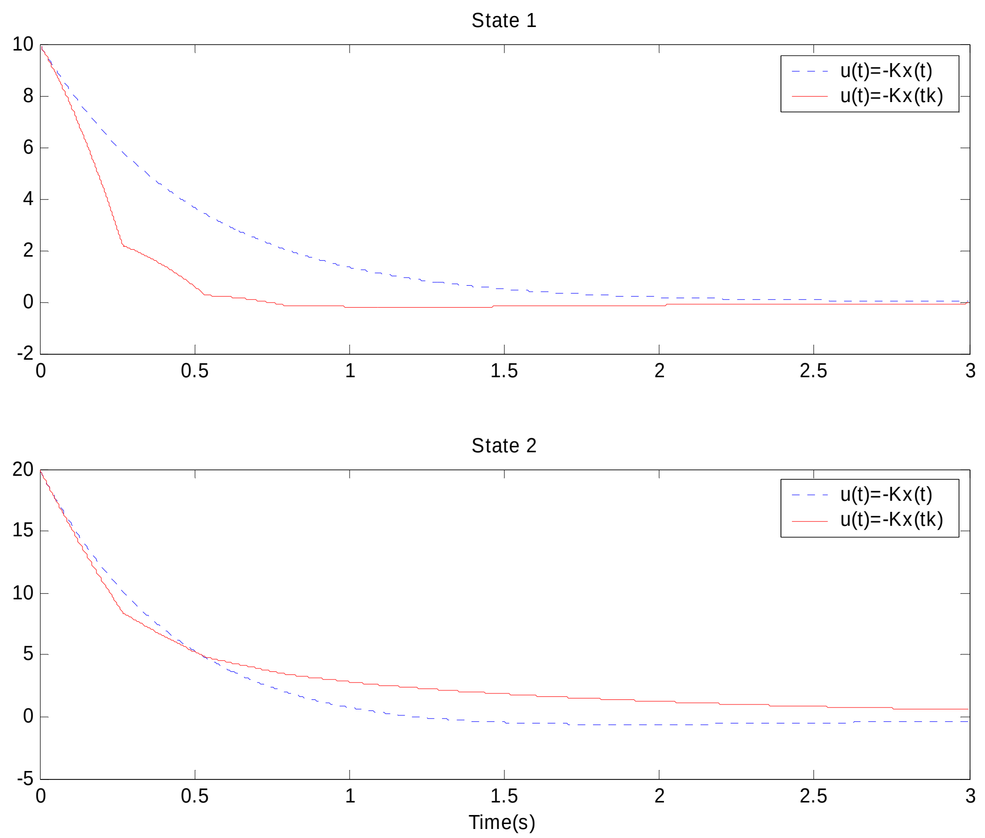

Figure 3-3-1(b) States of system Ⅰ under controlled.

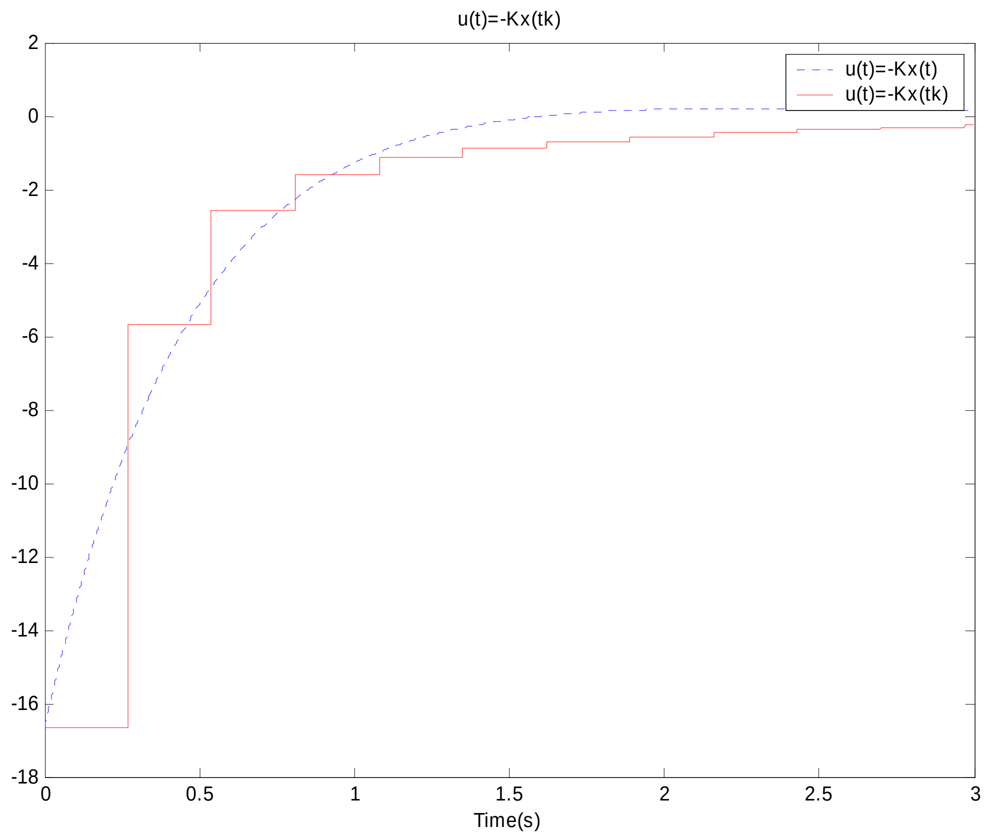

Figure 3-3-1 (c) Control input in system Ⅰ.

System Ⅱ.

The system is $\dot{x}(t) = Ax(t) + Bu(t)$, where $u(t) = -Kx(t)$, $x(t) \in \Re^{2\times1}$ is the state of the system, $u(t) \in R$ is the control signal, and $A = \begin{bmatrix} -1 & 3 \\ 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $x(0) = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

The poles of matrix A are -2.57 and 8.57. When we choose $K = [1.296 \quad 2.556]$, then eigenvalues of $\bar{A} = A - BK$ are -1, -2. The closed-loop system is stable.

From Lemma 4, we can obtain τ_m is the smallest one of 0.2272s and 0.07s, hence $\tau_m = 0.07s$ for this system.

The feedback state is updated in a period of 0.07 sec. *Fig. 3-3-2(a)* shows the states of original system Ⅱ. The meaning of OFF and ON is shown in *Fig.3-3(a)* and *Fig.3-3(b)*. *Fig. 3-3-2(b)* shows two kinds of controlled states in system Ⅱ. One of them is real-time control. The other one is as controller updated in a period of 0.07s. *Fig. 3-3-2(c)* shows the above control inputs $u(t) = -Kx(t)$ and $u(t) = -Kx(t_k)$.

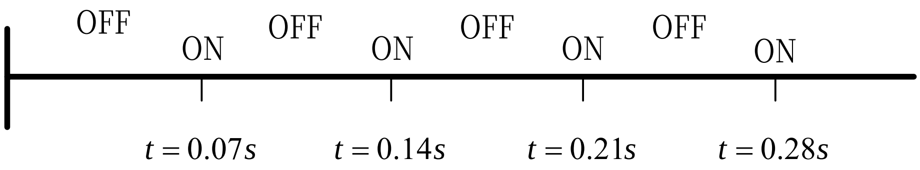

Figure 3-3-2 The switch assignment in system Ⅱ.

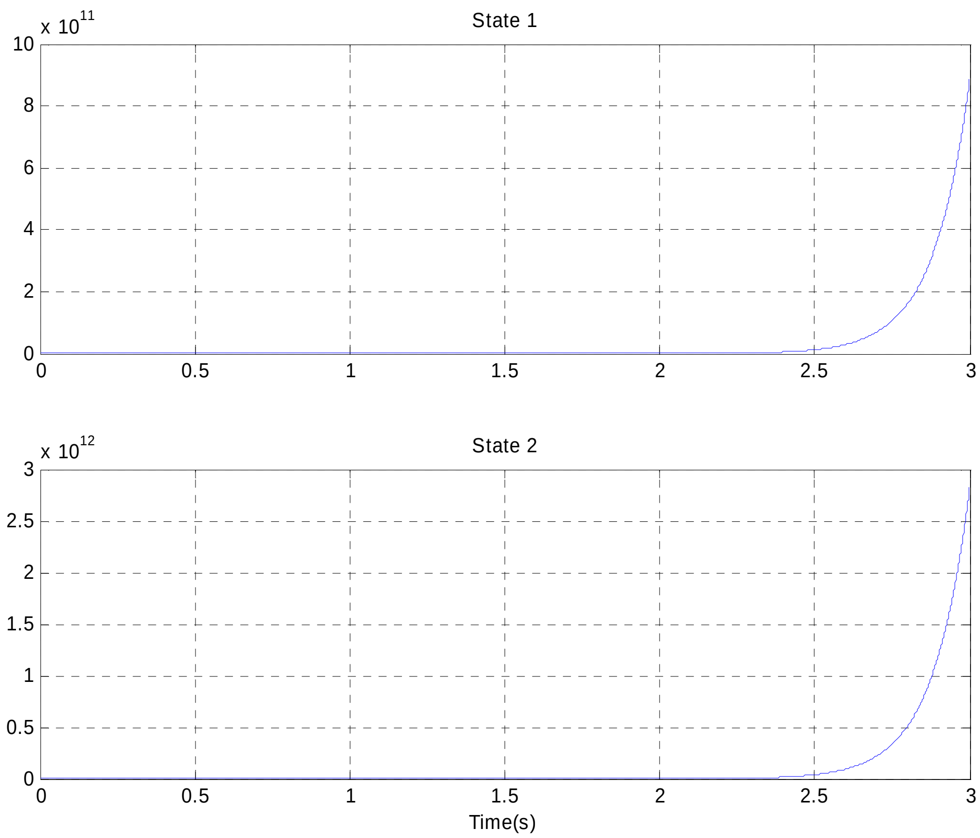

Figure 3-3-2(a) Original states in system Ⅱ.

