國 立 交 通 大 學

應用數學系

碩　士　論　文

邊著色的空間熵以及最小週期生成

Spatial Entropy and Minimal Cycle of Edge Coloring

研　究　生：陳晉育

指導老師：林松山　教授

中　華　民　國　九　十　九　年　六　月

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最小週期花樣邊著色的熵

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摘　　　要

這篇研究在邊著色的平面磁磚的複雜性。在平面上對邊著色，邊有 p 種顏色選擇的單位方塊並肩排著，相鄰的邊必須要有一樣的顏色，在[12]王浩猜測任意可以拼成全平面的磁磚集合就可以週期性的拼成全平面。

在兩個顏色的邊著色時，胡文貴學長和林松山老師證明王浩的猜測是成立的，任意可以拼成全平面的磁磚集合就可以週期性的拼成全平面。更精確的說，$\Sigma(B)\neq\phi$ 充要 B 有一個最小週期生成的子集。所有最小週期生成的集合 C(2) 包含 38 個元素。

本篇論文討論給定一個王浩磁磚(tiles)集合，熵(spatial entropy)是正值或零可由集合裡的最小週期生成子集(minimal cycles)決定；當集合中最小週期生成子集的子集合個數大於四組，除了O∪I∪J∪K 以外，此集合有正的熵。

Spatial Entropy and Minimal Cycle of Edge Coloring

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ABSTRACT

his investigation studies the complexity problems of plane square tiling with colored edges . In the edge coloring of a plane, unit squares with colored edges of p colors are arranged side by side such that adjacent tiles have the same colors.In [12],Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically.

W.G. Hu and S.S.Lin proved that Wang's conjecture holds when p=2 that any set of tiles that can tile a plane can tile the plane periodically. More precisely, $\Sigma(B) \neq \phi$ if and only if B has a subset of minimal cycle generator. The set of all minimal cycle generators $C(2)$ contains 38 elements.

In this paper we consider the given a basic set of Wang tiles, the spatial entropy is positive or zero can be determined by studying a subset of minimal cycles. When the number of minimal cycles in the basic set more than four except O∪I∪J∪K, the basic set have positive entropy .

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謝謝林松山老師，這兩年來給我的指導與教誨。

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謝謝我的朋友們，有你們無私的幫助及陪伴，

讓我的生活可以得到慰藉。

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有你們真好

陳晉育

謹誌于交通大學

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目錄

1 Introduction

The coloring of unit squares on $\mathbb{Z}^2$ has been studied for many years [6]. In 1961, in studying proving theorem by pattern recognition, Wang [10] started to study the square tiling of a plane. The unit squares with colored edges are arranged side by side so that the adjacent tiles have the same color; the tiles cannot be rotated or reflected. Today, such tiles are called Wang tiles or Wang dominos [4] [6].

The 2×2 unit squares is denoted by $\mathbb{Z}_{2\times 2}$. Let $\mathcal{S}_p$ be a set of p (≥ 1) colors. The total set of all Wang tiles is denoted by $\Sigma_{2\times 2}(p) \equiv \mathcal{S}_p^{\mathbb{Z}_{2\times 2}}$. A set $\mathcal{B}$ of Wang tiles, such that $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, is called a basic set (of Wang tiles). Let $\Sigma(\mathcal{B})$ be the set of all global patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$. Clearly, $\mathcal{P}(\mathcal{B}) \subseteq \Sigma(\mathcal{B})$. The nonemptiness problem is to determine whether or not $\Sigma(\mathcal{B}) \neq \emptyset$. In [10], Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically, i.e.,

$$if \quad \Sigma(\mathcal{B}) \neq \emptyset \quad then \quad \mathcal{P}(\mathcal{B}) \neq \emptyset. \tag{1.1}$$

However, W.G. Hu and S.S. Lin proved that Wang's conjecture holds provide $p = 2$: any set of Wang tiles with two colors that can tile a plane can tile the plane periodically.

First,the minimal cycle generator is introduced. $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ whenever $\mathcal{B}' \subset \mathcal{B}$. Denote the set of all minimal cycle generators by $C(p)$. Indeed,In [11] $C(2)$ has 38 members.Furthermore, under the symmetry group D_4 of $\mathbb{Z}_{2\times 2}$ and the permutation group S_p of colors of horizontal and vertical edges separately, $C(2)$ can be classified into six classes.

Notably,the nonemptiness problem can easily be determined by studying $\mathcal{P}(\mathcal{B})$, as in the case $p = 2$. More precisely, $\Sigma(\mathcal{B}) \neq \emptyset$ if and only if $\mathcal{B}$ has a subset of minimal cycle generator.

This work show that the complexity of the set of global patterns . In this study, the first, we show that spatial entropy $h(\mathcal{B}) \equiv \lim_{m\to\infty, n\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{m\times n} > 0$ then $\mathcal{B}$ contain more than two minimal cycles, where $\Gamma_{m\times n}(\mathcal{B})$ be the cardinal number of $\Sigma_{m\times n}(\mathcal{B})$. The second, we study all case of choose arbitrarily two, three, four and five minimal cycles from $C(2)$. It show that for any $\mathcal{B}$, if $\mathcal{B}$ contain more than four minimal cycles except $\mathcal{B} = \{O, I, J, K\}$ than $h(\mathcal{B}) > 0$, i.e. if $\mathcal{B} = \cup_i C_i$ $i \geq 4$ and $\mathcal{B} = \cup_i C_i \neq \{O, I, J, K\}$ then $h(\mathcal{B}) > 0$ where $\{O, I, J, K\}$: the first classe of minimal cycle generators in $C(2)$.

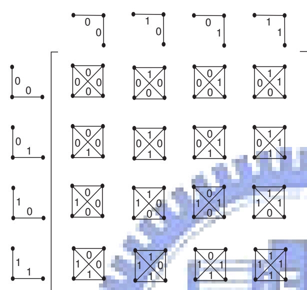
The third, show for any $\mathcal{B} = \cup_i C_i \cup \mathcal{N}$ then $h(\mathcal{B}) = 0$ where $2 \leq i \leq 4$, if $h(\cup_i C_i) = 0$.

If $\mathcal{B} = \cup_i \mathcal{C}_i \cup \mathcal{N}$. where $\mathcal{C}_i \in \mathcal{C}(2)$ and $\mathcal{N}$: $\mathcal{B}$ add tiles but can't produce new minimal cycle. then the complexity be determined by $\mathcal{B} = \cup_i \mathcal{C}_i$.

2 Tiles and Minimal cycles

This section discusses edge coloring (Wang tiles). In this section, the unit square is still denoted by $\mathbb{Z}_{2\times2}$. The left, right, bottom and top edges of the unit square $\mathbb{Z}_{2\times2}$ are given by $h_1(\mathbb{Z}_{2\times2})$, $h_2(\mathbb{Z}_{2\times2})$, $v_1(\mathbb{Z}_{2\times2})$ and $v_2(\mathbb{Z}_{2\times2})$, respectively. Denote the set of all local patterns with colored edges on $\mathbb{Z}_{2\times2}$ (Wang tiles) over $\mathcal{S}_2$ by $\Sigma_{2\times2}(2)$.

$\Sigma_{2\times2}(2)$ is given as follows:

$\mathbf{x}_{w;2\times2} =$  $= \begin{bmatrix} O & E_2 & E_4 & R \\ E_3 & J & B & \overline{E}_1 \\ E_1 & T & I & \overline{E}_3 \\ L & \overline{E}_4 & \overline{E}_2 & E \end{bmatrix},$

Given $\mathcal{B} \subset \Sigma_{2\times2}(2)$, let $\Sigma_{m\times n}(\mathcal{B})$ be the set of all local patterns on $\mathbb{Z}_{m\times n}$ generated by $\mathcal{B}$; $\Sigma(\mathcal{B})$ be the set of all global patterns generated by $\mathcal{B}$, and $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns generated by $\mathcal{B}$.

