國立交通大學

應用數學系
數學建模與科學計算碩士班

碩　士　論　文

多重網格與自調適法於 Laplace 方程角奇異解的數值計算

Multigrid and Adaptive Methods for Computing Singular Solutions of Laplace Equation on Corner Domains

研 究 生：李偉任
指導老師：吳金典　教授

中　華　民　國　九　十　九　年　七　月

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研 究 生：李偉任　　Student：Wei-Jen Lee

指導教授：吳金典　　Advisor：Chin-Tien Wu

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學生：李偉任　　指導老師：吳金典 教授

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摘　要

橢圓邊界值問題在凹角的地方會有奇異的行為，而這個奇異的行為對於用有限元素法離散的精確度會受到影響。對於給定 Dirichlet 邊界條件的 Poisson 方程式和在定義域有凹角的情況之下，本論文利用一個奇異解的表示法 $u=w+\sum_{j=1}^{J}\kappa_j s_j$ 算出較準確的近似值，其中 $\kappa_j=\frac{1}{\pi}\{\int_\Omega f s_{-1}dx+\int_\Omega u\Delta s_{-1}dx\}$ 在工程上稱之為應力強度因子。這些量的精確計算在許多實際的工程問題上，是一門很重要的課題。

Multigrid and Adaptive Methods for Computing Singular Solutions of Laplace Equation on Corner Domains

Student: Wei-Jen Lee Advisor: Chin-Tien Wu

Institute of Mathematical Modeling and Scientific Computing

National Chiao Tung University

Abstract

Elliptic boundary value problems on domain with corners have singular behavior near the corners. Such singular behavior affect the accuracy of the finite element method throughout the whole domain. For the Poisson equation with homogeneous Dirichlet boundary conditions defined on a polygonal domain with re-entrant corners, it is well known that the solution has the singular function representation $u=w+\sum_{j=1}^{J}\kappa_j s_j$,where w is the regular part of the solution and s_j are known as singular functions that depend only on the corresponding re-entrant angles. Coefficients κ_j known as the stress intensity factors in the context of mechanics can be expressed in terms of u by extraction formula $\kappa_j=\frac{1}{\pi}\{\int_\Omega f s_{-1}dx+\int_\Omega u\Delta s_{-1}dx\}$, where s_{-j} are known as dual singular funciton. Accurate calculation of these quantities is of great importance in many practical engineering problems. Similar singular function representations hold for the solutions of interface,biharmonic,elasticity, and evolution problems in [1, 2].

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在論文口試中，承蒙口試委員賴明治博士與李國明博士於百忙之中撥冗審閱，提供了相關多寶貴的意見和指正，使本論文得以更臻嚴謹完整，特此表示由衷的感謝之意。

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李偉任　謹誌

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Introduction

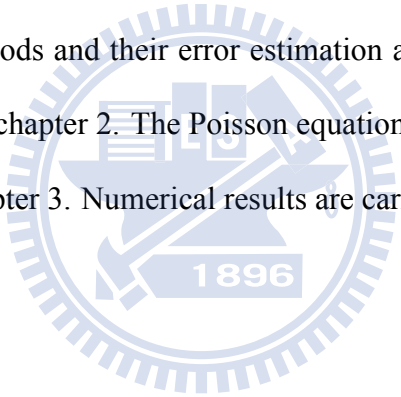
The finite element method has become one of the most popular and effective methods for the numerical solution of partial differential equations, particularly for elliptic equations. In practice many important problems involve polygonal domains. Previously, the geometry of a problem would be restricted so that the triangulation elements fit the polygonal boundary exactly. Form a theoretical standpoint, under the assumption that solutions were sufficiently smooth, this case has been thoroughly analyzed. Unfortunately, in practice one is not likely to achieve the smoothness required for these previous analyses. It is the basic behavior of elliptic equations that solutions possess singularities at corners. These singularities substantially affect the rate of convergence of numerical approximations.

To handle this problem , here the two main procedures which have been proposed to overcome this difficulty. The first is based on mesh refinements and has been analysed by Babuska and kellogg [21],Raugel, Schatz and Wahlbin, Thatcher for instance. This method may be applied to most of the practical problems since it requires only a qualitative knowledge of the behaviour of the solution near the corners. The second consists in augmenting the space of trail funcitons in which one looks for the approximate solution. This is done by adding some of the singular solutions of the problem to the usual spaces of piecewise polynomial funcitons. For instance, S.C. Brenner and L.Y. SUNG [9], Babuska and Rosenzweig [22], Kellogg, Lelievre, Djaoua and Ladeveze and Peyret.

In this thesis ,the first we introduce the singular element to capture the singular point at the corner and see the accuracy reduced.The advantage of the singular element is that

there are many small tranguler near the singular point , the solution near the singular point can be approximation efficiency, but the disadvantage is that create large linear systems . To solve the large linear systems , we introduce the Multigrid method.The second,we applied the S.C. Brenner and L. Y. SUNG's method as a standard. The advantage of S.C. Brenner's method is that the stress intensity factors κ_j can be represent by the simple expression and correct calculation , but the disadvantage is that the lack of accuracy for the whole domain .We will improve the accuracy by introducing the adaptive mesh-refinement and adaptive cut-off function .

Finite element methods and their error estimation are given in section 1. Multigrid method are introduced in chapter 2. The Poisson equation and the singular funciton representation are given in chapter 3. Numerical results are carried out in chapter 4.



Chapter 1
Finite Element Method

1.1 Introduction of Finite Element Method

The basic idea in any numerical method for solving a differential equation is first to discretize given continuous problem with infinite degrees of freedom to a discrete problem or with only finite degrees of freedom such that the differential equation is transformed into a system of linear equations which can be solved by using a computer.

Finite element method start from a reformulation of a given differential equation as an equivalent variational problem. In the case of elliptic equations this variational problem in basic case is a minimization problem of the form

$$Find\ u \in V\ such\ that\ F(u) \leqq F(v)\ \ for\ all\ v \in V \tag{1.1}$$

where V is a given set of admissible functions and $F : V \rightarrow R$ is a functional. $F(v)$ is the total energy associated with v and (1.1) corresponds to an equivalent characterization of the solution of the differential equation as the function in V that minimizes the total energy of the considered system. In general the dimension of V is infinite and thus in general the problem (1.1) can't be solved exactly. To obtain a problem that can be solved on a computer the idea in the finite element method is to replace V by a set V_h consisting of simple function only depending on finitely many parameters. This leads to a finite-

dimensional minimization problem of the form:

$$Find\ u_h \in V_h\ such\ that\ F(u_h) \leqq F(v)\ \ for\ all\ v \in V_h \tag{1.2}$$

This problem is equivalent to a linear or nonlinear system of equations. We hope that the solution u_h of this problem is sufficiently good approximation of the solution of the original minimization problem (1.1). Usually one chooses V_h to be a subset of V and in this case (1.2) corresponds to the classical Ritz-Galerkin method.

The advantage of finite element methods as compared with finite difference methods is that complicated geometry,general boundary conditions and variable or non-linear material properties can be handled relatively easily.In all these cases one meets unnecessary artificial complications with finite difference methodology.Further,the finite element method has a solid theoretical foundation which gives added reliability and in many cases makes it possible to mathematically analyze and estimate the error in the approximate finite element solution.

To solve a given differential or integral equation approximately using the finite element method, one has to go through basically the following steps:

1. variational formulation of the given problem
2. Mesh Generator
3. discretization using FEM: construction of the finite dimensional space V_h and choose basis function
4. Assemble the element matrix to obtain global matrix

From step 1~4, one obtain a linear systems we will introduce at section 1.2.

1.2 Variational Formulation

We will now consider the following boundary value problem for the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where Ω is a bounded domain in the $\mathbb{R}^2$ with boundary $\partial\Omega$,g is a constant, f is a given function, where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1.4}$$

the equivalent variational problem is

$$-\int_\Omega (\Delta u) v dx = \int_\Omega f v dx \tag{1.5}$$

where v is test function in $H_0^1(\Omega)$, $v|_{\partial\Omega} = 0$ .By taking integration by parts ,

$$\int_\Omega \nabla v \nabla u dx = \int_\Omega v f dx + \overrightarrow{n} \cdot \nabla u v|_{\partial\Omega} = \int_\Omega v f dx \tag{1.6}$$

1.2.1 Existence and Uniqueness of Solution

Let a:$V \times V \rightarrow R$ be a bilinear mapping with following properties:

(1) $a(.,.)$ is symmetric

(2) (Continuity) $a(.,.)$ is continuous,ie,there is a constant $\beta > 0$ such that $|a(v,w)| \leq \beta ||v||_V ||w||_V \quad \forall v, w \in V$

(3) (Coercivity) $a(.,.)$ is V-elliptic,ie,there is a constant $\alpha > 0$ such that $a(v,v) \geq \alpha||v||_V^2 \qquad \forall v \in V$

(4) L is continuous,ie, there is a constant $\Lambda > 0$ such that $|L(v)| \leq \Lambda||v||_V \qquad \forall v \in V$

Theorem (Lax-Milgram theorem) Let V be a Hilbert space with scalar product $(.,.)_V$ and corresponding norm $||\cdot||_V$(the $V-norm$). Suppose that $a(.,.)$ is a bilinear form on $V \times V$ and L a linear form on V such that under the assumptions (1)-(4), there exists a unique $u \in V$ such that

$$a(u,v) = L(v), \text{ for all } v \in V$$

1.3 Finite Element Discretization

Let $T_h = \{K\}$ is a triangulation of $\Omega \subset R^2$, the integral equation can be rewritten as

$$\sum_{K \in T_h(\Omega)} \int_K \nabla v \nabla u dx = \sum_{K \in T_h(\Omega)} \int_K v f dx \tag{1.7}$$

The finite element method is then employed to discretize the terms $\int \nabla v \nabla u dx$ and $\int v f dx$ on element, we first look the germetry on an element.

The geometry of the 3-node triangle is specified by the location of its three corner nodes on the $\{x,y\}$ plane. The nodes are labeled 1, 2, 3 while traversing the sides in counterclockwise fashion. The location of the corners is defined by their coordinates:

$$x_i, y_i, \ i = 1,2,3$$

the area of triangle is denoted by A and is given by:

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = x_{21}y_{31} - x_{31}y_{21}$$

where $x_{ij} = x_i - x_j, y_{ij} = y_i - y_j$ for $i, j = 1, 2, 3\ i \neq j$.

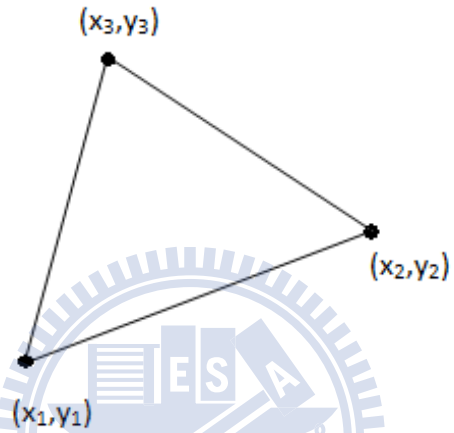


Fig1.3.1

1.3.1 Linear Interpolation

One can choose a piecewise ploynomial function to approximate the exact solution u and the test function v .For example , if one choose linear piecewise funciton ,then the function $u\,(x, y)$ may be expressed as

$$u\,(x, y) = a_0 + a_1 x + a_2 y \tag{1.8}$$

where a_0, a_1 and a_2 are coefficients to be determined from three conditions. In finite element work such conditions are often the nodal values taken by u at the corners:

$$u_1, u_2, u_3$$

The expression in triangular coordinates makes direct use of these three values:

$$\begin{aligned} u(\zeta,\eta) &= u_1(1-\zeta-\eta)+u_2\zeta+u_3\eta = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 1-\zeta-\eta \\ \zeta \\ \eta \end{bmatrix} \\ &= \begin{bmatrix} 1-\zeta-\eta & \zeta & \eta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned} \tag{1.9}$$

equation (1.9) is called a linear interpolant for u.