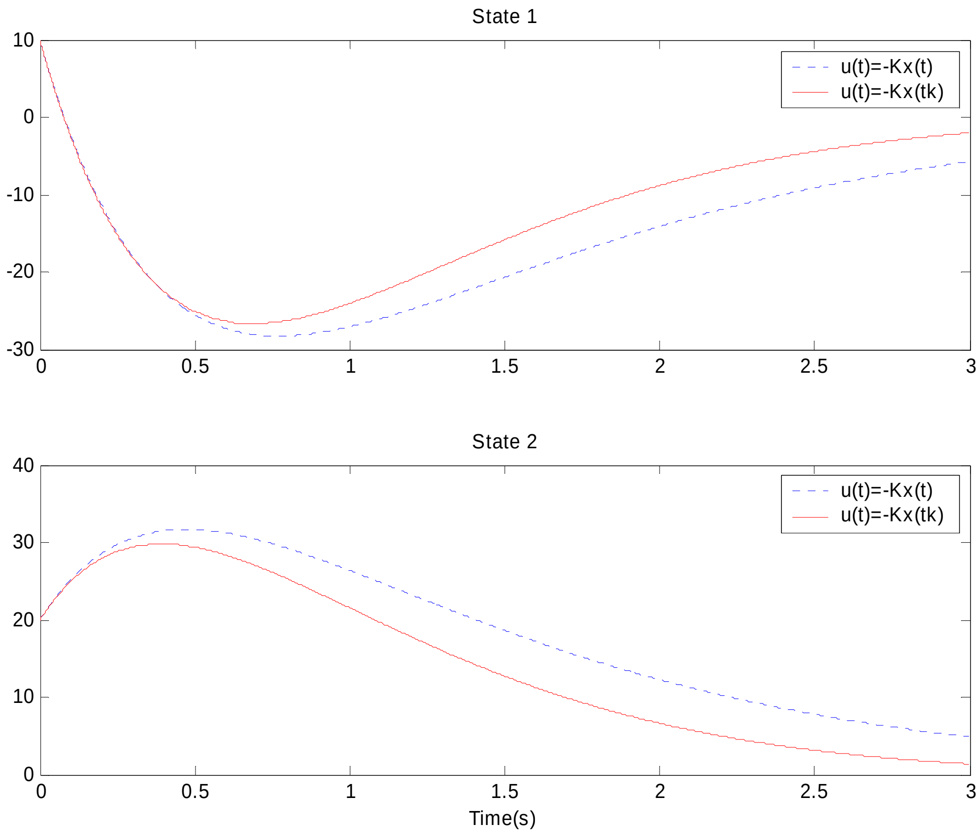

Figure 3-3-2(b) **States of system Ⅱ under controlled.**

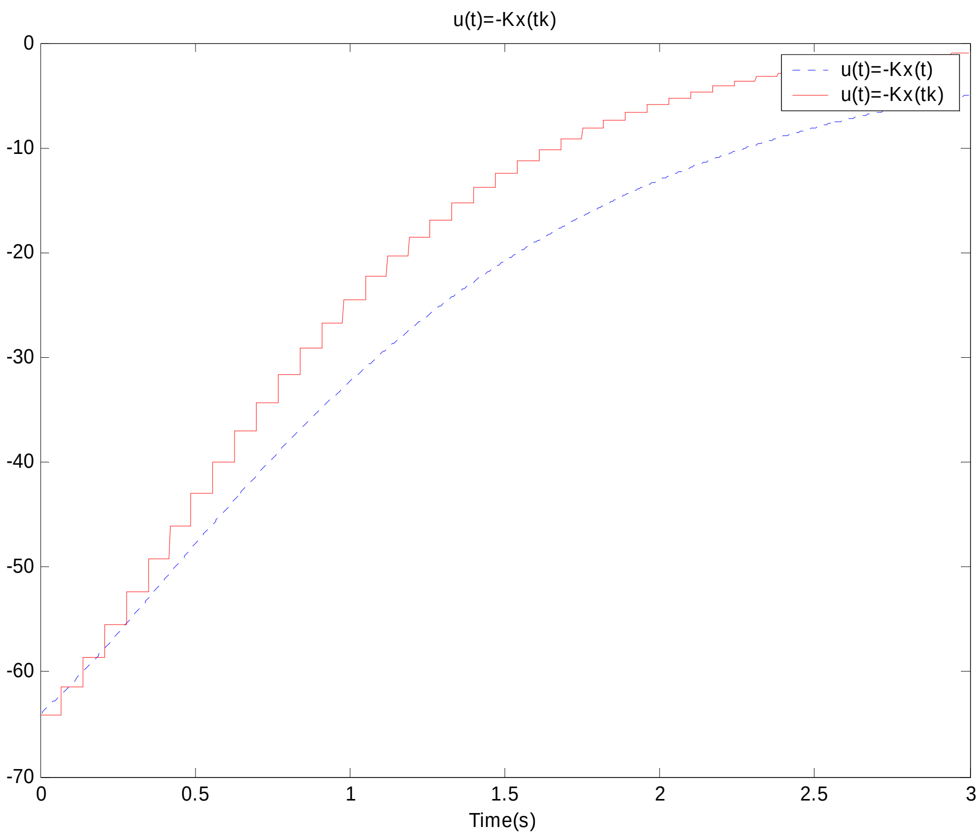

Figure 3-3-2(c) **Control input in system Ⅱ.**

System Ⅲ.

Now we consider a 4-order system described by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \Re^{4\times 1}$ is the state of the system, $u(t) \in R$ is the control signal, and

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -4 & 3 & 0 \\ 0.3 & 0.2 & -6 & 1 \\ 0 & -0.4 & 0.2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -8 \\ 2 \\ 5 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 10 \\ 20 \\ -5 \\ -2 \end{bmatrix}.$$

The poles of matrix A are $0.526 \pm 0.6023i$, -6.37 and -2.6814. Choosing $K = \begin{bmatrix} 0.26 & -0.24 & -0.015 & 0.87 \end{bmatrix}$, then eigenvalues of $\bar{A} = A - BK$ are $-1, -2, -4, -7$. The closed-loop system is stable.

From Lemma 4, we can obtain τ_m is the smallest one of 0.3444s and 0.08s, hence $\tau_m = 0.08s$ for this system.

The feedback state is updated in a period of 0.08 sec. *Fig. 3-3-3(a)* shows the states of original system Ⅲ. The meaning of OFF and ON is shown in *Fig.3-3-3(a)* and *Fig.3-3-3(b). Fig. 3-3-3(b)* shows two kinds of controlled states in system Ⅲ. One of them is real-time control. The other one is as controller updated in a period of 0.07s. *Fig. 3-3-3(c)* shows the above control inputs $u(t) = -Kx(t)$ and $u(t) = -Kx(t_k)$.

Figure 3-3-3 The switch assignment in system Ⅲ.

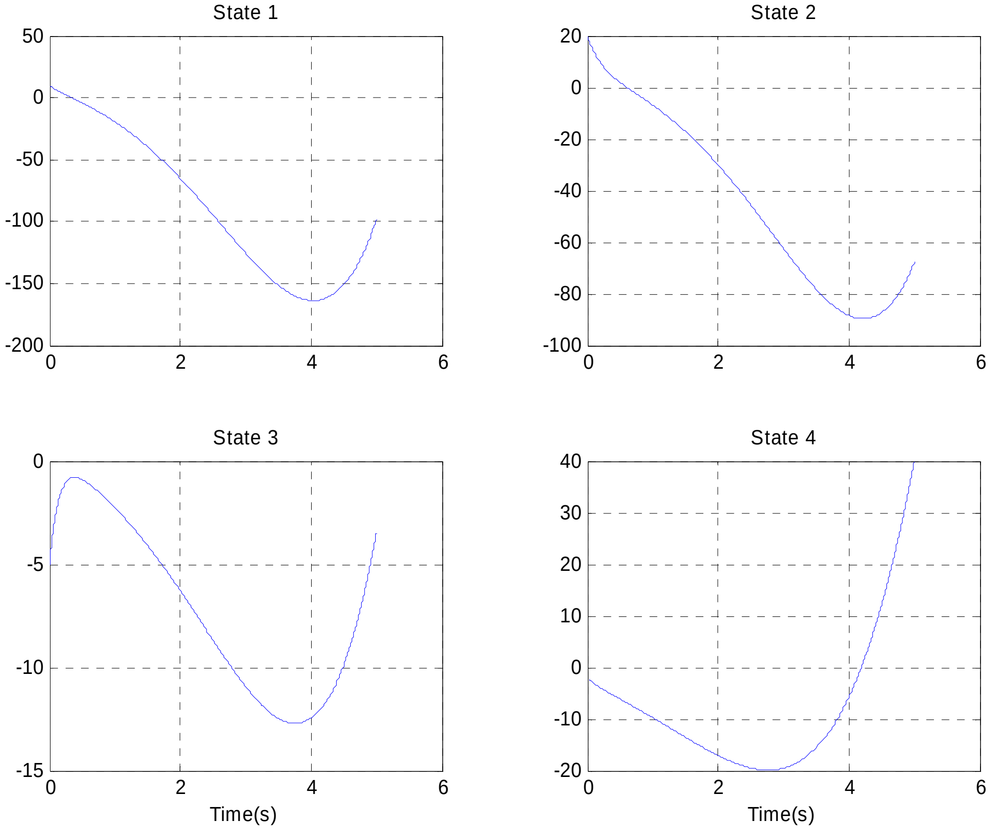

Figure 3-3-3(a) Original states in system Ⅲ.

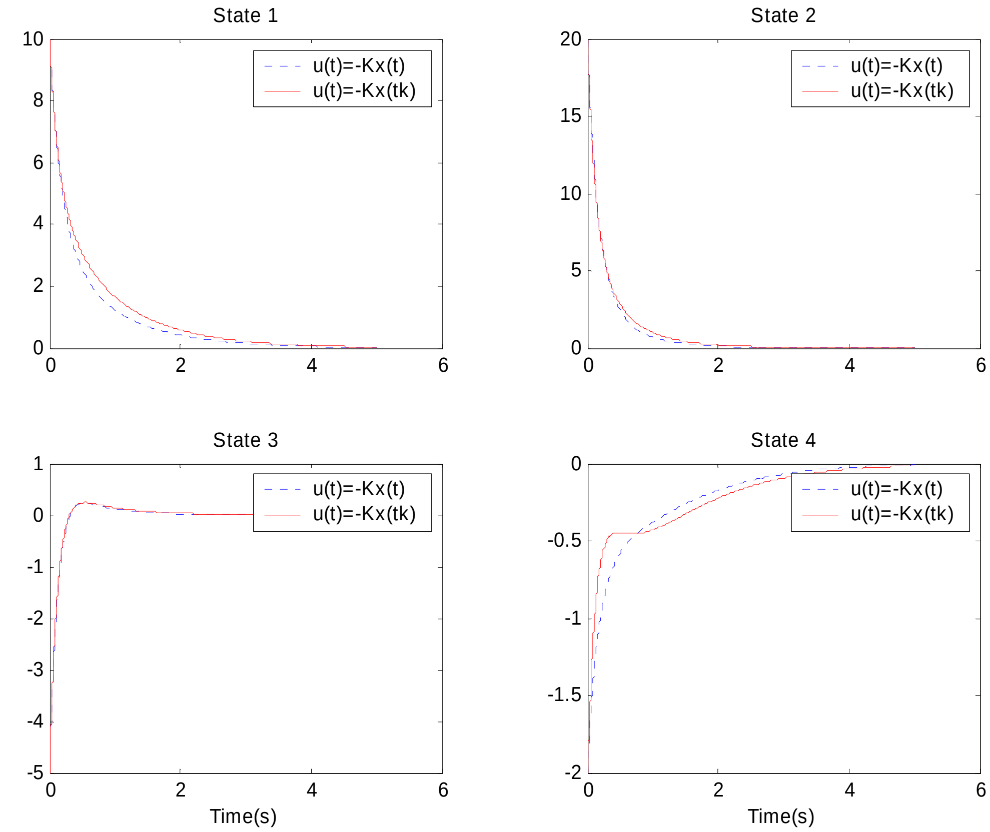

Figure 3-3-3(b) States of system Ⅲ under controlled.

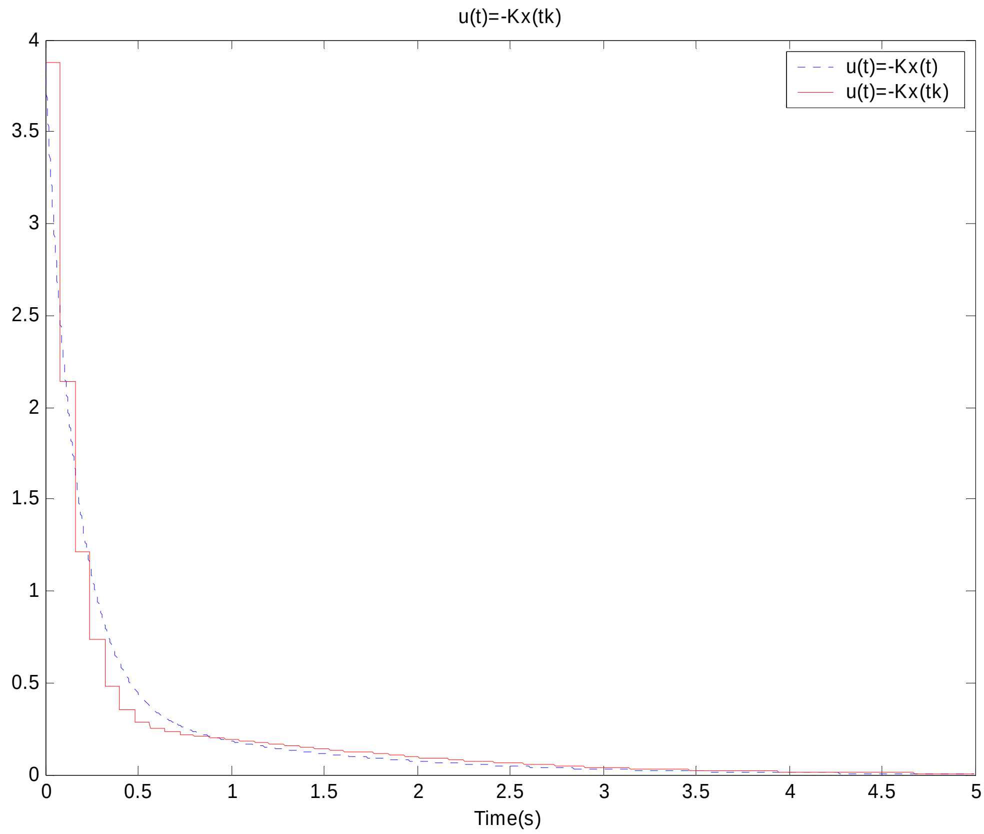

Figure 3-3-3(c) Control input in system Ⅲ.

3.3.2 Remark

The updated interval of each above system is listed *Table 3-1*.

Table 3-1 List of the updated interval.

	System Ⅰ	System Ⅱ	System Ⅲ
Original poles	$\pm 1.5i$	$-2.57,\ 8.57$	$0.526 \pm 0.6023i, -6.37,\ -2.6814$
New poles	$-1,\ -2$	$-1,\ -2$	$-1, -2, -4, -7$
Updated interval	0.27 sec	0.07 sec	0.08 sec

From system Ⅰ and system Ⅱ, we can obtain that if the original system is more unstable (the poles is more positive), then in general the updated interval is shorter. From system Ⅲ, we can get Lemma 4 also suit to use in high order system.

Chapter4

Networked Control Systems with Disturbance

4.1 System Model

4.1.1 Normal Control Systems

Now we model the NCSs for system with disturbance. If there is no network, the sketch map of this system can be shown as in *Fig. 4-1-1*.