Now, in [11] the symmetry of the unit square $\mathbb{Z}_{2\times2}$ is introduced. The symmetry group of the rectangle $\mathbb{Z}_{2\times2}$ is D_4, the dihedral group of order eight. The group D_4 is generated by the rotation ρ, through $\frac{\pi}{2}$, and the reflection m about the y-axis. Denote by $D_4 = \{I, \rho, \rho^2, \rho^3, m, m\rho, m\rho^2, m\rho^3\}$.

Since, in edge coloring, the permutations of colors in the horizontal and vertical directions are mutually independent, denote the permutations of colors in the horizontal and vertical edges by $\eta_h \in S_p$ and $\eta_v \in S_p$, respectively. Then, for any $\mathcal{B} \subset \Sigma_{2\times2}(p)$, define the equivalent class $[\mathcal{B}]$ of $\mathcal{B}$ by

$$[\mathcal{B}] = \left\{\mathcal{B}' \subset \Sigma_{2\times2}(p) : \mathcal{B}' = \left(((\mathcal{B})_\tau)_{\eta_h}\right)_{\eta_v}, \tau \in D_4 \text{ and } \eta_h, \eta_v \in S_p\right\}.$$

More definitions are required.

Definition 2.1. [11] For any $\mathcal{B} \subset \Sigma_{2\times2}(p)$,

(i) $\mathcal{B}$ is called a cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$.

(ii) $\mathcal{B}$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ for all $\mathcal{B}' \subsetneqq \mathcal{B}$.

(iii) $C(p)$ is the set of all minimal cycle generators that are subsets of $\Sigma_{2\times 2}(p)$.

From now on, only the case $p = 2$ is considered. The ordering matrix $\mathbf{Y}_{2;i} = [y_{2;i;j,k}]$ of all local patterns in $\Sigma_{2\times 2}(2)$ is denoted by

$$\mathbf{Y}_{2;1} = \begin{bmatrix} y_{2;1;1,1} & y_{2;1;1,2} \\ y_{2;1;2,1} & y_{2;1;2,2} \end{bmatrix} = \begin{bmatrix} O & E_2 \\ E_3 & J \end{bmatrix}.$$

$$\mathbf{Y}_{2;2} = \begin{bmatrix} y_{2;2;1,1} & y_{2;2;1,2} \\ y_{2;2;2,1} & y_{2;2;2,2} \end{bmatrix} = \begin{bmatrix} E_4 & R \\ B & \overline{E}_1 \end{bmatrix}.$$

$$\mathbf{Y}_{2;3} = \begin{bmatrix} y_{2;3;1,1} & y_{2;3;1,2} \\ y_{2;3;2,1} & y_{2;3;2,2} \end{bmatrix} = \begin{bmatrix} E_1 & T \\ L & \overline{E}_4 \end{bmatrix}.$$

$$\mathbf{Y}_{2;4} = \begin{bmatrix} y_{2;4;1,1} & y_{2;4;1,2} \\ y_{2;4;2,1} & y_{2;4;2,2} \end{bmatrix} = \begin{bmatrix} I & \overline{E}_3 \\ \overline{E}_2 & E \end{bmatrix}.$$

Table A.7. present the tile's details.

$$\mathbf{Y}_2 = \textstyle\sum_{i=1}^{4} \mathbf{Y}_{2;i}.$$

Now, the following result gives the six classes of 38 minimal cycle generators in $C(2)$. Tables A.1. present the details of six equivalent classes of $C(2)$.

The six classes of minimal cycle generators in $C(2)$ are given as follows.

(1) $[\{O\}]$,

(2) $[\{E_1, E_4\}]$,

(3) $[\{E_1, \overline{E}_1\}]$,

(4) $[\{B, T\}]$,

(5) $[\{E_1, B, R\}]$,

(6) $[\{E_1, E_2, B\}]$.

3 Recursive Formula and Transition Matrix

For $m \geq 2$

The recursive formula are given as follows:

$$\mathbf{Y}_{m+1;1} = \begin{bmatrix} y_{2;1;1,1}\mathbf{Y}_{m;1} + y_{2;2;1,1}\mathbf{Y}_{m;3} & y_{2;1;1,2}\mathbf{Y}_{m;1} + y_{2;2;1,2}\mathbf{Y}_{m;3} \\ y_{2;1;2,1}\mathbf{Y}_{m;1} + y_{2;2;2,1}\mathbf{Y}_{m;3} & y_{2;1;2,2}\mathbf{Y}_{m;1} + y_{2;2;2,2}\mathbf{Y}_{m;3} \end{bmatrix}$$

$$\mathbf{Y}_{m+1;2} = \begin{bmatrix} y_{2;1;1,1}\mathbf{Y}_{m;2} + y_{2;2;1,1}\mathbf{Y}_{m;4} & y_{2;1;1,2}\mathbf{Y}_{m;2} + y_{2;2;1,2}\mathbf{Y}_{m;4} \\ y_{2;1;2,1}\mathbf{Y}_{m;2} + y_{2;2;2,1}\mathbf{Y}_{m;4} & y_{2;1;2,2}\mathbf{Y}_{m;2} + y_{2;2;2,2}\mathbf{Y}_{m;4} \end{bmatrix}$$

$$\mathbf{Y}_{m+1;3} = \begin{bmatrix} y_{2;3;1,1}\mathbf{Y}_{m;1} + y_{2;4;1,1}\mathbf{Y}_{m;3} & y_{2;3;1,2}\mathbf{Y}_{m;1} + y_{2;4;1,2}\mathbf{Y}_{m;3} \\ y_{2;3;2,1}\mathbf{Y}_{m;1} + y_{2;4;2,1}\mathbf{Y}_{m;3} & y_{2;3;2,2}\mathbf{Y}_{m;1} + y_{2;4;2,2}\mathbf{Y}_{m;3} \end{bmatrix}$$

$$\mathbf{Y}_{m+1;4} = \begin{bmatrix} y_{2;3;1,1}\mathbf{Y}_{m;2} + y_{2;4;1,1}\mathbf{Y}_{m;4} & y_{2;3;1,2}\mathbf{Y}_{m;2} + y_{2;4;1,2}\mathbf{Y}_{m;4} \\ y_{2;3;2,1}\mathbf{Y}_{m;2} + y_{2;4;2,1}\mathbf{Y}_{m;4} & y_{2;3;2,2}\mathbf{Y}_{m;2} + y_{2;4;2,2}\mathbf{Y}_{m;4} \end{bmatrix}$$

$$\mathbf{Y}_{m+1} = \sum_{i=1}^{4} \mathbf{Y}_{m+1;i}$$

Given $\mathcal{B} \subset \Sigma_{2\times2}(2)$, the associated transition matrix $\mathbf{V}_m(\mathcal{B})$ is obtained from $\mathbf{Y}_m$. Indeed,for $i = 1 \sim 4$, $\mathbf{V}_{2;i}(\mathcal{B}) = [v_{2;i;j,k}]$ where $v_{2;i;j,k} = 1$ if and only if $y_{2;i;j,k} \in \mathcal{B}$. As in edge coloring, the recursive formula of $\mathbf{Y}_{m+1}$ can also be applied to $\mathbf{V}_{m+1}(\mathcal{B})$ as follows.

