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碩 士 論 文

混沌理論之速返斥子在生態學上的應用

Snapback Repellers in Chaos Theory and their Application in Ecology

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摘　要

本論文簡短地回顧混沌理論(chaos theory)的歷史，並利用關係圖描述混沌(chaos)和黎阿普諾夫指數(Lyapunov exponent)、拓樸熵(topological entropy)、奇異吸引子(strange attractor)、速返斥子(snapback repeller)以及薩可夫斯基定理(Sarkovskii's theorem)之間的關係。這些數學和電腦輔助的理論及工具能夠藉由計算某個量或證明其存在性來決定某個系統是否會有混沌現象。在生態學上，佐竹曉子(Akiko Satake)和巖佐庸(Yoh Iwasa)修改了井鷺裕司(Yuji Isagi)的能量預算模型(resource budget model)，並建立更一般化的能量預算模型 (generalized resource budget model)，利用正的黎阿普諾夫指數證明如果消耗係數(depletion coefficient)大於一，則系統會產生混沌現象。然而，正的黎阿普諾夫指數只意謂德瓦尼(Devaney)混沌定義中的敏感性(sensitivity)而已。因此，本論文從數學和數值的角度去分析佐竹曉子和巖佐庸的模型，利用速返斥子方法去證明此模型會有德瓦尼定義的混沌現象。此外，本論文也修正了他們論文在探討消耗係數是整數時所遺漏的部份，並更進一步區分奇數與偶數的差異性。

關鍵詞：李-約克混沌、德瓦尼混沌、拓樸熵、速返斥子、薩可夫斯基定理、能量預算模型。

Snapback Repellers in Chaos Theory and their Application in Ecology

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Abstract

This work briefly reviews the history of chaos theory and elucidates the relationship among chaos, Lyapunov exponent, topological entropy, strange attractor, snapback repeller, and Sarkovskii's theorem, connecting them to each other using a relational graph. Mathematical and computer-assisted tools can be used to determine whether maps or systems are chaotic by finding a quantity or sometimes identifying the existence of a property. In ecology, Satake's generalized resource budget model that modified from Isagi's resource budget model, Satake and Iwasa proved by computing the positive Lyapunov exponent that if the depletion coefficient k is greater than one, then the system is chaotic. However, a positive Lyapunov exponent means only sensitivity in Devaney's chaos. Therefore, this work presents mathematical views and a numerical analysis on Satake's model, using the "snapback repeller method" to prove that the model is chaotic in Devaney's sense (involving transitivity, density, and sensitivity). Moreover, this work also overcomes the omission of Satake's paper (Satake & Iwasa, 2000) when the depletion coefficient k is a positive integer. Furthermore, the end of this work investigates the difference between odd depletion coefficients and even depletion coefficients.

Keywords: Li-Yorke's chaos, Davaney's chaos, topological entropy, snapback repeller, Sarkovskii's theorem, resource budget model.

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陳勛暉

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Symbols

$S^{(t)}$	:	the amount of energy reserved at the beginning of year t
P_S	:	the annual amount of photosynthate
L_T	:	energy threshold of a tree
$Y^{(t)}$	:	non-dimensionalized variable of $(S(t) + P_S - L_T)/P_S$
k	:	depletion coefficient
k_p	:	critical point with respect to Satake's generalized resource budget model under the iterative number 2^p
C_f	:	a cost of flowering
C_a	:	a cost of fruiting
R_C	:	the ratio of C_a/C_f

1 Introduction

In recent decades, chaotic theory has advanced rapidly. This work discusses definitions of chaos and their relationships. Many mathematical tools can be used to measure or describe the "chaotic" condition of systems. This work introduces the commonly utilized Lyapunov exponent and topological entropy, and most importantly connects these tools to chaos. The strange attractor is considered. Like the Lorenz attractor, although it has an attract property, it exhibits "unstable", meaning that its interior exhibits a "sensitive dependence on initial conditions", which phenomena are together identified as "strange". Finally, a computer-assisted method, involving snapback repellers, is introduced: if a system has a snapback repeller, then it is chaotic.

1.1 History of Chaos

In 1960, the meteorologist Edward Lorenz was working on the problem of weather prediction. In 1961, he discovered the "butterfly effect" while trying to forecast the weather. This phenomenon, the "butterfly effect" refers to the sensitive dependence on initial conditions in chaos theory. Lorenz called the image he obtained when he graphed the associated equations the "Lorenz attractor". Lorenz was the discoverer of chaos and its first true experimental researcher.

A decade later (1971), David Ruelle and Floris Takens elucidated a phenomenon they called a "strange attractor", and in so doing, gave birth to a whole new area of chaos theory. The association of "turbulence" with a strange attractor was revolutionary [77].

The word "chaos" was coined by James A. Yorke [55]. According to one investigation [52], many scientists believe that the "existence" and "uniqueness" of a solution to a ordinary difference equation if the system is smooth is required by the Poincarè-Bendixson theorem, even when the system has more than over two dimensions. They considered that such a solution was "regular" and that convergence to such an (almost) periodic or quasi-periodic solution is unaffected by noise. Whenever the solution was irregular, they regarded the problem as a computing error. However, in 1974, Li and Yorke solved the logistic model problems of the ecologist Robert May, and an increasing number of scientists no longer thought that these problems involved computing mistakes. In the following year, Li and Yorke revised their draft, which had been written one year earlier and published it in 1975 [55].

At that time, chaos was not a science, or even a cohesive theory. During the 1970s, several brave scientists such as Robert May, Mitchell Feigenbaum, and Benoit Mandelbrot, were so intrigued with the new concept of chaos that they began to research it.

In 1989, Devaney explicitly defined chaos [21]. Diamond [22, 23], Marotto [23, 57], Wiggins [98], Robinson [74] and Martelli [59] also provided their definitions of chaos in 1976, 1978, 1991, 1998, and 1999, respectively.

The existence of chaotic dynamics has been discussed in the mathematical literature for many decades with important contributions by Poincaré, Birkhoff, Cartwright, Littlewood, Levinson, Smale, and Kolmogorov and his students, among others. The field is now undergoing explosive growth, and has various applications across a wide range of scientific disciplines, including–ecology [20, 33, 34, 61, 80, 86, 95, 102], economics [82], physics [73], sociology [78], anthropology [70, 71], biology [18, 65], chemistry, engineering, fluid mechanics, and many others. Specific examples of chaotic time-dependence include the convection of a fluid that is heated from below, simple models of the annual variation of insect populations, stirred chemical reactor systems, and determination of the limits on the length of reliable weather forecasting. The number of such examples continues to increase [31].

Many studies have applied these definitions, and some have emphasized their mutual relationships [7, 8, 30, 35, 46, 47, 60, 97].

1.2 Definitions of Chaos

The chaos of a map has been defined in several ways [46]. Although the comment "so many authors, so many definitions," is true, a basic component of all definitions is the unpredictability of the behavior of the trajectory which is determined with some certain error. (The associated phenomenon is usually described in terms of sensitive dependence on initial conditions.)

The definitions of chaos of Devaney and of Li and Yorke are considered herein because they are fundamental and widely accepted:

Definition 1.1 (Devaney [21]). *Let X be a metric space. A continuous map $f : X \to X$ is said to be chaotic on X if*

(D1) *f is topologically transitive. That is, for any pair of nonempty open sets $U, V \subset X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$;*

(D2) *periodic points are dense in X;*

(D3) *f has sensitive dependence on initial conditions,meaning that, there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exists $y \in N$ and $n \in \mathbb{N}$ such that $|f^n(x) - f^n(y)| > \delta$.*

Intuitively, a topologically transitive map has points that eventually move under iteration from one arbitrarily small neighborhood to any other. Consequently, a dynamical system cannot be decomposed into two disjoint open sets that are invariant under the map. Notably, if a map possesses a dense orbit, then it is necessarily topologically transitive. The converse is also true (for compact subsets of $\mathbb{R}$ or S^1).

A map possesses sensitive dependence on initial conditions if there exist points arbitrarily close to x that are eventually separated from x by at least δ under iteration of f. Importantly, not all points close to x need eventually to separate from x under iteration, but at least one such point must exist in every neighborhood of x. If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defy numerical computation. Small rounding errors in the computation become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever to the real orbit.

In summary, a chaotic map possesses three ingredients, which are: indecomposability, an element of regularity, and unpredictability. A chaotic system cannot be broken down or decomposed into two subsystems (two invariant open subsets) that do not interact under f because of topologically transitivity. In the midst of this random behavior, however, is an element of regularity, which is exhibited by the periodic points that are dense. The system is unpredictable because of the sensitive dependence on initial conditions [21].

Banks, Brooks, Cairns, Davis, and Stacey proved that **(D1)** and **(D2)** imply **(D3)**.

Theorem 1.2 ([8])**.** *If $f : X \to X$ is transitive and has dense periodic points, then f has sensitive dependence on initial conditions.*

Vellekoop and Berglund proved that **(D1)** alone can imply **(D2)** and **(D3)** on intervals.

Theorem 1.3 ([97])**.** *Let I be a (not necessarily finite) interval and $f : I \to I$ a continuous function and topologically transitive map. Then (1) the periodic points of f are dense in I and (2) f has sensitive dependence on initial conditions.*

Remark 1.4 ([7])**.** *Assaf and Gadbois demonstrated that **(D1)** and **(D3)** do not imply **(D2)** by giving the example $X = S^1 \setminus \left\{e^{i2\pi p/q} | p, q \in \mathbb{Z}, q \neq 0\right\}$ equipped with the usual arclength metric d and $f : X \to X$ defined by $f(e^{i\theta}) = e^{i2\theta}$. They also demonstrated that **(D2)** and **(D3)** do not imply **(D1)** by giving the examples $X = S^1 \times [0, 1]$, where S^1 is the unit circle and $[0, 1]$ is the unit interval with the standard metrics and $f : X \to X$ defined by $f(e^{i\theta}, t) = (e^{i2\theta}, t)$.*

The definition of chaos in the sense of Li and Yorke is now introduced.

Definition 1.5 (Li and Yorke [55]). *Let I be an interval of the real line and $f : I \to I$ a continuous function. f is chaotic if f has an uncountable scrambled set $S \subset I$ which satisfies the following condition:*

(i) *for every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0 \quad \text{and} \quad \liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0;$$

(ii) *for every $p \in S$ and periodic point $q \in I$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0.$$

Li and Yorke defined only "chaos" on an interval, Huang and Ye extended the interval to general metric space.

Definition 1.6 ([35]). *Let (X, f) be a dynamical system, where X is a compact metric space with metric d, and $f : X \to X$ is continuous and surjective. A subset $S \subset X$ is a scrambled set (for f), if any different points x and y from S are proximal and not asymptotic:*

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \quad \text{and} \quad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

The function f is said to be chaotic in the sense of Li-Yorke, if there exists an uncountable scrambled set.

Li and Yorke proved that if a map has a periodic point with period 3, then the map has Li-Yorke chaos and implies the existence of all other periods. (Detailed discussions see **Appendix A**)

Theorem 1.7 ([21, 55]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point with period 3, then*

(1) *for every $k = 1, 2, \ldots$, there is a periodic point having period k;*

(2) *f is chaotic in the sense of Li-Yorke.*

Chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke. The relationship is given by the following two theorems.

Theorem 1.8 ([46]). *Suppose $f : I \to I$ is continuous, where I is an interval on the real line. If f is topologically transitive (in the sense of Devaney's chaos), then it is Li-Yorke chaotic.*

Theorem 1.9 ([35, 90]). *Let V be a compact set of a metric space (X, d), containing infinitely many points. If a map $f : V \to V$ is continuous, surjective, and chaotic in the sense of Devaney on V, then it is chaotic in the sense of Li-Yorke.*

Remark 1.10. *The converse statement is incorrect [46].*

Various versions of "chaos", such as those defined by Diamond, Marotto, Martelli, Robinson, Wiggins and many others, also exist. Detailed discussions have been presented elsewhere [22, 23, 57, 59, 74, 98].

1.3 Topological Entropy

Definition 1.11 ([53]). *Let X be a sample space where X contains n events $w_1, w_2, ..., w_n$ with probability $p_1, p_2, ..., p_n$, respectively. The* ***Shannon's entropy*** *with respect to X is defined by*

$$H(p_1, ..., p_n) = -\sum_{i=1}^{n} p_i \log(p_i).$$

The following holds.

Theorem 1.12 ([53]). *In the above definition,*

$$\begin{aligned} H\left(\frac{1}{n}, ..., \frac{1}{n}\right) &= \log n \\ &= \max\left\{H(p_1, ..., p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\right\}. \end{aligned}$$

Based on studies [53, 54], conducted in 1948, Shannon introduced a definition of entropy for use in information theory to establish the concept of "uncertain quantities" [83]. Later, in 1958, Kolmogorov used Shannon's entropy to formalize the entropy on dynamical systems and opened the door to established the foundation of ergodic theory. This "Kolmogorov's entropy" was improved and revised by his student Sinai in 1959. The revised version is called "Kolmogorov-Sinai entropy" or "Measure-theoretic entropy".

Definition 1.13 ([53, 75]). *Suppose (X, Σ, μ) is a probability measure space, that is, X is a set, Σ is a σ-algebra of subsets of X, and μ is a probability on Σ. Let $f : X \to X$ be a measurable function. Arbitrarily choose a finite partition $\bar{A} = \{A_1, ..., A_n\}$, that is, $A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, and $\bigcup_{i=1}^{n} A_i = X$. The entropy of f with respect to $\bar{A}$ is defined by*

$$h_\mu\left(f, \bar{A}\right) = \limsup_{n\to\infty} H\left(f^{-n}(\bar{A}) | \bigvee_{i=0}^{n-1} f^{-i}(\bar{A})\right),$$

where $\bar{A} \vee \bar{B} = \left\{A \cap B : A \in \bar{A}, B \in \bar{B}\right\}$, *and* $\bar{A}$ *and* $\bar{B}$ *are finite partitions. The* ***Kolmogorov-Sinai entropy*** *of* f *is defined by*

$$h_\mu(f) = \sup\left\{h_\mu(f,\bar{A}) : \bar{A} \text{ is the finite partition of } X\right\}.$$

Remark 1.14. *If* f *is measure-preserving transformation, then*

$$\begin{aligned} h_\mu(f,\bar{A}) &= \limsup_{n\to\infty} H\left(f^{-n}(\bar{A}) | \bigvee_{i=0}^{n-1} f^{-i}(\bar{A})\right) \\ &= \lim_{n\to\infty} H\left(f^{-n}\left(\bar{A}\right) | \bigvee_{i=0}^{n-1} f^{-i}\left(\bar{A}\right)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}(\bar{A})\right) \end{aligned}$$

for any finite partition $\bar{A}$ *of* X*. Although this equation is not more easily understood or interpreted than the foregoing definition, its equivalent condition can be more conveniently computed [53].*

Modeled after Kolmogorov-Sinai entropy, "topological entropy" was defined by Adler, Konhein, and McAndrew for topologically conjugate invariance in 1965 [2].