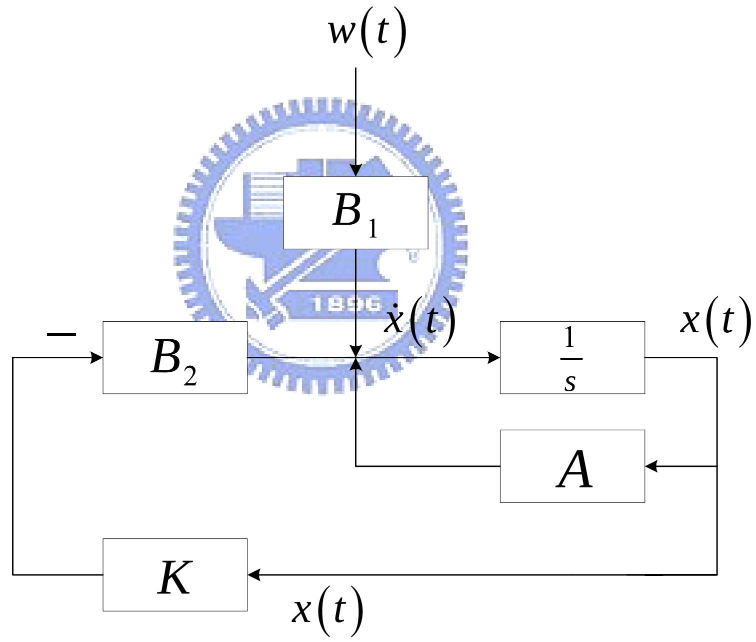

Figure 4-1-1 System $\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t)$, where $u(t) = -Kx(t)$.

Consider the system

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t)\,, \tag{4-1}$$

where $x(t) \in \Re^n$ is the state of the system, $u(t) \in \Re^m$ is the control signal, $w(t) \in R^p$ is the disturbances satisfying $\|w(t)\| \le \rho$ (ρ is a positive scalar), and A, B_1, B_2 are known matrices with proper dimensions. Suppose the original system ($\dot{x}(t) = Ax(t)$) has eigenvalues in the right-half-plane. We must design a state feedback controller such that the system to be stable. For convenience, we define $\bar{A} = A - B_2 K$, which is stable, and the controlled system can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ &= \left(A - B_2 K\right) x(t) + B_1 w(t) \\ &= \bar{A}x(t) + B_1 w(t) \end{aligned} \tag{4-2}$$

The sketch map of this system is shown in *Fig. 4-1-1.* Now, we want to find the condition of the system (4-2) to be stable via Theorem1 and Theorem 2. Let $V(x(t)) = x^T(t)Px(t)$, where P is the solution of (3-2). Then,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &= x^T(t)(\bar{A}^T P + P\bar{A})x(t) + w^T(t)B_1^T Px(t) + x^T(t)PB_1 w(t) \\ &= -x^T(t)x(t) + w^T(t)B_1^T Px(t) + x^T(t)PB_1 w(t) \\ &= -\left\|x(t)\right\|^2 + w^T(t)B_1^T Px(t) + x^T(t)PB_1 w(t) \\ &\le -\left\|x(t)\right\|^2 + 2\rho\left\|PB_1\right\|\left\|x(t)\right\| \end{aligned}$$

We can observe if $-\left\|x(t)\right\|^2 + 2\rho\left\|B_1 P\right\|\left\|x(t)\right\| < 0$, that is if

$$\left\|x(t)\right\| > 2\rho\left\|PB_1\right\|, \tag{4-3}$$

the system *(4-2)* converges to neighborhood of origin.

4.1.2 Networked Control Systems

When we connect the feedback channel of sensors to the network, the system can be given as in *Fig. 4-1-2*. Then, $u(t) = -Kx(t_k)$ as $t \in \left[t_k, t_{k+1}\right)$. The closed-loop system becomes

$$\dot{x}(t) = Ax(t) + B_1 w(t) - B_2 Kx(t_k), \; as \; t \in \left[t_k, t_{k+1}\right). \tag{4-4}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, and $w(t) \in R^p$.

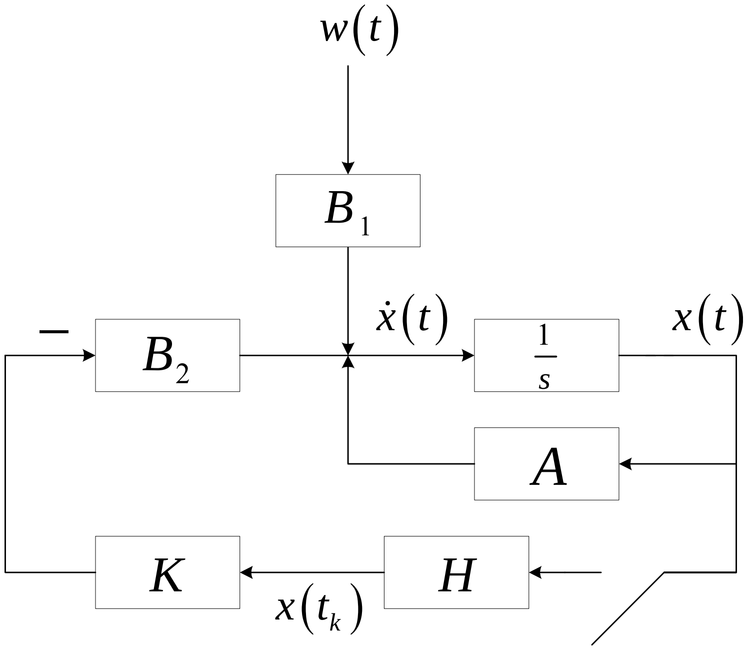

Figure 4-1-2 **System** $\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$ **, where** $u(t) = -Kx(t_k)$ **,** H **denotes a zero-order-hold stage.**

Let $V(x(t)) = x^T(t)Px(t)$ be a Lyapunov function of the networked control system, where P is the solution of (3-2). For the system given in (4-4),

$$
\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= (Ax(t) - B_2Kx(t_k) + B_1w(t))^T Px(t) + x^T(t)P(Ax(t) - B_2Kx(t_k) + B_1w(t)) \\
&= x^T(t)(\bar{A}^T P + P\bar{A})x(t) + x^T(t)(K^T B_2{}^T P + PB_2K)x(t) \\
&\qquad - x^T(t_k)K^T B_2{}^T Px(t) - x^T(t)PB_2Kx(t_k) + w^T(t)B_1^T Px(t) + x^T(t)PB_1w(t) \\
&= -\|x(t)\|^2 + \left[x(t) - x(t_k)\right]^T K^T B_2{}^T Px(t) - x^T(t)PB_2K\left[x(t) - x(t_k)\right] \\
&\qquad + w^T(t)B_1^T Px(t) + x^T(t)PB_1w(t) \\
&= -\|x(t)\|^2 + e^T(t)K^T B_2{}^T Px(t) - x^T(t)PB_2Ke(t) + w^T(t)B_1^T Px(t) + x^T(t)PB_1w(t) \\
&\le -\|x(t)\|^2 + 2\|PB_2K\|\|x(t)\|\|e(t)\| + 2\rho\|B_1P\|\|x(t)\|
\end{aligned}
$$

It means that the system (4-4) converges to neighborhood of origin, if

$$-\|x(t)\| + 2\|PB_2K\|\|e(t)\| + 2\|\rho\|\|B_1P\| < 0 . \qquad (4\text{-}6)$$

4.2 Transmission Error Upper Bound

From now on, the state is not real time feedback, and it would be interesting to find out how much the error is created. The system we consider here is (4-4), and then we define $e(t) = x(t) - x(t_k)$ where $t_k \le t < t_{k+1}$. We call $\|e(t)\|$ as "transmission error". Furthermore, we will derive the upper bound of transmission error in Lemma 5, and we call it as 'transmission error upper bound' shown in *Fig. 4-2-2*.

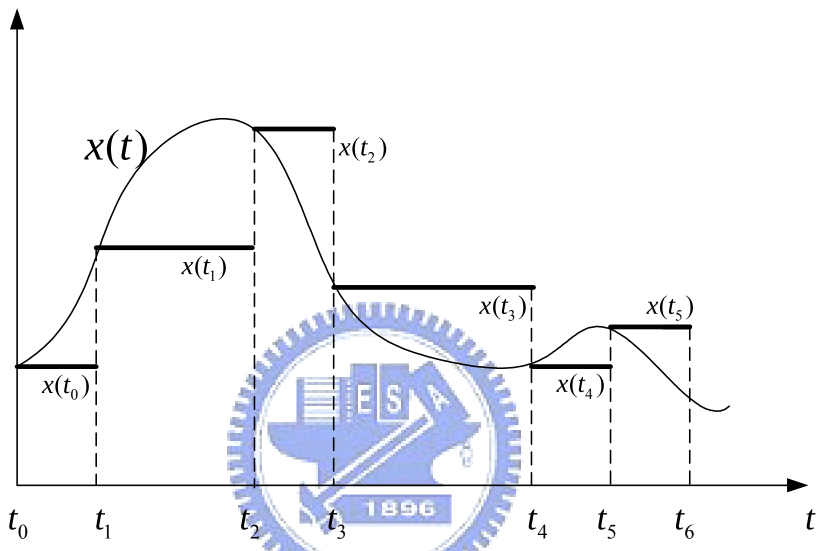

Figure 4-2-1 Signification of $x(t)$ and $x(t_k)$.

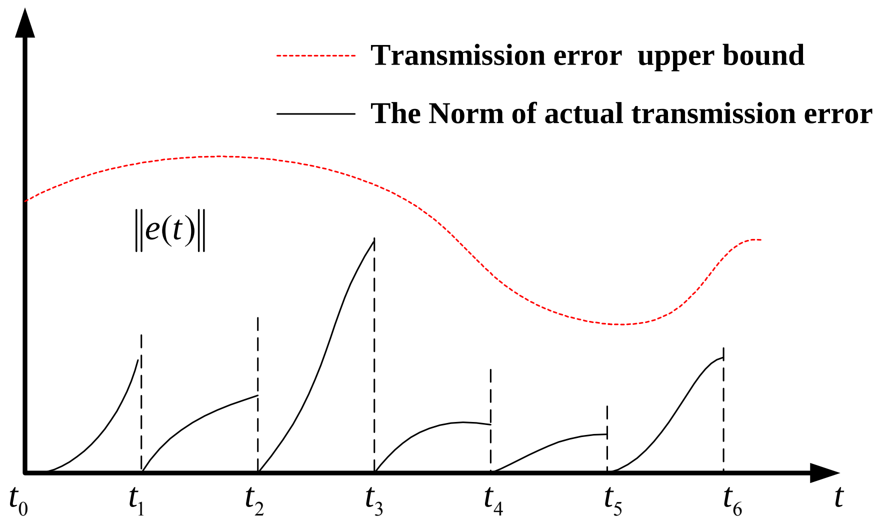

Figure 4-2-2 Signification of transmission error and transmission error upper bound.

Lemma 5.

(Transmission Error Upper Bound of Networked Control Systems with Disturbances)

The system is $\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$, *where* $\|w(t)\| \le \rho$, and the transmission error, $e(t)$, defined as $e(t) = \left[x(t) - x(t_k)\right]$ is bounded by

$$\|e(t)\| \le \frac{(e^{\|A\|\tau} - 1)}{\|A\|}\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right], \qquad t \in [t_k, t_{k+1})$$

between two successive transmissions, where $\bar{A} = A - B_2 K$, $\tau = t - t_k$.

Proof.

From (4-4), and let $\bar{A} = A - B_2 K$ we can derive

$$\dot{e}(t) = \dot{x}(t) = A\big(x(t) - x(t_k)\big) + Ax(t_k) - B_2 K x(t_k) + B_1 w(t) = Ae(t) + \bar{A}x(t_k) + B_1 w(t).$$

Taking the integral on both sides, we have

$$\int \dot{e}(t)dt = \int \left[Ae(v) + \bar{A}x(t_k) + B_1 w(v)\right] dv.$$

Then,

$$e(t) - e(t_k) = \int_{t_k}^{t} \left[Ae(v) + \bar{A}x(t_k) + B_1 w(t)\right] dv = \bar{A}x(t_k)(t - t_k) + \int_{t_k}^{t} \left[B_1 w(v) + Ae(v)\right] dv.$$

Substituting $t = t_k$, we get $e(t_k) = x(t_k) - x(t_k) = 0$. Therefore,

$$\|e(t)\| \le \underbrace{\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right](t - t_k)}_{\lambda(t)} + \int_{t_k}^{t} \underbrace{\|A\|}_{k(v)}\|e(v)\| \, dv.$$

Let $\lambda(t) = \left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right](t - t_k)$ and $k(v) = \|\bar{A}\|$. Using Lemma 1, hence

$$\|e(t)\| \le \lambda(t_k)e^{\int_{t_k}^{t}\|A\|dv} + \int_{t_k}^{t}\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right]e^{\|A\|(t-v)}dv.$$

where $\lambda(t) = \left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right](t - t_k)$. Then, $\lambda(t_k) = \left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right](t_k - t_k) = 0$. So

$$\begin{aligned}
\|e(t)\| &\le \int_{t_k}^{t}\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right]e^{\|A\|(t-v)}dv \\
&= \frac{\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right]}{\|A\|}(e^{\|A\|(t-t_k)} - 1) \\
&= \frac{(e^{\|A\|\tau} - 1)}{\|A\|}\left[\|\bar{A}\|\|x(t_k)\| + \|B_1\|\rho\right], \qquad t \in [t_k, t_{k+1}).
\end{aligned}$$

4.3 Simulation

4.3.1 Validating Lemma 5

There are two systems chosen in this section for verifying Lemma 5.