$$\mathbf{V}_{m+1;1}(\mathcal{B}) = \begin{bmatrix} v_{2;1;1,1}\mathbf{V}_{m;1} + v_{2;2;1,1}\mathbf{V}_{m;3} & v_{2;1;1,2}\mathbf{V}_{m;1} + v_{2;2;1,2}\mathbf{V}_{m;3} \\ v_{2;1;2,1}\mathbf{V}_{m;1} + v_{2;2;2,1}\mathbf{V}_{m;3} & v_{2;1;2,2}\mathbf{V}_{m;1} + v_{2;2;2,2}\mathbf{V}_{m;3} \end{bmatrix}.$$

$$\mathbf{V}_{m+1;2}(\mathcal{B}) = \begin{bmatrix} v_{2;1;1,1}\mathbf{V}_{m;2} + v_{2;2;1,1}\mathbf{V}_{m;4} & v_{2;1;1,2}\mathbf{V}_{m;2} + v_{2;2;1,2}\mathbf{V}_{m;4} \\ v_{2;1;2,1}\mathbf{V}_{m;2} + v_{2;2;2,1}\mathbf{V}_{m;4} & v_{2;1;2,2}\mathbf{V}_{m;2} + v_{2;2;2,2}\mathbf{V}_{m;4} \end{bmatrix}.$$

$$\mathbf{V}_{m+1;3}(\mathcal{B}) = \begin{bmatrix} v_{2;3;1,1}\mathbf{V}_{m;1} + v_{2;4;1,1}\mathbf{V}_{m;3} & v_{2;3;1,2}\mathbf{V}_{m;1} + v_{2;4;1,2}\mathbf{V}_{m;3} \\ v_{2;3;2,1}\mathbf{V}_{m;1} + v_{2;4;2,1}\mathbf{V}_{m;3} & v_{2;3;2,2}\mathbf{V}_{m;1} + v_{2;4;2,2}\mathbf{V}_{m;3} \end{bmatrix}.$$

$$\mathbf{V}_{m+1;4}(\mathcal{B}) = \begin{bmatrix} v_{2;3;1,1}\mathbf{V}_{m;2} + v_{2;4;1,1}\mathbf{V}_{m;4} & v_{2;3;1,2}\mathbf{V}_{m;2} + v_{2;4;1,2}\mathbf{V}_{m;4} \\ v_{2;3;2,1}\mathbf{V}_{m;2} + v_{2;4;2,1}\mathbf{V}_{m;4} & v_{2;3;2,2}\mathbf{V}_{m;2} + v_{2;4;2,2}\mathbf{V}_{m;4} \end{bmatrix}.$$

$$\mathbf{V}_{m+1}(\mathcal{B}) = \sum_{i=1}^{4} \mathbf{V}_{m+1;i}(\mathcal{B}), \Gamma_{(m+1)\times(n+1)}(\mathbf{B}) = |\mathbf{V}^{n}_{m+1}(\mathcal{B})|$$
$$\Gamma_{m\times n}(\mathcal{B}) bethecardinalnumberof \Sigma_{m\times n}(\mathcal{B}).$$

4 Computation of Spatial Entropy

Definition 4.1. Given $\mathcal{B} \subset \Sigma_{2\times 2}(2)$, define the spatial entropy

$$h(\mathcal{B}) \equiv \lim_{m\to\infty, n\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{m\times n}$$

where $\Gamma_{m\times n}(\mathcal{B})$ be the cardinal number of $\Sigma_{m\times n}(\mathcal{B})$.

Now, we use the Theorem 4.2. to estimate the lower bound of spatial entropy.

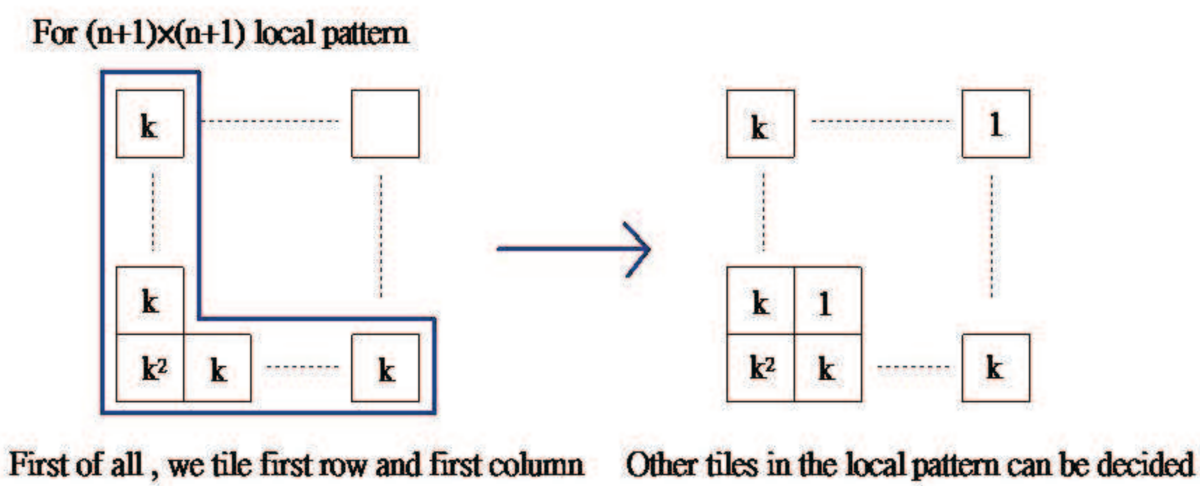
Theorem 4.2. *Given $\mathcal{B} \subset \Sigma_{2\times 2}(2)$ Let $\alpha_1, \alpha_2 ..., \alpha_k$,where $\alpha_j \in \{1, 4\}, 1 \le j \le k$ Then, for any $m \ge 2$*
$h(\mathcal{B}) \ge \frac{1}{(m-1)\times(k)} \log \rho(\mathbf{V}_{m;\alpha_1}\mathbf{V}_{m;\alpha_2}\cdots\mathbf{V}_{m;\alpha_k})$

And use the following propositions to estimate the upper bound of spatial entropy.

Proposition 4.3. *For $p = k$, $k \in N$,if the tiles of $\mathcal{B}$ satisfy*

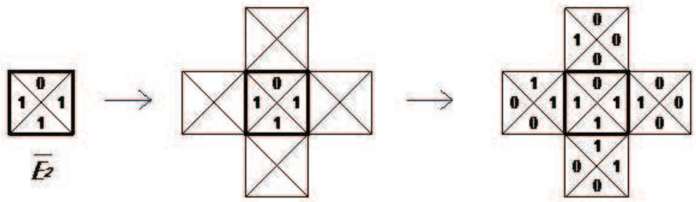
then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \le k^{(2\times n)}$, $h(\mathcal{B}) = 0$

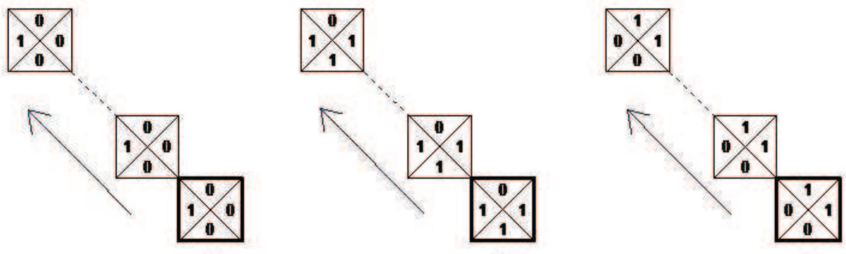
Proof:
$\Gamma_{4\times 4}(\mathcal{B}) \le k^{(2\times(4-1))}$, first of all, we tile first row and column then other tiles in 4×4 local pattern can be decided



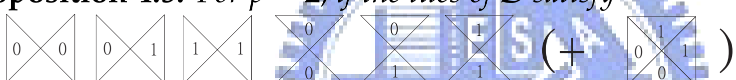
Proposition 4.4. *In the case of choose two minimal cycles, the spatial case $\mathcal{B} = \{O, E_1, \overline{E_2}, R\}$ is the only class can not use Theorem 4.1 and Proposition 4.2 to*

determine the spatial entropy.Now, show the entropy $h(\mathcal{B}) = 0$, To observe the stable tile $\overline{E_2}$,and other development is dependent on it .