Definition 1.15 ([2, 53, 54])**.** *Suppose* (X,T) *is a dynamical system, where* X *is a compact Hausdorff space and* $T : X \to X$ *is a continuous map. For an open cover* $\bar{A}$ *of* X*, let* $N(\bar{A})$ *be defined as the smallest cardinality of a subcover of* $\bar{A}$*. If* $\bar{A}$ *and* $\bar{B}$ *are open covers of* X*, then their common refinement is defined as* $\bar{A} \vee \bar{B} = \{A \cap B : A \in \bar{A}, B \in \bar{B}\}$*. Let*

$$\bigvee_{i=0}^{n} T^{-i}(\bar{A}) = \bar{A} \vee T^{-1}(\bar{A}) \vee \cdots \vee T^{-(n-1)}(\bar{A}),$$

where $T^{-k}(\bar{A}) = \{T^{-k}(A) : A \in \bar{A}\}$*, and define*

$$h_{\text{top}}(T,\bar{A}) = \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n} T^{-i}(\bar{A})\right).$$

The ***topological entropy*** *of* (X,T) *is defined as the supremum*

$$h_{\text{top}}(T) = \sup\left\{h_{\text{top}}(T,\bar{A}) : \bar{A} \text{ is the finite open cover of } X\right\}.$$

Remark 1.16. $N(\bar{A})$ *is finite because of compactness, and the limit exists for any open cover* $\bar{A}$*.*

If the space is compact metric, then the following definition is equivalent to the above notion [12] and it is more useful [4].

Definition 1.17 ([12, 15, 74])**.** *Let $f : X \to X$ be a continuous map on the space X with metric d. A set $S \subset X$ is called (n, ϵ)-separated for f for n a positive integer and $\epsilon > 0$ provided that for every pair of distinct points $x, y \in S$, $x \neq y$, there is at least one k with $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$. The number of different orbits of length n (as measured by ϵ) is defined by*

$$r(n, \epsilon, f) = \{\#(S) : S \subset X \textit{ is a } (n, \epsilon)\textit{-separated set for } f\},$$

where $\#(S)$ is the cardinality of elements in S. Let

$$h_{\text{top}}(\epsilon, f) = \limsup_{n\to\infty} \frac{\log(r(n, \epsilon, f))}{n},$$

*and define the **topological entropy** of f as*

$$h_{\text{top}}(f) = \lim_{\epsilon\to 0, \epsilon>0} h_{\text{top}}(\epsilon, f).$$

Remark 1.18. *Here, only two definitions are introduced. See other references for topological entropy [9, 13, 29, 96].*

Example 1.19 ([74])**.** *Let $f : S^1 \to S^1$ have a covering map $F : \mathbb{R} \to \mathbb{R}$ given by $F(x) = 2x$. This map is called the doubling map. The distance on S^1 is the one inherited from $\mathbb{R}$ by taking x to $x \mod 1$. Therefore, points near 1 are close to points near 0. Two points x and y stay within ϵ of each other for $n-1$ iterations of f if and only if $|x-y| \leq \epsilon 2^{-(n-1)}$ (as points in $\mathbb{R}$). If points are separated by a distance of exactly $\epsilon 2^{-(n-1)}$ in $[0, 1)$, then the maximum number of points is $[\epsilon^{-1}2^{n-1}]$. However, the last point to the right is close to the first point on the left when considered modulo 1 in S^1, so there are $[\epsilon^{-1}2^{n-1}] - 1$ points in S^1. These points can be spread apart slightly to make them (n, ϵ)-separated. Therefore, $r(n, \epsilon, f) = [\epsilon^{-1}2^{n-1}] - 1$, where $[a]$ is the integer part of a. Then,*

$$\begin{aligned} h_{\text{top}}(\epsilon, f) &= \limsup_{n\to\infty} \frac{\log\left([\epsilon^{-1}2^{n-1}] - 1\right)}{n} \\ &= \limsup_{n\to\infty} \frac{\log\left(\epsilon^{-1}\right) + (n-1)\log(2)}{n} \\ &= \log(2) \end{aligned}$$

for any $\epsilon > 0$, so $h_{\text{top}}(f) = \log(2)$.

The following theorem relates the topological entropy to the Kolmogorov-Sinai entropy:

Theorem 1.20 ([24])**.** *If K is compact and $f : K \to K$ is continuous, then $h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu$ is an ergodic measure with respect to $f\}$.*

Remark 1.21. *An invariant measure* μ *satisfies the equation* $\mu(f^{-t}(K)) = \mu(K)$, $t > 0$. *An invariant probability measure* μ *may be decomposable into several pieces, each of which is again invariant. If it is not so decomposable, then it is said to be ergodic.*

Once the topological entropy has been computed, whether systems are chaotic or not is known.

Theorem 1.22 ([51])**.** *If a continuous map of the interval has positive topological entropy, then it is chaotic according to the definition of Li and Yorke.*

Remark 1.23 ([28, 51, 92])**.** *The converse of this theorem is incorrect. Xiong [101] and Smítal [92] given counterexamples.*

In general metric space, a similar result is obtained:

Theorem 1.24 ([10])**.** *If the dynamical system* (X, T) *has positive topological entropy, where* X *is compact metric space and* T *is surjective and continuous, then it is chaotic in the sense of Li and Yorke.*

Consider the continuous map on the compact interval, the relationship between positive topological entropy and Devaney's chaos is equivalent:

Theorem 1.25 ([48, 50, 51, 63])**.** *Let* f *be a continuous map of a compact interval* I *to itself.* f *has positive topological entropy if and only if* f *is chaotic in the sense of Devaney.*

Let $f : X \to X$ and $g : Y \to Y$ be two maps. f and g are said to be **topologically conjugate** if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism h is called a **topological conjugacy**. Mappings that are topologically conjugate have completely equivalent dynamics [21].

Adler, Konhein, and McAndrew established "topological entropy" for invariant of topologically conjugate, such that if two maps are topologically conjugate, then their topological entropies are equal as follows:

Theorem 1.26 ([2, 54, 74])**.** *Let* $f : X \to X$ *and* $g : Y \to Y$ *be two continuous maps, where* X *and* Y *are invariant compact sets under* f *and* g*, respectively. That is,* $f(X) \subset X$ *and* $g(Y) \subset Y$*. If* f *and* g *are topologically conjugate, then* $h_{\text{top}}(f) = h_{\text{top}}(g)$.

This basic result that is used to help calculate the entropy, relates the entropy of a map f to a power f^k of f.

Theorem 1.27 ([74])**.** *Assume* $f : X \to X$ *is uniformly continuous or* X *is compact, and* k *is an integer with* $k \geq 1$*. Then* $h_{\text{top}}(f^k) = k \cdot h_{\text{top}}(f)$.

1.4 Lyapunov Exponent

The Lyapunov exponent is usually computed to measure the exponential rate at which nearby orbits are moving apart. It is used in a diagnostic method that has been proven to be the most useful dynamical diagnostic method for chaotic systems in chaos theory [1, 74, 99, 100]. Therefore, an expression for the growth rate of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ as the number of iterations increases is sought. Let x_0 be an initial value and $\delta > 0$ (δ can sufficiently small). By Taylor's expansion [74], $|f^n(x_0+\delta) - f^n(x_0)| \approx |(f^n)'(x_0)|\delta$. If $|(f^n)'(x_0)| \sim L^n$, where L is a linear function, then $\dfrac{1}{n}\log(|(f^n)'(x_0)|) \sim \log(L)$.

Definition 1.28 ([74]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. For each point x_0, define the* ***Lyapunov exponent*** *of x_0, $\lambda(x_0)$, as follows:*

$$\begin{aligned}\lambda(x_0) &= \limsup_{n\to\infty} \frac{1}{n}\log(|(f^n)'(x_0)|)\\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\log(|f'(x_k)|),\end{aligned}$$

where $x_j = f^j(x_0)$.

Remark 1.29. *Oseledec [68] showed that the limit exists for almost all points.*

Remark 1.30. *Assume $\lambda(x_0) > 0$, which implies that*

$$\log(|(f^n)'(x_0)|) \approx n\lambda(x_0) \quad \text{or} \quad |(f^n)'(x_0)| \approx e^{n\lambda(x_0)} = L(x_0)^n,$$

where $L(x_0) = e^{\lambda(x_0)} > 1$, and

$$|f^n(x_0+\delta) - f^n(x_0)| \approx |(f^n)'(x_0)|\delta \approx L(x_0)^n\delta \to \infty$$

as $n \to \infty$. Therefore, a positive Lyapunov exponent means sensitive dependence on initial conditions, this result is very important and useful since it enables a single quantity to be computed to determine whether a chaotic process is highly sensitive to initial conditions [74, 100].

In the case of m variables with $x \in \mathbb{R}^m$, the derivative $\dfrac{df}{dx}$ is replaced by the Jacobian matrix, which is evaluated at x: $D_x f = (\dfrac{\partial f_i}{\partial x_j})$, and is called the **spectrum of Lyapunov exponents** [24]. In a deterministic system, the positivity of the largest Lyapunov exponent is a necessary but not sufficient condition to guarantee the existence of chaos. For example, suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x$, it is sensitive dependence on initial conditions, but it is not transitive. However, fulfillment of this condition is often sufficient [100].

Katok proved the following theorem, which relates the Lyapunov exponent to topological entropy:

Theorem 1.31 ([41])**.** *If a $C^{1+\alpha}$-diffeomorphish f, $\alpha > 0$, of a compact manifold has a Borel probability invariant continuous ergodic measure with non-zero Lyapunov exponents, then $h_{\text{top}}(f) > 0$. (That is, the map is chaotic in the sense of Devaney and of Li and Yorke).*

Remark 1.32.

(1) *X is called a compact manifold if it is a manifold that is compact as a topological space, where the manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space;*

(2) *$f : X \to X$ is a C^r-diffeomorphish if f is a C^r-homeomorphism such that f^{-1} is also C^r.*

Pesin also proved the relationship in other situations:

Theorem 1.33 ([24])**.** *If ρ is an invariant ergodic measure with compact support under the diffeomorphism f of a finite-dimensional manifold and ρ has smooth density with respect to the Lebesgue measure, then $h_{\text{top}}(\rho) = \sum$ positive λ_i.*

1.5 Strange Attractor

Roughly, an attractor is an invariant set to which all nearby orbits converge. Hence, attractors are the sets that one "sees" when a dynamical system is iterated on a computer [21]. First, the following is defined:

Definition 1.34 ([21, 74])**.** *A compact region $N \subset M$ is called a* ***trapping region*** *for f if $f(N) \subset int(N)$, where M is a (smooth) compact manifold and $f : M \to M$ is a diffeomorphism.*

Since $f(N)$ is compact and $f(N) \subset int(N)$, the sets $f^n(N)$ are all compact and nested for $k \geq 0$. Therefore

$$\Lambda = \bigcap_{n \geq 0} f^n(N)$$

is a compact, nonempty set. Λ is the set of points whose full orbits, both forward and backward, remain in N for all time [21].

Definition 1.35 ([21, 74])**.** *A set Λ is called an* ***attracting set*** *for f if there exists a trapping region N such that $\Lambda = \bigcap_{k \geq 0} f^k(N)$.*

Remark 1.36. *Λ is an invariant set [21].*

Definition 1.37 ([21, 74])**.** *An ϵ-chain or a pseudo-orbit of length n from x to y for a map f is a sequence $\{x = x_0, ..., x_n = y\}$ such that for all $1 \leq j \leq n$, $d(f(x_{j-1}), x_j) < \epsilon$. Let X be a set. The ϵ-chain limit set of X for f is the set*

$$\Omega^+_\epsilon(f) = \{x \in X : \forall\, n \geq 1,\ \exists\, y \in X \text{ and } \epsilon\text{-chain from } y \text{ to } x \text{ of length greater than } n\}.$$

Then, the forward chain limit set of X is the set

$$\Omega^+(f) = \bigcap_{\epsilon>0} \Omega^+_\epsilon(f).$$

Finally, the chain recurrent set of f is the set

$$R(f) = \{x : \text{there exists an } \epsilon\text{-chain from } x \text{ to } x \text{ for all } \epsilon > 0\}.$$

Definition 1.38 ([74])**.** *The following relation $\sim$ on $R(f)$ is defined: $x \sim y$ if $y \in \Omega^+(x)$ and $x \in \Omega^+(y)$. Two such points are called chain equivalent, and the relation is an equivalence relation. The equivalence classes are called the chain components of $R(f)$. If f has a single chain component on an invariant set Λ, then f is said to be* ***chain transitive*** *on Λ.*

Definition 1.39 ([74])**.** *A set Λ is called an* ***attractor*** *if it is an attracting set and $f|_\Lambda$ is chain transitive, so $\Lambda \subset R(f)$.*

Remark 1.40. *In the above definition,*

(1) *Sometimes, $f|_\Lambda$ may wish to be assumed to be topologically transitive [74].*

(2) *Other definitions of attractors are in common use [21, 24, 62, 64, 76].*

The omega limit set $\omega(x)$ of a point $x \in M$ is the collection of all accumulation points for the sequence $\{x,\ f(x),\ f^2(x),\ ...\}$ of successive images of x. If some metric for the topological space M is chosen, then $\omega(x)$ can also be described as the smallest closed set S such that the distance from $f^n(x)$ to the nearest point of S tends to zero as $n \to \infty$. A set Λ is said to be the likely limit set if it is the smallest closed of M with $\omega(x) \subset \Lambda$ for every point $x \in M$ outside of a set of measure zero [28, 62]. Interestingly, the following properties are identified:

Theorem 1.41 ([62])**.** *This likely limit set Λ is well defined and is an attractor for f. In fact, Λ is the unique maximal attractor, which contains all others.*

Theorem 1.42 ([62])**.** *If S is a compact set of positive measure with the property with $f(S) \subset S$, then S necessarily contains at least one attractor.*

Now, the definition can be presented:

Definition 1.43 ([74]). *An invariant set Λ is called a* ***strange attractor*** *if it is an attractor and f has positive Lyapunov exponent on Λ (meaning that it has sensitive dependence on initial conditions).*

1.6 Snapback Repeller

Generally, proving that a dynamical system has chaotic behavior is difficult. Most techniques for making such a determination involve computer simulations, which apply the arithmetic of the Lyapunov exponent, find a period doubling bifurcation, and perform other tasks that are associated with numerical dynamical systems. However, obtaining such results by rigorous mathematical proofs is difficult.

A dynamical system with diffeomorphism has chaotic behavior that can be proved by using known methods, such as the existence of Smale horseshoe, transversal homoclinic orbits, or heteroclinic orbits. Noninvertible maps have chaotic behavior that can be identified by the existence of snapback repellers. However, for general focus problems, applying the above methods without computer-assistance is difficult. In most cases, the verification must be carried out with the aid of a computer and many investigations on the verification of chaotic behavior using computer-assisted techniques for continuous dynamical systems and discrete dynamical systems have been published [69].

In 1978, Marotto defined the snapback repeller. The existence of snapback repellers is adopted to determine whether a system is chaotic or not.