System Ⅳ.

Consider the following system:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$

where $x(t) \in \Re^{2\times 1}$ is the state of the system, $u(t) \in R$ is the control signal, and $A = \begin{bmatrix} -1 & 5 \\ 3 & 7 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\tau = 0.1\,\text{s}$.

The poles of matrix A are 8.6 and -2.6. The eigenvalues of $\bar{A} = A - BK$ are -1, -2 with choosing $u(t) = -Kx(t)$, where $K = \begin{bmatrix} -4.2 & -13.2 \end{bmatrix}$.

Suppose the disturbance $w(t) = \sin(20t)$, as $t \in [0\,\text{sec}, 2\,\text{sec}]$ is shown in *Fig. 4-3-1(a)*. The system state $x(t)$ is shown in *Fig. 4-3-1(b)*. The upper bound of $\|e(t)\|$ and actual $\|e(t)\|$ is shown in *Fig. 4-3-1(c)*.

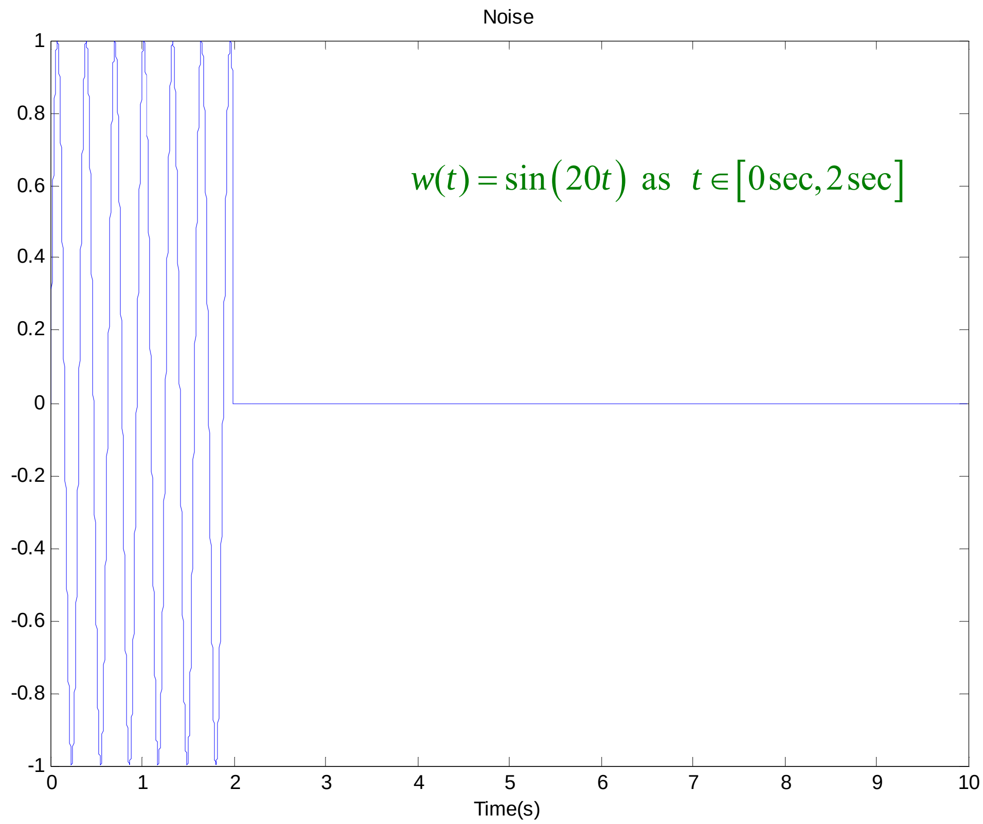

Figure 4-3-1(a) Noise of system Ⅳ.

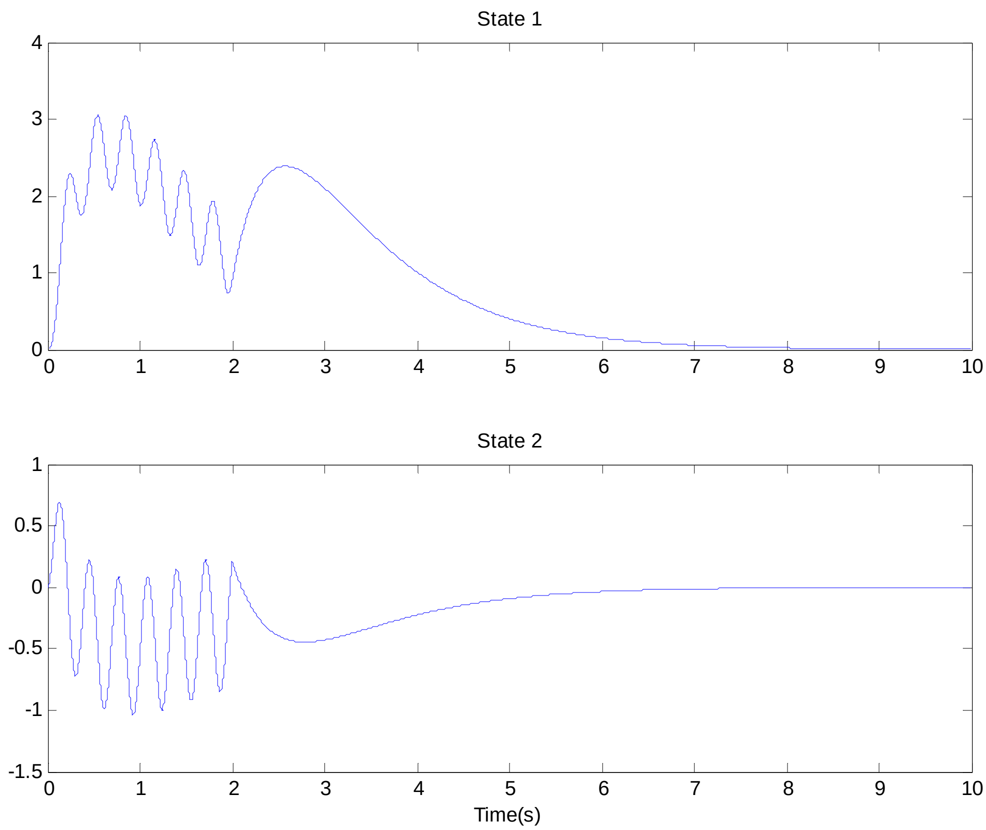

Figure 4-3-1(b) States of system Ⅳ under controlled.

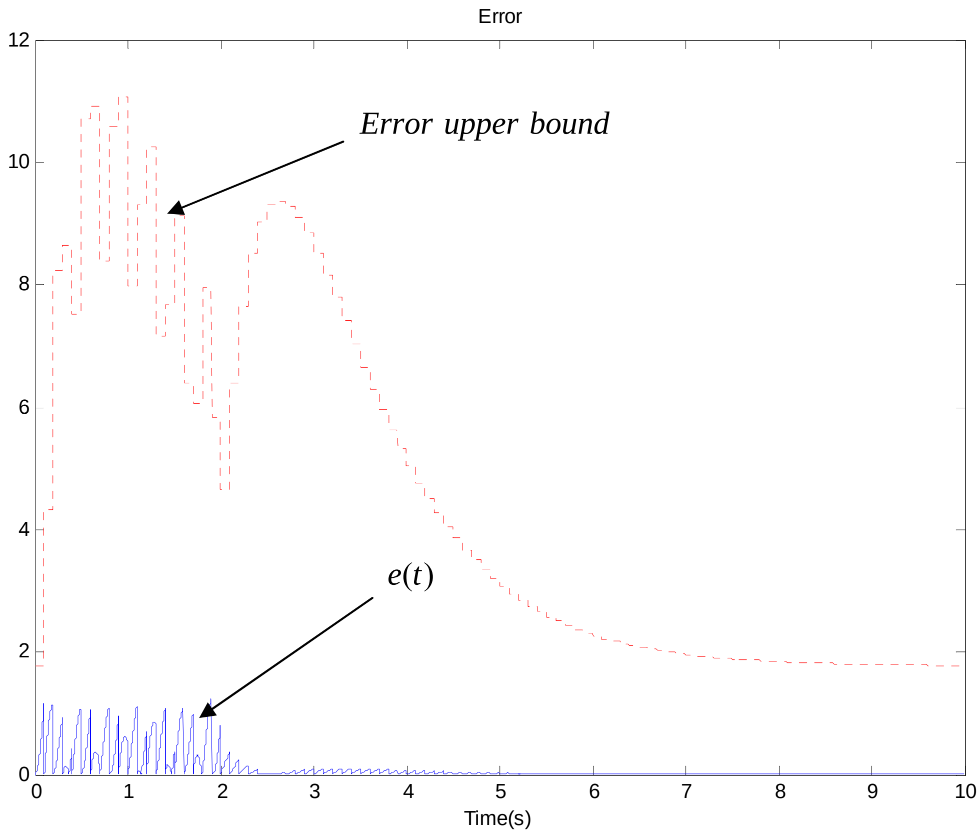

Figure 4-3-1(c) Error upper bound of system Ⅳ as $w(t) = sin(20t)$.

System Ⅴ.

Consider the following system:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$

where $x(t) \in \Re^{2\times1}$ is the state of the system, $u(t) \in R$ is the control signal, and

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \tau = 0.1\,\text{s}.$$

The poles of matrix A are 1.5 and 0.5. Choosing $u(t) = -Kx(t)$, where $K = [12 \quad 7.5]$, then the eigenvalues of $\bar{A} = A - BK$ are -1, -2. Suppose the disturbance $w(t) = 10\sin(20t)$, as $t \in [0\sec, 2\sec]$ as shown in *Fig.4-3-2(a)*. The system state $x(t)$ is shown in *Fig. 4-3-2(b)*. The upper bound of $\|e(t)\|$ and actual $\|e(t)\|$ are shown in *Fig. 4-3-2(c)*.

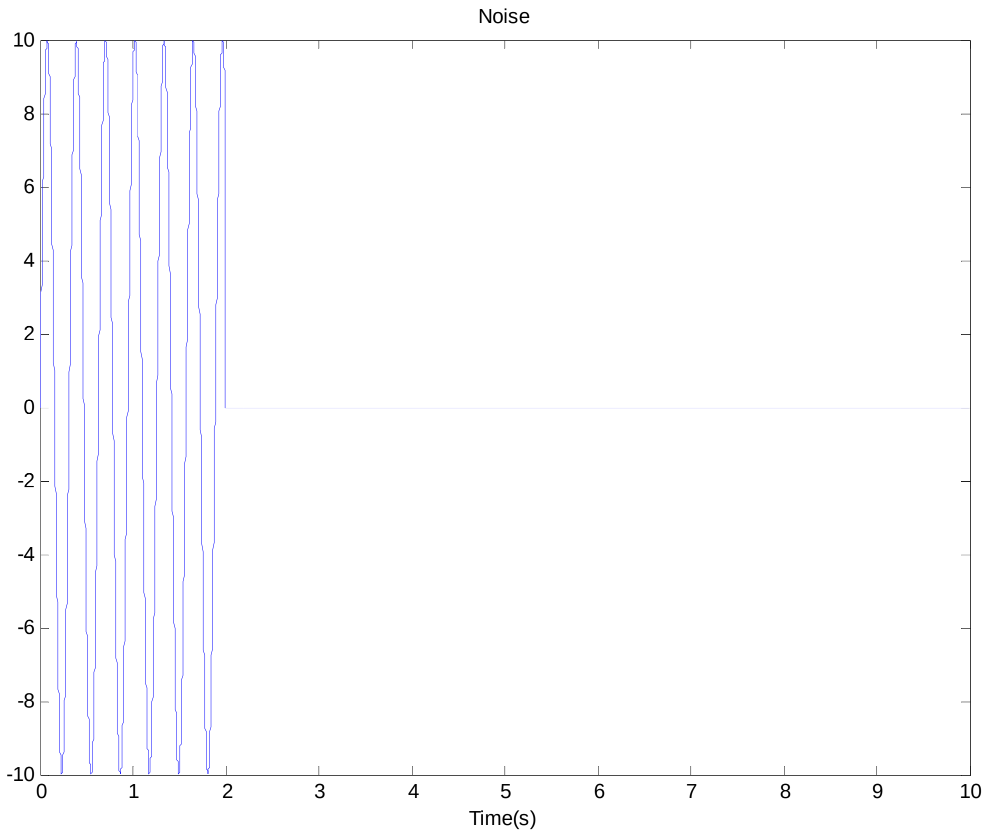

Figure 4-3-2(a) Noise of system Ⅴ.