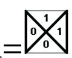


Proposition 4.5. *For $p = 2$, if the tiles of $\mathcal{B}$ satisfy*

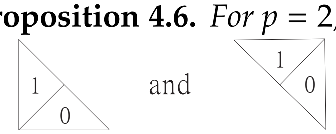


then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq (n+2)^{3n}, h(\mathcal{B}) = 0$

Proof:

For $(n+1)\times(n+1)$ local pattern, we first focus on the horizontal direction, tile  every row most $n+2$ possible. Note the spatial tile R=. Tile R is a trouble for vertical direction but every row most can contain one tile R. $\Sigma_{(n+1)\times(n+1)}(\mathcal{B})$ have most n tile R. Next, we focus on the vertical direction. We have n column and most n tile R. So it can be considered as $2\times n$ column. Then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq (n+2)^{3\times n}, h(\mathcal{B}) = 0$

Proposition 4.6. *For $p = 2$, if the tiles of $\mathcal{B}$ not contain the following form :*

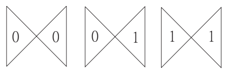


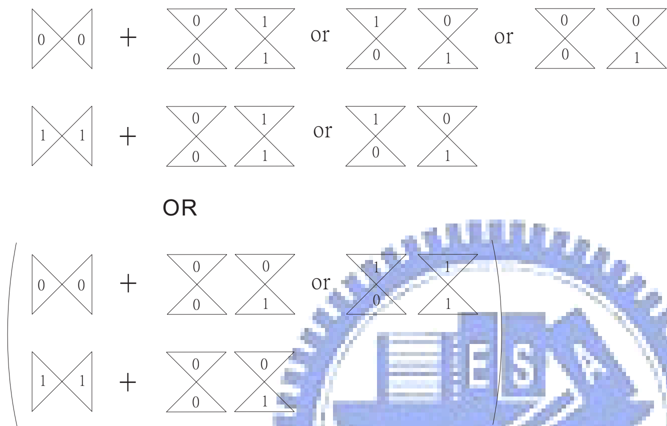
then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq (2n+2)^{2n}, h(\mathcal{B}) = 0$

Proof:

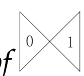
$\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq (2n+2)^{2n}, h(\mathcal{B}) = 0$

Proposition 4.7. *For $p = 2$, if the tiles of $\mathcal{B}$ satisfy*



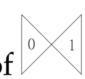


then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq c^n \times (n+2)^n \times k^n$,$h(\mathcal{B}) = 0$

where c: the tile's number of $\mathcal{B}$ and k: the tile's number of 

Proof:

For $(n+1)\times(n+1)$ local pattern , if we first focus on first row then $\Gamma_{(n+1)\times 1}(\mathcal{B}) \leq c^n$. Next, we focus on row just like Proposition 4.5. Now, when we focus on the vertical direction, it decide on front row. Every following rows a most k possible. Then $\Gamma_{(n+1)\times(n+1)}(\mathcal{B}) \leq c^n \times (n+2)^n \times k^n$,$h(\mathcal{B}) = 0$

where c :the tile's number of $\mathcal{B}$ and k:the tile's number of 

5 The Complexity of $\mathcal{B}$

In this studies, we show if entropy $h(\mathcal{B}) > 0$, then $\mathcal{B}$ at least contain two minimal cycles.

Lemma 5.1. *For any $\mathcal{B} \subseteq \Sigma_{2\times 2}$,if $h(\mathcal{B}) > 0$ then $\mathcal{B} = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}$, $n \geq 2$*
where $C_i \in \mathcal{C}(2)$ and $\mathcal{N}$: $\mathcal{B}$ add tiles can't produce new minimal cycle .

A disproof show for any the maximal basic set $\mathcal{B}$ contain only one minimal cycle that $h(\mathcal{B}) = 0$.

In the proof, choose a representative from the six classes of minimal cycle generators in C(2) representative. The other by the symmetry group D_2 of $\Sigma_{2\times2}$ and the permutation group S_p of colors of horizontal and vertical edges separately. Table A.2 present the details.

Now, we discuss the complexity of $\mathcal{B} = C_1 \cup C_2$ for any C_1, C_2 from $C(2)$. The method that we choose a representative from the six classes of minimal cycle generators in $C(2)$ representative. And make every representative to match other 37 minimal cycle. The other by the symmetry group D_2 of $\Sigma_{2\times2}$ and the permutation group S_p of colors of horizontal and vertical edges separately. We determine the entropy for every pair $C_1 \cup C_2$ of by Theorem 4.2. and Proposition 4.3. Table A.3. present the details.

Following this way, we discuss complexity of $\mathcal{B} = C_1 \cup C_2 \cup C_3$ for any C_1,C_2, C_3 from $C(2)$. The method that we choose by

(1) $C_1 \cup C_2 \cup C_3$ in the same classe from the six classes of minimal cycle

(2) $C_1 \cup C_2, C_3$ in the different two classes from the six classes of minimal cycle

(3) C_1, C_2, C_3 in the different classes from the six classes of minimal cycle

And fix C_1 to match other all possible C_2, C_3. The other by the symmetry group D_2 of $\Sigma_{2\times2}$ and the permutation group S_p of colors of horizontal and vertical edges separately. Table A.4. present the details.

And then, we discuss complexity of $\mathcal{B} = C_1 \cup C_2 \cup C_3 \cup C_4$ for any C_1, C_2, C_3, C_4 from $C(2)$. The method that we choose by

(1) $C_1 \cup C_2 \cup C_3 \cup C_4$ in the same classe from the six classes of minimal cycle

(2) $C_1 \cup C_2 \cup C_3$, C_4 in the different two classes from the six classes of minimal cycle

(3) $C_1 \cup C_2$, $C_3 \cup C_4$ in the different two classes from the six classes of minimal cycle

(4) $C_1 \cup C_2$, C_3, C_4 in the different three classes from the six classes of minimal cycle

(5) C_1, C_2, C_3, C_4 in the different classes from the six classes of minimal cycle

And fix C_1 or the representative of $C_1 \cup C_2$ to match other all possible other. The other by the symmetry group D_2 of $\Sigma_{2\times 2}$ and the permutation group S_p of colors of horizontal and vertical edges separately.

Particularly, we have that if $\mathcal{B} = \cup_{i=1}^{i=4} C_i$ then If $\mathcal{B} \neq \{O, I, J, E\}$ and the spatial case $\mathcal{B} \neq \{E_1, E_4\} \cup \{\overline{E}_1, \overline{E}_4\} \cup \{E_1, \overline{E}_1\} \cup \{E_4, \overline{E}_4\}$ then $h(\mathcal{B}) > 0$ Table A.5 present the details.

Now we choose $\mathcal{B} = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$. we just discuss complexity of $\mathcal{B} = O \cup I \cup J \cup E$,the spatial case $\mathcal{B} = \{E_1, E_4\} \cup \{\overline{E}_1, \overline{E}_4\} \cup \{E_1, \overline{E}_1\} \cup \{E_4, \overline{E}_4\}$ already be considered by $\mathcal{B} = \{E_1, E_4\} \cup \{\overline{E}_1, \overline{E}_4\}$,we have $h(\mathcal{B}) > 0$ by Theorem 4.2. . Particularly, we have that if $\mathcal{B} = \cup_{i=1}^{i=n} C_i$ then if $n = 5$ then $h(\mathcal{B}) > 0$.

Now, given a basic set $\mathcal{B} \subseteq \Sigma_{2\times 2}(2)$ that we know

(1) If $\Sigma(\mathcal{B}) \neq \emptyset$ if and only if $\mathcal{B}$ has a subset of minimal cycle generator.

(2) If entropy $h(\mathcal{B}) > 0$ then $\mathcal{B}$ at least contain two minimal cycles.

(3) we understand the complexity of $\mathcal{B} = \cup_{i=1}^{i=n} C_i$ where $C_i \in \mathcal{C}(2)$ for any $n \in N$.

Next step,we want to know how about the spatial entropy of $\mathcal{B}$ when $\mathcal{B}$ add some other new tiles but these tiles cann't add new minimal cycles. The theorem are given as follows.

Theorem 5.2. *$\mathcal{B} = \cup_{i=1}^{i=n} C_i$, $\mathcal{B}' = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}$, where $\mathcal{N}$ is the set of tiles which add to $\mathcal{B}$ but can not produce new minimal cycle.*

If $h(\mathcal{B}) = 0$ then $h(\mathcal{B}') = 0$.