1.3.2 Coordinate Transformations

Consider triangular on regular triangular, points of the triangle may also be located in terms of a parametric coordinate system ζ, η

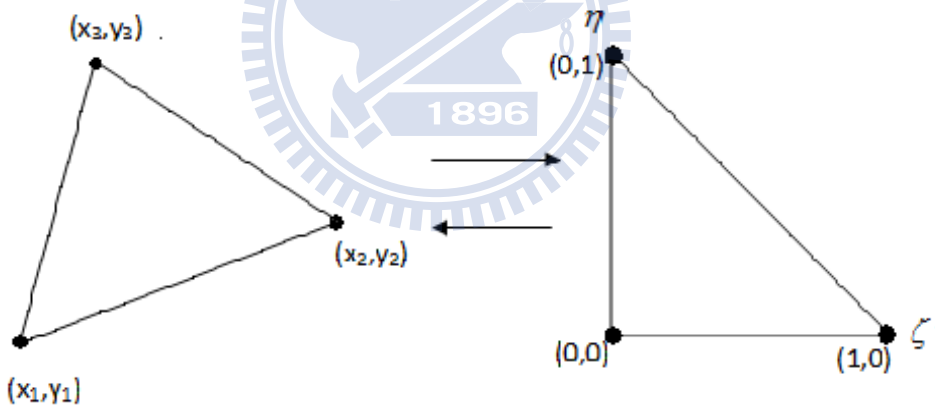


Fig1.3.2

Cartesian coordinates and triangular coordinates are linked by the relation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}[1-\zeta-\eta] + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\zeta + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\eta$$
$$\begin{bmatrix} x_{21} & x_{31} \\ y_{21} & y_{31} \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{1.10}$$

These simply apply the linear interpolant formula to the Cartesian coordinates: $x = x_1(1-\zeta-\eta)+x_2\zeta+x_3\eta$ and $y = y_1(1-\zeta-\eta)+y_2\zeta+y_3\eta$.

Inversion of (1.10) yields

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{2A} \begin{bmatrix} y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{1.11}$$

1.3.3 Linear FEM Discretization

After generating mesh e_j, we shall now construct a finite dimensional subspace V_h of the space V defined above consisting of piecewise linear function. We now let V_h be the set of functions v such that v is linear on each subinterval e_j, v is continuous on domain Ω and $v = 0$ on $\partial\Omega$.We observe that $V_h \subset V$. As parameter to describe a function $u_j = v(x_j)$ we may choose the values $u_j = v(x_j)$ at the node points $x_j, j = 0, ..., m+1$. Let us introduction the basis function $\phi_j \in V_h, j = 0, ..., m+1$. defined by

$$\phi_j(x_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j, \ i, j = 1, ..., M \end{cases} \tag{1.12}$$

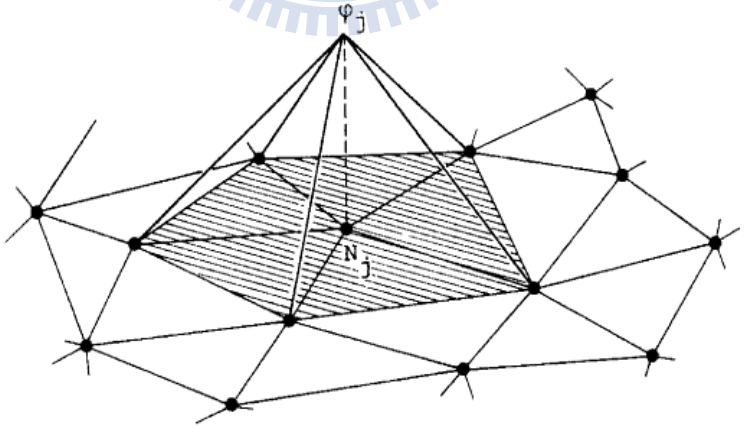


Fig1.3.3

i.e., ϕ_j is the continuous piecewise linear function that take the value 1 at node point x_j and the value 0 at other node points. A function $v \in V_h$ then has the representation

$$v(x) = \sum_{i=1}^{m} u_i \phi_i(x), \quad x \in \Omega \tag{1.13}$$

where $u_j = v(x_j)$, i.e., each $u_j = v(x_j)$ can be written in a unique way as a linear combination of the basis function ϕ_j. In particular it follow that V_h is a linear space of dimension m with basis $\{\phi_j\}_{i=1}^{m}$.

1.3.4 Partial Derivatives

From equations (1.10) and (1.11) we immediately obtain the following relations between partial derivatives:

$$\frac{\partial x}{\partial \zeta} = x_{21}, \frac{\partial x}{\partial \eta} = x_{31}, \frac{\partial y}{\partial \zeta} = y_{21}, \frac{\partial y}{\partial \eta} = y_{31}.$$

$$\frac{\partial \zeta}{\partial x} = \frac{y_{31}}{2A}, \frac{\partial \eta}{\partial x} = \frac{y_{12}}{2A}, \frac{\partial \zeta}{\partial y} = \frac{x_{13}}{2A}, \frac{\partial \eta}{\partial y} = \frac{x_{21}}{2A}.$$

The derivatives of function $u(\zeta, \eta)$ is

$$\begin{aligned} \nabla u &= \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \end{bmatrix} \\ &= \frac{1}{2A}\begin{bmatrix} \frac{\partial u}{\partial \zeta}y_{31} + \frac{\partial u}{\partial \eta}y_{12} \\ \frac{\partial u}{\partial \zeta}x_{13} + \frac{\partial u}{\partial \eta}x_{21} \end{bmatrix} \\ &= \frac{1}{2A}\begin{bmatrix} y_{31} & y_{12} \\ x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} \\ &= \frac{1}{2A}\Phi\begin{bmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} \end{aligned} \tag{1.14}$$

,where $\Phi = \begin{bmatrix} y_{31} & y_{12} \\ x_{13} & x_{21} \end{bmatrix}$.

Similarly,the derivatives of test function $v(\zeta, \eta)$ is

$$\begin{aligned} \nabla v &= \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{31} & y_{12} \\ x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} \frac{\partial v}{\partial \zeta} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \\ &= \frac{1}{2A}\begin{bmatrix} \frac{\partial v}{\partial \zeta} & \frac{\partial v}{\partial \eta} \end{bmatrix}\Phi^T \end{aligned} \tag{1.15}$$

So,we let

$$u = \sum_{i=1}^{3} u_i \phi_i \qquad ,v = \sum_{i=1}^{3} v_i \phi_i \tag{1.16}$$

where $\phi_1 = 1 - \zeta - \eta, \phi_2 = \zeta, \phi_3 = \eta,$

$$\begin{aligned} \begin{bmatrix} \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \phi_1}{\partial \zeta} & \frac{\partial \phi_2}{\partial \zeta} & \frac{\partial \phi_3}{\partial \zeta} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = D \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned} \tag{1.17}$$

where $D = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Similarity,

$$\begin{bmatrix} \frac{\partial v}{\partial \zeta} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_1}{\partial \zeta} & \frac{\partial \phi_1}{\partial \eta} \\ \frac{\partial \phi_2}{\partial \zeta} & \frac{\partial \phi_2}{\partial \eta} \\ \frac{\partial \phi_3}{\partial \zeta} & \frac{\partial \phi_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} D^T.$$

for one element,we get

$$\int_K \nabla v \nabla u dx = \int_{\widehat{K}} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} D^T \Phi^T \frac{1}{4A^2} \Phi D \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} |J| d\zeta d\eta \qquad (L.H.S) \tag{1.18}$$

$K_K = \int_{\widehat{K}} D^T \Phi^T \frac{1}{4A^2} \Phi D |J| d\zeta d\eta$ is called element stiffness matrix.

and

$$\int_K v f dx = \int_{\widehat{K}} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} |J| d\zeta d\eta \quad (R.H.S) \tag{1.19}$$

$M_K = \int_{\widehat{K}} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} |J| d\zeta d\eta$ is called mass matrix.

1.3.5 Assembling the element matrix

Assembling all the element matrices K_K, M_k into global matrices,here $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{f}$ are the global nodal vector.$\sum_{K\in T_h(\Omega)} K_K = K$, $\sum_{K\in T_h(\Omega)} M_K = M$. The equation (1.7) becomes

$$\overrightarrow{v} \cdot K \cdot \overrightarrow{u} = \overrightarrow{v} \cdot M \cdot \overrightarrow{f}, \quad \text{for all } v \in V_h$$

As a result,we obtain a linear system

$$K \cdot \overrightarrow{u} = M \cdot \overrightarrow{f} \tag{1.20}$$

1.4 Error Estimation

(Cea Lemma)

Let $u \in V$ be the solution of $a(u,v) = L(v)$, $\forall v \in V$ and $u_h \in V_h \subset V$.Then

$$||u - u_h||_V \leq \frac{\beta}{\alpha} \inf ||u - v_h||_V \qquad \forall v_h \in V_h \tag{1.21}$$

where α is coercivity scalar and β is continuity scalar.

For a typical elliptic problem satisfying the conditions (1)-(4) in the section 1.2.1,we choosing $v_h = \pi_h u \in V_h$ to be a suitable interpolant of u and estimate the interpolation error $||u - \pi_h u||_V$.

1.4.1 Interpolation Error with Piecewise Linear Functions

We first consider the case $V = H^1(\Omega)$ and $V_h = \{v_h \in V : v_h|_K \in P_1(K), \forall K \in T_h\}$ where $T_h = \{K\}$ is a triangulation of $\Omega \subset R^2$,ie,V_h is the standard finite element space of piecewise linear functions on triangles K.For $K \in T_h$ we define

$h_k =$ the dimeter of K=the longest side of K,

$\rho_k =$ the dimeter of the circle inscribed in K,

$h = \max_{K \in T_h} h_K$.

We shall assume that there is a positive constant β independent of h ,such that

$$\frac{\rho_K}{h_K} \geq \beta \qquad \forall K \in T_h \tag{1.22}$$

This condition means that the angles of the triangles K are not allowed to be arbitrarily small;the constant β is a measure of the smallest angle in any $K \in T_h$.Let $N_i, i = 1, ..., M$, be the nodes of T_h.Given $u \in C^0(\Omega)$ we define the interpolant

$$\pi_h u(N_i) = u(N_i) \quad i = 1, ..., M \tag{1.23}$$

,thus $\pi_h u$ is the piecewise linear function agreeing with u at the nodes of T_h.

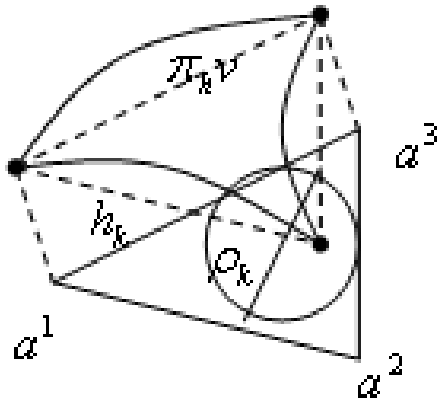


Fig1.4.1

For j=1,2 and x$\in K$ we have

$$\sum_{i=1}^{3}\phi_i(x) = 1, \tag{1.24}$$

$$\sum_{i=1}^{3}p_i(x)\phi_i(x) = 0, \tag{1.25}$$

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_j}\phi_i(x) = \frac{\partial}{\partial x_j}\sum_{i=1}^{3}\phi_i(x) = 0, \tag{1.26}$$

$$\sum_{i=1}^{3}p_i(x)\frac{\partial\phi_i}{\partial x_j}(x) = \frac{\partial v}{\partial x_j}(x). \tag{1.27}$$

Theorem 1 *Let $K \in T_h$ be a triangle with vertices a^i,i=1,2,3.Given $v \in C^0(K)$,let the interpolant $\pi v \in P_1(K)$ be defined by*

$$\pi v(a^i) = v(a^i), i = 1, 2, 3. \tag{1.28}$$

then

$$||v - \pi v||_{L^\infty(K)} \leq 2h_K^2 \max_{|\alpha=2|}||D^\alpha v||_{L^\infty(K)}, \tag{1.29}$$

$$\max_{|\alpha=1|}||D^\alpha(v - \pi v)||_{L^\infty(K)} \leq 6\frac{h_K^2}{\rho_K}\max_{|\alpha=2|}||D^\alpha v||_{L^\infty(K)}, \tag{1.30}$$

where

$$||v||_{L^\infty(k)} = \max_{x\in K}|v(x)|. \tag{1.31}$$

Proof. *Let $\phi_i, i = 1, 2, 3$, be the basis functions for $P_1(K)$.A general function $w \in P_1(K)$ then has the representation*

$$w(x) = \sum_{i=1}^{3}w(a^i)\phi_i(x), \quad x \in K, \tag{1.32}$$

so that in particular

$$\pi v(x) = \sum_{i=1}^{3}v(a^i)\phi_i(x), \quad x \in K, \tag{1.33}$$

since $\pi v(a^i) = v(a^i)$, *we now using the Taylor expansion at* $x \in K$:

$$v(y) = v(x) + \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x)(y_j - x_j) + R(x, y), \tag{1.34}$$

where

$$R(x, y) = \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j}(\xi)(y_i - x_i)(y_j - x_j), \tag{1.35}$$

is the remainder term of order 2 and ξ *is a point on line segment between x and y.In particular by choosing* $y = a^i$,*we have*

$$v(a^i) = v(x) + p_i(x) + R_i(x) \tag{1.36}$$

where

$$\begin{aligned} p_i(x) &= \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x)(a_j^i - x_j), a^i = (a_1^i, a_2^i), \\ R_i(x) &= R(x, a^i). \end{aligned} \tag{1.37}$$