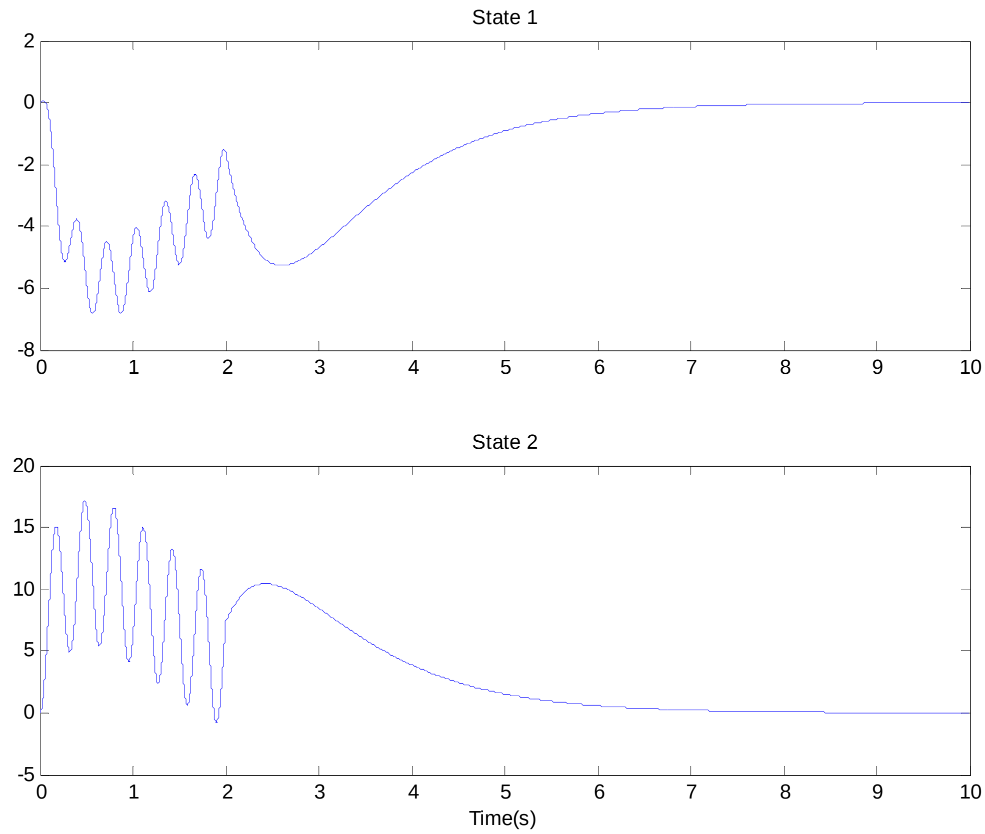

Figure 4-3-2(b) States of system Ⅴ under controlled.

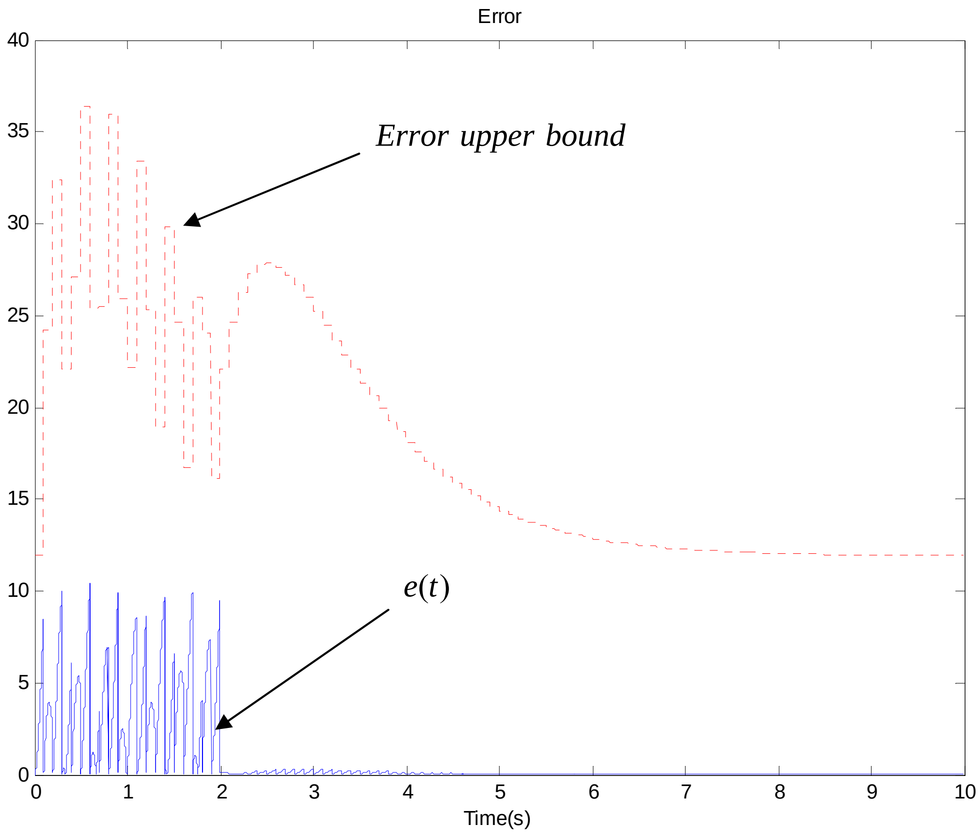

Figure 4-3-2(c) Error upper bound of system V as $w(t)=10\sin(20t)$.

4.3.2 Remark

We take a view of Lemma 6: $\|e(t)\| \le \frac{(e^{\|A\|\tau}-1)}{\|A\|}\left[\|\bar{A}\|\|x(t_k)\|+\|B_1\|\rho\right]$. When the value of τ is small, according to the Taylor Expansion, the term $e^{\|A\|\tau}$ can be rewritten as $1+\|A\|\tau+\ldots$, and then the upper bound is decided mainly by $\|\bar{A}\|\|x(t_k)\|+\|B_1\|\rho$. However, the upper bound will become extreme large if τ is large. At this time, the bound will be useless.

4.4 H_∞ Control

4.4.1 Basic H_∞ Control Concept [13]

For linear system, the system is defined as follows ,

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{cases}$$

where $x(t) \in \Re^n$ is the state of the system, $u(t) \in \Re^m$ is the control, $w(t) \in R^p$ is the disturbances with $w \in L^2[0,\infty)$, $z(t) \in R^q$ is the output vector, $y(t) \in R^v$ is the measurement vector, and A, B_1, B_2, C_1, D_{11}, D_{12}, C_2, D_{21}, D_{22} are known matrices with proper dimensions. If we can find a positive definite function $V(x) = x^T(t)Px(t)$ which satisfies

$$V(x(t)) - V(x(0)) + \int z^T(t)z(t)dt - \gamma^2 \int w^T(t)w(t)dt \le 0$$

then we can get the results that $\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$. Taking derivative on both sides, we get

$$V_x\left[Ax(t) + B_1 w(t) + B_2 u(t)\right] + z^T(t)z(t) - \gamma^2 \omega^T(t)\omega(t) \le 0 .$$

We can obtain

$$\begin{aligned} &2x^T(t)PAx(t) + 2x^T(t)PB_1 w(t) + 2x^T(t)PB_2 u(t) \\ &+x^T(t)C_1^T C_1 x(t) + x^T(t)C_1^T D_{11} w(t) + x^T(t)C_1^T D_{12} u(t) \\ &+w^T(t)D_{11}^T C_1 x(t) + w^T(t)D_{11}^T D_{11} w(t) + w^T(t)D_{11}^T D_{12} u(t) \\ &+u^T(t)D_{12}^T C_1 x(t) + u^T(t)D_{12}^T D_{11} w(t) + u^T(t)D_{12}^T D_{12} u(t) - \gamma^2 w^T(t)w(t) \le 0. \end{aligned}$$

Let the left term of the inequality above is $H(t)$, that is

$$\begin{aligned} H(t) = {} & 2x^T(t)PAx(t) + 2x^T(t)PB_1 w(t) + 2x^T(t)PB_2 u(t) + x^T(t)C_1^T C_1 x(t) \\ & + x^T(t)C_1^T D_{11} w(t) + x^T(t)C_1^T D_{12} u(t) + w^T(t)D_{11}^T C_1 x(t) \\ & + w^T(t)D_{11}^T D_{11} w(t) + w^T(t)D_{11}^T D_{12} u(t) + u^T(t)D_{12}^T C_1 x(t) \\ & + u^T(t)D_{12}^T D_{11} w(t) + u^T(t)D_{12}^T D_{12} u(t) - \gamma^2 w^T(t)w(t). \end{aligned} \tag{4-7}$$

Consequently, by setting, $\frac{\partial H(t)}{\partial w(t)} = 0$, and $\frac{\partial H(t)}{\partial u(t)} = 0$, we get

$$\frac{\partial H(t)}{\partial w(t)} = 2B_1^T Px(t) + 2D_{11}^T C_1 x(t) + 2D_{11}^T D_{11} w(t) + 2D_{11}^T D_{12} u(t) - 2\gamma^2 w^T(t) = 0 .$$

$$\frac{\partial H(t)}{\partial u(t)} = 2B_2^T Px(t) + 2D_{12}^T C_1 x(t) + 2D_{12}^T D_{11} w(t) + 2D_{12}^T D_{12} u(t) = 0 .$$

It can be denoted as

$$\begin{bmatrix} D_{12}^T D_{12} & D_{12}^T D_{11} \\ D_{11}^T D_{12} & D_{11}^T D_{11} - \gamma^2 I \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} -B_2^T P - D_{12}^T C_1 \\ -B_1^T P - D_{11}^T C_1 \end{bmatrix} x(t) .$$

So

$$\begin{bmatrix} u^*(t) \\ w^*(t) \end{bmatrix} = \begin{bmatrix} D_{12}^T D_{12} & D_{12}^T D_{11} \\ D_{11}^T D_{12} & D_{11}^T D_{11} - \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} -B_2^T P - D_{12}^T C_1 \\ -B_1^T P - D_{11}^T C_1 \end{bmatrix} x(t) .$$

In order to simplify the results, assume $D_{11} = 0, \quad C_1^T D_{12} = 0, \quad D_{12}^T D_{12} = I$. Then, we have

$u^*(t) = -B_2^T Px(t), \quad \omega^*(t) = \frac{1}{\gamma^2} B_1^T Px(t)$.

Replacing to (4-7), we have

$$\begin{aligned} H(t) = {} & 2x^T(t)PAx(t) + 2x^T(t)PB_1 w(t) + 2x^T(t)PB_2 u(t) + x^T(t)C_1^T C_1 x(t) \\ & + u^T(t)D_{12}^T C_1 x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t). \end{aligned} \qquad (4\text{-}8)$$

Replacing $u(t)$ and $w(t)$ of (4-8) by $u^*(t)$ and $w^*(t)$, we can get

$$H(x(t), \omega^*(t), u^*(t)) = x^T(t)(A^T P + PA + \frac{1}{\gamma^2} PB_1 B_1^T P + C_1^T C_1 - PB_2 B_2^T P)x(t) .$$

We obtain the conclusion that only if there exists a positive symmetric matrix P satisfying

$$A^T P + PA + \frac{1}{\gamma^2} PB_1 B_1^T P + C_1^T C_1 - PB_2 B_2^T P < 0 \qquad (4\text{-}9)$$

if choosing controller $u(t) = u^*(t)$ then $\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$.

4.4.2 H_∞ Control of Networked Control Systems

In this section, we derive the condition such that $\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$,

where the system is $\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t_k) \\ z(t) = C_1 x(t) + D_{12} u(t_k) \end{cases}$, $D_{11} = 0, \quad C_1^T D_{12} = 0, \quad D_{12}^T D_{12} = I$.

Lemma 6. **(H_∞ Control of Networked Control Systems)**

Choosing controller $u(t) = -B_2^T Px(t_k)$, if the Norm of transmission error satisfies:

$$\|e(t)\| \le \sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(PB_2B_2^T P)}}\|x(t)\|$$

where Q is a positive symmetrical matrix, P is the solution of

$$A^T P + PA + \frac{1}{\gamma^2} PB_1B_1^T P + C_1^T C_1 - PB_2B_2^T P = -Q .$$

Then, $\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$.

Proof.

Replacing $u^*(t) = -B_2^T Px(t_k)$, $\omega^*(t) = \frac{1}{\gamma^2} B_1^T Px(t)$ to (4-8), we can obtain

$$\begin{aligned} H(x(t),\omega^*(t),u^*(t)) &= x^T(t)(A^T P + PA + \frac{1}{\gamma^2} PB_1B_1^T P + C_1^T C_1 - PB_2B_2^T P)x(t) \\ &\quad + x^T(t)PB_2B_2^T P(x(t) - x(t_k)) - (x(t) - x(t_k))^T PB_2B_2^T Px(t_k). \end{aligned} \quad (4\text{-}10)$$

From (4-9), we set

$$A^T P + PA + \frac{1}{\gamma^2} PB_1B_1^T P + C_1^T C_1 - PB_2B_2^T P = -Q$$ where Q is a positive matrix.