Given the proof of the theorem, choose the maximal set $\mathcal{N}$ and claim $h(\mathcal{B}') = 0$ where $\mathcal{B}' = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}$. Table A.6. present the details. After all, given a basic set $\mathcal{B} = \cup_{i=1}^{i=n} C_i \cup \mathcal{N} \subseteq \Sigma_{2\times 2}(2)$, if we want to understand the complexity of $\mathcal{B}$ then we just focus on $\mathcal{B} = \cup_{i=1}^{i=n} C_i$.

Theorem 5.3. *Given any basic set $\mathcal{B} \subseteq \Sigma_{2\times 2}(2)$,$\mathcal{B} = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}$ $\mathcal{N}$: the set of tiles in $\mathcal{B}$ can't produce minimal cycle.*

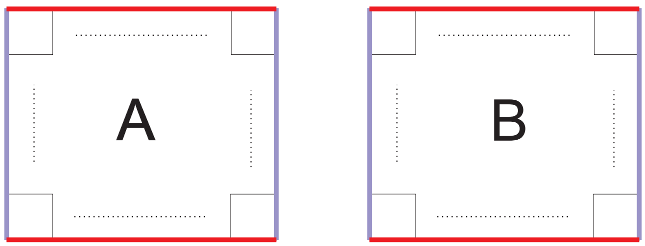
(i) if $h(\cup_{i=1}^{i=n} C_i) = 0$ if and only if $h(\mathcal{B}) = 0$.

(ii) if $\cup_{i=1}^{i=n} C_i$, $n \geq 4$, $\cup_{i=1}^{i=n} C_i \neq \{O, I, J, E\}$ then $h(\mathcal{B}) > 0$.

(iii) if $\cup_{i=1}^{i=n} C_i$ $n \geq 5$ then $h(\mathcal{B}) > 0$.

Notably, then the complexity problem can easily be determined by studying $\mathcal{B} = \cup_{i=1}^{i=n} C_i$, as in the case $p = 2$. More precisely, whether $h(\mathcal{B}) > 0$ or $h(\mathcal{B}) = 0$ decided by minimal cycles of $\mathcal{B}$.

A basic set $\mathcal{B}$ have building blocks if tiles can produce (more than) two rectangular patterns A,B such that $h_1(A)$=$h_2(A)$ = $h_1(B)$=$h_2(B)$ and $v_1(A)$=$v_2(A)$ = $v_1(B)$=$v_2(B)$ where $\max\{m,n\} \geq 2$



building blocks

Theorem 5.4. *Given a basic set $\mathcal{B}$, if $h(\mathcal{B}) > 0$ then $\mathcal{B}$ have building blocks.*

Proof:

$\forall\mathcal{B}$,$h(\mathcal{B}) > 0$ then $\exists m, k > 0$, let $\alpha_1, \alpha_2 ..., \alpha_k$,where $\alpha_j \in \{1,4\}, 1 \leq j \leq k$ such that at least a number on diagonal of $(\mathbf{V}_{m;\alpha_1}\mathbf{V}_{m;\alpha_2}\cdots\mathbf{V}_{m;\alpha_k})$ greater than one .

6 APPENDIX A

6.1

The details of six equivalent classes of $\mathcal{C}(2)$ are listed in Table A.1.

$[\{O\}]$	=	$\{\{O\}, \{I\}, \{J\}, \{E\}\}$
$[\{E_1, E_4\}]$	=	$\left\{\{E_1, E_4\}, \{E_2, E_3\}, \{\overline{E}_1, \overline{E}_4\}, \{\overline{E}_2, \overline{E}_3\}\right\}$
$\left[\{E_1, \overline{E}_1\}\right]$	=	$\left\{\{E_1, \overline{E}_1\}, \{E_2, \overline{E}_2\}, \{E_3, \overline{E}_3\}, \{E_4, \overline{E}_4\}\right\}$
$[\{B, T\}]$	=	$\{\{B, T\}, \{L, R\}\}$
$[\{E_1, B, R\}]$	=	$\left\{\begin{array}{llll} \{E_1, B, R\}, & \{E_2, B, L\}, & \{E_3, T, R\}, & \{E_4, T, L\}, \\ \{\overline{E}_1, T, L\}, & \{\overline{E}_2, T, R\}, & \{\overline{E}_3, B, L\}, & \{\overline{E}_4, B, R\} \end{array}\right\}$
$[\{E_1, E_2, B\}]$	=	$\left\{\begin{array}{llll} \{E_1, E_2, B\}, & \{E_1, E_3, R\}, & \{E_2, E_4, L\}, & \{E_3, E_4, T\}, \\ \{E_1, \overline{E}_2, R\}, & \{E_1, \overline{E}_3, B\}, & \{E_2, \overline{E}_1, L\}, & \{E_2, \overline{E}_4, B\}, \\ \{E_3, \overline{E}_1, T\}, & \{E_3, \overline{E}_4, R\}, & \{E_4, \overline{E}_2, T\}, & \{E_4, \overline{E}_3, L\}, \\ \{\overline{E}_1, \overline{E}_2, T\}, & \{\overline{E}_1, \overline{E}_3, L\}, & \{\overline{E}_2, \overline{E}_4, R\}, & \{\overline{E}_3, \overline{E}_4, B\} \end{array}\right\}$

Table A.1

6.2

The details that if entropy $h(\mathcal{B}) > 0$,then $\mathcal{B}$ at least contain two minimal cycles are listed in Table A.2.

$\mathcal{B}'$:the maximal basic set $\mathcal{B}$ contain only one minimal cycle

$\mathcal{B}$: the maximal basic set that it's tiles can be actual used to tile a plane and $\mathcal{B} \subseteq \mathcal{B}'$.

class	representative	$\mathcal{B}'$	$h(\mathcal{B}')$
$[\{O\}]$	$\{O\}$	$\{O, E_1, E_2, \overline{E_3}, \overline{E_4}, T, R\}$	$h(\mathcal{B}') = 0$; $\mathcal{B} = \{O\}$
$[\{E_1, E_4\}]$	$\{E_1, E_4\}$	$\{E_1, E_4, E_2, \overline{E_3}, T, R\}$	$h(\mathcal{B}') = 0$; $\mathcal{B} = \{E_1, E_4\}$
$[\{E_1, \overline{E}_1\}]$	$\{E_1, \overline{E}_1\}$	$\{E_1, \overline{E_1}, E_2, \overline{E_3}, T, R\}$	$h(\mathcal{B}') = 0$; $\mathcal{B} = \{E_1, \overline{E}_1\}$
$[\{B, T\}]$	$\{B, T\}$	$\{B, T, E_1, E_3, \overline{E_2}, \overline{E_4}, L\}$	$h(\mathcal{B}') = 0$; $\mathcal{B} = \{B, T\}$
$[\{E_1, B, R\}]$	$\{E_1, B, R\}$	ϕ	$h(\mathcal{B}') = 0$
$[\{E_1, E_2, B\}]$	$\{E_1, E_2, B\}$	ϕ	$h(\mathcal{B}') = 0$

Table A.2.

6.3

The details of the complexity of $\mathcal{B} = C_1 \cup C_2$ for any C_1,C_2 from $C(2)$ are listed in Table A.3.

By the **Theorem4.2**. to estimate the lower bound of spatial entropy and the **Proposition4.3**. to estimate the upper bound of spatial entropy.

(i) $\mathbf{P_i}$:$h(\mathcal{B}) > 0$ by **Theorem4.2**. .