Since

$$|a_j^i - x_j| \leq h_K, i = 1, 2, 3, \quad j = 1, 2, \tag{1.38}$$

the estimate of the remainder term $R_i(x)$

$$R_i(x) \leq 2h_K^2 \max_{|\alpha|=2} ||D^\alpha v||_{L^\infty(K)}, i = 1, 2, 3. \tag{1.39}$$

Now (1.33) *and* (1.36) *combine to give*

$$\pi v(x) = v(x) \sum_{i=1}^{3} \phi_i(x) + \sum_{i=1}^{3} p_i(x)\phi_i(x) + \sum_{i=1}^{3} R_i(x)\phi_i(x), \quad x \in K. \tag{1.40}$$

By(1.24) ,(1.25) *in Lamma 2 and*(1.40) ,*we have*

$$\pi v(x) = v(x) + \sum_{i=1}^{3} R_i(x)\phi_i(x), \tag{1.41}$$

which gives us the following representation of the interpolation error:

$$v(x) - \pi v(x) = -\sum_{i=1}^{3} R_i(x)\phi_i(x). \tag{1.42}$$

Since $0 \le \phi_i(x) \le 1$,*if* $x \in K, i = 1, 2, 3$,*we can use the previous estimate* (1.39) *of the remainder term* R_i *to get*

$$\begin{aligned} |v(x) - \pi v(x)| &\le \sum_{i=1}^{3} |R_i(x)||\phi_i(x)| \\ &\le \max_i |R_i(x)| \sum_{i=1}^{3} |\phi_i(x)| \le 2h_K^2 \max_{|\alpha|=2} ||D^\alpha v||_{L^\infty(K)}, x \in K \end{aligned} \tag{1.43}$$

we proves (1.29) . ■

Proof. *To prove* (1.30) *we differentiate* (1.33) *with respect to* x_1 *to get*

$$\frac{\partial \pi v}{\partial x_1}(x) = \sum_{i=1}^{3} v(a^i)\frac{\partial \phi_i}{\partial x_1}(x), \tag{1.44}$$

which together with (1.36) *shows that*

$$\frac{\partial \pi v}{\partial x_1}(x) = v(x)\sum_{i=1}^{3} \frac{\partial \phi_i}{\partial x_1}(x) + \sum_{i=1}^{3} p_i(x)\frac{\partial \phi_i}{\partial x_1}(x) + \sum_{i=1}^{3} R_i(x)\frac{\partial \phi_i}{\partial x_1}(x). \tag{1.45}$$

Hence,by (1.26) *and* (1.27) *we have*

$$\frac{\partial \pi v}{\partial x_1}(x) = \frac{\partial v}{\partial x_1}(x) + \sum_{i=1}^{3} R_i(x)\frac{\partial \phi_i}{\partial x_1}(x), \tag{1.46}$$

which gives the following representation o the error $\frac{\partial v}{\partial x_1} - \frac{\partial \pi v}{\partial x_1}$:

$$\frac{\partial v}{\partial x_1}(x) - \frac{\partial \pi v}{\partial x_1}(x) = -\sum_{i=1}^{3} R_i(x)\frac{\partial \phi_i}{\partial x_1}(x), \quad x \in K. \tag{1.47}$$

and

$$\max_{x \in K} |\frac{\partial \phi_i}{\partial x_1}(x)| \le \frac{1}{\rho_K}, \tag{1.48}$$

which together with (1.39) *finally gives*

$$|\frac{\partial v}{\partial x_1}(x) - \frac{\partial \pi v}{\partial x_1}(x)| \le 6\frac{h_K^2}{\rho_K} \max_{|\alpha=2|} ||D^\alpha v||_{L^\infty(K)}. \tag{1.49}$$

In the same way we estimate $\frac{\partial v}{\partial x_2} - \frac{\partial \pi v}{\partial x_2}$ *and thus* (1.30) *follows.The proof of the theorem is now complete once the lemma is established.* ■

Theorem 2 *Under the assumptions of Theorem 1 there is an absolute constant C such that*

$$||v - \pi v||_{L_2(K)} \leq C h_K^2 |v|_{H^2(K)}, \tag{1.50}$$

$$|v - \pi v|_{H^1(K)} \leq C \frac{h_K^2}{\rho_K} |v|_{H^2(K)}. \tag{1.51}$$

Theorem 3 *The global interpolation errors*

$$||u - \pi_h u||_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}, \tag{1.52}$$

$$||u - \pi_h u||_{L^2(\Omega)} \leq Ch^2|u|_{H^2(\Omega)} \tag{1.53}$$

Proof. We have by summing over $K \in T_h$,

$$\begin{aligned} ||u - \pi_h u||^2_{L^2(\Omega)} &= \sum_{K \in T_h} ||u - \pi_h u||^2_{L^2(K)} \leq \sum_{K \in T_h} C^2 h_K^4 |u|^2_{H^2(K)} \\ &\leq C^2 h^4 \sum_{K \in T_h} |u|^2_{H^2(K)} = C^2 h^4 |u|^2_{H^2(\Omega)}, \end{aligned} \tag{1.54}$$

and similarly using (1.22) ,ie, $\frac{h_K}{\rho_K} \leq \frac{1}{\beta}$,

$$\begin{aligned} ||u - \pi_h u||^2_{H^1(\Omega)} &\leq \sum_{K \in T_h} C^2 \frac{h_K^4}{\rho_K^2} |u|^2_{H^2(K)} \leq \sum_{K \in T_h} C^2 \frac{h_K^2}{\beta^2} |u|^2_{H^2(K)} \\ &\leq C^2 \frac{h^2}{\beta^2} |u|^2_{H^2(\Omega)} \end{aligned} \tag{1.55}$$

so that

$$||u - \pi_h u||_{H^1(\Omega)} \leq \frac{Ch}{\beta} |u|_{H^2(\Omega)} = Ch|u|_{H^2(\Omega)}, \tag{1.56}$$

if the constant β is included in the constant C,and

$$||u - \pi_h u||_{L^2(\Omega)} \leq Ch^2 |u|_{H^2(\Omega)}. \tag{1.57}$$

■

1.4.2 A Priori Error Estimation

The bilinear form $a(u,v) = \int \nabla u \nabla v dx$, the finite element space of piecewise linear element is $H^1(\Omega) = V$, the test function space is $H_0^1(\Omega)$. The priori error estimate of Possion equation

$$||u - u_h||_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}$$

Proof. by Céa Lamme, so the inequality is

$$||u - u_h||_{H^1(\Omega)} \leq \frac{\beta}{\alpha} \inf ||u - v_h||_{H^1(\Omega)} \quad \text{for all } v_h \in V_h$$

where α is coercivity scalar and β is continuity scalar ,than we choose $v_h = \pi_h u$, where $\pi_h u$ is interpolant by the piecewise linear basis function

$$\frac{\beta}{\alpha} \inf ||u - v_h||_{H^1(\Omega)} \leq \frac{\beta}{\alpha} ||u - \pi_h u||_{H^1(\Omega)}$$

by interpolation error estimate

$$\frac{\beta}{\alpha} ||u - \pi_h u||_{H^1(\Omega)} \leq C \frac{\beta}{\alpha} h|u|_{H^2(\Omega)} = C_2 h|u|_{H^2(\Omega)}$$

■

1.4.3 A Posterior Error Estimation and Adaptive Mesh-Refinement Techniques

On the other way to refine the mesh ,we introduce a posterior error estimation and the adaptive mesh-refinement techniques. Before presenting the a posteriori error estimators, we introduce some notations.

For any open subset ω of Ω with Lipschitz boundary γ ,we denote by $L^2(\omega)$,$H^k(\omega)$, and $L^2(\gamma)$,$k \geq 1$, the standard Lebesque and Sobolev spaces ,respectively,equipped with the norms $||\cdot||_{0,\omega} := ||\cdot||_{L^2(\omega)}$, $||\cdot||_{k,\omega} := ||\cdot||_{H^k(\omega)}$, $||\cdot||_{0,\gamma} := ||\cdot||_{L^2(\gamma)}$.

For $T \in \overline{T}_h$ we denote by $\overline{E}(T)$ and $\overline{N}(T)$ the set of its edges and vertices,respectively.Let

$$\overline{E}_h := \bigcup_{T\in\overline{T}_h} \overline{E}(T), \qquad \overline{N}_h := \bigcup_{T\in\overline{T}_h} \overline{N}(T),$$

be the set of all edges and vertices,respectively,in the triangulation. We split $\overline{E}_h$ and $\overline{N}_h$ in the form

$$\overline{E}_h = \overline{E}_{h,\Omega} \bigcup \overline{E}_{h,D}, \qquad \overline{N}_h = \overline{N}_{h,\Omega} \bigcup \overline{N}_{h,D},$$

with

$$\overline{E}_{h,D} \ : \ = \{E \in \overline{E}_h : E \subset \Gamma_D\}$$

$$\overline{N}_{h,D} \ : \ = \{x \in \overline{N}_h : x \subset \Gamma_D\}$$

For $T \in \overline{T}_h$ and $E \in \overline{E}_h$, we denote by h_T and h_E their diamter and length,respectively.

For $T \in \overline{T}_h$ and $E \in \overline{E}_h$ and $x \in \overline{N}_h$,let

$$\omega_T := \bigcup_{\overline{E}(T)\cap\overline{E}(T')\neq 0} T', \ \ \omega_E := \bigcup_{E\in\overline{E}(T')} T', \ \ \omega_x := \bigcup_{x\in\overline{N}(T')} (T'),$$

Figure1.4.2: Domain ω_T, ω_E and ω_x

and put

$$\begin{aligned} V_T &: = \{\phi \in C(T) : \phi \in \prod\nolimits_{\max(k+1,3),2}, \phi|_E \in \prod\nolimits_{k+1,1}, \forall E \in \overline{E}(T) \\ \phi(x) &= 0, \forall x \in \overline{N}(T), \phi|_E = 0, \forall E \in \overline{E}(T) \cap \overline{E}_{h,D}\}, \end{aligned} \tag{1.58}$$

$$V_x := \{\phi \in C(\overline{\omega}_x) : \phi|_{T'} \in V_{T'}, \forall T' \subset \omega_x\}. \tag{1.59}$$

Given $E \in \overline{E}_{h,\Omega}$ and $\phi \in L^2(\omega_E)$ with $\phi|_{T'} \in C(T'), \forall T' \subset \omega_E$,we denote by $[\phi]_E$ the jump of ϕ across E in an arbitrary,but fixed direction.Put

$$R_E(u_h) := \begin{cases} -[\dfrac{\partial u_h}{\partial n}]_E, & \forall E \in \overline{E}_{h,\Omega}\ , \\ 0, & \forall E \in \overline{E}_{h,D}\ . \end{cases} \tag{1.60}$$

and

$$R_T := \prod\nolimits_T f + \triangle u_h, \ \ \forall T \in \overline{T}_h, \tag{1.61}$$

where $\prod_T$ is the L^2 projectors of $L^2(\Omega)$ onto the space of piecewise constant functions with respect to $\overline{T}_h$.

The first error estimator simply is a weighted combination of the residuals $R_T(u_h)$ and $R_E(u_h)$. It is given by

$$\eta_{T,R} := \{h_T^2 ||R_T(u_h)||_{0,T}^2 + \sum_{E \in \overline{E}(T) \setminus \overline{E}_{h,D}} h_E ||R_E(u_h)||_{0,E}^2\}^{1/2}, \ \ \forall T \in \overline{T}_h \tag{1.62}$$

The second error estimator is based on the solution of local Dirichlet problems. For any $x \in \overline{N}_{h,\Omega}$, it is given by

$$\eta_{x,D} := ||\nabla(u_{x,D} - u_h)||_{0,\omega}\ , \tag{1.63}$$

For more detialed proof of the lower bound and upper bound for the error estimators is in [17].

According to the error estimator η_T we obtained above, we want to decide a refinement of T, there is a strategy that we refine all T with $\eta_T \geq \gamma \max_{T\in\overline{T}_h} \eta_T$. Here $0 \leq \gamma \leq 1$ is a given threshold,typically $\gamma = 0.5$. This strategy is very cheap and often satisfactory results.

Having decided which elements should be refined, we refine them by connecting the mid-points of their edges.Triangles may be cut into four new ones by connecting the mid-points of their edges .

Figure1.4.3: triangles with adaptive mesh-refinement

1.5 Finite Element Approximation for Singular Funcitons

The problem on hand is to devise singular elements within which the field variable would vary as $R^t(0 \leq t \leq 1)$ where R is the radial distance from the point where the derivative of the function would have a singularity of the type R^{t-1}.For example,let (x_1, y_1) be the desired points of singularity in two-dimensional problem,such that R is equal to $[(x-x_1)^2+$

$(y - y_1)^2]^{1/2}$.The singularity may be introduced in the geometric transformation between (x, y) and (ζ, η) systems.