Then (4-10) becomes

$$\begin{aligned} H(x(t),\omega^*(t),u^*(t)) &= -x^T(t)Qx(t) \\ &\quad + x^T(t)PB_2B_2^T P(x(t) - x(t_k)) - (x(t) - x(t_k))^T PB_2B_2^T Px(t_k). \end{aligned}$$

If we want to make the the signal $\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$, then $H(x(t),\omega^*(t),u^*(t))$ must be less than zero. Thus

$$x^T(t)PB_2B_2^T P(x(t) - x(t_k)) - (x(t) - x(t_k))^T PB_2B_2^T Px(t_k) \le x^T(t)Qx(t). \quad (4\text{-}11)$$

Since $PB_2B_2^T P$ is a symmetric matrix, and $x^T(t)PB_2B_2^T Pe(t)$ is a scalar

$$x^T(t)PB_2B_2^T Pe(t) = \left(x^T(t)PB_2B_2^T Pe(t)\right)^T = e^T(t)PB_2B_2^T Px(t) .$$

Therefore, (4-11) can be simplied as

$$e^T(t)PB_2B_2^T P\left(x(t) - x(t_k)\right) \le x^T(t)Qx(t) . \quad (4\text{-}12)$$

So if $\lambda_{\max}(PB_2B_2^T P)\|e(t)\|^2 \le \lambda_{\min}(Q)\|x(t)\|^2$, that is if $\|e(t)\| \le \sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(PB_2B_2^T P)}}\|x(t)\|$, then

(4-11) holds. It means that $\int_0^{\infty} z^T(t)z(t)dt \le \gamma^2 \int_0^{\infty} w^T(t)w(t)dt$.

4.5 Simulation

For convenience, we call $\dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(PB_2B_2^TP)}$ of Lemma 6 as "weight". Now we try to find out 'weight' of the following system.

4.5.1 Validating Lemma 6

System Ⅵ.

Consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t) \\ z(t) = C_1x(t) + D_{12}u(t) \end{cases},$$

where $C_1^TD_{12} = 0$, $D_{12}^TD_{12} = I$, $u(t) \in R$ is the control signal, $x(t) \in \Re^{2\times 1}$ is the state of the system, and $A = \begin{bmatrix} -1 & 5 \\ 3 & -2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C_i = \begin{bmatrix} -4 & 8 \\ 3 & -6 \end{bmatrix}$, $D_{i1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $D_{i2} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$, choose $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, $\gamma = 10$.

Suppose the disturbance signal $w(t) = 5\sin 20t$, as $t \in [0\sec, 0.7\sec]$ is shown in *Fig. 4-5-1(a).*

The poles of matrix A are 2.405 and -5.405. Using H_∞ control approach, we can obtain the control input signal which makes $\int_0^{\infty} z^T(t)z(t)dt \le 100\int_0^{\infty} w^T(t)w(t)dt$ as $u(t) = -B_2^TPx(t_k) = [4.53 \quad -8.46]x(t_k)$. And then we calculate the value of 'weight'= $\sqrt{\dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(PB_2B_2^TP)}} = 0.3296$.

Fig. 4-5-1(b) shows the original states of system Ⅵ. *Fig. 4-5-1(c)* shows the states of controlled system Ⅵ under instantly updated control or updated hold control under the 'weight'=0.3296 of Lemma 6. The control input signal as updated instantly or as 'weight'=0.3296 is shown in *Fig. 4-5-1(d).*

The requirement of Lemma 6, $\|e(t)\| \le 0.3296\|x(t)\|$, is shown in *Fig. 4-5-1(e).*

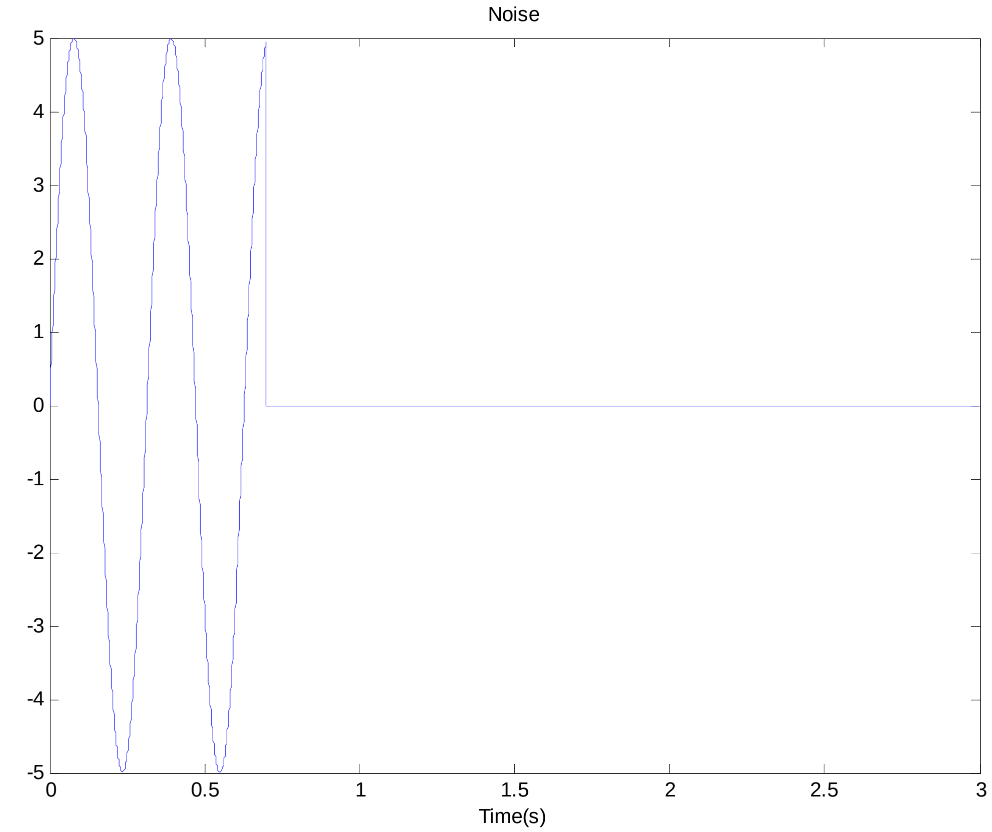

Figure 4-5-1 (a) Noise of system Ⅵ.

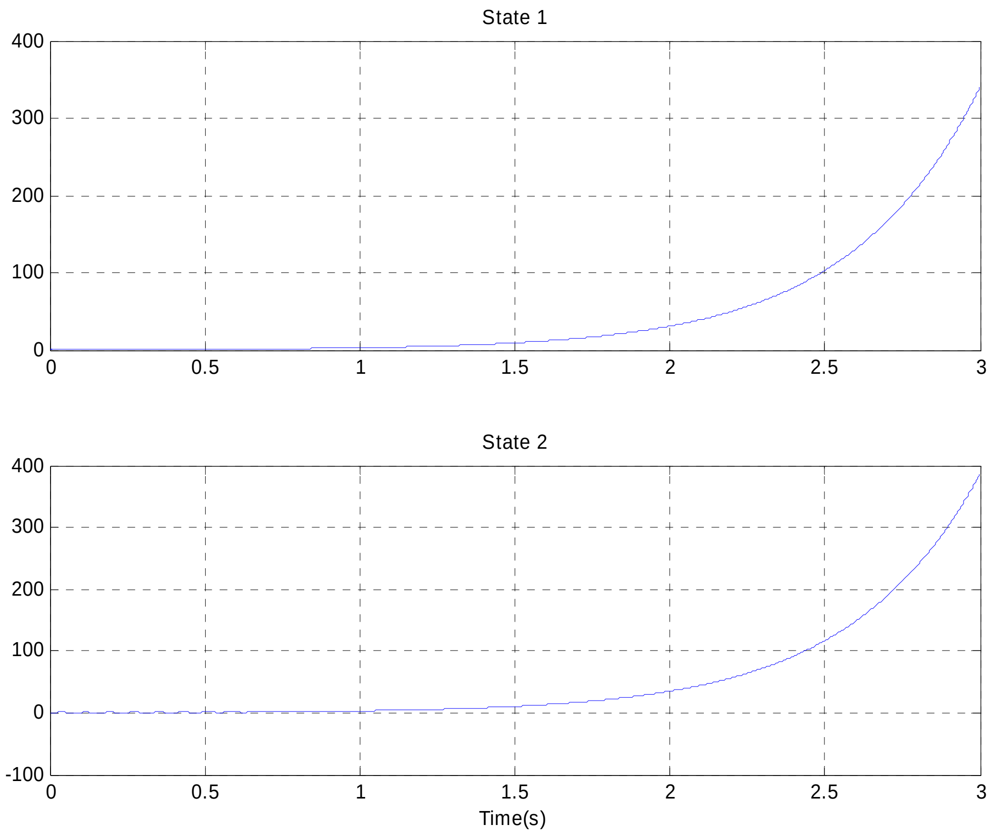

Figure 4-5-1 (b) Original states in system Ⅵ.

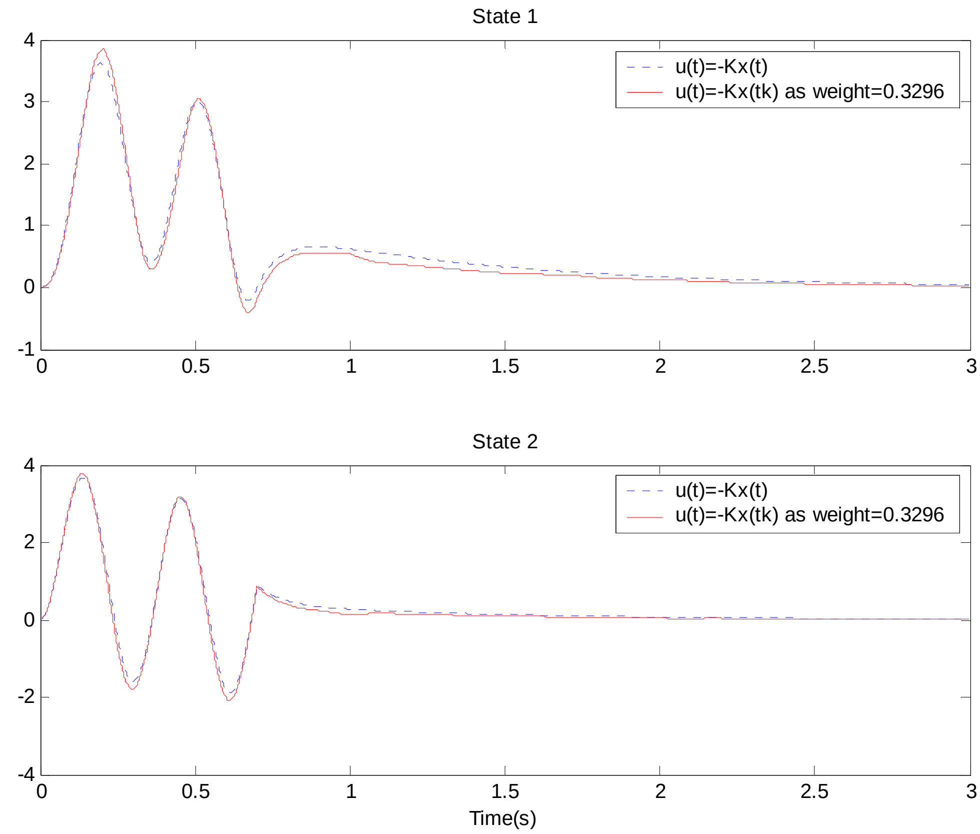

Figure 4-5-1 (c)　States of system Ⅵ under controlled.

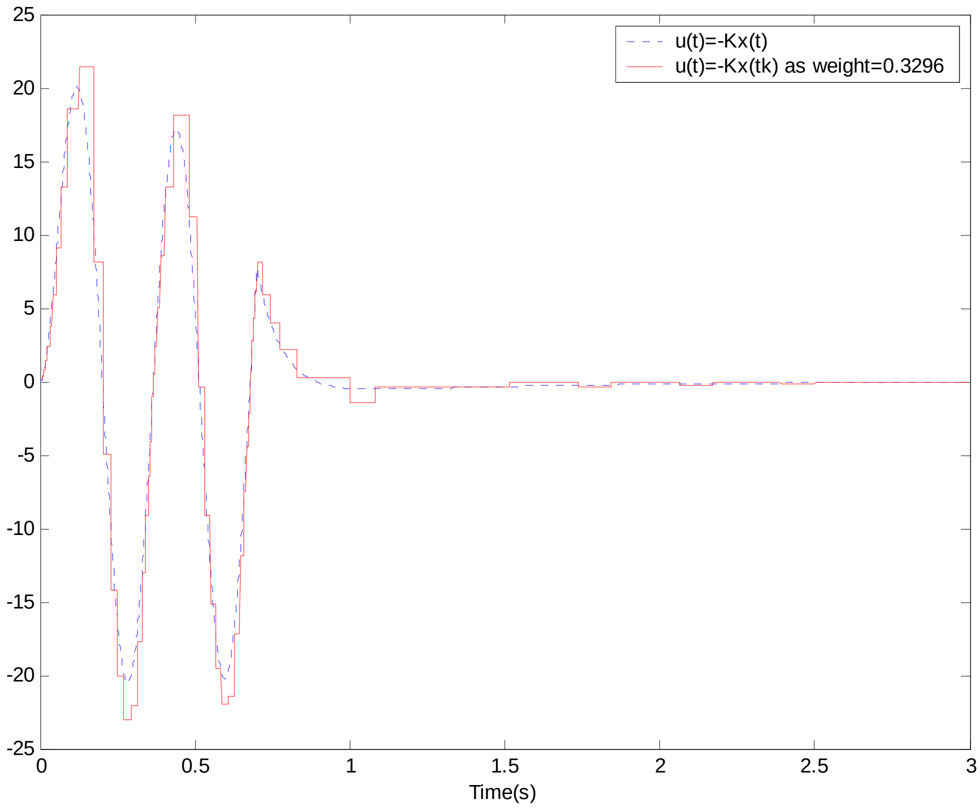

Figure 4-5-1 (d)　Control input in System Ⅵ.