(1) $P_1 : \rho(\mathcal{V}_{3,1}) = 1.414$

(2) $P_2 : \rho(\mathcal{V}_{3,1}) = 1.618$

(3) $P_3 : \rho(\mathcal{V}_{3,1}) = 2.618$

(4) $P_4 : \rho(\mathcal{V}_{3,1}) = 2$

(5) $P_5 : \rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,4}) = 1.618$

(6) $P_6 : \rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,4}) = 2$

(7) $P_7 : \rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,4}) = 4$

(8) $P_8 : \rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,4}) = 1.618$

(9) $P_9 : \rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,4} \times \mathcal{V}_{3,4}) = 1.618$

(10) $P_{10}: \rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,4}) = 6.854$

(11) $P_{11}: \rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,4}) = 1.480$

(12) $P_{12}: \rho(\mathcal{V}_{6,1} \times \mathcal{V}_{6,4} \times \mathcal{V}_{6,1}) = 3.732$

(13) P_0 : the $\mathcal{B} \supset \mathcal{B}'$ where $\mathcal{B}' \in P(i)$, $i = 1 \sim 11$.

(ii) **0** : $h(\mathcal{B}) = 0$ by **Proposition4**.**3**.

in the same class	representative	ρ
1	$[\{O, I\}]$	0
1	$[\{O, E\}]$	0
2	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	0
2	$[\{E_1, E_2, E_3, E_4\}]$	P_2
3	$[\{E_1, \overline{E}_1, E_2, \overline{E}_2\}]$	0
4	$[\{B, T, R, L\}]$	0
5	$[\{E_1, \overline{E}_4, B, R\}]$	0
5	$[\{E_1, \overline{E}_2, B, T, R\}]$	P_1
5	$[\{E_1, \overline{E}_1, B, T, R, L\}]$	P_3
6	$[\{E_1, E_2, \overline{E}_3, B\}]$	0
6	$[\{E_1, E_2, \overline{E}_1, \overline{E}_2, B, L\}]$	P_{10}
6	$[\{E_1, E_2, \overline{E}_3, \overline{E}_4, B\}]$	P_{11}

in different two classes	representative	ρ
1&2	$[\{O, E_1, E_4\}]$	P_0
1&2	$[\{O, \overline{E}_1, \overline{E}_4\}]$	0
1&3	$[\{O, E_1, \overline{E}_1\}]$	0
1&4	$[\{O, B, T\}]$	0
1&5	$[\{O, E_1, B, R\}]$	P_2
1&5	$[\{O, E_1, T, L\}]$	0
1&6	$[\{O, E_1, \overline{E}_2, R\}]$	P_2
1&6	$[\{O, \overline{E}_1, \overline{E}_2, T\}]$	0
2&3	$[\{E_1, E_2, E_4, \overline{E}_2\}]$	P_6
2&3	$[\{E_1, E_4, \overline{E}_1\}]$	0
2&4	$[\{E_1, E_4, B, T\}]$	P_7
2&5	$[\{E_1, E_4, B, R\}]$	P_2
2&6	$[\{E_1, E_3, E_4, \overline{E}_1, T\}]$	P_{12}
2&6	$[\{E_1, E_3, E_4, \overline{E}_1, T\}]$	P_5
2&6	$[\{E_1, E_4, \overline{E}_1, \overline{E}_2, T\}]$	P_5
3&4	$[\{E_1, \overline{E}_1, B, T\}]$	0
3&5	$[\{E_1, \overline{E}_1, B, R\}]$	P_2
3&5	$[\{E_1, E_2, \overline{E}_1, B, L\}]$	P_2
3&5	$[\{E_1, E_4, \overline{E}_1, T, L\}]$	P_2
3&6	$[\{E_1, E_2, \overline{E}_1, B\}]$	0
3&6	$[\{E_1, E_2, E_4, \overline{E}_1, L\}]$	P_5
4&5	$[\{E_1, B, T, R\}]$	0
4&6	$[\{E_1, E_3, B, T, R\}]$	P_1
4&6	$[\{E_1, E_2, B, T\}]$	0
5&6	$[\{E_1, E_2, B, R\}]$	0
5&6	$[\{E_1, E_2, \overline{E}_4, B, R\}]$	P_8
5&6	$[\{E_1, \overline{E}_3, \overline{E}_4, B, R\}]$	P_9
5&6	$[\{E_1, E_3, \overline{E}_1, B, T, R\}]$	P_4

Table A.3

6.4

The details of the complexity of $\mathcal{B} = C_1 \cup C_2 \cup C_3$ for any C_1,C_2,C_3 from $C(2)$. are listed in Table A.4.

$\rho(\mathcal{V}_{2,1} \times \mathcal{V}_{2,4}) = 4$

P_0 : the $\mathcal{B} \supset \mathcal{B}'$: $\mathcal{B}' = C_1 \bigcup C_2$ and $h(\mathcal{B}') > 0$

0 : $h(\mathcal{B}) = 0$ by **Proposition4.3**.
In order to convenience, we let

1 : $[\{O\}]$

2 : $[\{E_1, E_4\}]$

3 : $\left[\{E_1, \overline{E}_1\}\right]$

4 : $[\{B, T\}]$

5 : $[\{E_1, B, R\}]$

6 : $[\{E_1, E_2, B\}]$

in the same class	representative	ρ
1	$[\{O, I, J, E\}]$	0
2	$[\{E_1, E_2, E_3, E_4, \overline{E}_2, \overline{E}_3\}]$	P_0
3	$[\{E_1, \overline{E}_1, E_2, \overline{E}_2, E_3, \overline{E}_3\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,4}) = 2.618$
4	ϕ	
5	Table A.3.	P_0
6	$[\{E_1, E_2, \overline{E}_3, \overline{E}_4, B\}]$	P_0

Note:In different two classes:

1&2 : two minimal cycles in 1 and one minimal cycle in 2.

2&1 : two minimal cycles in 2 and one minimal cycle in 1.

in different two classes	representative	ρ
1&2	$[\{O, J, \overline{E}_2, \overline{E}_3\}]$	0
1&2	$[\{O, I, E_1, E_4\}]$	P_0
2&1	$[\{O, E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	P_0
1&3	$[\{O, E, E_1, \overline{E}_1\}]$	0
1&3	$[\{O, I, E_1, \overline{E}_1\}$	$\rho(\mathcal{V}_{4,1}) = 2$
1&3	$[\{O, J, E_1, \overline{E}_1\}]$	$\rho(\mathcal{V}_{2,1} \times \mathcal{V}_{2,2}) = 2$
3&1	$[\{O, E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	$\rho(\mathcal{V}_{3,1}) = 2$
3&1	$[\{O, E_1, E_2, \overline{E}_1, \overline{E}_2\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,2}) = 3$
1&4	$[\{O, E, B, T\}]$	0
1&4	$[\{O, I, B, T\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 2.618$
4&1	$[\{O, B, T, R, L\}]$	$\rho(\mathcal{V}_{3,1}) = 1.618$
1&5	$[\{O, I, E_1, B, R\}]$	P_0
1&5	$[\{O, J, E_1, B, R\}]$	P_0
1&5	$[\{O, E, E_1, B, R\}]$	P_0
1&5	$[\{I, E, E_1, B, R\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 2.618$
1&5	$[\{J, E, E_1, B, R\}]$	$\rho(\mathcal{V}_{4,1}) = 1.414$
5&1	$[\{O, E_1, \overline{E}_4, B, R\}]$	$\rho(\mathcal{V}_{3,1}) = 1.618$
1&6	$[\{O, J, E_1, E_2, B\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,2}) = 3$
1&6	$[\{O, E, E_1, E_2, B\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,2} \times \mathcal{V}_{5,2}) = 8$
6&1	$[\{O, E_1, E_2, \overline{E}_3, B\}]$	P_0
6&1	$\{O, E_1, \overline{E}_3, \overline{E}_4, B\}$	$\rho(\mathcal{V}_{5,2} \times \mathcal{V}_{5,1}) = 1.732$
6&1	$[\{O, E_1, E_2, \overline{E}_3, \overline{E}_4, B\}]$	P_0
2&3	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	0
2&3	$[\{E_1, E_2, E_3, \overline{E}_1, \overline{E}_2, \overline{E}_3\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,2} \times \mathcal{V}_{3,2}) = 7.873$
3&2	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	0
3&2	$[\{E_1, E_2, E_3, E_4, \overline{E}_2, \overline{E}_3\}]$	P_0
3&2	$[\{E_1, E_2, E_4, \overline{E}_1, \overline{E}_2\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,2} \times \mathcal{V}_{3,2}) = 2$
2&4	Table A.3.	P_0
4&2	Table A.3.	P_0
2&5	Table A.3.	P_0
5&2	Table A.3.	P_0
2&6	Table A.3.	P_0
6&2	Table A.3.	P_0
3&4	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4, B, T\}]$	P_0
3&4	$[\{E_1, E_2, \overline{E}_1, \overline{E}_2, B, T\}]$	P_0
4&3	$[\{E_1, \overline{E}_1, B, T, R, L\}]$	P_0
3&5	Table A.3.	P_0
5&3	Table A.3.	P_0