1.5.1 Lagrange Interpolation

For one-dimensional domain, consider a straight line along x-axis in a Cartesain reference, such that $x_0 \leq x \leq x_M$.Consider the dimensionless coordinate $\xi = (x - x_0)/(x_M - x_0)$ such that $0 \leq \xi \leq 1$.A mth-order polynomial $F^{(m)}(\xi)$ can be represented by interpolating the discrete values of the function at $(m + 1)$ equidistant points $\xi_i[i = 0, ..., m; \xi_i(i/m)]$ using the Lagrange interpolation function, $L_i^{(m)}(\xi)$, as

$$F^{(m)}(\xi) = \sum_{i=o}^{m} F_i L_i^{(m)}(\xi) \tag{1.64}$$

where

$$L_i^{(m)}(\xi) = \prod_{j=0, j\neq i}^{m} (\frac{\xi - \xi_j}{\xi_i - \xi_j}). \tag{1.65}$$

It is seen that the functions in (1.65) have the property that

$$L_i^{(m)}(\xi_k) = \delta_{ik} \tag{1.66}$$

where $\delta_{ik} = 0$ $(i \neq k)$ and $\delta_{ik} = 1(i = k)$.

For a simple example, let we consider second-order polynomial $F^{(2)}(\xi)$ and the transformation reduces to

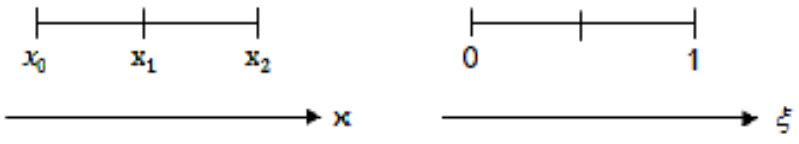


Fig1.5.1: x and ξ coordinate systems

$$(x - x_0) = (x_M - x_0)\xi \tag{1.67}$$

the Lagrange function can be represent as

$$\begin{aligned} L_0^{(2)} &= 2(\xi - \frac{1}{2})(\xi - 1) \\ L_1^{(2)} &= -4\xi(\xi - 1) \\ L_2^{(2)} &= 2\xi(\xi - \frac{1}{2}) \end{aligned}$$

1.5.2 Singular Element

Apart from problems of fracture,at a given point in the domain,say P,the solution funciton F,whould have a singularity in its derivative.If R is the radial length measured from the point P,the solution function F would behave as R^{λ} $(0 < \lambda < 1)$,so that the first derivative of F at P would be infinite. Such singularities may arise at re-entrant corners, tips of sharp crack,etc.

For One-dimensional cases,consider a straight line along the x-axis,$x_0 \leq x \leq x_M$.Let the geometry of this line be described by an rth-order polynomial as

$$x = \sum_{i=o}^{r} x_i L_i^{(r)}(\xi) \tag{1.68}$$

where $x_i = x(\xi_i)$ and $\xi_i = (i/r)(i = 0, 1, ..., r)$ are $(r + 1)$ equidistant points in the parametric domain $0 \leq \xi \leq 1$.(x_i themselves may not necessarily be equidistant).Also let the dependent variable,say u,be represented by an mth-order polynomial.

$$u(\xi) = \sum_{j=0}^{m} u_j L_j^{(m)}(\xi) \tag{1.69}$$

where $u_j = u(\xi_j)$, but now $\xi_j = (j/m)(j = 0, ..., m)$ are $(m + 1)$ equidistant points in $0 \leq \xi \leq 1$.

If the geometric transformation is defined by

$$(x - x_0) = (x_M - x_0)\xi^t \tag{1.70}$$

where t is a positive integer greater than unity, the function $u(x)$ corresponding to $u(\xi)$ of Eq.(1.69) would be an mth-order polynomial in the variable $(x - x_0)^{1/t}$ or $R^{1/t}$,and hence the derivative $(\partial u/\partial x)$ has a term which varies as $R^{(1-t)/t}$,for $t > 1$,would have a singularity at $x = x_0$.

For instance when $r = 2$ and $t = 2$, the positioning of nodes at $x = x_0; x = x_0 + (x_M - x_0)(\frac{1}{2})^2 = x_0 + (x_M - x_0)(\frac{1}{4})$, and $x = x_M$.it would lead to $x \sim \xi^2$ and hence $\partial u/\partial x$ would have a singularity of the type $R^{-1/2}$.

In general,singularities in $(\partial u/\partial x)$ of the type $R^{(1-t)/t}, t \leq r$ (t and r are integer),can be created by suitably choosing the valus x_i in an rth-order geometric transformation of the type (1.68) .

For two-dimensional domain, consider a standard dimensionless triangle $(0 \leq \eta, \zeta \leq 1)$ as shown in Fig1.5.2. The triangle $P_1P_2P_3$ with Cartesain coordinate $(x_i, y_i)(i = 1, 2, 3)$ is then mapped into the standard triangle ,through the relations:

$$\begin{aligned} (x - x_1) &= (x_2 - x_1)\zeta + (x_3 - x_1)\eta \\ (y - y_1) &= (y_2 - y_1)\zeta + (y_3 - y_1)\eta \end{aligned} \tag{1.71}$$

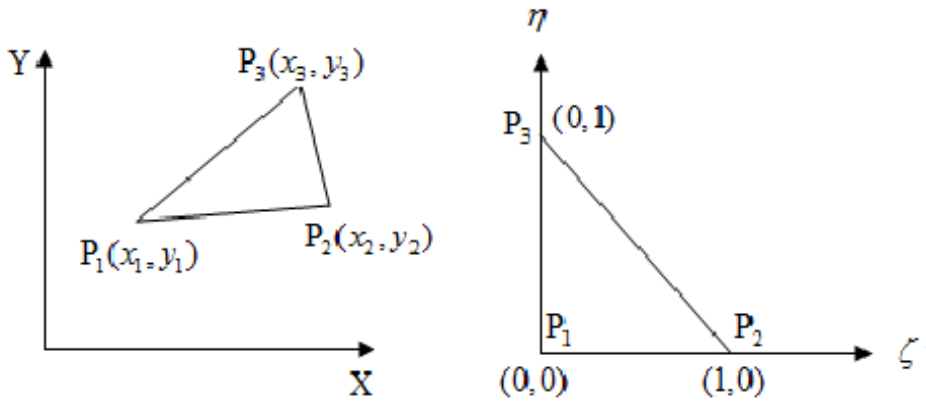


Fig1.5.2: Triangular element in (x, y) and (ζ, η) coordinate systems

Consider $(m+1)$ equidistant point along the line $\eta = 0$ and $\zeta = 0$,respectively,such that $\eta_i = (i/m)$ and $\zeta_j = (j/m)$ $(i, j = 0, 1, ..., m)$.A mth-order polynomial (m being an integer) , $F^{(m)}(\eta, \zeta)$, can be represented by interpolating the values of the function in the standard triangle,as:

$$F^{(m)}(\eta, \zeta) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} F_{ij} L_{ij}^{(m)}(\eta, \zeta) \tag{1.72}$$

where

$$L_{ij}^{(m)} = \left[\prod_{r=0}^{i-1} \left(\frac{\eta - \eta_r}{\eta_i - \eta_r}\right)\right] \left[\prod_{s=0}^{j-1} \left(\frac{\zeta - \zeta_s}{\zeta_j - \zeta_s}\right)\right] \left[\prod_{t=0}^{m-(i+j+1)} \left(\frac{1 - (\eta + \zeta) - (t/m)}{1 - (\eta_i + \zeta_j) - (t/m)}\right)\right]. \tag{1.73}$$

The function also have the property that

$$L_{ij}^{(m)}(\eta_k, \zeta_l) = \delta_{ik}\delta_{jl}. \tag{1.74}$$

For the detail of singular element geometric transformation in two-dimensional domain, see the reference [8].

Chapter 2
Multigrid Method

2.1 Introduction of Multigrid Method

Multigrid methods(MG) are among the most efficient methods of solving the linear systems arising from discretization of eilliptic partial differential equations. There has been intensive research on the convergence of MG since it was introduced by Fedorenko. For symmetric positive-definite eilliptic problems, thanks to many researchers, such as Bank, Brandt, Braess, Bramble, Dupont, Hackbusch, Mandel and McCormick, etc, the convergence theory has matured. The major ingredients for convergence analysis of MG are called the $approximation\ property$ and the $smoothing\ property$. One approach for convergence analysis is the so-called $compact\ perturbation\ technique$, which relies on a strong approximation property and treats the lower order terms as a small perturbation of the symmetric positive define term. The technique has been successfully applied to diffusion problems and Bramble, Pasciak, Wang, Xu have shown robust MG uniform convergence. In these studies, uniform convergence of MG can be established with one step of standard Jacobi or Gauss-Seidel smoothing even without regularity assumptions. Another approach requires a strong smoothing property to compensate for poor approximation property. In this direction, it is very important to find a robust smoother.Robust smoothers such as the block Jacobi , block Gauss-Seidel method and the incomplete LU factorization (ILU) method are commonly used.

The efficiency of the multigrid algoritthm is achieved from an elegant combination of the smoothing procedure and the coarse grid correction procedure. The smoothing procedure plays the role of reducing highly oscillatory error modes, and the coarse grid is used to correct the remaining smooth error modes. Hackbusch and Braess give the first rigorous proof on the multigrid convergence and identify that the $smoothing\ property$ and the $approximation\ property$ are the cornerstones for the convergence analysis of multigrid methods. The smoothing property achieved by stationary iterative method. The approximation property is achieved by choosed intergrid interpolation operator.

In general,the stationary iterative methods can effective eliminating the high-frequency or oscillatory components of the error,but leave the low-frequency or smooth components of the error as we will shown in experiment 1 in section 2.6.This process is called the relaxation process in MG. Moreover, the smooth error modes on a fine grid appear more oscillatory on the coarse grid. As a result, repeating the relaxation process may consequently remove various oscillartory modes of error. To achieve the efficiency of the MG algorithm, we only have to concern how to restrict smooth error to the coarse grid and how to prolong the errors remained in coarse grid to the fine grid. This process is called the inter-grid interpolation, we will introduce the relaxation and interpolation process in the fillowing sections.

2.2 Relaxation Process

For relaxation process, we introduce the stationary iteration methods.Stationary iterative methods solve a linear system with an operator approximating the original one and based

on a measurement of the error in the result (the residual), form a "correction equation" for which this process is repeated. While these methods are simple to derive, implement, and analyze, convergence is only guaranteed for a limited class of matrices. Examples of stationary iterative methods are the Jacobi method, Gauss–Seidel method and the Successive over-relaxation method.

2.2.1 Jacobi Mehtod

One of the methods to solve the linear system is the Jacobi method . The Jacobi method is derived by examining each of the $N-1$ equations in the linear system $Av = f$ in isolation .If in the $i-th$ equation

$$\sum_{j=1}^{N-1} a_{i,j} v_j = f_i, \tag{2.75}$$

we solve for the value of v_i while assuming the other entries of v remain fixed ,we obtain

$$v_i = (f_i - \sum_{j \neq i} a_{i,j} v_j)/a_{i,i} \quad , 1 \leq j \leq N-1 \tag{2.76}$$

this suggests an iterative method defined by

$$v_i^{(k)} = (f_i - \sum_{j \neq i} a_{i,j} v_j^{(k-1)})/a_{i,i} \quad , 1 \leq j \leq N-1 \tag{2.77}$$

which is the Jacobi method.

In matrix terms, the definition of the Jacobi method can be expressed as

$$v^{(k)} = D^{-1}(L+U)v^{(k-1)} + D^{-1}f, \tag{2.78}$$

where the matrices D, $-L$ and $-U$ represent the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of A,respectively.

The pseudocode for the Jacobi method is given by following

```
Choose an initial guess v^(0) to the solution u
for k = 1, 2, ···
    for i = 1, 2, ··· , N − 1
        v_i = 0
        for j = 1, 2, ··· , i − 1, i + 1, ··· , N − 1
            v_i = v_i + a_{i,j} v_j^(k−1)
        end
        v_i = (f_i − v_i)/a_{i,i}
    end
    v^(k) = v
    check convergence;continue if necessary
end
```

There is a simple modification which can be made to the Jacobi iteration.

$$v_j^* = (f_i - \sum_{j \neq i} a_{i,j} v_j^{(k-1)})/a_{i,i} \quad , 1 \leq j \leq N-1$$

However,v_j^* is now only an intermediate value. The new approximation is given by the weighted average

$$v_j^{(k)} = (1-\omega) v_j^{(k-1)} + \omega v_j^* \quad , 1 \leq j \leq N-1$$

where $\omega \in \mathbb{R}$ is a weighting factor which may be chosen.This iteration is called weighted Jacobi method.