Definition 1.44 ([57]). *Let $p^* \in \mathbb{R}^n$, $B_r(p^*)$ be the open ball in $\mathbb{R}^n$ of radius r centered at the point p^*, and $Df(p^*)$ be the Jacobi matrix of f evaluated at p^*. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in $B_r(p^*)$, the point p^* is an* ***expanding fixed point*** *of f in $B_r(p^*)$ if $f(p^*) = p^*$ and all eigenvalues of $Df(x)$ exceed 1 in norm for all $x \in B_r(p^*)$.*

Definition 1.45 ([57]). *Suppose p^* is an expanding fixed point of f in $B_r(p^*)$ for some $r > 0$. p^* is said to be a* ***snapback repeller*** *of f if there exists a point $x_0 \in B_r(p^*)$ with $x_0 \neq p^*$ and some positive integer m such that $f^m(x_0) = p^*$ and $\det(Df^m(x_0)) \neq 0$.*

The surprising main related result [57] is the following theorem, proven formerly by Marotto in 1978.

Theorem 1.46 ([57]). *If $f : \mathbb{R}^n \to \mathbb{R}^n$ has a snapback repeller, then f is chaotic in the sense of Li and Yorke.*

The above theorem states that if snapback repellers can be found, then the system exhibits Li-Yorke's chaos. However, there exists an error in Marotto's paper [57]. Proving Theorem 1.46, Marotto used the statement that if

(A): f satisfies that "expanding fixed point" condition in Definition 1.44, then

(B): there exists a constant $s > 1$ such that $\|f(x) - f(y)\| > s\|x - y\|$ for all $x, y \in B_r(p^*)$.

But, **(A)** does not imply **(B)** (the converse is true). Chen, Hsu & Zhou [16] and Li & Chen [49] gave two counterexamples to point out that the above statement is not true. As a result, the proof of Theorem 1.46 is in error.

In order to modify Marotto's definition and theorem, many authors like Shi, Chen, and Yu [87, 89, 90] proposed their improved version of Definition 1.44, Definition 1.45, and Theorem 1.46. Furthermore, Marotto also redefined the definition of snapback repeller as follows.

Definition 1.47 ([58])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in $B_r(p^*)$ and $p*$ be a fixed point of f with all eigenvalues of $Df(p^*)$ exceeding 1 in norm, and there exists a constant $s > 1$ such that $\|f(x) - f(y)\| > s\|x - y\|$ for all $x, y \in B_r(p^*)$. Suppose there exists a point $x_0 \in B_r(p^*)$ with $x_0 \neq p^*$ and some positive integer m such that $f^m(x_0) = p^*$ and $\det(Df^m(x_0)) \neq 0$. Then p^* is called a* ***snapback repeller*** *of f.*

Remark 1.48.

(1) *The result of Theorem 1.46 remains essentially unchanged, as long as one refers to the above revised definition [58].*

(2) *Actually, Definition 1.45 and Definition 1.47 are all true and have no problems in above discussions in one-dimensional space $\mathbb{R}$ which can be verified by applying the mean value theorem [87]. In principle this work still uses original definition of snapback repeller (Definition 1.45).*

(3) *In one-dimensional space $\mathbb{R}$, the Jacobi matrix $Df(p^*) = f'(p^*)$ and*

$$\begin{aligned}\det(Df^m(x_0)) &= (f^m)'(x_0) \\ &= f'(f^{m-1}(x_0)) \cdot f'(f^{m-2}(x_0)) \cdots f'(f(x_0)) \cdot f'(x_0) \\ &= f'(x_{m-1}) \cdot f'(x_{m-2}) \cdots f'(x_1) \cdot f'(x_0),\end{aligned}$$

where $x_j = f^j(x_0)$, $1 \leq j \leq m-1$.

Definition 1.49 ([87, 89]). *Let snapback repeller p^*, f, m, and x_0 be the same as Definition 1.47. p^* is said to be a* ***nondegenerate snapback repeller*** *of f if there exist positive constants μ and δ_0 such that $B_{\delta_0}(x_0) \subset B_{r_0}(p^*)$ and $\|f^m(x) - f^m(y)\| \geq \mu\|x - y\|$ for all $x, y \in B_{\delta_0}(x_0)$; p^* is called a* ***regular snapback repeller*** *of f if $f(B_{r_0}(p^*))$ is open and there exists a positive constant δ_0^* such that $B_{\delta_0^*}(x_0) \subset B_{r_0}(p^*)$ and p^* is an interior point of $f^m(B_\delta(x_0))$ for any positive constant $\delta \leq \delta_0^*$.*

The snapback repeller in Marotto's theorem is nondegenerate and regular. If f is continuously differentiable in some neighborhood of x_j and $\det(Df(x_j)) \neq 0$ with $0 \leq j \leq m-1$, then f is chaotic in the sense of Devaney.

Theorem 1.50 ([56, 87, 88, 89, 90]). *Let snapback repeller p^*, f, m, and x_0 be the same as Definition 1.47. If f is C^1 in some neighborhood of x_j, $\det(Df(x_j)) \neq 0$, $0 \leq j \leq m-1$, and f has a snapback repeller p^*, then f is chaotic in the sense of Devaney.*

Remark 1.51. *Chen, Hsu and Zhou [16] proved that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and it has a snapback repeller, then f is chaotic in the sense of Devaney. The ecological model in this paper is not C^1, hence we prefer using Theorem 1.50 to using the theorem of Chen, Hsu and Zhou.*

Based on Devaney's chaos and Li-Yorke's chaos in Section 1.2, this work establishes a relational graph of chaos and relative checking methods (Figure 6). In Theorem 1.22, Theorem 1.24, and Theorem 1.25. A continuous map on a compact interval (or a surjective and continuous map on a compact matric space) has positive topological entropy implying that the map has both Devaney's chaos and Li-Yorke's chaos. At the same time, in Theorem 1.46 and Theorem 1.50, if a map could find a snapback repeller, then it also exhibits Devaney's chaos and Li-Yorke's chaos.

In Remark 1.30, a positive Lyapunov exponent exhibits only sensitivity but transitivity, but in Theorem 1.31, if a $C^{1+\alpha}$-diffeomorphism map ($\alpha > 0$) of a compact manifold has a Borel probability invariant continuous ergodic measure and a positive Lyapunov exponent, then its topological entropy is positive. Furthermore, it exhibits both Devaney's chaos and Li-Yorke's chaos. On the other hand, in the hypothesis of Theorem 1.33, it has a positive Lyapunov exponent when the map has a positive topological entropy. Furthermore, it is a strange attractor when the map has a positive Lyapunov exponent and an attractor.

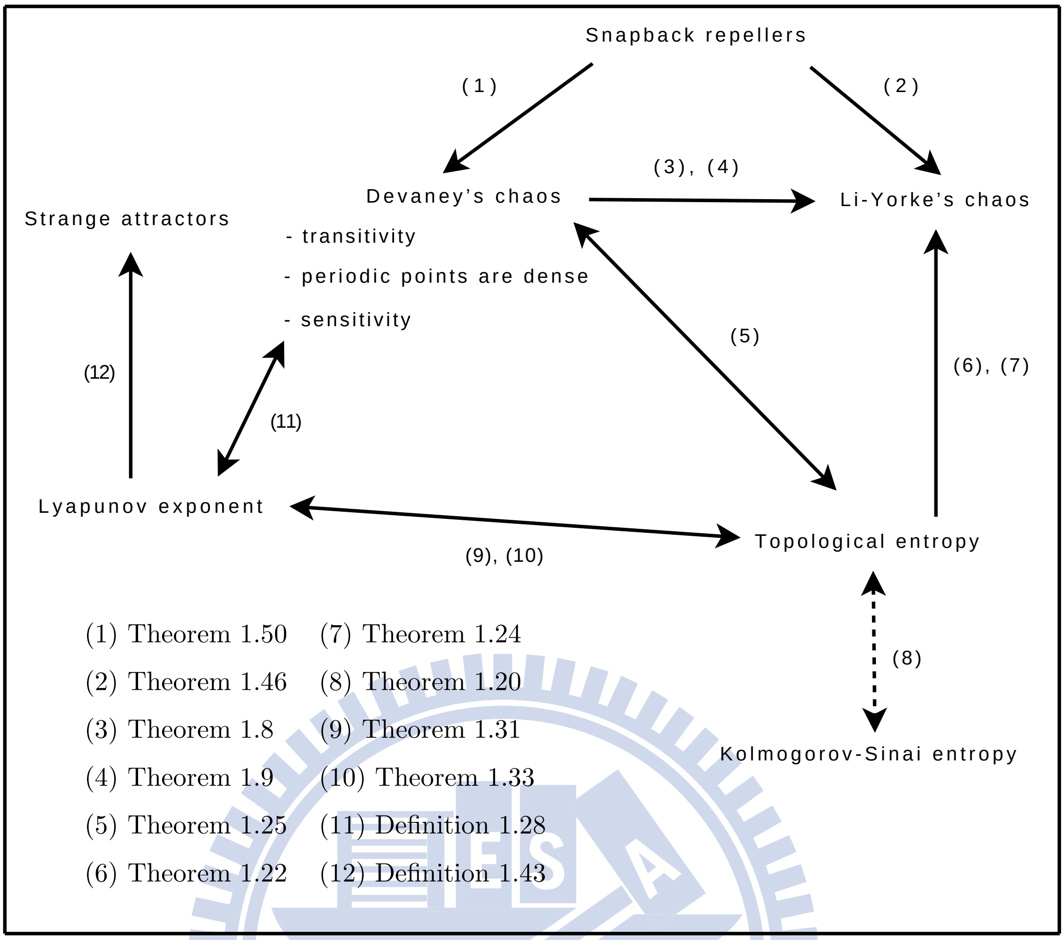

Figure 1: Relational graph of chaos and relative checking methods.

2 Ecological Model

Many trees in forests reproduce intermittently, rather than at a constant rate [79]. A number of flowers and fruits are produced in a particular year (called a mast year) but very little reproductive activity occurs during several subsequent years until the next mast year. This "synchronous production of highly variable amount of seeds from year to years by a population of plants" is called masting [79]. Perfect periodicity in reproduction is rarely observed, and the intervals between masting are rather irregular.

Several explanations of the masting phenomenon have been proposed [3, 5, 6, 17, 19, 25, 26, 27, 32, 36, 37, 39, 40, 43, 44, 45, 67, 72, 85, 91, 93, 94]. They involve environmental fluctuations, weather conditions, swamping predators, the weight of young deer, bird populations, the reproductive success of bears, increased efficiency of wind pollination, attraction to seed distributions, cue masting, and the dispersing animals. However, most of these hypotheses explain neither the mechanism of masting nor the mechanism by which the timing of reproduction varies among individuals [79].

2.1 Isagi's Resource Budget Model

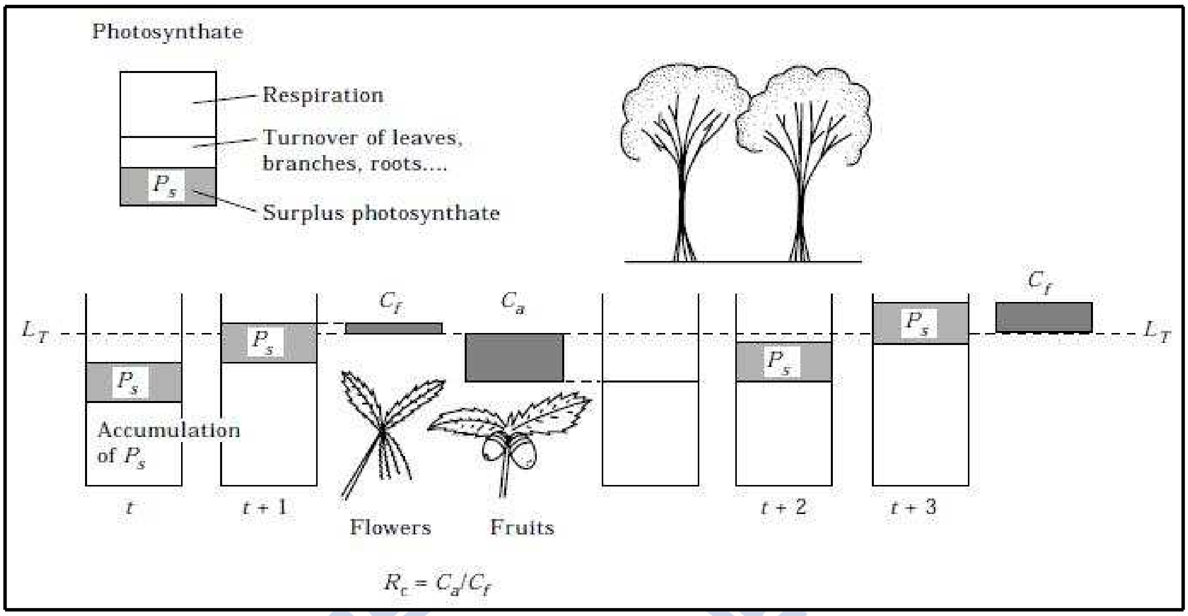

Figure 2: Resource budget model of an individual plant [38].

Isagi, Sugimura, Sumidaa and Ito proposed a simple model of the mechanism of masting that was based on the resource budget of an individual tree [38]. They assumed that a constant amount of photosynthate is produced by each tree annually, given that the environmental conditions are constant from year to year. In Figure 2, photosynthate (P_S) is consumed for the growth and the maintenance of the tree; any that is not used by the plant is stored in a pool within the tree. The amount of P_S was constant from year to year. In one year when the accumulated P_S exceeded a threshold (L_T), the amount of accumulated P_S minus L_T was used for flowering, and is regarded as the cost of flowering C_f. Hence, whenever the amount of photosynthate accumulated in preceding years was large, the tree was inclined to flower more, and the amount of flowering in a year also depended on the amount of photosynthetic products that had accumulated in the previous years. The amount of accumulated P_S was decreased to L_T after the flowering. The flowers were pollinated and bore fruits at a cost of C_a. The ratio C_a/C_f was assumed to be constant R_C. After the fruiting had been completed, the amount accumulated was $L_T - C_a = L_T - R_C C_f$. In the model, P_S accumulates annually, until the tree flowers again when the amount exceeds L_T.

Isagi et al. performed several numerical simulations [38], which R_C was an important role in their model. In Figure 3, when $R_C < 1$, the amount of seed production was constant every year. When $R_C \geq 1$, mast years appeared. A higher R_C value was associated with larger intervals between mast years, and therefore, a higher rate of crop failure. When $R_C = 1$, the autocorrelation values between seed production and lagged values of the prior seed production reciprocated between two specific values. In this case, the mast and non-mast years are predictable because the interval is regular. As R_C increases, the model is less able to predict masting.