in different two classes	representative	ρ
3&6	$[\{E_1, E_2, \overline{E}_1, \overline{E}_2, B\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2} \times \mathcal{V}_{4,2}) = 2$
6&3	$[\{E_1, E_2, \overline{E}_1, \overline{E}_3, B\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,1} \times \mathcal{V}_{5,2} \times \mathcal{V}_{5,2}) = 3$
6&3	$[\{E_1, E_2, \overline{E}_1, \overline{E}_4, B\}]$	P_0
6&3	$[\{E_1, E_2, \overline{E}_1, \overline{E}_3, \overline{E}_4, B\}]$	P_0
6&3	$[\{E_1, \overline{E}_1, \overline{E}_3, \overline{E}_4, B\}]$	P_0
6&3	$[\{E_1, E_2, E_3, \overline{E}_1, \overline{E}_4, B\}]$	P_0
4&5	$[\{E_1, B, T, R, L\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,1} \times \mathcal{V}_{3,2} \times \mathcal{V}_{3,2} \times \mathcal{V}_{3,1}) = 10.2268$
5&4	$[\{E_1, \overline{E}_4, B, T, R\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,2}) = 2.618$
4&6	$[\{E_1, E_2, B, T, R, L\}]$	P_0
6&4	$[\{E_1, E_2, \overline{E}_3, B, T\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,2}) = 2.618$
6&4	$[\{E_1, E_2, \overline{E}_3, \overline{E}_4, B, T\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,2}) = 2.148$
5&6	$[\{E_1, E_2, \overline{E}_4, B, R\}]$	P_0
5&6	$[\{E_1, E_2, E_4, \overline{E}_1, T, L\}]$	P_0
6&5	$[\{E_1, E_2, \overline{E}_3, B, R\}]$	$\rho(\mathcal{V}_{6,2}) = 1.414$
6&5	$[\{E_1, E_2, \overline{E}_3, \overline{E}_4, B, R\}]$	P_0

in different three classes	representative	ρ
1&2&3	$[\{O, E_2, \overline{E}_2, \overline{E}_3\}]$	0
1&2&3	Table A.3.	P_0
1&2&4	Table A.3.	P_0
1&2&5	Table A.3.	P_0
1&2&6	Table A.3.	P_0
1&3&4	$[\{O, E_1, \overline{E}_1, B, T\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 2$
1&3&5	Table A.3.	P_0
1&3&6	$[\{O, E_1, \overline{E}_1, \overline{E}_2, R\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2} \times \mathcal{V}_{4,2}) = 2$
1&3&6	$[\{O, E_1, E_2, \overline{E}_2, R\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,2} \times \mathcal{V}_{5,2}) = 2.618$
1&4&5	$[\{O, \overline{E}_1, B, T, L\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 2$
1&4&6	$[\{O, E_1, E_2, B, T\}]$	$\rho(\mathcal{V}_{3,1}) = 1.618$
1&4&6	$[\{O, E_1, \overline{E}_3, B, T\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 4$
1&4&6	$[\{O, \overline{E}_3, \overline{E}_4, B, T\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 3$
1&4&6	$[\{O, E_2, \overline{E}_4, B, T\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,1} \times \mathcal{V}_{4,2}) = 8$
1&5&6	$[\{E, E_1, E_2, B, R\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,1}) = 2$
1&5&6	$[\{E, E_1, \overline{E}_3, B, R\}]$	$\rho(\mathcal{V}_{4,1} \times \mathcal{V}_{4,2} \times \mathcal{V}_{4,2} \times \mathcal{V}_{4,2} \times \mathcal{V}_{4,2}) = 2$
2&3&4	Table A.3.	P_0
2&3&5	Table A.3.	P_0
2&4&5	Table A.3.	P_0
2&4&6	Table A.3.	P_0
2&5&6	Table A.3.	P_0
3&4&5	Table A.3.	P_0
3&4&6	$[\{E_1, E_2, \overline{E}_1, B, T\}]$	$\rho(\mathcal{V}_{6,1} \times \mathcal{V}_{6,1} \times \mathcal{V}_{6,2}) = 3.7321$
3&4&6	$[\{E_1, \overline{E}_1, \overline{E}_3, B, T\}]$	$\rho(\mathcal{V}_{6,2} \times \mathcal{V}_{6,1} \times \mathcal{V}_{6,2}) = 3.7321$
3&4&6	$[\{E_1, E_2, \overline{E}_1, \overline{E}_4, B, T\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,2}) = 4.4495$
3&4&6	$[\{E_1, \overline{E}_1, \overline{E}_3, \overline{E}_4, B, T\}]$	$\rho(\mathcal{V}_{3,1} \times \mathcal{V}_{3,2}) = 4.4495$
4&5&6	$[\{E_1, E_2, B, T, R\}]$	$\rho(\mathcal{V}_{6,1} \times \mathcal{V}_{6,1} \times \mathcal{V}_{6,2}) = 3$
4&5&6	$[\{E_1, \overline{E}_3, B, T, R\}]$	$\rho(\mathcal{V}_{6,1} \times \mathcal{V}_{6,2} \times \mathcal{V}_{6,2}) = 3$

Table A.4.

6.5

The details of the complexity of $\mathcal{B} = C_1 \cup C_2 \cup C_3 \cup C_4$ for any C_1,C_2,C_3,C_4 from $C(2)$. are listed in Table A.5.

P_0 : the $\mathcal{B} \supset \mathcal{B}'$: $\mathcal{B}' = C_1 \cup C_2 \cup C_3$ and $h(\mathcal{B}') > 0$

0 : $h(\mathcal{B}) = 0$ by **Proposition4.3**.
In order to convenience,we let

$1 : [\{O\}]$

$2 : [\{E_1, E_4\}]$

$3 : \left[\{E_1, \overline{E}_1\}\right]$

$4 : [\{B, T\}]$

$5 : [\{E_1, B, R\}]$

$6 : [\{E_1, E_2, B\}]$

in the same class	representative	ρ
1	$[\{O, I, J, E\}]$	0
other	Table A.3.	P_0

Note:In different two classes:

1&2 : three minimal cycles in 1 and one minimal cycle in 2.

2&1 : three minimal cycles in 2 and one minimal cycle in 1.

in different two classes	representative	ρ
1&2	$[\{O, I, J, E_1, E_4\}]$	P_0
1&3	$[\{O, I, J, E_1, \overline{E}_1\}]$	P_0
1&4	$[\{O, I, J, B, T\}]$	P_0
other	Table A.3. and A.4.	P_0

Note:In different two classes:

1&2 : two minimal cycles in 1 and two minimal cycle in 2.

2&1 : two minimal cycles in 2 and two minimal cycle in 1.

in different two classes	representative	ρ
1&3	$[\{O, E, E_2, E_1, \overline{E}_1, \overline{E}_2\}]$	$\rho(\mathcal{V}_{5,1} \times \mathcal{V}_{5,2}) = 6$
1&4	$[\{O, I, B, T, R, L\}]$	P_0
2&3	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	0
2&3	other	P_0
other	Table A.3. and A.4.	P_0

Note:In different three classes:

1&2&3 : two minimal cycles in 1 and one minimal cycle in 2 and one minimal cycle in 3.