2.2.2 Gauss-Seidel Method

Consider again the linear systems $Av = f$,now assume that the equations are examined one at a time in sequence,and that previously computed results are used as soon as they are available ,we obtain the Gauss-Seidel method:

$$v_i^{(k)} = (f_i - \sum_{j<i} a_{i,j} v_j^{(k)} - \sum_{j>i} a_{i,j} v_j^{(k-1)})/a_{i,i} \quad , 1 \leq j \leq N-1 \tag{2.79}$$

In matrix form,

$$Dv^{(k)} - Lv^{(k)} - Uv^{(k-1)} = f$$

$$(D - L)v^{(k)} = Uv^{(k-1)} + f$$

$$v^{(k)} = (D - L)^{-1}Uv^{(k-1)} + (D - L)^{-1}f$$

where the matrices D, $-L$ and $-U$ represent the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of A,respectively. The pseudocode for the Gauss-Seidel method is given by following

Choose an initial guess $v^{(0)}$ to the solution u
for $k = 1, 2, \cdots$
  for $i = 1, 2, \cdots, N-1$
    $v_i = 0$
    for $j = 1, 2, \cdots, i-1$
      $v_i = v_i + a_{i,j}v_j^{(k)}$
    end
    for $j = i+1, \cdots, N-1$
      $v_i = v_i + a_{i,j}v_j^{(k-1)}$
    end
    $v_i = (f_i - v_i)/a_{i,i}$
  end
  $v^{(k)} = v$
  check convergence;continue if necessary
end

2.2.3 Successive Overrelaxation Method

The Successive Overrelaxation Method, or SOR ,is devised by applying extrapolation to the Gauss-Seidel method. This extrapolation takes the form of a weighted average between the

previous iterate and the computed Gauss-Seidel iterate successively for each component:

$$v_i^{(k)} = \omega \overline{v_i}^{(k)} + (1-\omega) v_i^{(k-1)} \tag{2.80}$$

,where$\overline{v_i}^{(k)}$ denotes a Gauss-Seidel iterate ,and ω is the extrapolation (weighting) factor.The idea is to choose a value for ω that will accelerate the rate of convergence of the iterates to the solution.

In matrix form, the SOR is written as

$$\begin{aligned} Dv^{(k)} &= \omega D\overline{v}^{(k)} + (1-\omega)Dv^{(k-1)} \\ &= \omega[f + Lv^{(k)} + Uv^{(k-1)}] + (1-\omega)Dv^{(k-1)} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad & (D-\omega L)v^{(k)} = [\omega U + (1-\omega)D]v^{(k-1)} + \omega f \\ v^{(k)} \quad = \quad & (D-\omega L)^{-1}[\omega U + (1-\omega)D]v^{(k-1)} + (D-\omega L)^{-1}\omega f \end{aligned}$$

where the matrices $D, -L$ and $-U$ represent the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of A,respectively.

If the extrapolation factor ω is choosing by one,the SOR method simplifies to the Gauss-Seidel method.If $0 < \omega < 1$,its called underrelaxation.If $1 < \omega < 2$, its called overrelaxation and SOR fails to converge when ω is outside the interval $(0, 2)$.

The pseudocode for the SOR method is given by following

```
Choose an initial guess v^(0) to the solution u
for k = 1, 2, ···
    for i = 1, 2, ··· , N − 1
        σ = 0
        for j = 1, 2, ··· , i − 1
            σ = σ + a_{i,j} v_j^(k)
        end
        for j = i + 1, ··· , N − 1
            σ = σ + a_{i,j} v_j^(k−1)
        end
        σ = (f_i − σ)/a_{i,i}
        v_i^(k) = v_i^(k−1) + ω(σ − v_i^(k−1))
    end
    check convergence;continue if necessary
end
```

2.3 Inter-grid Interpolation : Restriction and Prolongation

First,we consider the linear prolongation, the operator will be denoted I_{2h}^{h}.It takes coarse grid vectors and produces find grid vectors according to the rule $I_{2h}^{h} v^{2h} = v^{h}$

$$\begin{aligned} v_{2j}^{h} &= v_{j}^{2h} \\ v_{2j+1}^{h} &= \frac{1}{2}(v_{j}^{2h} + v_{j+1}^{2h}), \quad 0 \le j \le \frac{N}{2} - 1 \end{aligned} \tag{2.81}$$

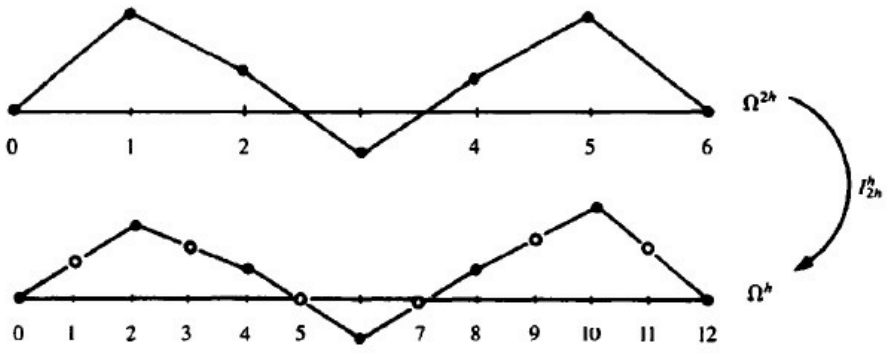


Figure 2.3.1: Prolongation of a vector on the coarse grid to the fine grid.

At even-numbered fine grid points ,the values of the vector are transferred directly from Ω^{2h} to Ω^h. At odd-numbered find grid points,the value of the vector is the average of the adjacent coarse grid values.

For the case $N = 8$, this operator has the form

$$I_{2h}^h v^{2h} = \frac{1}{2}\begin{bmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{2h} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}_h = v^h$$

The second ,we consider moving vectors from a fine grid to a coarse grid.There are called restriction operators and are denoted by I_h^{2h}.It is defined by $I_h^{2h} v^h = v^{2h}$,where

$$v_j^{2h} = \frac{1}{4}(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h), \quad 1 \le j \le \frac{N}{2} - 1. \tag{2.82}$$

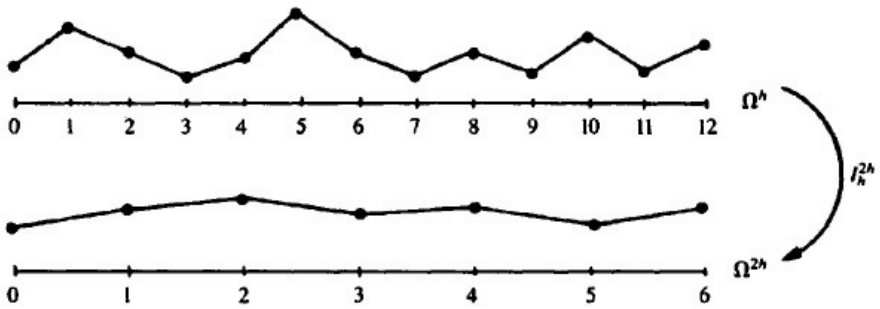


Figure 2.3.2: Restriction by full weighting of a find grid vector to the coarse grid.

the values of the coarse grid vector are a weighted average of values at neighboring find grid points.

In the case of $N = 8$,the operator has the form

$$I_h^{2h} v^h = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 & & & & & \\ & & 1 & 2 & 1 & & & \\ & & & & 1 & 2 & 1 & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}_h = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{2h} = v^{2h}.$$

2.4 Multigrid Algorithm

After kowning the procedure , we compose all the components aobve to make the correction scheme. First , relaxation on the fine grid will eliminate the oscillatory components of the error,leaving a relatively smooth error,then transfer the residual on the fine grid to the coarse grid an solve the residual equation exactly on the Ω^{2h}.Since the error is smooth on the Ω^{2h},we can prolongation the error accurately back to the fine grid.Finally, relaxation on the fine grid use the better initial which is correction before.

The procedure is given by following

$v^h \leftarrow MG(v^h, f^h)$
Relax ν_1 times on $A^h u^h = f^h$ on Ω^h with initial guess v^h.
Compute $r^{2h} = I_h^{2h}(f^h - A^h v^h)$.
Solve $A^{2h} e^{2h} = r^{2h}$ on Ω^{2h}.
Correct fine grid approximation : $v^h \leftarrow v^h + I_{2h}^h e^{2h}$.
Relax ν_2 times on $A^h u^h = f^h$ on Ω^h with initial guess v^h.

Here,in practice, the number of relaxation times ν_1 and ν_2 are often 1,2 or 3.

Moreover, the correction scheme is not just two level, it can be more level,by recursive within itself.The algorithm is called the $V - cycle$.The definition which is given

by

V-cycle Scheme

$$v^h \leftarrow MV^h(v^h, f^h)$$

1.Relax ν_1 times on $A^h u^h = f^h$ with a given initial guess v^h.
2.If $\Omega^h =$ coarsest grid, then go to 4.
Else $f^{2h} \leftarrow I_h^{2h}(f^h - A^h v^h)$.
$v^{2h} = 0$
$v^{2h} \leftarrow MV^{2h}(v^{2h}, f^{2h})$.
3.Correct $v^h \leftarrow v^h + I_{2h}^h v^{2h}$.
4.Relax ν_2 times on $A^h u^h = f^h$ with initial guess v^h.

The $V - cycle$ is just one of a family of multigrid cycling schemes.The entire family is called the $\mu - cycle$ method and is defined by

μ-cycle Scheme

$$v^h \leftarrow M\mu^h(v^h, f^h)$$

1.Relax ν_1 times on $A^h u^h = f^h$ with a given initial guess v^h.
2.If $\Omega^h =$ coarsest grid, then go to 4.
Else $f^{2h} \leftarrow I_h^{2h}(f^h - A^h v^h)$.
$v^{2h} = 0$
$v^{2h} \leftarrow M\mu^{2h}(v^{2h}, f^{2h})$ μ times.
3.Correct $v^h \leftarrow v^h + I_{2h}^h v^{2h}$.
4.Relax ν_2 times on $A^h u^h = f^h$ with initial guess v^h.

If $\mu = 1$,which gives the $V - cycle$ and $\mu = 2$ is called the $W - cycle$.

Here are other ideas,if it starts on the coarsest discretization with an exact solver, the results are interpolated to the next finer grid with a few cycles (V or W) of the multigrid method are applied .The result is again interpolated to the next finer grid, where again a few cycles of multigrid method are applied.If this is used recursively, the so-called $full - multigrid$ method.The algorithm with $V - cycle$ is given by

Full Multigrid V-cycle Scheme

$v^h \leftarrow FMV^h(v^h, f^h)$

Initialize f^h,f^{2h},...;v^f,v^{2h},...to zero.

Solve or relex on coarsest grid

$\vdots$

$v^{4h} \leftarrow v^{4h} + I_{8h}^{4h} v^{8h}$

$v^{4h} \leftarrow MV^{4h}(v^{4h}, f^{4h})$

$v^{2h} \leftarrow v^{2h} + I_{4h}^{2h} v^{4h}$

$v^{2h} \leftarrow MV^{2h}(v^{2h}, f^{2h})$

$v^h \leftarrow v^h + I_{2h}^h v^{2h}$

$v^h \leftarrow MV^h(v^h, f^h)$

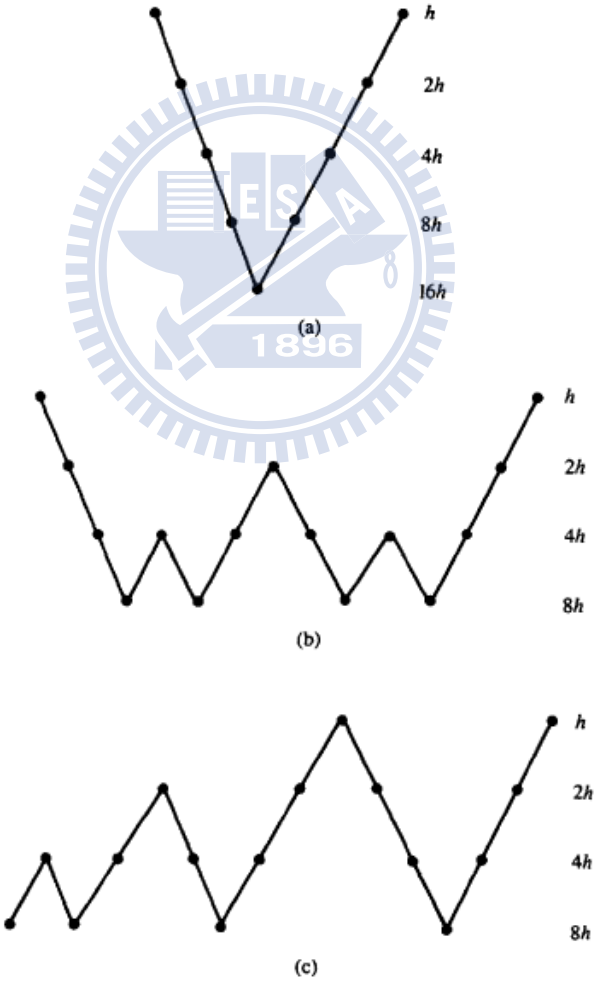


Figure 2.4.1: Schedule of grids for (a) V-cycle, (b) W-cycle and (c) FMV scheme

2.5 Complexity

There are various method to solve the linear systems,including stationary iterative methods or nonstationary iterative methods ,the following table gives us the complexity of various method :