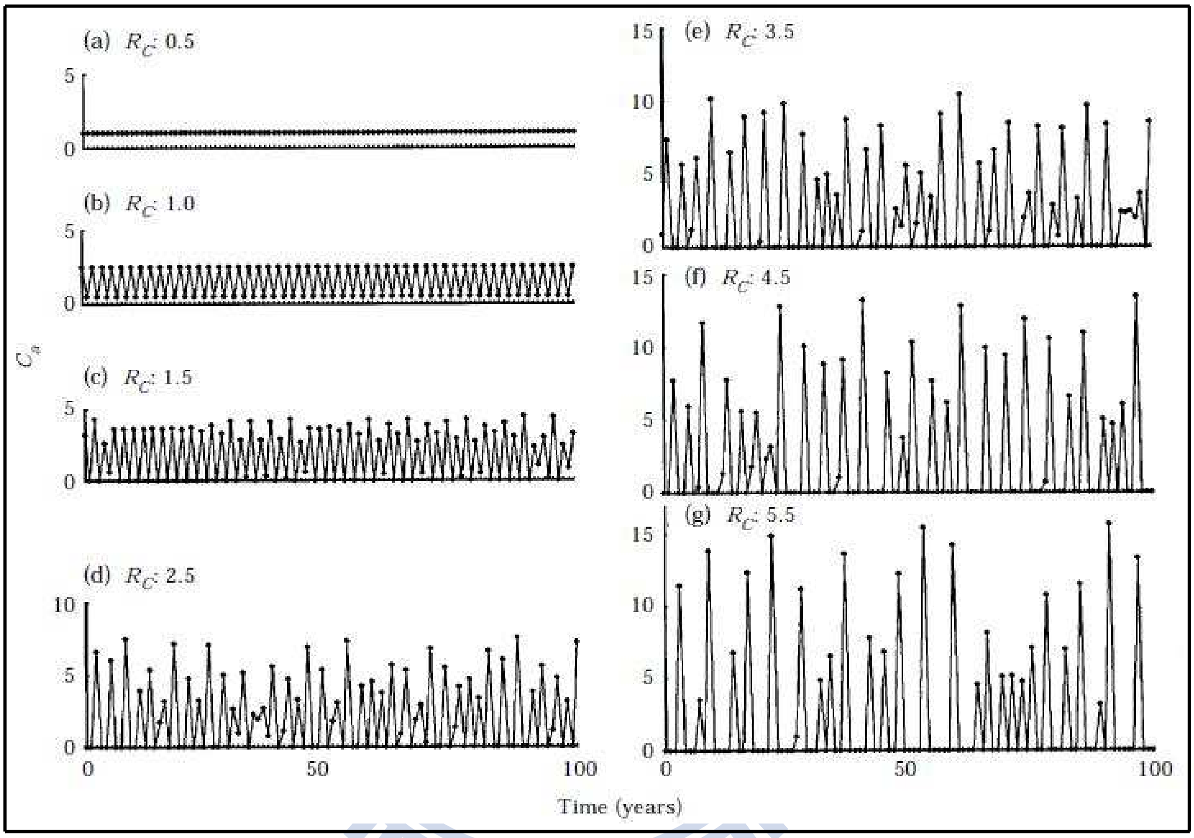

Figure 3: Time series of C_a with various R_cs. For all plates, P_s, L_T, and $C_f(0)$ were 3.0, 6.0, and 2.5, respectively. Results were excluded for the initial 50 years [38].

2.2 Satake's Generalized Resource Budget Model

Let $S^{(t)}$ be the amount of energy reserved at the beginning of year t. If the sum $S^{(t)} + P_S$ is below the threshold L_T, then the tree does not reproduce and saves all of its reserved energy for the following year. If the sum exceeds L_T, then the tree uses energy for flowering.

Isagi et al. assumed that the energy expenditure for flowering exactly equals the excess, $S^{(t)} + P_S - L_T$. Satake and Iwasa generalized Isagi's model, the amount of energy expenditure for flowering is proportional to the excess, $a(S^{(t)} + P_S - L_T)$, where a is a positive constant. Flowering trees may be pollinated and set seeds and fruits. The cost for fruits is assumed to be proportional to the cost of flowers, and is expressed as $R_C a(S^{(t)} + P_S - L_T)$. After the reproductive stage, the energy reserves of the tree have fallen to

$$S^{(t)} + P_S - a(S^{(t)} + P_S - L_T) - R_C a(S^{(t)} + P_S - L_T) = S^{(t)} + P_S - a(R_C + 1)(S^{(t)} + P_S - L_T).$$

Therefore,

$$S^{(t+1)} = \begin{cases} S^{(t)} + P_S, & \text{if } S^{(t)} + P_S \leq L_T, \\ S^{(t)} + P_S - a(R_C + 1)(S^{(t)} + P_S - L_T), & \text{if } S^{(t)} + P_S > L_T. \end{cases} \tag{1}$$

Define the non-dimensionalized variable $Y^{(t)} = (S^{(t)} + P_S - L_T)/P_S$, equation (1) is now rewritten as

$$Y^{(t+1)} = \begin{cases} Y^{(t)} + 1, & \text{if } Y^{(t)} \leq 0, \\ -kY^{(t)} + 1, & \text{if } Y^{(t)} > 0, \end{cases} \tag{2}$$

where $k = a(R_C + 1) - 1$. The parameter k denotes the degree of resource depletion after a reproductive year divided by the excess amount of energy in reserve before that year, and is called the depletion coefficient [79]. Notably, the quantity $Y^{(t)}$ is positive if and only if the tree exhibits some reproductive activity in year t.

After rescaling, the dynamics (2) include only a single parameter k. Other parameters such as P_S or L_T do not affect the essential features of the dynamics if k remains the same. In Isagi's model, $a = 1$ is assumed and the depletion coefficient is the same as the ratio of the fruiting cost to the flowering cost, $k = R_C$. Since the maximum value of $Y^{(t+1)}$ equals one when $Y^{(t)} = 0$, the minimum value of $Y^{(t+1)}$ equals $-k + 1$. Thus, the possible range of $Y^{(t)}$ contained in $[-k + 1, 1]$.

3 Mathematical Analysis

The following definitions are present for convenience later:

Definition 3.1. *The composition of two functions is denoted by $f \circ g(x) = f(g(x))$. The n-fold composition of f with itself recurs repeatedly in the sequel. The function f is denoted by $f^n(x) = f \circ \cdots \circ f(x)$, where n is an* ***iterative number****.*

Now, Satake's model (2) is analyed mathematically.

Proposition 3.2. *If $k \leq -1$, then $Y^{(t)}$ tends to infinity.*

Proof. By definition of Y, if $Y^{(t)} \leq 0$, then there exists a $\bar{t} > 0$ such that $Y^{(\bar{t})} \in (0,1]$. Let $Y_0 = Y^{(\bar{t})}$ and $Y_m = Y^{(\bar{t}+m)}$. When $k = -1$, $Y_m = Y_0 + m$ for all $m > 0$ is to be shown. When $m = 1$, $Y_1 = -kY_0 + 1 = Y_0 + 1$. Suppose $Y_j = Y_0 + j$, since $Y_j > 0$,

$$Y_{j+1} = -kY_j + 1 = (Y_0 + j) + 1 = Y_0 + (j+1).$$

By mathematical induction, $Y_m = Y_0 + m$ for all $m > 0$. Then,

$$\lim_{m\to\infty} Y_m = \lim_{m\to\infty} Y_0 + m = \infty.$$

When $k < -1$, since $k < -1$ and $-k > 1$, $Y_m > 0$ for all $m > 0$. Next, $Y_m = (-k)^m Y_0 + (-k)^{m-1} + \cdots + (-k) + 1$ for all $m > 0$ is to be shown. Now, $Y_1 = -kY_0 + 1$ is known. Suppose $Y_j = (-k)^j Y_0 + (-k)^{j-1} + \cdots + (-k) + 1$, since $Y_j > 0$,

$$\begin{aligned} Y_{j+1} &= -kY_j + 1 \\ &= -k\left[(-k)^j Y_0 + (-k)^{j-1} + \cdots + (-k) + 1\right] + 1 \\ &= (-k)^{j+1} + (-k)^j + \cdots + (-k)^2 + (-k) + 1. \end{aligned}$$

Again, by mathematical induction, $Y_m = (-k)^m Y_0 + (-k)^{m-1} + \cdots + (-k) + 1$ for all $m > 0$, and

$$Y_m = (-k)^m Y_0 + (-k)^{m-1} + \cdots + (-k) + 1 = (-k)^m Y_0 + \frac{1-(-k)^m}{1-(-k)}. \tag{3}$$

Hence,

$$\lim_{m\to\infty} Y_m = \lim_{m\to\infty}\left[(-k)^m Y_0 + \frac{1-(-k)^m}{1+k}\right] = \infty.$$

□

Proposition 3.3. *If $-1 < k < 1$, then $Y^{(t)}$ converges to the stable equilibrium $\dfrac{1}{k+1}$.*

Proof. When $0 < k < 1$, since $Y_0 \in (0,1]$, $Y_1 = -kY_0 + 1 \in (0,1)$. Assume $Y_j \in (0,1)$, $Y_{j+1} = -kY_j + 1$ is also contained in $(0,1)$. By mathematical induction, $Y_m \in (0,1)$ for all $m > 0$. Since

$$Y_m = (-k)^m Y_0 + (-k)^{m-1} + \cdots + (-k) + 1 = (-k)^m Y_0 + \frac{1-(-k)^m}{1-(-k)}$$

and $0 < k < 1$, $|-k| < 1$, let m tend to infinity,

$$\lim_{m\to\infty} Y_m = \frac{1}{1+k}.$$

If $k = 0$, $Y_m = 1$ for all $m > 0$, then

$$\lim_{m\to\infty} Y_m = 1 = \frac{1}{1+k}.$$

When $-1 < k < 0$, $0 < -k < 1$, $Y_1 = -kY_0 + 1 > 0$. Suppose $Y_j > 0$, now $Y_{j+1} = -kY_j + 1 > 0$ because $-k > 0$ and $Y_j > 0$. Hence, $Y_m > 0$ for all $m > 0$. From (3),

$$\lim_{m\to\infty} Y_m = \lim_{m\to\infty}\left[(-k)^m Y_0 + \frac{1-(-k)^m}{1-(-k)}\right] = \frac{1}{1+k}.$$

□

Since k is the depletion coefficient, $k > 0$ can be assumed. Therefore, from Proposition 3.3, if $0 < k < 1$, then the tree reproduces every year at a constant rate.

Proposition 3.4. *If $k = 1$, then there exists a number of periodic points with period 2 corresponding to different initial conditions.*

Proof. By hypothesis $Y_0 \in (0, 1]$,

$$Y_1 = -kY_0 + 1 = -Y_0 + 1 \in [0, 1).$$

If $Y_1 = 0$, then $Y_2 = 1$ and $Y_3 = 0 = Y_1$. If $Y_1 \in (0, 1)$, then $Y_2 = -Y_1 + 1 = -(-Y_0 + 1) + 1 = Y_0$. □

Hence, if $k = 1$, there are a number of two-point cycles corresponding to different initial conditions. (Isagi's model satisfies Proposition 3.3 and Proposition 3.4.)

In one study [79], Satake and Iwasa identified chaos by finding a positive Lyapunov exponent if $k > 1$. Of course, some authors regard the positive Lyapunov exponent as the definition of chaos because sensitivity is the most important property of chaotic systems and is easily observed. However, a positive Lyapunov exponent means only that the model is sensitive dependence on initial conditions. The goal here is to prove chaos by identifying dense periodic subsets and transitivity rather than sensitivity (as in the chaos of Devaney). In this work, the model is proven to exhibit Devaney's chaos by identifying snapback repellers.

Theorem 3.5. *If $k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.3247$, then the system is chaotic in Devaney's sense.*

Proof. First, $p^* = \frac{1}{1+k}$ is a fixed point of Y; let $g = Y^{-1}$. Since $|g'(p^*)| < 1$, there exists $r > 0$ with $U = (p^* - r, p^* + r)$, $U \subset (0, 1)$ such that $\lim_{m\to\infty} g^m(x) = p^*$ if $x \in U$. Choose

$$g(p^*) = \frac{-k}{1+k} < 0 \quad \text{and} \quad g^2(p^*) = \frac{2k+1}{k^2+k} > 0.$$

Let $g(p^*) > -k + 1$ and check $g^2(p^*) < 1$, then $k^2 - k - 1 > 0$. Solve the inequality,

$$k > \frac{1+\sqrt{5}}{2} \approx 1.6180 \quad \text{or} \quad k < \frac{1-\sqrt{5}}{2} \approx -0.6180.$$

Choosing $k > \dfrac{1+\sqrt{5}}{2}$ allows j to be found such that

$$g^j(p^*) > 0 \quad \text{for all } j \geq 3$$

by the definition of Y. Computing $|g^j(p^*) - p^*|$, yield $|g(p^*) - p^*| = 1 = \dfrac{1}{k^0}$, and

$$|g^j(p^*) - p^*| = \frac{1}{k^{j-1}}$$

for all $j \geq 2$ by mathematical induction. Hence, $|g^j(p^*) - p^*|$ decreases to 0 as j tends to infinity. That is, for this r, there exists a natural number $J > 0$ such that

$$g^j(p^*) \in U \quad \text{as } j \geq J.$$

Fix J and let $x_0 = g^J(p^*)$, then $x_0 \in U$ and $Y^J(x_0) = p^*$. Since $|Y'(p)| = k > 1$ for all $p \in U$, and $(Y^J)'(x_0) \neq 0$, p^* is a snapback repeller of Y.

Next, choose the (upper-right) fixed point $p^{**} = \dfrac{2}{1+k}$ of Y^2. Suppose $h = (Y^2)^{-1}$ and choose

$$h(p^{**}) < \frac{1}{k} \quad \text{and} \quad h^2(p^{**}) > \frac{1}{k}.$$

Let

$$h(p^{**}) > -k + 2 \quad \text{and} \quad h^2(p^{**}) < 1,$$

yielding

$$k^4 - k^3 - k^2 + 1 = (k-1)(k^3 - k - 1) > 0.$$

In **Appendix B**, the cubic equation is solved exactly. Hence, solving this inequality,

$$k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.3247 \quad \text{or} \quad k < 1.$$

Selecting $k > \left(\dfrac{1}{2} + \sqrt{\dfrac{23}{108}}\right)^{1/3} + \left(\dfrac{1}{2} - \sqrt{\dfrac{23}{108}}\right)^{1/3}$, yields j such that

$$h^j(p^{**}) > \frac{1}{k} \quad \text{for all } j \geq 3$$

by the definition of Y^2. By mathematical induction,

$$|h^j(p^{**}) - p^{**}| = \frac{k-1}{k^{j+1}}$$

for all $j \geq 1$. Therefore, $|h^j(p^{**}) - p^{**}|$ is decreasing and

$$\lim_{j\to\infty} |h^j(p^{**}) - p^{**}| = \lim_{j\to\infty} \frac{k-1}{k^{j+1}} = 0. \tag{4}$$

Since $|h'(p^{**})| < 1$, there exists $r > 0$ with $V = (p^{**} - r, p^{**} + r)$, $V \subset (\frac{1}{k}, 1)$ such that

$$\lim_{m\to\infty} h^m(x) = p^{**} \quad \text{if } x \in V.$$

For this r, from (4), there exists a natural number $J' > 0$ such that

$$h^j(p^{**}) \in V \quad \text{as } j \geq J'.$$

Fix this J', let $y_0 = h^{J'}(p^{**})$, then $y_0 \in V$ and $(Y^2)^{J'}(y_0) = p^{**}$. Since $|(Y^2)'(p)| = k > 1$ for all $p \in V$, and $\left[(Y^2)^{J'}\right]'(y_0) \neq 0$, p^{**} is a snapback repeller of Y^2.