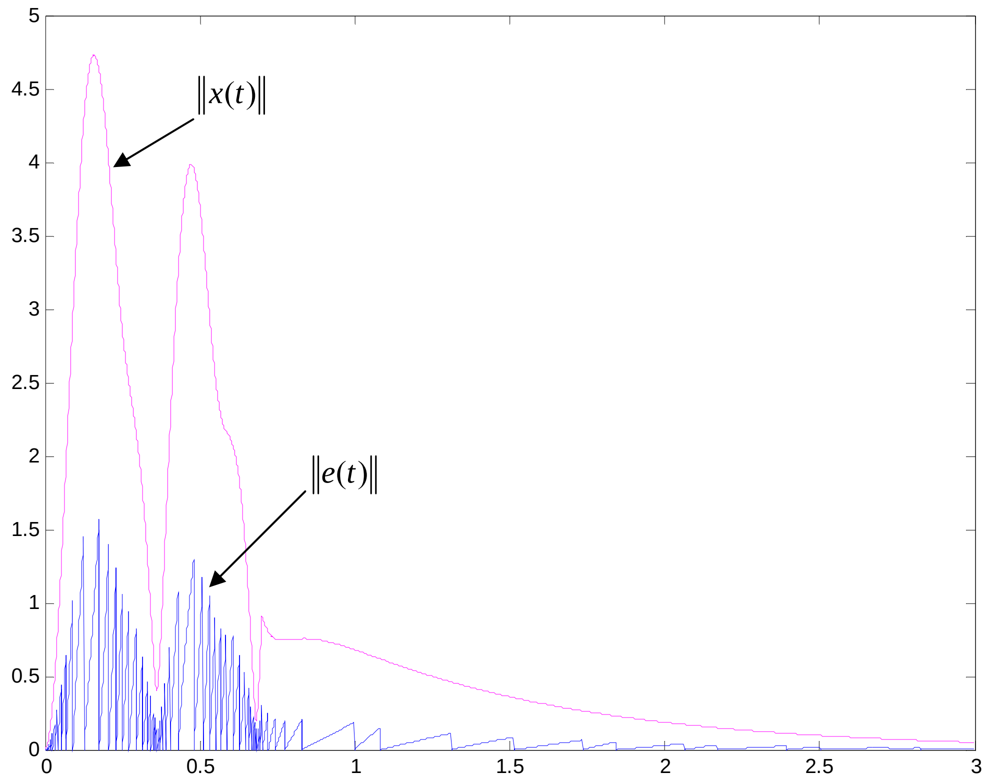

Figure 4-5-1 (e) $\|e(t)\| \le 0.3296 \|x(t)\|$ **in System Ⅵ.**

System Ⅶ.

Consider the following system :

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{12} u(t) \end{cases},$$

where $C_1^T D_{12} = 0$, $D_{12}^T D_{12} = I$, $u(t) \in R$ is the control signal, $x(t) \in \Re^{4\times 1}$ is the state of the system, and $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -4 & 3 & 0 \\ 0.3 & 0.2 & -6 & 1 \\ 0 & -0.4 & 0.2 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0.5 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$C_i = \begin{bmatrix} 24 & -7 \\ -2.4 & 0.7 \\ -12 & 3.5 \\ 7.2 & -2.1 \end{bmatrix}^T$, $D_{i1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $D_{i2} = \begin{bmatrix} 0.28 \\ 0.96 \end{bmatrix}$, choose $Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$, $\gamma = 10$.

Suppose the disturbance signal $w(t)$, is defined as in *Fig.4-5-2(a)*.

The poles of matrix A are $0.5259 \pm 0.6023i$, -2.6814 and -6.3704. Because two of them are in the right-half-plane, it is an unstable system Using H_∞ control approach, we can obtain the control input signal which makes $\int_0^\infty z^T(t)z(t)dt \le 100\int_0^\infty w^T(t)w(t)dt$ as $u(t) = -B_2^T Px(t_k) = \begin{bmatrix} -26.22 & 3.8 & 11.56 & -10.58 \end{bmatrix} x(t_k)$. And then we calculate the value of 'weight'= $\sqrt{\dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(PB_2B_2^T P)}} = 0.1027$.

Fig. 4-5-2(b) shows the original states of system VII. *Fig. 4-5-2(c)(d)* shows the states of controlled system VII under instantly updated control or updated hold control under the 'weight'=0.1027 of Lemma 6. The control input signal as updated instantly or as 'weight'=0.1027 is shown in *Fig. 4-5-2(e)*.

The requirement of Lemma 6, $\|e(t)\| \le 0.1027\|x(t)\|$, is shown in *Fig. 4-5-2(f)*.

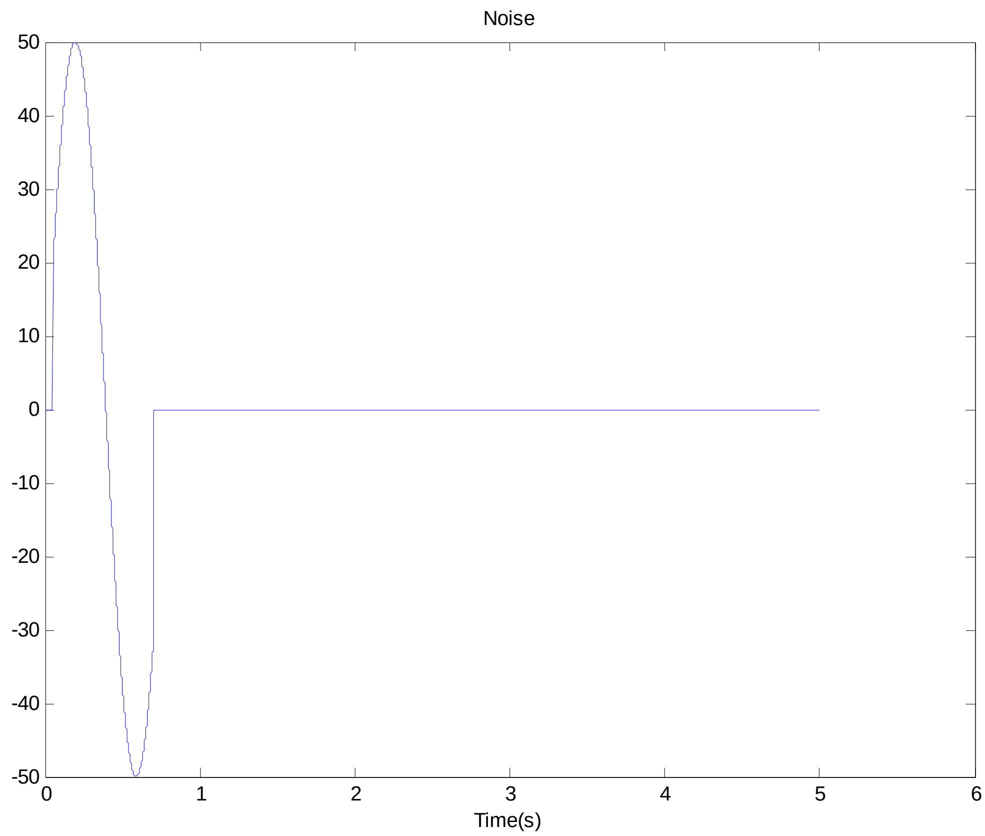

Figure 4-5-2(a) Noise of system Ⅶ.

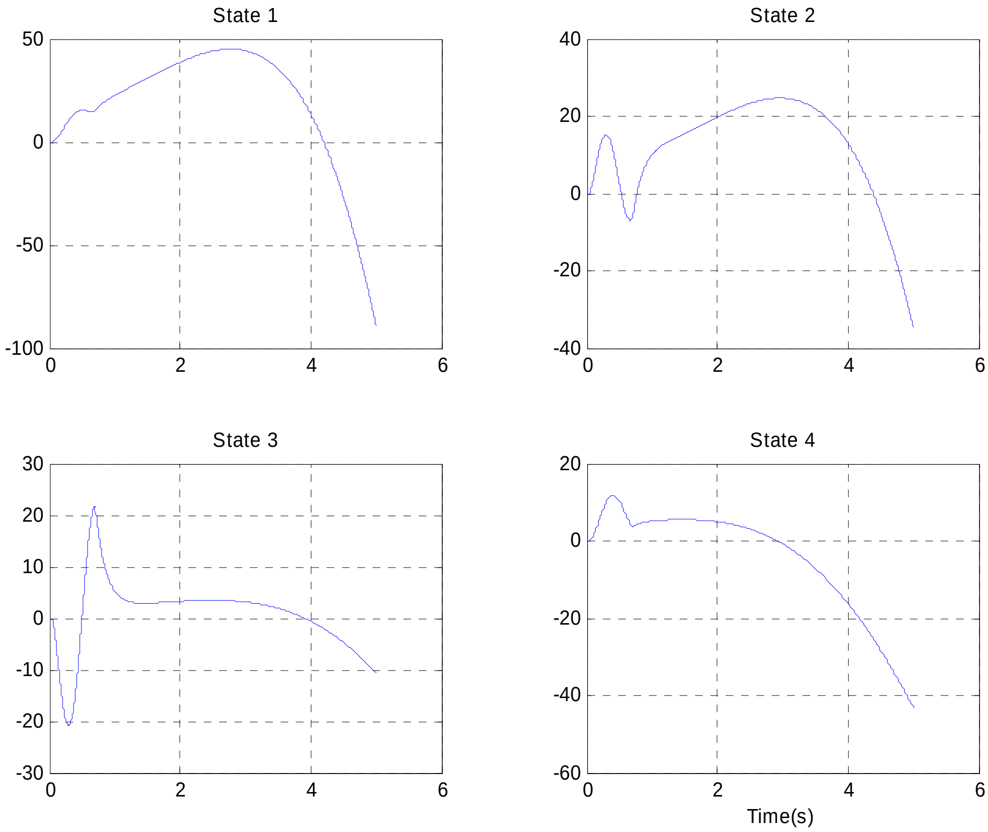

Figure 4-5-2(b)　Original states in system Ⅶ.

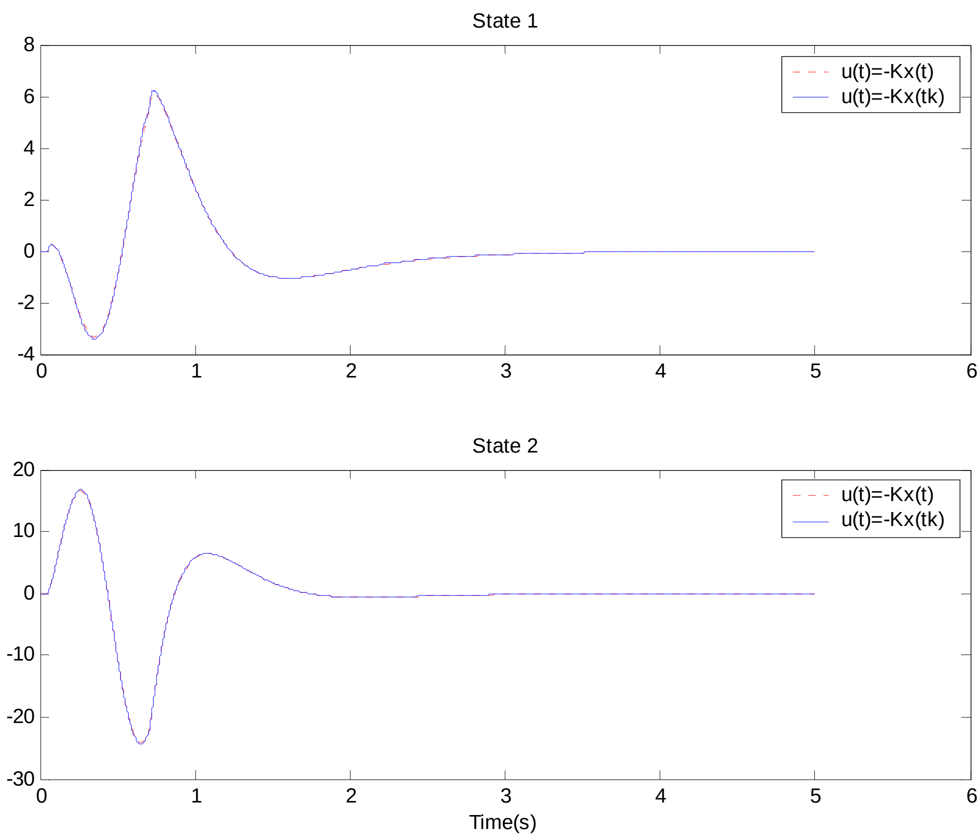

Figure 4-5-2(c)　State 1,2 of system Ⅶ under controlled.