Table 3.3		
Complexity of Various Methods		
Method	2D	3D
JM	$O(N^2)$	$O(N^{\frac{5}{3}})$
GS	$O(N^2)$	$O(N^{\frac{5}{3}})$
SOR	$O(N^{\frac{3}{2}})$	$O(N^{\frac{4}{3}})$
CG	$O(N^{\frac{3}{2}})$	$O(N^{\frac{4}{3}})$
Multigrid	$O(N \log N)$	$O(N \log N)$

2.6 Numerical Experiments

Experiment 1

In this experiments we consider the weighted Jacobi method with $\omega = 2/3$ applied to the one-dimensional problem $Au = 0$ on a grid N=48. We use an initial guess,

$$v_j^h = \frac{1}{2}\left[\sin(\frac{12j\pi}{N}) + \sin(\frac{30j\pi}{N})\right]$$

the results of this calculation are given in Figs 2.6.1(a)-(e). The initial guess is shown in Fig 2.6.1(a) . In Fig 2.6.1(b), the approximation v^h after one relaxation sweep is superimposed on the initial guess. Much of the oscillatory component of the initial guess has already been removed. The maximum norm of the error has decreased signification. Fig. 2.6.1(c) shows the approximation after three relaxation sweeps, superimposed on the previous approximations. Further relaxations on the fine grid would provide only a slow improvement at this

point. This signals that it is time to move to the coarse grid.Fig. 2.6.1(d) shows the fine grid error after one relaxation sweep on the coase grid and the error after three coarse grid relaxation sweeps is shown in Fig. 2.6.1(e)

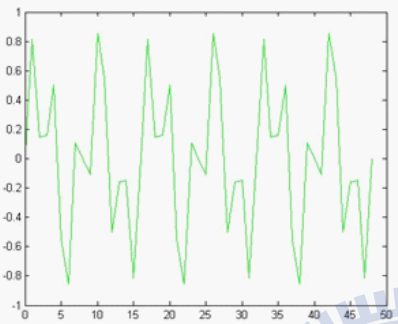


Figure 2.6.1(a): $||e|| = 0.8536$

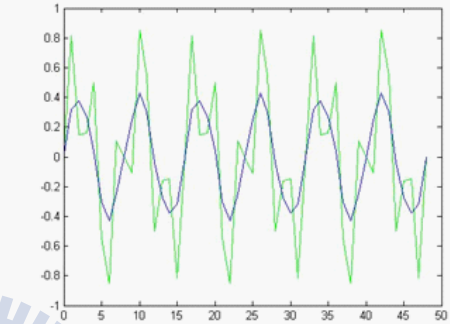


Figure 2.6.1(b): $||e|| = 0.4300$

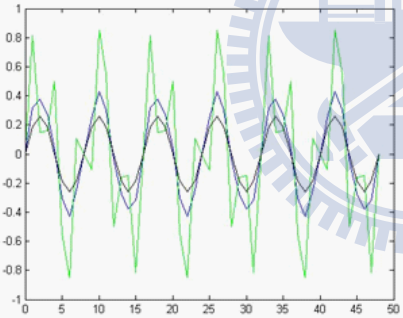


Figure 2.6.1(c): $||e|| = 0.2607$

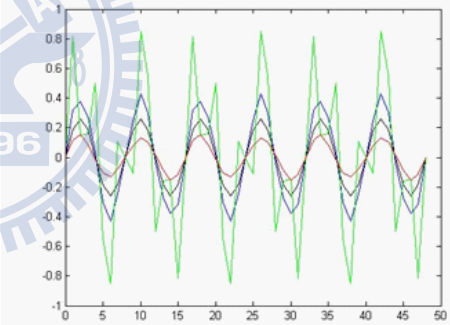


Figure 2.6.1(d): $||e|| = 0.1564$

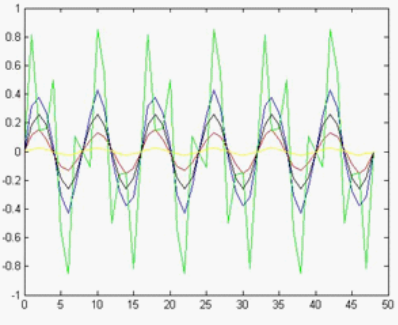


Figure 2.6.1(e): $||e|| = 0.0243$

Experiments 2

Consider the following two-point boundary value problem. It is given by the second-order ordinary differential equation

$$\begin{cases} (\alpha(2+\cos(\beta x))u'(x))' = f(x), \quad 0 < x < \pi\ , \ \ \alpha \text{ and } \beta \text{ are constant.} \\ u(0) = 0, \ \ u(\pi) = \pi^2 \end{cases} \tag{2.83}$$

we use finite difference method to discrete the domain.The domain of the problem $\{x : 0 \le x \le \pi\}$ is partitioned into N subintervals by introducing the grid points $x_j = jh$,where $h = \pi/N$ is the constant width of the subintervals. At each of the $N-1$ interior grid points, the differential equation is replaced by a second-order finite difference approximation. We also introduce v_j as an approximation to the exact solution $u(x_j)$.

$$\begin{cases} \dfrac{[2\alpha+\alpha\cos(\beta x_{j-\frac{1}{2}})]v_{j-1} - [4\alpha+\alpha\cos(\beta x_{j+\frac{1}{2}})+\alpha\cos(\beta x_{j-\frac{1}{2}})]v_j}{h^2} + \\ \dfrac{[2\alpha+\alpha\cos(\beta x_{j+\frac{1}{2}})]v_{j+1}}{h^2} = f(x_j), \quad 1 \le j \le N-1. \\ v_0 = 0, \ v_N = \pi^2 \end{cases} \tag{2.84}$$

Let a represent the entry $4\alpha+\alpha\cos(\beta x_{j+\frac{1}{2}})+\alpha\cos(\beta x_{j-\frac{1}{2}})$, b represent the entry $2\alpha+\alpha\cos(\beta x_{j+\frac{1}{2}})$, c represent the entry $2\alpha+\alpha\cos(\beta x_{j-\frac{1}{2}})$,the system can be represent in matrix form as $Av = f$.

$$\frac{1}{h^2}\begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ \vdots \\ f_{N-1} \end{bmatrix}$$

The matrix A is tridagonal,symmetric positive definite and has dimension $(N-1)\times(N-1)$.

We use multigrid method to solve the linear system.There are several comments in order.First, we choose α and β the constant one.Second, we use three level $v-cycle$ and

the number of relaxation times ν_1 and ν_2 is the constant three.Third, we use $v = \pi x$ as an initial guess.

The solution of this boundary value problem is $u(x) = x^2$ .The error e^h is defined by $e^h = ||u^h - v^h||_{L^2}$,where u^h is the exact solution and v^h is the approximation on Ω^h.The rate of convergence σ is computed by

$$\sigma = \frac{e^{2h}}{e^h} \tag{2.85}$$

Table2.6.1		
N	e	σ
128	2.7315e-004	
256	6.8301e-005	3.9992
512	1.7076e-005	3.9998

The results are tabulated in above .The theoretical convergence rate is 4 for σ.

Experiments 3

In experiment two , we consider the boundary value problem in two-dimension given by

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega : [0,1] \times [0,1] \\ u &= 0 \quad \text{on boundary} \end{aligned} \tag{2.86}$$

and finite element method is used to discretize the domain.To solve the linear systems $Ax = b$,we applied the Jacobi Method,Gauss-Seidel Method,Successive Overrelaxation Method, Conjugate Gradient Method and Multigrid Method. Compare the iteration numbers and the relative residual error between the multigrid method and other methods ,we can find the advantages of using multigrid method as the linear systems solver.

The exact solution is

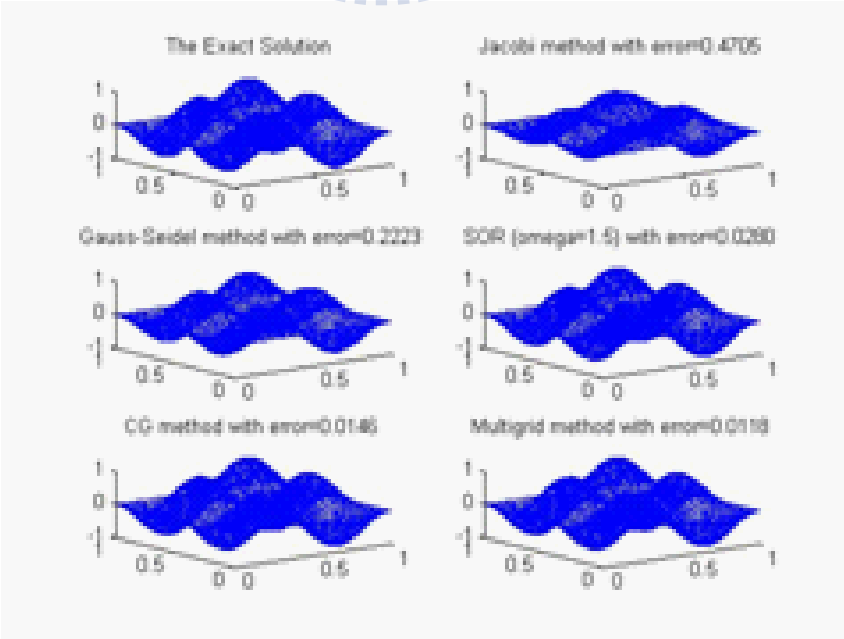
$$u(x, y) = \sin(2\pi x)\sin(4\pi y) \tag{2.87}$$

The tolerance is 10e-16.The error is defined by

$$error = ||u_{ex} - \widehat{u}||_{\infty} \tag{2.88}$$

and the relative residue is defined by

$$Relative\ residue = \frac{||b - Ax||}{||b||} \tag{2.89}$$



Figrue2.6.2: Approximate solutions

Figure3.6.1 show the exact solution and five kinds of approximation.

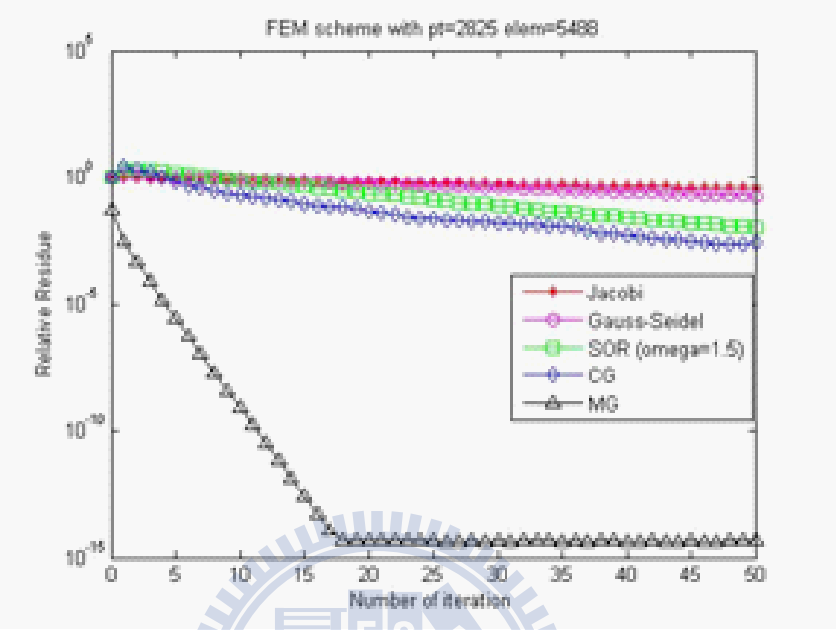


Figure2.6.3: Iteration numbers of five methods

Figure3.6.2 show us the relation between the number of iteration and the log relative residue.With five kinds of iteration method, Multigrid Method gets the best performance.

Chapter 3
Research Method

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with re-entrant angle.Consider the Poisson equation with homogeneous boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.90}$$

where $f \in L^2(\Omega)$.

When (3.90) is solved by the P_1 finite element method on a quasi-uniform grid ,the convergence rate in the energy norm is therefore of order $O(h^{(\pi/\omega)-\epsilon})$, where h is the mesh size of the triangulation and ω is the re-entrant angle.

In 1996, S.C.Brenner improves the convergence rate developed in [3].It is based on the full multigrid iteration technique and the following singular function representation of u [4, 5, 6, 7]

$$u = \kappa s + w \tag{3.91}$$

where $w \in H^2(\Omega)$ and the s are the singular function associated with the re-entrant angle. Note that the coefficient κ is known as stress intensity factors in elasticity problems.