Finally, Y^2 has an (upper-right) snapback repeller $\dfrac{2}{1+k}$ as

$$k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}.$$

By Theorem 1.50, Y^2 is chaotic in the Devaney sense. Then, from Theorem 1.25, $h_{\text{top}}(Y^2) > 0$. Since $h_{\text{top}}(Y^2) = 2 \cdot h_{\text{top}}(Y)$ by Theorem 1.27, $h_{\text{top}}(Y) > 0$. From Theorem 1.25, Y is chaotic in Devaney's sense as

$$k > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}.$$

$\square$

Remark 3.6. *Let* $k_0 = \dfrac{1+\sqrt{5}}{2} \approx 1.6180$ *and* $k_1 = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.3247$. *The term* k_p *will be used later; it denotes the critical point with respect to the system (2) under the iterative number* 2^p.

In the above theorem, the "snapback repeller method" is used when the iterative number is two to find k_1 such that the system has Devaney's chaos when $k > k_1$. Next, when k is between 1 and k_1, a snapback repeller is still required as the iterative number increases. However, when iterative number of Y is odd and $1 < k \leq k_1$, the system has only one fixed point, and the "snapback repeller method" fails, but when iterative number of Y is even but not two to the power of any natural number, Theorem 3.5 can not be improved upon. Therefore, the last case to be considered is that in which the iterative number is two to the power of any natural number in system (2).

Theorem 3.7. *For* $1 < k \leq k_1$*, the system is chaotic in Devaney's sense.*

Proof. When the iterative number is two to the power of any natural number, the general form of Y^{2^p} can be represented to

$$Y^{2^p}(x) = \begin{cases} L_{2^p}(x), & x \in \left[C_{p-3}\left(\frac{1}{k}\right), C_{p-2}\left(\frac{1}{k}\right)\right], \\ R_{2^p}(x), & x \in \left[C_{p-2}\left(\frac{1}{k}\right), 1\right], \end{cases}$$

where $p \in \mathbb{N}$,

$$L^{2^p}(x) = \begin{cases} -kR_{2^p}(x) + k + 1, & p \text{ is odd}, \\ \dfrac{-R_{2^p}(x) + k + 1}{k}, & p \text{ is even}, \end{cases}$$

$$R^{2^p}(x) = L_{2^{p-1}}R_{2^{p-1}}(x),$$

and $j \in \mathbb{N}$

$$C_j = \begin{cases} C_{j-1}AAC_{j-1}, & j \text{ is odd}, \\ C_{j-1}BC_{j-1}, & j \text{ is even} \end{cases}$$

with $C_0 = C_{-1}BC_{-1}$, where $C_{-1}(x) \equiv x$ and $C_{-2} \equiv 0$. It is similar to Theorem 3.5 in which the snapback repeller of system (2) can be found by numerical computation for $p \geq 2$.

p	k_p
0	**1.6180**33988749895 $(\equiv k_0)$
1	**1.3247**17957244745 $(\equiv k_1)$
2	**1.1347**24138401520 $(\equiv k_2)$
3	**1.0682**97188920740 $(\equiv k_3)$
4	**1.0327**70966453956 $(\equiv k_4)$
5	**1.0164**43864419055 $(\equiv k_5)$
6	**1.0081**40050503278 $(\equiv k_6)$
7	**1.0041**60992268882 $(\equiv k_7)$
8	**1.0036**64292317828 (C1)
9	**1.0037**95792338565 (C2)

Table 1: When $k > k_p$, system (2) is chaotic in Devaney's sense as determined by numerical computation in **Matlab**. The result of Y^1 and Y^2 in Theorem 3.5 is above dotted line, and the iterative number n greater than two is below the dotted line.

Therefore, do the same steps as in Theorem 3.5 are used, and **Matlab** and **Maple** are used

p	k_p
0	**1.61803**3988749894848204586834365638117720309179805762862135448622705261
1	**1.32471**7957244746025960908854478097340734404056901733364534015050302828
2	**1.13472**4138401519492605446054506472840279667226382801485925149551668237
3	**1.06829**7188920841276369429588323878282093631016920833444507611946647007
4	**1.03277**0966441042909329492888334744856652058371140403253917031540208661
5	**1.01644**3864059417072092280201941787277910662321454134609733959043245535
6	**1.00814**0032021166342336675311408118208893644908964048997902342844304787
7	**1.00407**3666388692740274952354135845754211121309836120298287534443071976
8	**1.00203**1776333416997088893271971142972647918937489170894541068546238239
9	**1.00101**6116350239987853959635630193675245706270323947596435520337219342
10	**1.00050**7743074500114948189347177723859179135821018512700930688524462566
11	**1.00025**3885799306497646948038000941319259507014651397354037337963961327

Table 2: When $k > k_p$, system (2) is chaotic in Devaney's sense as determined by numerical computation in **Maple**. The result of Y^1 and Y^2 in Theorem 3.5 is above dotted line, and the iterative number n greater than two is below the dotted line.

to perform numerical computation and establish the following table to determine the iteration and the regions of k where the system is chaotic in Devaney's sense. Including the result of Y^1 and Y^2 in Theorem 3.5, we have Table 1 and Table 2 (see **Appendix C**).

The k's regions of Y^{2^p} are found by determining the roots of polynomial with degree 2^{p+1}. However, in Table 1, since the limitation of computer's binary representation only four bytes in **Matlab**, the results of $(C1)$ and $(C2)$ have large errors. Hence, in Table 2, the representation extended to 100 digits to reveal more accurate results in **Maple**.

In the numerical analysis, the speed at which a convergent sequence approaches its limit is called the rate of convergence. A sequence k_p converges linearly to L if there exists a number $M \in (0,1)$ such that $\lim_{p\to\infty} \dfrac{k_{p+1}-L}{k_p-L} = M$, and the number M is called the rate of convergence [81]. The sequence k_p is provided in Table 2, and the value $\dfrac{k_{p+1}-L}{k_p-L}$ is presented in Table 3. Decreasing trend point of view of k_1, k_1,..., k_p in Table 2 and the result in Table 3 demonstrate that the sequence k_p converges linearly to the greatest lower bound 1 at a rate of convergence of 0.5. Hence, the system (2) is necessarily chaotic as long as $k > 1$. A computer

p	$(k_{p+1}-1)/(k_p-1)$
0	**0.52540**469157943422769003322478664651351376649047830696763492605159518883
1	**0.41489**586699997431299925477792765572121310751072840368371184368616710443
2	**0.50694**099610638878113952349805380330694715113829015656080921537994903761
3	**0.47982**892061671168115785962262331467488250827627624491318606863118921204
4	**0.50178**148053706772234152456827894503210609893138022208365602874183801158
5	**0.49501**941829205944983109846713194236053127665283014625698323313050315463
6	**0.50044**844763510476499969515141640000606898362052434282373474249083429693
7	**0.49875**864627908427712294912377763749255500672526772548623019825923679742
8	**0.50011**230740694058357812570761070978087582752392863114119242456520421925
9	**0.49968**989710695575574326476020722865297776557761856147868459186532735834
10	**0.50002**808912057385358698942088743772381022001594295705733873847542234191

Table 3: Rate of convergence of k_p.

that can manipulate a number with more digits and that has a larger memory can yield more accurate result. □

Therefore, Theorem 3.5 and Theorem 3.7 prove that Satake's generalized resource budget model is chaotic in the sense of Devaney when the depletion coefficient $k > 1$. This section mathematically interprets the dynamics of system (2) when $k > 1$. The next section will analyze Satake's model by calculation for $k > 1$.

4 Numerical Simulation

The bifurcation diagram (Figure 4) of system (2) with iterations given by the same random initial values that the theoretical results of Proposition 3.2–3.4 satisfy for $k \leq 1$. For $k > 1$, Theorem 3.5 and Theorem 3.7 yield rigorous mathematical and numerical results that show that system (2) is chaotic in Devaney's sense. However, the system (2) eventually converges to periodic points when the initial value is a rational number and the depletion coefficient is a natural number.

Theorem 4.1. *For any initial value $x \in \mathbb{Q}$ and $k \in \mathbb{N}$, $Y^{(t)}(x)$ is eventually periodic.*

Proof. Without loss of generality, $x \in \mathbb{Q} \cap [-k+1, 1]$. Let $x = \dfrac{q}{p} \in \mathbb{Q}$ with $p \in \mathbb{N}$ and $q \in \mathbb{Z}$.

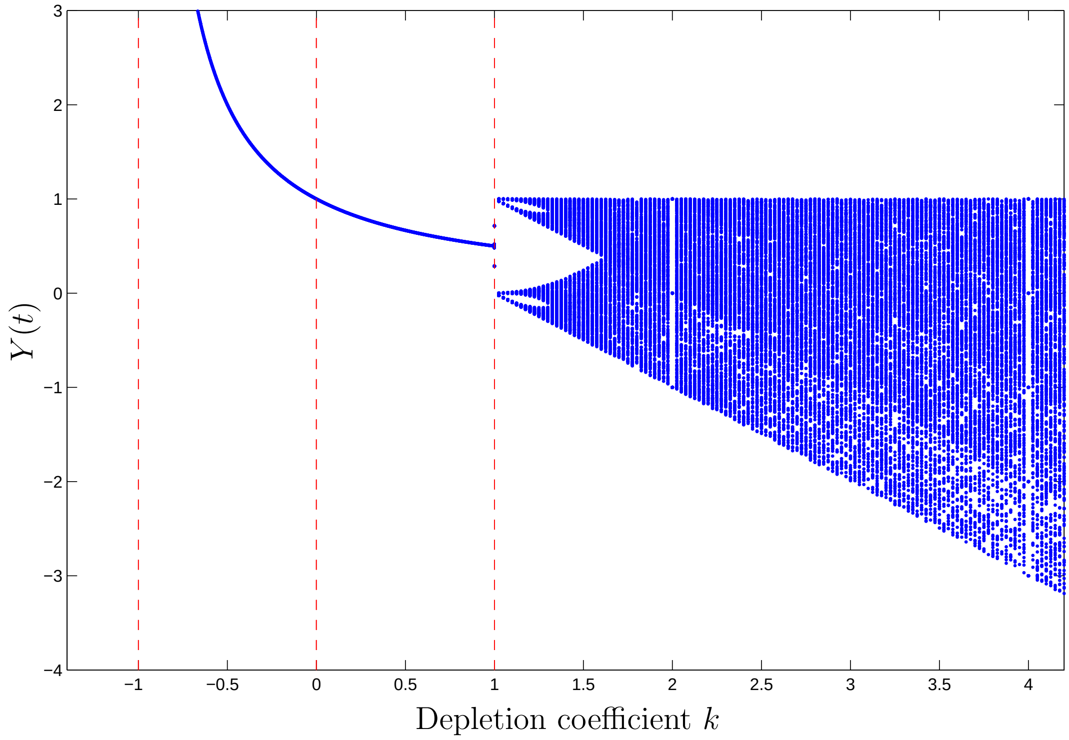

Figure 4: Bifurcation diagram of a single tree. The horizontal axis represents the depletion coefficient k, and the vertical axis represents $Y^{(t)}$ for many units of time t.

Suppose $S = \left\{ \frac{j}{p} \in [-k+1, 1] \,\middle|\, j \in \mathbb{Z} \right\}$, then $x = \frac{q}{p} \in S$. The cardinality of S is denoted by $|S|$, and

$$\begin{aligned} |S| &= \left| \left\{ \frac{j}{p} \in [-k+1, 1] \,\middle|\, j \in \mathbb{Z} \right\} \right| \\ &= k \left| \left\{ \frac{j}{p} \in (0, 1] \,\middle|\, j \in \mathbb{Z} \right\} \right| + 1 \\ &= kp + 1. \end{aligned}$$

For any $x = \frac{j}{p} \in S$,

$$Y(x) = \begin{cases} \dfrac{j}{p} + 1 = \dfrac{j+p}{p}, & \text{if } x \in [-k+1, 0], \\ \dfrac{(-k)j}{p} + 1 = \dfrac{(-k)j + p}{p}, & \text{if } x \in (0, 1], \end{cases}$$

then $Y(x) \in S$, that implies $Y(S) \subseteq S$. Hence, $Y^n(x) \in S$ for all $n \in \mathbb{N}$ as $x = \frac{j}{p}$, and

$$S_1 \equiv \{Y^1(x), Y^2(x), \ldots, Y^{kp+2}(x)\} \subseteq S.$$

Since $S_1 \subseteq S$ and $|S| = kp + 1$, $|S_1| \leq |S|$ and there exists $Y^i(x) \in S$, for some i such that

$Y^i(x) = Y^{kp+2}(x)$ derived from the Pigeonhole Principle. It implies that the system has a periodic solution with the period at most $kp + 2 - i$. □

According to Theorem 4.1, when the initial value x is a rational number and k is a natural number, the initial value eventually converges to a periodic point independently of k. It is no doubt that x only can be expressed using finite digits in binary representation in the computer. Therefore, for any simulation in the computer the initial value is always a rational number such that system (2) eventually goes to a periodic solution under $k \in \mathbb{N}$.

Satake et al. [79] used the stochastic variable and the probability distribution density to elucidate in which situation k is a natural number. They assumed a stable distribution to show that the system converges to the periodic cycle $\{-k+1, \ldots, 0, 1\}$ for all k is an odd number or an even number.

In Figure 5 (Figure 4), the system indeed converges to a periodic point with period $k+1$ and the periodic cycle is $\{-k+1, \ldots, 0, 1\}$ when k is a positive even number (see Figure 5 (a) & (c)). This means that the even number under the computer's binary representation lets initial value x to carry that it converges to a "lower" period. However, the behavior is not like "lower" periodic when k is a positive odd number (see Figure 5 (b) & (d)). Moreover, whether the distribution that was proposed by Satake et al. is in fact stable is herein unknown. This section proposes an well explanation.

The following theorem show that why system (2) converges to a periodic point with period $k+1$, where the periodic cycle is $\{-k+1, -k+2, \ldots, 0, 1\}$ when k is a positive even number.