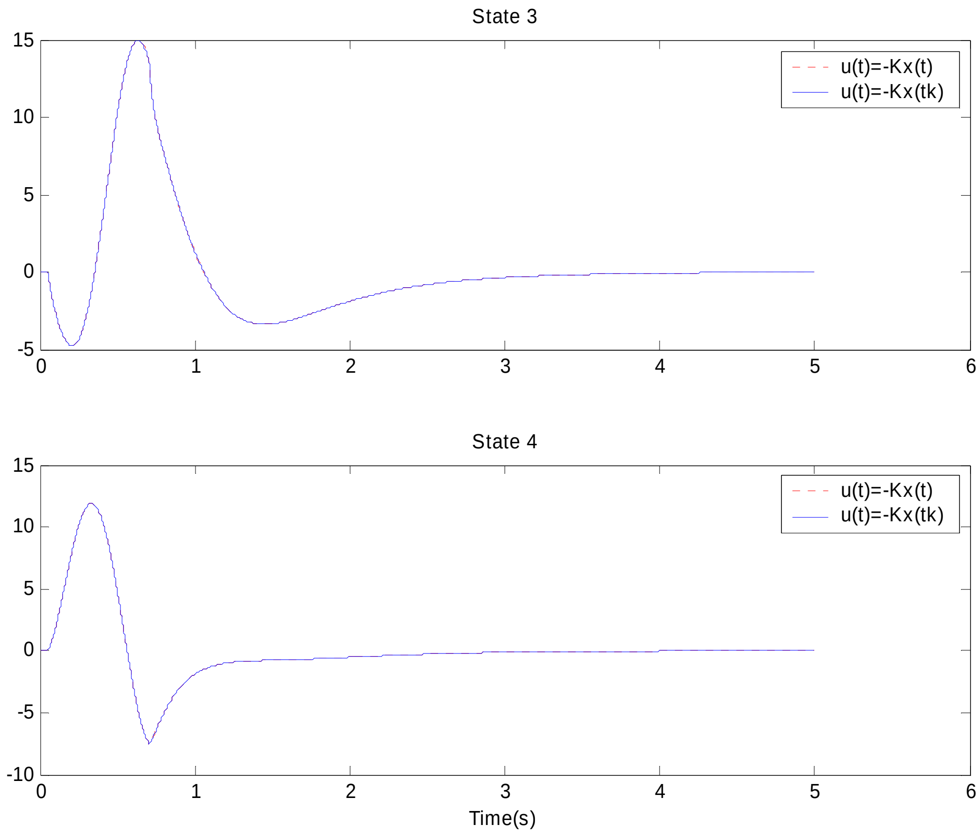

Figure 4-5-2(d)　State 3,4 of system Ⅶ under controlled.

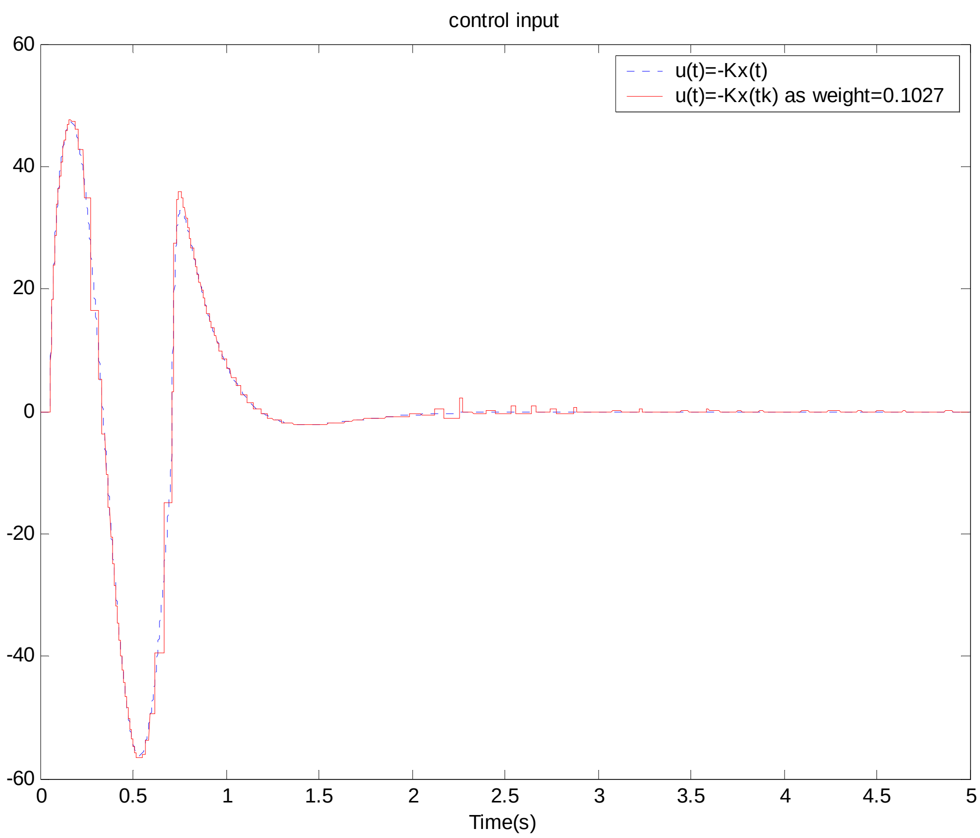

Figure 4-5-2(e)　Control input in system Ⅶ.

4.5.2 Remark

Table 4-1 List of the updated times.

	System VI	System VII
Times of real-time updated	3000	5000
Weight	0.3296	0.1027
Times of updated in Lemma 6	66	152

From *Table 4-1*, we can conclude that the usage of network is reduced by using the control method provided in Lemma 6.

Chapter5

Conclusions and Future Work

5.1 Conclusions

We investigate the stability of the linear system operating under limited communication. Using Lyapunov theory, we give a sufficient condition for stability of NCSs. In the system without disturbances, we really and truly get the maximum time interval of state updating which still guarantees the stability of the system. In the system with disturbances, we obtain the upper bound of state error caused by jamming in the network based on the H_∞ design. It not only ensures the stability of the controlled system, but also guarantees the closed-loop system satisfying the L^2-gain requirement. From these results, the time of state to be feedback will be reduced, so it minimizes the network usage.

5.2 Future Work

In the future, we hope to derive other corresponding results for NCSs about that time delay, scheduling, dropping packets out etc. Additionally, in this thesis, we only concentrated on the stability requirements. In future work, we can consider some other performance.

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Appendix

Proof. **(Lemma 1)**

Let $z(t) = \lambda(t) + \int_{t_i}^{t} k(w)y(w)dw, \qquad \forall t \ge t_i \ge 0$.

Form definition we can obtain $z(t)$ is differentiable and $z(t) \ge y(t)$. Then we get

$\dot{z}(t) = \dot{\lambda}(t) + k(t)y(t)$.

$$\begin{aligned} z(t_i) &= \lambda(t_i) + \int_{t_i}^{t_i} k(w)y(w)dw \\ &= \lambda(t_i). \end{aligned}$$

Let $v(t) = z(t) - y(t) \ge 0,$

Then we get

$$\begin{aligned} \dot{z}(t) &= \dot{\lambda}(t) + k(t)y(t) \\ &= \dot{\lambda}(t) + k(t)\left[z(t) - v(t)\right] \\ &= k(t)z(t) + \dot{\lambda}(t) - k(t)v(t). \end{aligned}$$

The state transition matrix is

$$\Phi(t_i, t) = e^{\int_{t_i}^{t} k(w)dw}.$$

Therefore,

$$z(t) = \Phi\left(t_i, t\right) z(t_i) + \int_{t_i}^{t} \Phi\left(t, s\right)\left[\dot{\lambda}(s) - k(s)v(s)\right]ds.$$

Since

$\int_{t_i}^{t} \Phi\left(t, s\right)k(s)v(s)ds \ge 0, \quad \forall t \ge t_i$ then it becomes

$$z(t) \le \Phi\left(t_i, t\right) z(t_i) + \int_{t_i}^{t} \Phi\left(t, s\right)\dot{\lambda}(s)ds.$$

Substitute $\Phi(t_i, t) = e^{\int_{t_i}^{t} k(w)dw}$. Thus it gives

$$y(t) \le z(t) \le \lambda(t_i)e^{\int_{t_i}^{t} k(w)dw} + \int_{t_i}^{t} \dot{\lambda}(s)e^{\int_{s}^{t} k(w)dw}ds, \qquad \forall t \ge t_i \ge 0.$$

Proof. **(Lemma 2)**

Let $z(t)=\lambda(t)+\int_{t}^{t_f}k(w)y(w)dw,\qquad \forall t_f\geq t\geq 0$.

Form definition, $z(t)$ is differentiable and $z(t)\geq y(t)$. Then we get

$$\dot{z}(t)=\dot{\lambda}(t)-k(t)y(t)$$

$$z(t_f)=\lambda(t_f)+\int_{t_f}^{t_f}k(w)y(w)dw$$
$$=\lambda(t_f).$$

Let $v(t)=z(t)-y(t)\geq 0$. Then we get

$$\dot{z}(t)=\dot{\lambda}(t)-k(t)y(t)$$
$$=\dot{\lambda}(t)-k(t)\left[z(t)-v(t)\right]$$
$$=-k(t)z(t)+\dot{\lambda}(t)+k(t)v(t).$$

The state transition matrix is

$$\Phi(t,t_f)=e^{\int_{t_f}^{t}-k(w)dw}=e^{\int_{t}^{t_f}k(w)dw}.$$

Therefore,

$$z(t)=\Phi\left(t,t_f\right)z(t_f)+\int_{t_f}^{t}\Phi\left(t,s\right)\left[\dot{\lambda}(s)+k(s)v(s)\right]ds.$$

Since

$\int_{t_f}^{t}\Phi\left(t,s\right)k(s)v(s)ds\leq 0,\quad \forall t_f\geq t$, we get

$$z(t)\leq\Phi\left(t,t_f\right)z(t_f)+\int_{t_f}^{t}\Phi\left(t,s\right)\dot{\lambda}(s)ds.$$

Substitute $\Phi(t,t_f)=e^{\int_{t}^{t_f}k(w)dw}$. Then it becomes

$z(t)\leq z(t_f)e^{\int_{t}^{t_f}k(w)dw}+\int_{t_f}^{t}\dot{\lambda}(s)e^{\int_{t}^{s}k(w)dw}ds$. This gives

$$y(t)\leq z(t)\leq\lambda(t_f)e^{\int_{t}^{t_f}k(w)dw}-\int_{t}^{t_f}\dot{\lambda}(s)e^{\int_{t}^{s}k(w)dw}ds,\qquad \forall t_f\geq t\geq 0.$$

Proof. **(Lemma 3)**

Because of $e(t) = x(t) - x(t_k)$

$\dot{e}(t) = Ae(t) + \bar{A}x(t_k)$.

Taking the integral on both sides,

$$\int_{t_k}^{t} \dot{e}(t)dt = \int_{t_k}^{t} \left[Ae(w) + \bar{A}x(t_k) \right] dw$$

$$\Rightarrow \; e(t) - e(t_k) = \bar{A}x(t_k)(t - t_k) + \int_{t_k}^{t} Ae(w)\, dw.$$

Substituting $t = t_k$, we get $e(t_k) = x(t_k) - x(t_k) = 0$.

This gives

$$e(t) = \bar{A}x(t_k)(t - t_k) + \int_{t_k}^{t} Ae(w)\, dw.$$

Taking *Norm* on both sides, we obtain

$$\|e(t)\| \le \underbrace{\|\bar{A}\| \|x(t_k)\| (t - t_k)}_{\lambda(t)} + \int_{t_k}^{t} \underbrace{\|A\|}_{k(w)} \|e(w)\| \; dw.$$

Setting

$\lambda(t) = \|\bar{A}\| \|x(t_k)\| (t - t_k)$

$k(w) = \|\bar{A}\|$

Using Lemma 1, we get

$$\|e(t)\| \le \lambda(t_k) e^{\int_{t_k}^{t} \|A\| dw} + \int_{t_k}^{t} \|\bar{A}\| \|x(t_k)\| e^{\|A\|(t-w)} dw.$$

Since from setting $\lambda(t) = \|\bar{A}\| \|x(t_k)\| (t - t_k)$, we can get

$$\begin{aligned} \lambda(t_k) &= \|\bar{A}\| \|x(t_k)\| (t_k - t_k) \\ &= 0. \end{aligned}$$

So

$$\begin{aligned} \|e(t)\| &\le \int_{t_k}^{t} \|\bar{A}\| \|x(t_k)\| e^{\|A\|(t-w)} dw \\ &= \frac{\|\bar{A}\|}{\|A\|} (e^{\|A\|(t-t_k)} - 1) \|x(t_k)\|. \end{aligned}$$