The multigrid method in [3] computes a solution of (3.91) in the form of

$$u_h = \kappa_h s + w_h$$

where w_h is a piecewise linear function.It is shown in [3] that

$$|u - u_h|_{H^1(\Omega)} \leq Ch||f||_{L^2(\Omega)}, \tag{3.92}$$

$$|\kappa - \kappa_h| \leq Ch^{1+\pi/\omega-\epsilon}||f||_{L^2(\Omega)}. \tag{3.93}$$

In 1997 ,S.C.Brenner and L.-Y.Sung [9] extend the results in [3] and [10] to the case of a polygonal domain with crack in $\mathbb{R}^2$(i.e. the re-entrant angle $\omega = 2\pi$.)

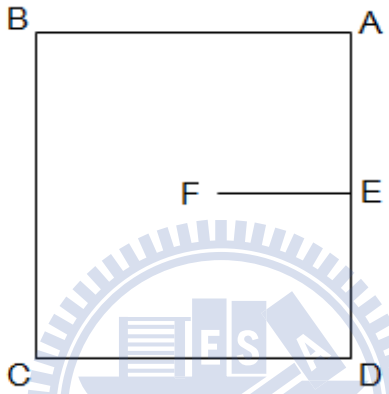


Figure3.1: A polygonal domain with cracks.

Let p be the vertice of Ω such that the angle ω associated with p satisfies $\omega > \pi$ (i.e. the vertice F in the figure1.1).Let polar coordinates (r, θ) be chosen at the vertex p so that the angle ω is spanned by the two rays $\theta = 0$ and $\theta = \omega$.

The singular function s is defined by

$$s(r, \theta) = \phi(r)r^{\pi/\omega} \sin(\frac{\pi}{\omega}\theta) \tag{3.94}$$

where $\phi(r)$ is smooth cut-off function which equal 1 identically in a neighborhood of 0,and the supports of the ϕ is small enough so that the singular function s vanish identically on $\partial\Omega$.

Then the solution u has the representation

$$u = \kappa s + w \tag{3.95}$$

The stress intensity fractors κ can be expressed in terms of u by the following extraction formula

$$\kappa = \frac{1}{\pi}\left\{\int_\Omega f s_{-1} dx + \int_\Omega u \triangle s_{-1} dx\right\}, \tag{3.96}$$

where the dual singular function s_{-1} is defined in the polar coordinate system (r, θ) as

$$s_{-1}(r,\theta) = \phi(r) r^{-\pi/\omega} \sin(\frac{\pi}{\omega}\theta)$$

([5, 7, 11, 12, 13, 14, 15, 16]).

There is a idea that we will take advantage of the singular function representation. We substitute the representation (3.95) into (3.90) to obtain the following boundary-value problem for w :

$$\begin{cases} -\triangle w = f + \kappa \triangle s & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.97}$$

If the κ were known,we could solve (3.97) using piecewise linear finite element method.Unfortunately the κ is unknown,so we apply the finite element method on the kth level to the following varational problem:

Find $w_k \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla \widehat{w}_k \nabla v dx = \int_\Omega (f + \kappa_k \triangle s)\, v dx \;\; \forall v \in H_0^1(\Omega) \tag{3.98}$$

where the approximate stress intensity factors κ_k is computed by the extraction formula (3.96) using the approximate solution u_{k-1} obtained in the $(k-1)st$ level, i.e.,

$$\kappa_k = \frac{1}{\pi}\left\{\int_\Omega f s_{-1} dx + \int_\Omega u_{k-1} \triangle s_{-1} dx\right\}. \tag{3.99}$$

We obtain, on the kth level, a piecewise linear approximate solution w_k to $\widehat{w}_k$ by applying the kth level iteration n times using w_{k-1} as the initial guess.The approximate solution u_k

to u is defined to be

$$u_k = \kappa_k s + w_k. \tag{3.100}$$

In other words we are really computing the regular part w of the solution. The improvement in the convergence rate is possible because w has better regularity than u.

The algorithm of S.C.Brenner's method is given by the following

for l = 1:level
  for k = 1:m
    if l = 1 , k= 1
      $\kappa_{l,k} = 0$
    else
      $\kappa_{l,k} = \frac{1}{\pi}\left\{\int_\Omega f s_{-1} dx + \int_\Omega u_{l,k-1} \triangle s_{-1} dx\right\}$
    end
    $\int_\Omega \nabla \widehat{w}_{l,k} \nabla v dx = \int_\Omega (f + \kappa_{l,k} \triangle s)\, v dx$
    direct solve $\widehat{w}_{l,k}$
    $u_{l,k} = \kappa s + \widehat{w}_{l,k}$
  end
end

Although we known that the singular function can replace the solution at the sigular point well, but there still have some problem we can observe.After leaving the singular point, the singular function is now not a correct solution ,so the error occured on this cut-off region.To improve this problem, we introduce two strategys,adaptive mesh-refinement techniques and adaptive cut-off function.

For the first strategy,adaptive mesh-refinement techniques, there is a relationship between the converge rate and the error estimator,we should find suitable order of h for the error estimator.The converge rate in the ragular domain is

$$\begin{aligned} ||u - u_h||_{L^2(\Omega)} &\leq Ch^2 |u|_{H^2(\Omega)} \\ ||u - u_h||_{H^1(\Omega)} &\leq Ch |u|_{H^2(\Omega)}, \end{aligned} \tag{3.101}$$

the order of error estimator is choose by

$$E(K) = \alpha||h(f - au)||_K + \beta(\frac{1}{2}\sum_{\tau\in\partial K} h_\tau[n_\tau \cdot (c\nabla u_h)]^2)^{1/2} \tag{3.102}$$

Now,in the sigular domain ,we also want to find the same relationship between the converge rate and the error estimator. Since the converge rate in singular domain is

$$\begin{aligned} ||u - u_h||_{L^2(\Omega)} &\leq Ch^{(3/2)-\epsilon}|u|_{H^2(\Omega)} \\ ||u - u_h||_{H^1(\Omega)} &\leq Ch^{(1/2)-\epsilon}|u|_{H^2(\Omega)}, \end{aligned} \tag{3.103}$$

the order of error estmator is choose by

$$E(K) = \alpha||h^{(3/4)}(f - au)||_K + \beta(\frac{1}{2}\sum_{\tau\in\partial K} h_\tau^{(1/4)}[n_\tau \cdot (c\nabla u_h)]^2)^{1/2} \tag{3.104}$$

we use this order as the rules of adaptive mesh-refinement techniques.

For the second strategy,adaptive cut-off function .Since we realize that the singular function replaced the solution successfully only near the singular point,so the range of cut-off function may affect the error.If we choose the range of cut-off function is too wide, most solution replace by the singular function , that is not a correct .Otherwise, if we choose the range of cut-off functions is too small, the error still big near the singular point.So, this adaptive cut-off function strategy is find the suitable cut range and move forward slowly .On the one hand, the error will not so big . On the other hand, the solution replace by the singular function more precision with respect to the mesh-refinement steps.

The adaptive cut-off funciton is defined by

$$\phi(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 \tag{3.105}$$

find the coefficient $a_0, a_1, a_2, a_3, a_4, a_5$ which is satisfy

$$\begin{cases} \phi(\frac{1}{3+i}) = 1 \ , \ \phi(\frac{2}{3+i}) = 0 \\ \phi'(\frac{1}{3+i}) = 0 \ , \ \phi'(\frac{2}{3+i}) = 0 \\ \phi''(\frac{1}{3+i}) = 0 \ , \ \phi''(\frac{2}{3+i}) = 0 \end{cases} \tag{3.106}$$

where $\frac{1}{3+i} \leq r \leq \frac{2}{3+i}, i = 0, 1, 2$ is the mesh-refinement steps.

The purpose of this range we choose is that we want the interval of cut range is reduce and close to singular point,also the cut range is move forward slowly.

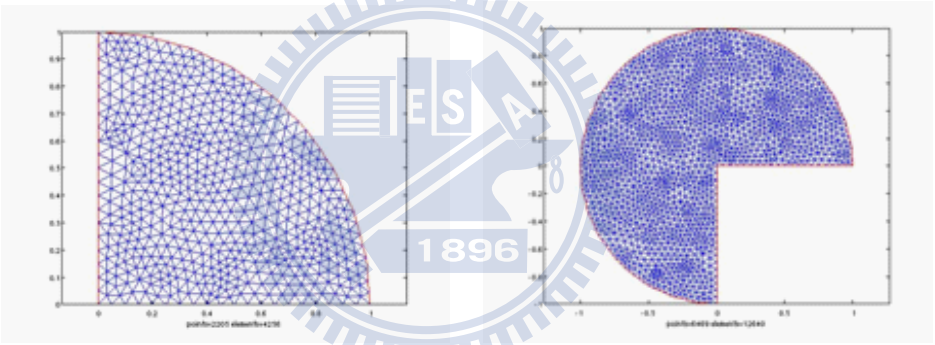
From the two strategy above , we can hold the stress intensity factorys and also improve the accuracy of the global error.The results shown in the experiment 4 in chapter 4.

Chapter 4
Numerical Results

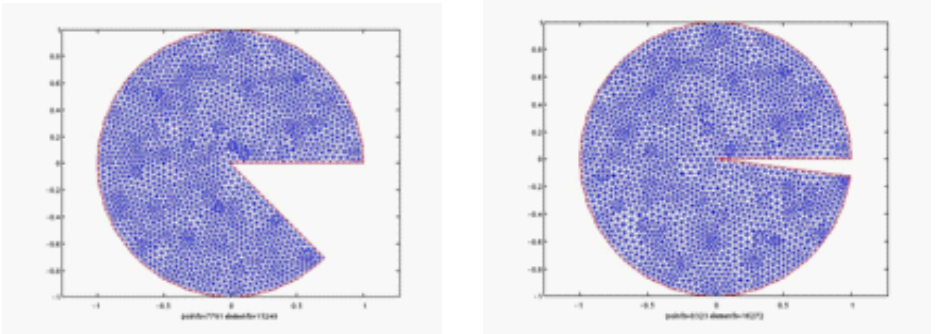
In this chapter we report the result of some numerical experiments. The first experiment we show the error arise from the singular point associated with the re-entrant angle.We consider the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.107}$$

where Ω is the circle with four angle $\omega = \pi/2, 3\pi/2, 7\pi/4$ and $\pi/0.51$, ω is the maximun re-entrant angle.



$4.1a$ $4.1b$



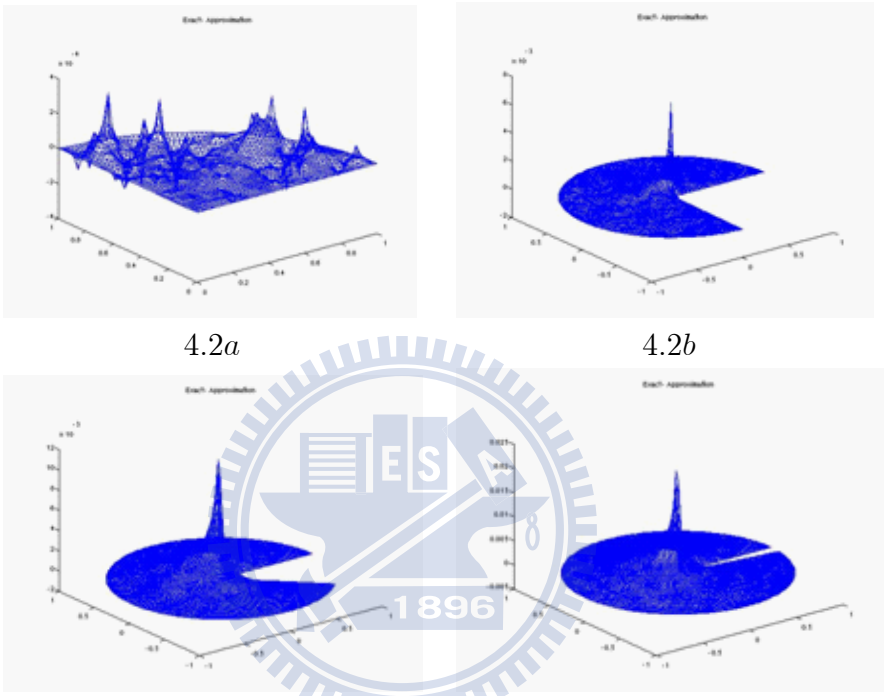
$4.1c$ $4.1d$

Figure4.1:Circle with four kinds angle.

Let the exact solution u be

$$u = (1 - r^2)r^{\beta}\sin(\beta\theta) \tag{4.108}$$

where $0 < r < 1, 0 < \theta < \omega, \beta = \frac{\pi}{\omega}, \omega$ is the maximun re-entrant angle.When is solved by piecewise linear finite element method on a quasi-uniform grid, the error is shown in the following:



$4.2a$ $4.2b$

$4.2c$ $4.2d$

Figure4.2:Error with four kinds of angle

Table 4.1 Results for four kinds of angle				
	$\omega = \frac{\pi}{2}$	$\omega = \frac{3\pi}{2}$	$\omega = \frac{7\pi}{4}$	$\omega = \frac{\pi}{0.51}$
$\|\|u - u_h\|\|_{H^1(\Omega)}$	2.6377E-02	4.1243E-02	7.0759E-02	9.6003E-02
# of points	2205	6469	7791	8323

we can see the Fig4.2a is not a re-entrant angle , so the error of Fig4.2a didn't have peak at the origin,but the others have.We also can find when the maximun re-entrant angle ω arise, the error also arise with the ω.