Theorem 4.2. *Under a binary representation with finite digits, if k is a positive even number, then $Y^{(t)}$ converges to the periodic cycle $\{-k+1, -k+2, \ldots, 0, 1\}$ with period $k+1$.*

Proof. Assume $k = 2n$, $n \in \mathbb{N}$. Let $x = 0.x_1x_2 \cdots x_p \in (0, 1)$ with $x_i \in \{0, 1\}$. Since

$$-kx + 1 = -(2n)x + 1 = -2(nx) + 1 = -2y + 1,$$

where $y = nx = y_1.y_2 \cdots y_py_{p+1}$ and a binary representation with a number of finite digits is

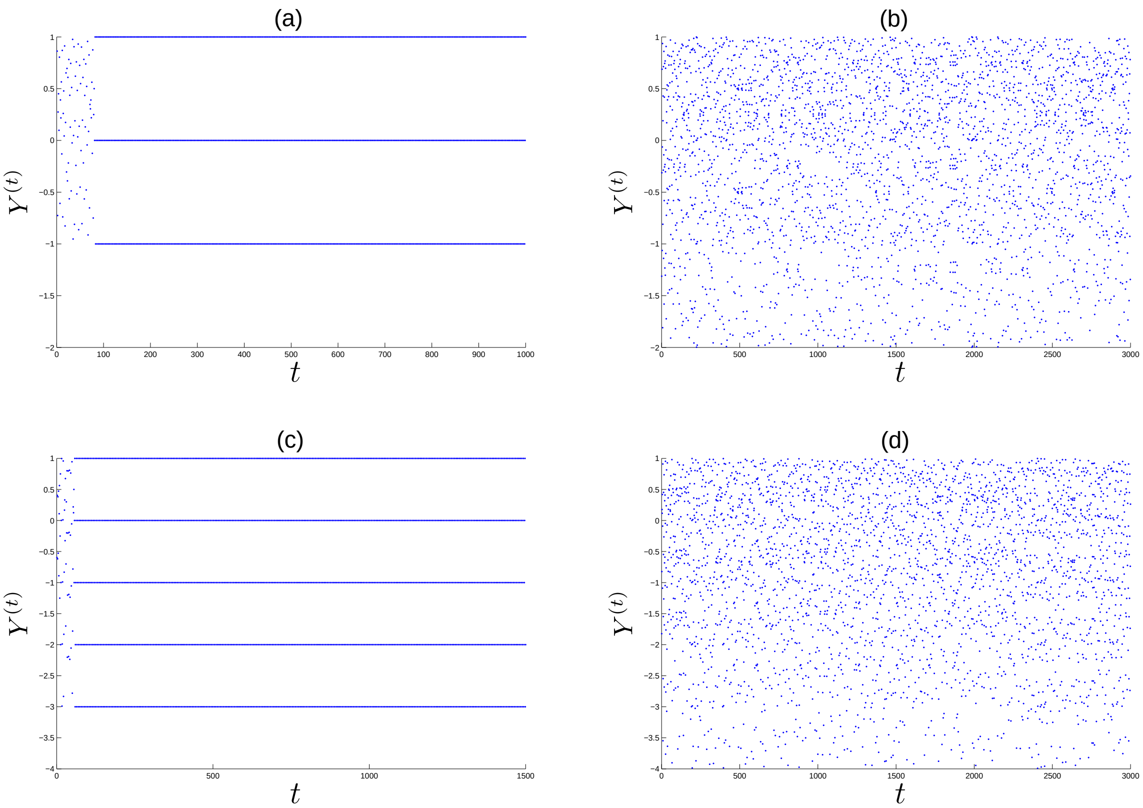

Figure 5: Iterative number t v.s $Y^{(t)}$ when k is a positive even number. The system converges to a lower periodic cycle with period $k+1$; however, when k is a positive odd number, the dynamics are not lower periodic. (a) $k=2$; (b) $k=3$; (c) $k=4$; (d) $k=5$.

used, only $k=2$ can be considered. Hence,

$$
\begin{aligned}
Y(x) &= -x_1.x_2x_3\cdots x_p0+1 \\
&= \begin{cases} -0.x_2x_3\cdots x_p0+1, & \text{if } x_1=0, \\ -1.x_2x_3\cdots x_p0+1, & \text{if } x_1=1 \end{cases} \\
&= \begin{cases} 0.(1-x_2)\cdots(1-x_{p-1})(2-x_p)\underbrace{0}_{1\ zero}, & \text{if } x_1=0, \\ -0.x_2x_3\cdots x_p0, & \text{if } x_1=1. \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Y^2(x) &= \begin{cases} -(1-x_2).(1-x_3)\cdots(1-x_{p-1})(2-x_p)00+1, & \text{if } x_1=0, \\ 0.(1-x_2)(1-x_3)\cdots(1-x_{p-1})(2-x_p)0, & \text{if } x_1=1 \end{cases} \\
&= \begin{cases} -0.(1-x_3)\cdots(1-x_{p-1})(2-x_p)00, & \text{if } x_1=0 \text{ and } x_2=0, \\ 0.x_3x_4\cdots x_p\underbrace{00}_{2\ zeros}, & \text{if } x_1=0 \text{ and } x_2=1, \\ 0.(1-x_2)\cdots(1-x_{p-1})(2-x_p)\underbrace{0}_{1\ zero}, & \text{if } x_1=1, \end{cases}
\end{aligned}
$$

$$Y^3(x) = \begin{cases} 0.x_3x_4\cdots x_p00, & \text{if } x_1 = 0 \text{ and } x_2 = 0, \\ -(1-x_2).(1-x_3)\cdots(1-x_{p-1})(2-x_p)00+1, & \text{if } x_1 = 1 \end{cases}$$

$$= \begin{cases} 0.x_3x_4\cdots x_p \underbrace{00}_{2\ zeros}, & \text{if } x_1 = 0,\ x_2 = 0 \ \text{ or } \ x_1 = 1,\ x_2 = 1, \\ -0.(1-x_3)\cdots(1-x_{p-1})(2-x_p)00, & \text{if } x_1 = 1 \text{ and } x_2 = 0, \end{cases}$$

and

$$Y^4(x) = 0.x_3x_4\cdots x_p \underbrace{00}_{2\ zeros}, \qquad \text{if } x_1 = 1 \text{ and } x_2 = 0.$$

Hence, there exists positive integers n_1 and n_2 such that

$$Y^{n_1}(x) = 0.(1-x_2)\cdots(1-x_{p-1})(2-x_p)\underbrace{0}_{1\ zeros} \quad \text{and} \quad Y^{n_2}(x) = 0.x_3x_4\cdots x_p\underbrace{00}_{2\ zeros}.$$

Suppose

$$Y^{n_k}(x) = \begin{cases} 0.\underbrace{(1-x_{k+1})\cdots(1-x_{p-1})(2-x_p)}_{(p-k)\ terms}\underbrace{00\cdots0}_{k\ zeros}, & \text{if } k \in 2\mathbb{N}-1, \\ 0.\underbrace{x_{k+1}x_{k+2}\cdots x_p}_{(p-k)\ terms}\underbrace{00\cdots0}_{k\ zeros}, & \text{if } k \in 2\mathbb{N}. \end{cases}$$

Then

$$\begin{aligned} Y^{n_k+1}(x) &= Y(Y^{n_k}(x)) \\ &= \begin{cases} -1.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}+1, & \text{if } k \in 2\mathbb{N}-1 \text{ and } x_{k+1} = 0, \\ -0.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}+1, & \text{if } k \in 2\mathbb{N}-1 \text{ and } x_{k+1} = 1, \\ -0.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}+1, & \text{if } k \in 2\mathbb{N} \text{ and } x_{k+1} = 0, \\ -1.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}+1, & \text{if } k \in 2\mathbb{N} \text{ and } x_{k+1} = 1 \end{cases} \\ &= \begin{cases} -0.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N}-1 \text{ and } x_{k+1} = 0, \\ 0.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N}-1 \text{ and } x_{k+1} = 1, \\ 0.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N} \text{ and } x_{k+1} = 0, \\ -0.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\ \underbrace{00\cdots0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N} \text{ and } x_{k+1} = 1, \end{cases} \end{aligned}$$

$$Y^{n_k+2}(x) = Y(Y^{n_k+1}(x))$$
$$= \begin{cases} 0.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\underbrace{00\cdots 0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N}-1 \text{ and } x_{k+1}=0, \\ 0.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\underbrace{00\cdots 0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N} \text{ and } x_{k+1}=1. \end{cases}$$

Hence, there is a $n_{k+1} \in \mathbb{N}$ such that

$$Y^{n_{k+1}}(x) = \begin{cases} 0.\underbrace{x_{k+2}x_{k+3}\cdots x_p}_{(p-k-1)\ terms}\underbrace{00\cdots 0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N}-1, \\ 0.\underbrace{(1-x_{k+2})\cdots(1-x_{p-1})(2-x_p)}_{(p-k-1)\ terms}\underbrace{00\cdots 0}_{(k+1)\ zeros}, & \text{if } k \in 2\mathbb{N}. \end{cases}$$

By mathematical induction,

$$Y^{n_j}(x) = \begin{cases} 0.\underbrace{(1-x_{j+1})\cdots(1-x_{p-1})(2-x_p)}_{(p-j)\ terms}\underbrace{00\cdots 0}_{j\ zeros}, & \text{if } j \in 2\mathbb{N}-1, \\ 0.\underbrace{x_{j+1}x_{j+2}\cdots x_p}_{(p-j)\ terms}\underbrace{00\cdots 0}_{j\ zeros} & \text{if } j \in 2\mathbb{N} \end{cases}$$

for all $j \in \mathbb{N}$ with $j \leq p$. When $j = p$, $Y^{n_p}(x) = 0.\underbrace{00\cdots 0}_{p\ zeros}$. Therefore, $Y(x)$ converges to the periodic cycle $\{-k+1, -k+2, \ldots, 0, 1\}$. □

Moreover, when the initial x could be represented in base-β with a finite number of digits, system (2) converges to periodic cycle $\{-k+1, \ldots, 0, 1\}$ under $k = \beta m$ with $m \in \mathbb{N}$ ($\beta \in \mathbb{N}$).

However, when k is a odd number, the following theorem explains that system (2) cannot converge to the periodic point with periodic cycle $\{-k+1, -k+2, \ldots, 0, 1\}$.

Theorem 4.3. *Under the binary representation with a finite number of digits, if k is a positive odd number, then $Y^{(t)}$ can not converge to the periodic cycle $S \equiv \{-k+1, -k+2, \ldots, 0, 1\}$ as the initial value $x \notin S$.*

Proof. Let x be the same as in Theorem 4.2 and consider only $k = 3$. Since $x \neq 0$, let the first nonzero digits be x_m, $1 \leq m \leq p$. That is,

$$x = 0.x_1 \cdots x_{m-1}x_m \underbrace{0\cdots 0}_{(p-m)\ zeros} = 0.x_1 \cdots x_{m-1}1 \underbrace{0\cdots 0}_{(p-m)\ zeros}.$$

Hence,

$$\begin{aligned}
Y(x) &= -x_1.(x_1+x_2)(x_2+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1 \\
&= \begin{cases} -0.x_2(x_2+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1, & \text{if } x_1=0, \\ -1.(1+x_2)(x_2+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1, & \text{if } x_1=1 \end{cases} \\
&= \begin{cases} 0.(1-x_2)(1-(x_2+x_3))\cdots(1-(1+x_{m-1}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=0, \\ -0.(1+x_2)(x_2+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=1 \end{cases} \\
&= \begin{cases} 0.(1-x_2)(1-(x_2+x_3))\cdots(1-(1+x_{m-1}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=0, \\ -0.1(x_2+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=1 \text{ and } x_2=0, \\ -1.0(1+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=1 \text{ and } x_2=1, \end{cases}
\end{aligned}$$

$$Y^2(x) = \begin{cases} 0.0(1-(x_2+x_3))\cdots(1-(1+x_{m-1}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=1 \text{ and } x_2=0, \\ -0.0(1+x_3)\cdots(1+x_{m-1})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1=1 \text{ and } x_2=1, \end{cases}$$

and

$$Y^3(x) = 0.1(1-(1+x_3))\cdots(1-(1+x_{m-1}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad \text{if } x_1=1 \text{ and } x_2=1.$$

That is, there exists an $n_1 \in \mathbb{N}$ such that

$$Y^{n_1}(x) = 0.x_1^{[n_1]}x_2^{[n_1]}\cdots x_{m-1}^{[n_1]}x_m^{[n_1]}\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad x_i^{[n_1]}\in\{0,1\}, \quad i\neq m,$$

whose first nonzero digit $x_m^{[n_1]}$ is also 1. Assume

$$Y^{n_k}(x) = 0.x_1^{[n_k]}x_2^{[n_k]}\cdots x_{m-1}^{[n_k]}x_m^{[n_k]}\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad x_i^{[n_k]}\in\{0,1\}, \quad i\neq m,$$

whose first nonzero digit $x_m^{[n_k]}$ is also 1, then

$$
\begin{aligned}
Y^{n_k+1}(x) &= -x_1^{[n_k]}.(x_1^{[n_k]}+x_2^{[n_k]})(x_2^{[n_k]}+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1 \\
&= \begin{cases} -0.x_2^{[n_k]}(x_2^{[n_k]}+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1, & \text{if } x_1^{[n_k]}=0, \\ -1.(1+x_2^{[n_k]})(x_2^{[n_k]}+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}+1, & \text{if } x_1^{[n_k]}=1 \end{cases} \\
&= \begin{cases} 0.(1-x_2^{[n_k]})(1-(x_2^{[n_k]}+x_3^{[n_k]}))\cdots(1-(1+x_{m-1}^{[n_k]}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=0, \\ -0.(1+x_2^{[n_k]})(x_2^{[n_k]}+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=1 \end{cases} \\
&= \begin{cases} 0.(1-x_2^{[n_k]})(1-(x_2^{[n_k]}+x_3^{[n_k]}))\cdots(1-(1+x_{m-1}^{[n_k]}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=0, \\ -0.1(x_2^{[n_k]}+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=1 \text{ and } x_2^{[n_k]}=0, \\ -1.0(1+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=1 \text{ and } x_2^{[n_k]}=1, \end{cases}
\end{aligned}
$$

$$
Y^{n_k+2}(x) = \begin{cases} 0.0(1-(x_2^{[n_k]}+x_3^{[n_k]}))\cdots(1-(1+x_{m-1}^{[n_k]}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=1 \text{ and } x_2^{[n_k]}=0, \\ -0.0(1+x_3^{[n_k]})\cdots(1+x_{m-1}^{[n_k]})1\underbrace{0\cdots0}_{(p-m)\ zeros}, & \text{if } x_1^{[n_k]}=1 \text{ and } x_2^{[n_k]}=1, \end{cases}
$$

and

$$
Y^{n_k+3}(x) = 0.1(1-(1+x_3^{[n_k]}))\cdots(1-(1+x_{m-1}^{[n_k]}))1\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad \text{if } x_1^{[n_k]}=1 \text{ and } x_2^{[n_k]}=1.
$$

Hence, there exists an $n_{k+1}\in\mathbb{N}$ such that

$$
Y^{n_{k+1}}(x) = 0.x_1^{[n_{k+1}]}x_2^{[n_{k+1}]}\cdots x_{m-1}^{[n_{k+1}]}x_m^{[n_{k+1}]}\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad x_i^{[n_{k+1}]}\in\{0,1\}, \quad i\neq m,
$$

whose first nonzero digit $x_m^{[n_{k+1}]}$ is also 1. By mathematical induction,

$$
Y^{n_j}(x) = 0.x_1^{[n_j]}x_2^{[n_j]}\cdots x_{m-1}^{[n_j]}x_m^{[n_j]}\underbrace{0\cdots0}_{(p-m)\ zeros}, \quad x_i^{[n_j]}\in\{0,1\}, \quad i\neq m,
$$

its first nonzero digit $x_m^{[n_j]}$ is still 1 for all $j\in\mathbb{N}$. That is, no $n\in\mathbb{N}$ exists such that $Y^n(x)=0$. Hence, Y can not converge to the periodic cycle $\{-k+1,-k+2,\ldots,0,1\}$. □

Therefore, from Theorem 4.3, even though x is represented in binary using a finite number of digits in computers, Y does not converge to the periodic cycle $\{-k+1,-k+2,\ldots,0,1\}$ when k is an odd number. The above theorem revises the statement of integers k in a investigation [79].