2&3&1 : two minimal cycles in 2 and one minimal cycle in 3 and one minimal cycle in 1.

3&1&2 :two minimal cycles in 3 and one minimal cycle in 1 and one minimal cycle in 2.

in different three classes	representative	ρ
1&2&3	$[\{O, J, E_1, \overline{E}_1, \overline{E}_2, \overline{E}_3\}]$	P_0
1&2&3	$[\{O, J, E_2, \overline{E}_2, \overline{E}_3\}]$	P_0
2&3&1	Table A.3.	P_0
2&3&1	Table A.3.	P_0
3&1&2	Table A.3.	P_0
other	Table A.3. and A.4.	P_0

in different four classes	ρ
1&2&3&4	P_0
1&2&3&5	P_0
1&2&3&6	P_0
other	P_0

Table A.5.

6.6

$h(\cup_{i=1}^{i=n} C_i) = 0$, $\mathcal{B} = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}$, where $\mathcal{N}$:$\mathcal{B}$ add tiles but can not produce new minimal cycle. Choose the maximal set $\mathcal{N}$ denote by $\mathcal{N}'$.

The details of the complexity of $\mathcal{B}' = \cup_{i=1}^{i=n} C_i \cup \mathcal{N}'$. are listed in Table A.6.

By the **Proposition4.5.**,**Proposition4.6.** and **Proposition4.7.** to estimate a upper bound of spatial entropy.

(i) (1) $\mathbf{0_1}$:$h(\mathcal{B}) = 0$ by **Proposition4.5.**.

(2) $\mathbf{0_2}$: $h(\mathcal{B}) = 0$ by **Proposition4.6.**.

(3) $\mathbf{0_3}$: $h(\mathcal{B}) = 0$ by **Proposition4.7.**.

(ii) ϕ : must add new minimal cycle when add any new tile

For n=2, $h(\cup_{i=1}^{i=n} C_i) = 0$

in the same class	representative	$\mathcal{B}'$	ρ
1	$[\{O, I\}]$	$[\{O, I, E_1, E_3, \overline{E}_2, \overline{E}_4, T, L\}]$	0_1
1	$[\{O, I\}]$	$[\{O, I, E_1, E_3, \overline{E}_2, \overline{E}_4, B, L\}]$	0_1
1	$[\{O, E\}]$	$[\{O, E, E_1, E_3, \overline{E}_2, \overline{E}_4, T, L\}]$	0_3
1	$[\{O, E\}]$	$[\{O, E, E_4, \overline{E}_1, \overline{E}_3, T, R\}]$	0_2
1	$[\{O, E\}]$	$[\{O, E, E_1, E_3, \overline{E}_2, \overline{E}_4, B, L\}]$	0_2
1	$[\{O, E\}]$	$[\{O, E, E_2, \overline{E}_1, \overline{E}_3, T, R\}]$	0_2
1	$[\{O, E\}]$	$[\{O, E, E_2, E_4, \overline{E}_1, \overline{E}_3, T\}]$	0_2
2	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$	$[\{E_1, E_3, E_4, \overline{E}_1, \overline{E}_2, \overline{E}_4, B, L\}]$	0_3
3	$[\{E_1, \overline{E}_1, E_2, \overline{E}_2\}]$		ϕ
4	$[\{B, T, R, L\}]$		ϕ
5	$[\{E_1, B, R, \overline{E}_2, T, R\}]$		ϕ
6	$[\{E_1, E_2, \overline{E}_3, B\}]$		ϕ

in different two classes	representative	$\mathcal{B}'$	ρ
1&2	$[\{O, \overline{E}_1, \overline{E}_4\}]$	$[\{O, E_3, \overline{E}_1, \overline{E}_2, \overline{E}_4, B, L\}]$	0_2
1&3	$[\{O, E_1, \overline{E}_1\}]$	$[\{O, E_1, E_3, \overline{E}_1, \overline{E}_2, B, L\}]$	0_3
1&4	$[\{O, B, T\}]$	$[\{O, E_1, E_3, \overline{E}_2, \overline{E}_4, B, T, L\}]$	0_2
1&5	$[\{O, \overline{E}_1, T, L\}]$		ϕ
1&6	$[\{O, E_1, \overline{E}_3, B\}]$		ϕ
1&6	$[\{O, E_1, \overline{E}_2, T\}]$		ϕ
2&3	$[\{E_1, E_4, \overline{E}_1\}]$	$[\{E_1, E_3, E_4, \overline{E}_1, \overline{E}_2, B, L\}]$	0_3
2&3	$[\{E_1, E_4, \overline{E}_1\}]$	$[\{E_1, E_2, E_4, \overline{E}_1, \overline{E}_3, T, R\}]$	0_3
3&4	$[\{E_1, \overline{E}_1, B, T\}]$		ϕ
4&5	$[\{E_1, B, T, R\}]$		ϕ
4&6	$[\{E_1, E_2, B, T\}]$		ϕ
5&6	$[\{E_1, E_2, B, R\}]$		ϕ

For n=3, $h(\cup_{i=1}^{i=n} C_i) = 0$

in the same class	representative	$\mathcal{B}'$	ρ
1	$[\{O, J, E\}]$	$[\{O, J, E, E_1, E_3, \overline{E}_2, \overline{E}_4, T, L\}]$	0_1
1	$[\{O, J, E\}]$	$[\{O, J, E, E_1, E_3, \overline{E}_2, \overline{E}_4, B, L\}]$	0_1
1	$[\{O, J, E\}]$	$[\{O, J, E, E_1, E_3, B, L\}]$	0_3
1	$[\{O, J, E\}]$	$[\{O, J, E, E_1, E_2, \overline{E}_3, R, T\}]$	0_3

Note:In different two classes:

1&2 : two minimal cycles in 1 and one minimal cycle in 2.

2&1 : two minimal cycles in 2 and one minimal cycle in 1.

in different two classes	representative	$\mathcal{B}'$	ρ
1&2	$[\{O, J, \overline{E}_2, \overline{E}_3\}]$	$[\{O, J, E_1, \overline{E}_2, \overline{E}_3, \overline{E}_4, T, L\}]$	0_3
1&3	$[\{O, E, E_1, \overline{E}_1\}]$	$[\{O, E, E_1, E_3, \overline{E}_1, \overline{E}_2, B, L\}]$	0_3
1&3	$[\{O, E, E_1, \overline{E}_1\}]$	$[\{O, E, E_1, E_3, \overline{E}_1, B, L\}]$	0_3
1&3	$[\{O, E, E_1, \overline{E}_1\}]$	$[\{O, E, E_1, E_2, \overline{E}_1, \overline{E}_3, T, R\}]$	0_3
1&3	$[\{O, E, E_1, \overline{E}_1\}]$	$[\{O, E, E_1, E_3, \overline{E}_1, \overline{E}_3, T, R\}]$	0_3
1&4	$[\{O, E, B, T\}]$	$[\{O, E, E_1, E_3, \overline{E}_2, \overline{E}_4, B, T, L\}]$	0_2
2&3	$[\{E_1, E_4, \overline{E}_1, \overline{E}_4\}]$		0_3
3&2	$[\{E_1, B, T, R\}]$		ϕ

in different three classes	representative	$\mathcal{B}'$	ρ
1&2&3	$[\{O, E_2, \overline{E}_2, \overline{E}_3\}]$	$[\{O, E_1, E_2, \overline{E}_2, \overline{E}_3, \overline{E}_4, T, L\}]$	0_3

For n=4, $h(\cup_{i=1}^{i=n} C_i) = 0$

in the same class	representative	$\mathcal{B}'$	ρ
1	$[\{O, I, J, E\}]$	$[\{O, I, J, E, E_1, E_3, \overline{E}_2, \overline{E}_4, T, L\}]$	0_1

Table A.6.

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