The second experiment we will concern about the mesh-refinement strategy.There are three different mesh-refinement strategy we used. The first strategy is uniform mesh

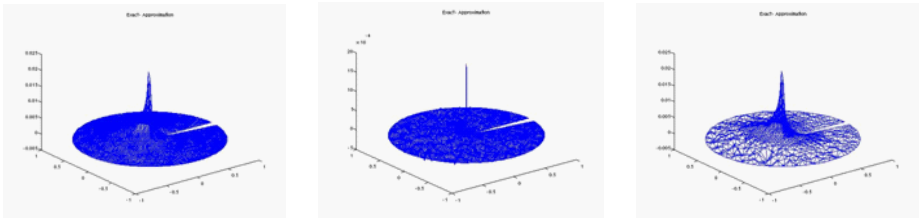
with uniformly refinement,the second is singular element mesh with uniformly refinement and the last is uniform mesh with adaptive short cut-region refinement.

The first mesh-refinement strategy is like as we refine mesh as usual.The second strategy , we changed the uniform mesh to the singular mesh. According to our mentioned in the singular element ,here the approach is like singular element,we put the exponential grid points (x, y) in the way

$$\begin{cases} (x, y) = (2^{-i}, 0) \\ (x, y) = (2^{-i}\cos\omega, 2^{-i}\sin\omega) \end{cases}, \ i = 1, ..., 10 \tag{4.109}$$

on the two rays.This may couse many grid points located near the singular point.The third strategy is using uniform mesh, but not refine the whole domain, only refine the region with radius $r = 0.5^j$ form the center points $(0, 0)$, where j represent the times of refinement.

We consider the maximun re-entrant angle $\omega = \frac{\pi}{0.51}$ case with three different mesh-refinement strategys .The times of mesh-refinement steps is 2, means that we have the original corasest domain Ω^{4h}, find domain Ω^{2h} and Ω^h.



Type A Type B Type C

Figure4.3: Error of three types of mesh-refinement

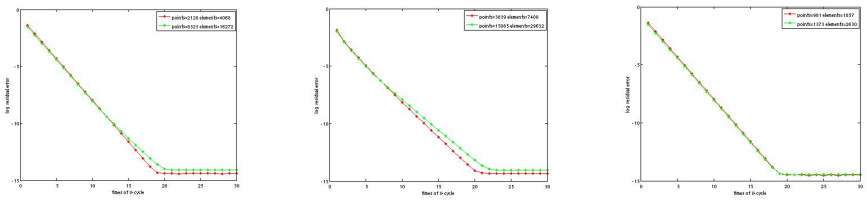
we caculate the H1-norm error and L2-norm in the following table

Table4.1			
	Type A	Type B	Type C
$\|\|u - u_h\|\|_{H^1(\Omega)}$	9.6003E-02	2.9259E-02	1.3185E-01
$\|\|u - u_h\|\|_{L^2(\Omega)}$	3.1643E-03	2.3718E-04	4.9345E-03
# of points	8323	15085	1373

From the Table4.1, we discuss some phenomena. In the second strategy, there are many points around of center.When we refine the mesh, the points around the center of all will be refined.Although we reduced the error, but we also pay a high price since too many points cause the large matrix systems.In the third strategy, we only refine the points inside the cut-region, the error outside the cut-region still not be reduced, therefore, the accuracy performance is not good. The convergence rate for type A and type B in the $H^1 - norm$ is therefore of order $O(h^{(\pi/\omega)-\epsilon})$.$(i.e.$the theoratical convergence rate is 1.42$)$

Table4.2						
Ω	Type A	ratio A	Type B	ratio B	Type C	ratio C
Ω^{4h}	2.3184E-01		1.1065E-01		2.1384E-01	
Ω^{2h}	1.4147E-01	1.5116	5.6813E-02	1.9476	1.6135E-01	1.3253
Ω^{h}	9.6003E-02	1.4737	2.9259E-02	1.9417	1.3185E-01	1.2237

Simultaneously, we apply the multigrid method for solving the linear systems and observed the benefits for three diffierent mesh-refinement. For the multigrid parameter, we apply V-cycle 2th level iteration and we choose the weight-Jacobi method to relaxation, the maximum relaxation number is 2.For the restriction operator ,we use full weighting as a restriction operator,and for the interpolation operator, we consider linear interpolation.



Type A Type B Type C

Figure4.4: Multigrid benefits for three different mesh-refinement

From the Figure4.4 shown in above,we can observe that although the mesh size is diffierent, the slope almost not changed,meens that the converge rate of multigrid method is indenpentent of mesh size.

The third experiment we use the method with S.C.Brenner and L.-Y.Sung.The singular function for Ω is

$$s(r,\theta) = \phi(r) r^{\pi/0.51} \sin(\frac{\pi}{0.51}\theta) \tag{4.110}$$

where the cut-off function $\phi(r)$ is defined to be

$$\begin{cases} 1, 0 \leq r \leq \frac{1}{3} \\ -1458r^5 + 3645r^4 - 3510r^3 + 1620r^2 - 360r + 32, \frac{1}{3} \leq r \leq \frac{2}{3} \\ 0, \frac{2}{3} \leq r \end{cases} \tag{4.111}$$

The κ_k are obtained by the extraction formula

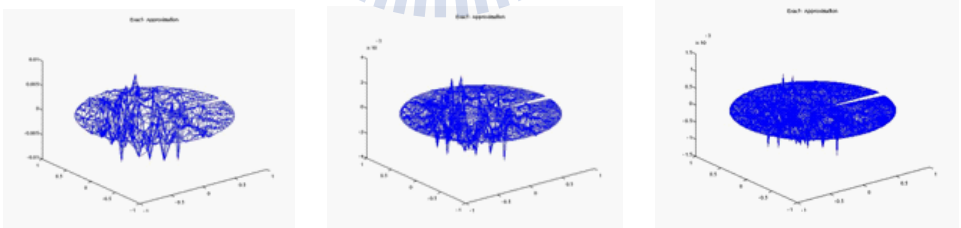
$$\kappa_k = \frac{1}{\pi}\left\{ \int_\Omega f s_{-1} dx + \int_\Omega u_{k-1} \triangle s_{-1} dx \right\}. \tag{4.112}$$

where

$$s_{-1}(r,\theta) = \phi(r) r^{-\pi/0.51} \sin(\frac{\pi}{0.51}\theta). \tag{4.113}$$

Here,for the approximate stress intensity factors κ_k,we choose 3-th level ($l = 3$) and iteration 5 times ($m = 5$)

The error is shown in the following



Ω^{4h} Ω^{2h} Ω^{h}

Figure4.5: Error of three level mesh

The theoretical number for the stress intensity factorys κ is 1.

Table4.3

Ω	κ	$\|\|u - u_h\|\|_{H^1(\Omega)}$	# of points
Ω^{4h}	9.9399E-01	2.3664E-01	556
Ω^{2h}	9.9815E-01	1.5423E-01	2128
Ω^{h}	9.9944E-01	1.0451E-01	8323

The fourth experiment we improve the method by including adaptive mesh-refinement techniques and adaptive cut-off function.The following table shown the stress intensity fac-

torys κ and the error in $H^1 - norm$.

Table4.4			
Ω	κ	$\|\|u - u_h\|\|_{H^1(\Omega)}$	# of points
1	9.9399E-01	2.3664E-01	556
2	1.0017E+01	2.0107E-01	913
3	9.9098E-01	1.7432E-01	1359
4	9.8648E-01	1.4946E-01	1842
5	9.8810E-01	1.3034E-01	2196
6	9.8722E-01	1.2721E-01	2747
7	9.8841E-01	1.1381E-01	3207
8	9.9204E-01	1.0953E-01	3533
9	9.9492E-01	1.0199E-01	4522
10	9.9341E-01	9.8286E-02	5494
11	9.9306E-01	9.7071E-02	6178
12	9.9560E-01	9.5075E-02	7217
13	9.9534E-01	9.5117E-02	7708
14	9.9596E-01	9.3322E-02	9533
15	9.9626E-01	9.3082E-02	10568

we can clearly compare the $||u - u_h||_{H^1(\Omega)}$ with these two results above.Fixed points on both sides of almost equal,we can clearly see the improvement in error in $H^1 - norm$.

Table4.5					
S.C.Brenner			Experiment		
Ω	$\|\|u - u_h\|\|_{H^1(\Omega)}$	# of points	Ω	$\|\|u - u_h\|\|_{H^1(\Omega)}$	# of points
1	2.3664E-01	556	1	2.3664E-01	556
2	1.5423E-01	2128	4	1.4946E-01	1842
3	1.0451E-01	8323	13	9.5117E-02	7708

References

[1]Kellogg RB.Singularities in interface problems .In Symposium on Numerical Solutions of Partial Differential Equations,vol.II. Acadamic Press: New York,1971;351-400

[2]Grisvard P.Singularities in Boundary Value Problems. Masson: Paris, Springer: Berlin, 1992.

[3]S.C.Brenner ,Multigrid methods for the computation of singular solutions and stress intensity factors I: Corner singularities,preprint ,1996.

[4]M.Dauge,Elliptic Boundary Value Problems on Corner Domains,Lecture Notes in Mathematics 1341,Springer-Verlag,Berlin-Heidelberg,1988.

[5]P.Grisvard,Elliptic Problrms in Non Smooth Domains,Pitman,Boston,1985.

[6]V.Kondratiev,Boundary value problems for elliptic equations in domains with conical or angular points, Tran.Moscow Math.Soc.,16(1967),pp.277-313.

[7]S.A.Nazarov and B.A.Plamenevsky,Elliptic Problems in Domains with Piecewise Smooth Boundaries,de Gruyter,Expositions in Mathematics ,13,Berlin,New York,1994.

[8]Satya N.Atluri, Computational Methods for Plane Problems of Fracture,Georgia Institue of Technology,U.S.A .

[9]S.C.Brenner and L.-Y.Sung ,Multigrid methods for the computation of singular solutions and stress intensity factors II: Crack singularities,1997.

[10]S.C.Brenner,Overcoming corner singularuties by multigrid methods ,preprint,1996.

[11]I.Babuska and A. Miller,The post-processing approach in the finite element method part 2 : The calculation of stress intensity factors,Int.J.Numer.Methods Engrg.,20(1984),pp.1111-1129.

[12]M.S.Birman and G.E.Skvorcov,On the square integrability of the highest derivatives of the solution of the Dirichlet problem on domain with piecewise smooth boundaries(in Russian),Izv.Vyssh.Uchebn.Zaved.Mat., Vol (1962),pp.136-155.

[13]H.Blum and M.Dobrowolski, On finite element methods for elliptic equations on domains with corners,Computing,28 (1982),pp.53-63.

[14]M.Dauge, M.-S. Lubuma, and S. Nicaise, Coefficient des singularities pour le probleme de Dirichlet sur un polygone,C.R.Acad. Sci.Paris Ser. I Math., 304 (1987),pp.483-486.

[15]M.Dauge, S. Nicaise, M.Bourlard, and M.-S. Lubuma ,Coefficients des singularities pour des problemes aux limites elliptiques sur un domaine a points coniques I: resultats generaux pour le probleme de Dirichlet, M^2AN,24 (1990),pp.27-52.

[16]M.Dauge, S. Nicaise, M.Bourlard, and M.-S. Lubuma ,Coefficients des singularities pour des problemes aux limites elliptiques sur un domaine a points coniques II: quelques operateurs particuliers, M^2AN,24 (1990),pp.343-367.

[17]R.Verfurth,A posteriori error estimation and adaptive mesh-refinement techniques, Journal of Computational and Applied Mathematics 50 (1994) 67-83.

[18]Claes Johnson, ”Numerical solution of partial differential equations by the finite element method”,Cambridge University Press, 1988.

[19]William L.Briggs,"A Multigrid Tutorial",Department of Mathematics University of Colorado at Denver Denver, Colorado.

[20]Richard Barrett,Michael Berry,Tony F.Chan,James Demmel ,June M. Donato,Jack Dongarra,Victor Eijkhout,Roldan Pozo,Charles Romine,and Henk Van der Vorst"Templates for the Solution of Linear Systems :Building Blocks for Iterative Methods".

[20]I.Babuska, R.B. kellogg, J,Pitkaranta, Direct and Inverse Error Estinates for Finite Elements with Mesh Refinements,1972.

[21]I.Babuska, Michael B. Rosenzweig, A finite element scheme for domains with corners,1971.