5 Conclusions

The relationship among chaos, Lyapunov exponent, topological entropy, strange attractors, and snapback repellers is elucidated. The conditions under which they are equivalent to each other or imply each other are identified.

Satake and Iwasa proved that the generalized budget resource model is chaotic when $k > 1$ by computing the Lyapunov exponent, but their proof was not clear. However, the model has no periodic points with period three when $k < \sqrt{2}$, so we can not to prove that it is chaotic by the existence of periodic points with period three. Therefore the existence of snapback repellers was used to prove theoretically and numerically that the model exhibits Devaney's chaos when $k > 1$.

Any rational number eventually converges to a periodic point when k is a natural number, and the periodic point are found. Under the binary representation with a finite number of digits, if k represents an even number, then Y converges to the periodic cycle $\{-k+1, -k+2, \ldots, 0, 1\}$ with period $k+1$, but if k represents an odd number, then Y cannot converge to the same periodic cycle even if its points are rational numbers.

Appendix A

Consider the following ordering of the natural numbers:

$$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \cdots$$
$$\triangleright 2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright 2^3 \cdot 7 \triangleright \cdots \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$

That is, first list all odd numbers except one, followed by 2 times the odds, 2^2 times the odds, 2^3 times the odds, etc. This exhausts all the natural numbers with the exception of the powers of two which we list last, in decreasing order. This is the Sarkovskii's ordering of the natural numbers. This ordering allows us to state Sarkovskii's Theorem.

Theorem A.1 (Sarkovskii's Theorem [84]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose f has a periodic point of prime period k. If $k \triangleright m$ in the Sarkovskii's ordering, then f also has a periodic point of period m.*

In above theorem, period 3 is the greatest period in the Sarkovskii's ordering and therefore implies the existence of all other periods, Devaney, Li and Yorke have the same result. Furthermore, Li and Yorke proved that if a map has a periodic point with period 3, then the map has Li-Yorke chaos.

Theorem A.2 ([21, 55]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point with period 3, then*

(1) *for every $k = 1, 2, \ldots$, there is a periodic point having period k;*

(2) *f is chaotic in the sense of Li-Yorke.*

Therefore, from Theorem A.1 and Theorem A.2, if f has only finitely many periodic points, then they all necessarily have periods which are powers of two. Conversely, if f has a periodic point whose period is not a power of two, then f necessarily has infinitely many period points.

Definition A.3 ([21]). *Let $f(p^*) = p^*$ and $f'(p^*) > 1$. A point x_0 is called* ***homoclinic*** *to p^* if $x_0 \in W^u_{loc}(p^*) = \{$the maximal such open interval about $p^*\}$ and there exists $n > 0$ such that $f^n(x_0) = p^*$.*

Remark A.4. *If p^* has a homoclinic point, then p^* is also called a snapback repeller.*

Definition A.5 ([21]). *A homoclinic point, together with its backward orbit and forward orbit, is called a* ***homoclinic orbit****. A homoclinic orbit is called* ***nondegenerate*** *if $f'(x) \neq 0$, for all points x on the orbit.*

Theorem A.6 ([21]). *Suppose f admits a nondegenerate homoclinic point to p^*. Then, every neighborhood of p^* contains infinitely many distinct periodic points.*

Comparing Theorem A.2 with Theorem A.6, if f has a periodic point whose period is not a power of two or f has a nondegenerate homoclinic point, then f necessarily has infinitely many period points. The relationship of periodic point whose period is not a power of two and homoclinic point is given by the following theorem.

Theorem A.7 ([11]). *f has a periodic point whose period is not a power of two if and only if f has a homoclinic point.*

From the above theorem, if f has homoclinic point, then f has a periodic point whose period is not a power of two. Bowen and Franks have shown in [14] that if f has a periodic point whose period is not a power of two, then the topological entropy of f is positive. Therefore, from Theorem 1.25, f is chaotic in the sense of Devaney. Naturally we have the following theorem.

Theorem A.8. *If f has a periodic point with period 3, then f has Devaney's chaos.*

Hence, combining the result of Theorem A.2 and Theorem A.8, we know that if f has a periodic point with period 3, then it is Devaney's chaotic and Li-Yorke's chaotic.

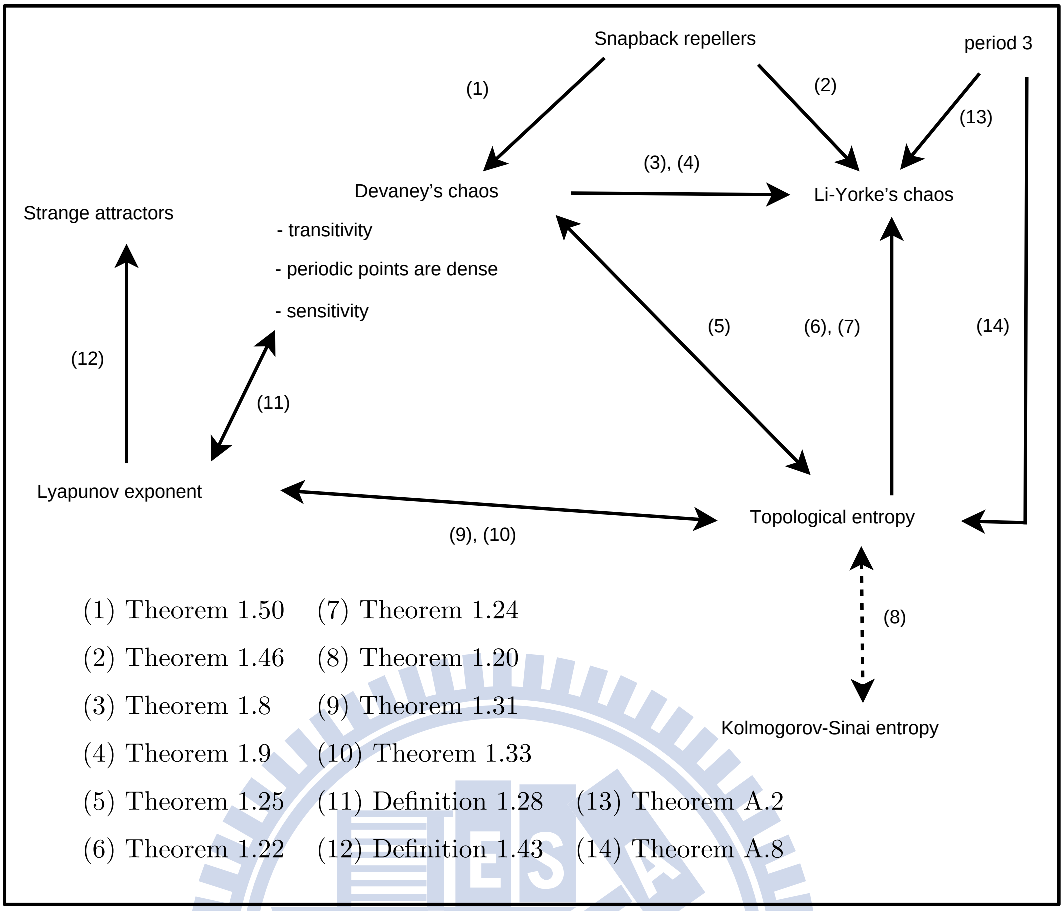

Figure 6: Relational graph of chaos and relative checking methods including period three.

Appendix B

In mathematics, a **cubic function** has the form

$$f(x) = ax^3 + bx^2 + cx + d,$$

where $a \neq 0$. If $f(x) = 0$, then the equation is called a **cubic equation**. The cubic equation will be solved exactly here [42, 66].

Suppose

$$x = y - \frac{b}{3a}. \tag{5}$$

Now,

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0. \tag{6}$$

Expanding and simplifying (6), yields

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0. \tag{7}$$

Converting (7) into the form

$$y^3 + ey + f = 0, \tag{8}$$

where

$$e = \frac{1}{a}\left(c - \frac{b^2}{3a}\right) \quad \text{and} \quad f = \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right).$$

Let

$$y = z - \frac{e}{3z}. \tag{9}$$

Substituting this equation into (8),

$$z^6 + fz^3 - \frac{e^3}{27} = 0.$$

One more substitution, $w = z^3$, yields the quadratic equation

$$w^2 + fw - \frac{e^3}{27} = 0. \tag{10}$$

Solving the quadratic equation (10),

$$w = z^3 = -\frac{f}{2} \pm \sqrt{\frac{f^2}{4} + \frac{e^3}{27}}. \tag{11}$$

Six values of z can be determined from (11), since a square root has two possible values ($\pm$), and a cubic root has three. The six possible values of z yield the six possible values of y (Equation (9)), but the three values of y will be identical to the other three. Therefore, three values of y, and three values of x (Equation (5)) are obtained. The three roots are

$$x_1 = -\frac{b}{3a} + \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}},$$

$$x_2 = -\frac{b}{3a} + \frac{-1 + \sqrt{3}\mathrm{i}}{2}\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \frac{-1 - \sqrt{3}\mathrm{i}}{2}\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}},$$

$$x_3 = -\frac{b}{3a} + \frac{-1 - \sqrt{3}\mathrm{i}}{2}\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \frac{-1 + \sqrt{3}\mathrm{i}}{2}\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}.$$

Appendix C

First, compute Y^{2^p}, $p = 2, 3, 4$, and 5. Choose the polynomial that passes through upper-right periodic point, and select its left polynomial at the same time. Then

$$Y^2(x) = \begin{cases} k^2x - k + 1, & x \in \left[0, \frac{1}{k}\right], \\ -kx + 2, & x \in \left[\frac{1}{k}, 1\right], \end{cases}$$

$$Y^{2^2}(x) = \begin{cases} k^2x - 2k + 2, & x \in \left[\frac{1}{k}, B\left(\frac{1}{k}\right)\right], \\ -k^3x + 2k^2 - k + 1, & x \in \left[B\left(\frac{1}{k}\right), 1\right], \end{cases}$$

$$Y^{2^3}(x) = \begin{cases} k^6x - 2k^5 + k^4 - k^3 + 2k^2 - k + 1, & x \in \left[B\left(\frac{1}{k}\right), \underbrace{BAAB}_{C_1}\left(\frac{1}{k}\right)\right], \\ -k^5x + 2k^4 - k^3 + k^2 - 2k + 2, & x \in \left[C_1\left(\frac{1}{k}\right), 1\right], \end{cases}$$

$$Y^{2^4}(x) = \begin{cases} k^{10}x - 2k^9 + k^8 - k^7 + 2k^6 - 2k^5 + 2k^4 - k^3 + k^2 - 2k + 2, \\ \qquad x \in \left[C_1\left(\frac{1}{k}\right), \underbrace{C_1BC_1}_{C_2}\left(\frac{1}{k}\right)\right], \\ -k^{11}x + 2k^{10} - k^9 + k^8 - 2k^7 + 2k^6 - 2k^5 + k^4 - k^3 + 2k^2 - k + 1, \\ \qquad x \in \left[C_2\left(\frac{1}{k}\right), 1\right], \end{cases}$$

and

$$Y^{2^5}(x) = \begin{cases} k^{22}x - 2k^{21} + k^{20} - k^{19} + 2k^{18} - 2k^{17} + 2k^{16} - k^{15} + k^{14} - 2k^{13} + k^{12} \\ -k^{11} + 2k^{10} - k^9 + k^8 - 2k^7 + 2k^6 - 2k^5 + k^4 - k^3 + 2k^2 - k + 1, \\ \qquad x \in \left[C_2\left(\frac{1}{k}\right), \underbrace{C_2AAC_2}_{C_3}\left(\frac{1}{k}\right)\right], \\ -k^{21}x + 2k^{20} - k^{19} + k^{18} - 2k^{17} + 2k^{16} - 2k^{15} + k^{14} - k^{13} + 2k^{12} - k^{11} \\ +k^{10} - 2k^9 + k^8 - k^7 + 2k^6 - 2k^5 + 2k^4 - k^3 + k^2 - 2k + 2, \\ \qquad x \in \left[C_3\left(\frac{1}{k}\right), 1\right], \end{cases}$$

where $A(x) = \frac{1}{k}(1-x)$, $B(x) = \frac{1}{k}(2-x)$. Next, to find the generated form Y^{2^p}, the following algorithm is established. Let $L_{2^1}(x) = k^2x - k + 1$, $R_{2^1}(x) = -kx + 2$, $R_{2^2}(x) = L_{2^1}R_{2^1}(x)$,

$L_{2^2}(x) = \dfrac{-R_{2^2}(x)+k+1}{k}$, $R_{2^3}(x) = L_{2^2}R_{2^2}(x)$, and $L_{2^3}(x) = -kR_{2^3}(x)+k+1$, then

$$Y^{2^1}(x) = \begin{cases} L_{2^1}(x), & x \in \left[0, \dfrac{1}{k}\right], \\ R_{2^1}(x), & x \in \left[\dfrac{1}{k}, 1\right], \end{cases}$$

$$Y^{2^2}(x) = \begin{cases} L_{2^2}(x), & x \in \left[\dfrac{1}{k}, B\left(\dfrac{1}{k}\right)\right], \\ R_{2^2}(x), & x \in \left[B\left(\dfrac{1}{k}\right), 1\right], \end{cases}$$

$$Y^{2^3}(x) = \begin{cases} L_{2^3}(x), & x \in \left[B\left(\dfrac{1}{k}\right), C_1\left(\dfrac{1}{k}\right)\right], \\ R_{2^3}(x), & x \in \left[C_1\left(\dfrac{1}{k}\right), 1\right], \end{cases}$$

and

$$Y^{2^p}(x) = \begin{cases} L_{2^p}(x), & x \in \left[C_{p-3}\left(\dfrac{1}{k}\right), C_{p-2}\left(\dfrac{1}{k}\right)\right], \\ R_{2^p}(x), & x \in \left[C_{p-2}\left(\dfrac{1}{k}\right), 1\right], \end{cases}$$

where $p \in \mathbb{N}$, $p \geq 4$,

$$L^{2^p}(x) = \begin{cases} -kR_{2^p}(x)+k+1, & p \text{ is odd}, \\ \dfrac{-R_{2^p}(x)+k+1}{k}, & p \text{ is even}, \end{cases}$$

$$R^{2^p}(x) = L_{2^{p-1}}R_{2^{p-1}}(x),$$

and $j \in \mathbb{N}$, $j \geq 2$,

$$C_j = \begin{cases} C_{j-1}AAC_{j-1}, & j \text{ is odd}, \\ C_{j-1}BC_{j-1}, & j \text{ is even}. \end{cases}$$

Therefore, the algorithm yields the general form Y^{2^p} in Theorem 3.7, and the "snapback repeller method" can be applied to the iterative number 2^p to identify the regions of k in which system (2) exhibits Devaney's chaos.

References

[1] H. D. I. Abarbanel, R. Brown, and P. Bryant. Computing the Lyapunov spectrum of a dynamical system from an observed time series. *Physical Review A*, 43(6):2787–2806, 1991.

[2] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Transactions of the American Mathematical Society*, 114(2):309–319, 1965.

[3] R. B. Allen and H. H. Platt. Annual seedfall variation in Nothofagus solandri (Fagaceae), New Zealand. *Oikos*, 57:199–206, 1990.

[4] L. Alseda, S. L. Kolyada, J. Llibre, and L. Snoha. Entropy and periodic points for transitive maps. *Transactions of the American Mathematical Society*, 351(4):1551–1573, 1999.

[5] S. Appanah. General flowering in the climax rain forests of South-east Asia. *J. Trop. Ecol.*, 1:225–240, 1985.

[6] P. S. Ashton, T. J. Givnish, and S. Appanah. Staggered flowering in the Dipterocarpaceae: new insights into floral induction and the evolution of mast fruiting in the aseasonal tropics. *Am. Nat.*, 132:44–66, 1988.

[7] IV D. Assaf and S. Gadbois. Definition of chaos. *American Mathematical Monthly*, 99(9):865, 1992.

[8] J .Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's definition of chaos. *American Mathematical Monthly*, 99(4):332–334, 1992.

[9] M. Barge and R. Swanson. Pseudo-orbits and topological entropy. *Proceedings of the American Mathematical Society*, 109(2):559–566, 1990.

[10] F. Blanchard, E. Glasner, S. F. Kolyada, and A. Maass. On Li-Yorke pairs. *J. Reine Angew. Math.*, 547:51–68, 2002.

[11] L. Block. Homoclinic points of mappings of the interval. *Proceedings of the American Mathematical Society*, 72(3):576–580, 1978.

[12] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Transactions of the American Mathematical Society*, 153:401–414, 1971.

[13] R. Bowen. Topological entropy for noncompact sets. *Transactions of the American Mathematical Society*, 184:125–136, 1973.

[14] R. Bowen and J. Franks. The periodic points of maps of the disc and the interval. *Topology*, 15:337–442, 1976.

[15] M. Brin and G. Stuck. *Introduction to dynamical systems.* Cambridge University Press, New York, 2002.

[16] G. Chen, S. B. Hsu, and J. Zhou. Snapback repellers as a cause of chaotic vibration of the wave equation with a van der pol boundary condition and energy injection at the middle of the span. *Journal of Mathematical Physics*, 39(12):6459–6489, 1998.

[17] K. M. Christensen and T. G. Whitham. Indirect herbivore mediation of avian seed dispersal in pinyon pine. *Ecology*, 72:534–542, 1991.

[18] B. J. Cole. Is animal behaviour chaotic? evidence from the activity of ants. *Proceedings: Biological Sciences*, 244(1311):253–259, 1991.

[19] R. T. Corlett. Flora and reproductive phenology of the rain forest at Bukit Timah, Singapore. *J. Trop. Ecol.*, 6:55–63, 1990.

[20] B. Dennis, R. A. Desharnais, J. M. Cushing, S. M. Henson, and R. F. Costantino. Estimating chaos and complex dynamics in an insect population. *Ecological Monographs*, 71(2):277–303, 2001.

[21] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems, Second Edition.* Addison-Wesley, Redwood City, Canada, 1989.

[22] P. Diamond. Chaotic behavior of systems of difference equations. *Internat. J. Systems Sci.*, 7:953–956, 1976.

[23] A. Dohtani. Occurrence of chaos in higher-dimensional discrete-time system. *SIAM Journal on Applied Mathematics*, 52(6):1707–1721, 1992.

[24] J. P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics*, 57(3):617–656, 1985.

[25] K. D. Elowe and W. E. Dodge. Factors affecting black bear reproductive success and cub survival. *J. Wildl. Manage.*, 53:962–968, 1989.

[26] G. A. Felhammer, T. P. Kilbane, and D. W. Sharp. Cumulative effect of winter on acorn yield and deer body weight. *J. Wildl. Manage.*, 53:292–295, 1989.

[27] P. P. Feret, R. E. Dreh, S. A. Merkle, and R. G. Oderwald. Flower abundance, premature acorn abscission, and acorn production in Quercus alba L. Botanical Gazette. 143:216–218, 1982.

[28] N. Franzova and J. Smítal. Positive sequence topological entropy characterizes chaotic maps. *Proceedings of the American Mathematical Society*, 112(4):1083–1086, 1991.

[29] P. Góra and A. Boyarsky. Computing the topological entropy of general one-dimensional maps. *Transactions of the American Mathematical Society*, 323(1):39–49, 1991.

[30] G. A. Gottwald and I. Melbourne. A new test for chaos in deterministic systems. *Proceedings: Mathematical, Physical and Engineering Sciences*, 460(2042):603–611, 2004.

[31] C. Grebogi, E. Ott, and J. A. Yorke. Chaos, strange attractors, and fractal Basin boundaries in nonlinear dynamics. *Science, New Series*, 238(4827):632–638, 1987.

[32] L. W. Gysel. A 10-year analysis of beechnut production and use in Michigan. *J. Wildl. Manag.*, 35:516–519, 1971.

[33] A. Hastings, C. L. Hom, S. Ellner, P. Turchin, and H. C. J. Godfray. Chaos in ecology: Is mother nature a strange attractor? *Annual Review of Ecology and Systematics*, 24:1–33, 1993.

[34] A. Hastings and T. Powell. Chaos in a three-species food chain. *Ecology*, 72(3):896–903, 1991.

[35] W. Huang and X. Ye. Devaney's chaos or 2-scattering implies Li-Yorke's chaos. *Topology and its Applications*, 117:259–272, 2002.

[36] R. A. Ims. The ecology and evolution of reproductive synchrony. *Trends Ecol. Evol.*, 5:135–140, 1990.

[37] R. A. Ims. On the adaptive value of reproductive synchrony as a predator-swamping strategy. *Am. Nat.*, 136:485–498, 1990.

[38] Y. Isagi, K. Sugimura, A. Sumida, and H. Ito. How does masting happen and synchronize? *J. theor. Biol.*, 187:231–239, 1997.

[39] D. H. Janzen. Seed predation by animals. *Ann. Rev. Ecol. Syst.*, 2:465–492, 1971.

[40] D. H. Janzen. Why bamboos wait so long to flower. *Ann. Rev. Ecol. Syst.*, 7:347–391, 1976.

[41] A. Katok. Lyapunov exponents, entropy, and periodic orbits for diffeomorphisms. *Publ. Math. I.H.E.S*, 51:137–173, 1980.

[42] A. Kaw. Exact solution to cubic equations. *Holistic Nuerical Methods Institute-nonlinear equations*, pages 03.02.1–03.02.10, 2009.

[43] H. Kawada and K. Maruyama. Effects of seed bearing of a natural beech (Fagus crenata Blume) forest on amount of litter fall and its nutrients. *Jpn. J. Ecol.*, 36:3–10, 1986.

[44] D. Kelly. The evolutionary ecology of mast seeding. *Trends Ecol. Evol.*, 9:465–471, 1994.

[45] W. D. Koenig, R. L. Mumme, W. J. Carmen, and M. T. Stanback. Acorn production by oaks in Central Coastal California: variation within and among years. *Ecology*, 75:99–109, 1994.

[46] S. F. Kolyada. Li-Yorke sensitivity and other concepts of chaos. *Ukrainian Mathematical Journal*, 56(8):1242–1257, 2004.

[47] S. F. Kolyada and L. Snoha. Some aspects of topological transitivity - a survey. *Grazer Math. Ber.*, 334:3–35, 1997.

[48] D. Kwietniak and M. Misiurewicz. Exact devaney chaos and entropy. *Qualitative Theory Dynamical Systems*, 6(95):169–179, 2005.

[49] C. Li and G. Chen. An improved version of the Marotto theorem. *Chaos, Solitons and Fractals*, 18:969–977, 2003.

[50] C. Li and G. Chen. Estimating the Lyapunov exponents of discrete systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 14(2):343–346, 2004.

[51] S. Li. ω-chaos and topological entropy. *Transactions of the American Mathematical Society*, 339(1):243–249, 1993.

[52] T. Y. Li. The story related to “Li-Yorke chaos”. *Mathmedia*, 12(3):13–16, 1988.

[53] T. Y. Li. 熵 (I). 數學傳播, 13(3):5–15, 1989.

[54] T. Y. Li. 熵 (II). 數學傳播, 13(4):35–40, 1989.

[55] T. Y. Li and J. A. Yorke. Period three implies chaos. *American Mathematical Monthly*, 82(10):985–992, 1975.

[56] Z. Li, Y. Shi, and W. Liang. Discrete chaos induced by heteroclinic cycles connecting repellers in Banach spaces. *Nonlinear Analysis*, 72(2):757–770, 2010.

[57] F. R. Marotto. Snap-back repellers imply chaos in $\mathbb{R}^n$. *J. Math. Anal. Appl.*, 63:199–223, 1978.

[58] F. R. Marotto. On redefining a snap-back repeller. *Chaos, Solitons and Fractals*, 25:25–28, 2005.

[59] M. Martelli. *Introduction to Discrete Dynamical Systems and Chaos*. John Wiley, 1999.

[60] M. Martelli, M. Dang, and T. Seph. Defining chaos. *Mathematics Magazine*, 71(2):112–122, 1998.

[61] R. M. May. Simple mathematical models with very complicated dynamics. *Nature*, 261:459–466, 1976.

[62] J. Milnor. On the concept of attractor. *Communications in Mathematical Physics*, 99:177–195, 1985.

[63] M. Misiurewicz. Horseshoes for mappings of the interval. *Bull. Acad. Polo. Sci. Ser. Sci. Math.*, 27:167–169, 1979.

[64] L. Mora and M. Viana. Abundance of strange attractors. *Acta Math.*, 171:1–71, 1993.

[65] A. Morozov, S. Petrovskii, and B. L. Li. Bifurcations and chaos in a predator-prey system with the Allee effect. *Proceedings: Biological Sciences*, 271(1546):1407–1414, 2004.

[66] R. W. D. Nickalls. A new approach to solving the cubic: Cardan's solution revealed. *The mathematical gazette*, 77:354–359, 1993.

[67] D. A. Norton and D. Kelly. Mast seeding over 33 years by Dacrydium cupressinum Lamb. (rimu) (podocarpaceae) in New Zealand: the importance of economies of scale. *Func. Ecol.*, 2:399–408, 1988.

[68] V. I. Oseledec. A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[69] C. C. Peng. Numerical computation of orbits and rigorous verification of existence of snapback repellers. *Chaos*, 17:013107, 2007.

[70] R. Pool. Chaos theory: How big an advance? *Science, New Series*, 245(4913):26–28, 1989.

[71] R. Pool. Is something strange about the weather? *Science, New Series*, 243(4896):1290–1293, 1989.

[72] G. E. Rehfeldt, A. R. Stage, and R. T. Bingham. Strobili development in western white pine: productivity, prediction, and association with water. *For. Sci.*, 17:454–461, 1971.

[73] A. L. Robinson. Physicists try to find order in chaos. *Science, New Series*, 218(4572):554–556, 1982.

[74] C. Robinson. *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, Second Edition*. CRC, Boca Raton, Florida, 1998.

[75] V. A. Rokhlin. New progress in the theory of transformations with invariant measure. *Russian Math. Surveys*, 15:1–22, 1960.

[76] D. Ruelle. Small random perturbations of dynamical systems and the definition of attractors. *Communications in Mathematical Physics*, 82:137–151, 1981.

[77] D. Ruelle and F. Takens. On the nature of turbulence. *Communications in Mathematical Physics*, 20:167–192, 1971.

[78] D. G. Saari. Chaos, but in voting and apportionments. *Proceedings of the National Academy of Sciences of the United States of America*, 96(19):10568–10571, 1999.

[79] A. Satake and Y. Iwasa. Pollen coupling of forest trees: Forming synchronized and periodic reproduction out of chaos. *J. theor. Biol.*, 203:63–84, 2000.

[80] W. M. Schaffer. Order and chaos in ecological systems. *Ecology*, 66(1):93–106, 1985.

[81] M. Schatzman. *Numerical analysis: a mathematical introduction*. Clarendon Press, Oxford, 2002.

[82] A. Serletis. Random walks, breaking trend functions, and the chaotic structure of the velocity of money. *Journal of Business and Economic Statistics*, 13(4):453–458, 1995.

[83] C. E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–423, 623–656, 1948.

[84] A. N. Sharkovskii. Coexistence of cycles of continuous mapping of the line into itself. *Ukrainian Math. J.*, 16:61–71, 1964.

[85] W. M. Sharp and V. G. Sprague. Flowering and fruiting in the white oaks. Pistillate flowering, acorn development, weather, and yields. *Ecology*, 48:243–251, 1967.

[86] J. A. Sherratt, M. A. Lewis, and A. C. Fowler. Ecological chaos in the wake of invasion. *Proceedings of the National Academy of Sciences of the United States of America*, 92(7):2524–2528, 1995.

[87] Y. Shi and G. Chen. Chaos of discrete dynamical systems in complete metric spaces. *Chaos, Solitons and Fractals*, 22:555–571, 2004.

[88] Y. Shi and G. Chen. Discrete chaos in Banach spaces. *Science in China Ser. A Mathematics*, 48(2):222–238, 2005.

[89] Y. Shi and P. Yu. Chaos induced by regular snap-back repellers. *J. Math. Anal. Appl.*, 337:1480–1494, 2008.

[90] Y. Shi, P. Yu, and G. Chen. Chaotification of discrete dynamical systems in Banach spaces. *Internat. J. Bifur. Chaos*, 16:2615–2636, 2006.

[91] J. W. Silvertown. The evolutionary ecology of mast seeding in trees. *Biol. J. Linnean. Soc.*, 14:235–250, 1980.

[92] J. Smítal. Chaotic functions with zero topological entropy. *Transactions of the American Mathematical Society*, 297(1):269–282, 1986.

[93] C. C. Smith, J. L. Hamrick, and C. L. Kramer. The advantage of mast years for wind pollination. *Am. Nat.*, 136:154–166, 1990.

[94] V. L. Sork, J. Bramble, and O. Sexton. Ecology of mast-fruiting in three species of North American deciduous oaks. *Ecology*, 74:528–541, 1993.

[95] L. Stone and S. Ezrati. Chaos, cycles and spatiotemporal dynamics in plant ecology. *The Journal of Ecology*, 84(2):279–291, 1996.

[96] F. Takens and E. Verbitski. Multifractal analysis of dimensions and entropies. *Regular and Chaotic Dynamics*, 5(24):361–382, 2000.

[97] M. Vellekoop and R. Berglund. On intervals, transitivity = chaos. *American Mathematical Monthly*, 101(4):353–355, 1994.

[98] S. Wiggins. *Chaotic Transport in Dynamical Systems*. Springer-Verlag, New York, 1991.

[99] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano. Determining Lyapunov exponents from a time series. *Physica*, 16D:285–317, 1985.

[100] Rodney C. L. Wolff. Local Lyapunov exponents: Looking closely at chaos. *Journal of the Royal Statistical Society. Series B (Methodological)*, 54(2):353–371, 1992.

[101] J. C. Xiong. A chaotic map with topological entropy 0. *Acta Math Scientia*, 6:439–443, 1986.

[102] C. Zimmer. Life after chaos. *Science, New Series*, 284(5411):83–86, 1999.

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