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A Bridge with Respect to the Compensated Compound Poisson Process or the Fractional Brownian Motion

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摘　　要

由 Föllmer, Wu, Yor, (1999) 中我們知道特定的隨機微分方程式的解會是一個布朗運動。在本論文中，我們討論有哪些隨機微分方程它們的解會是一個補償混合卜松過程。藉此，我們可以製造出新的補償混合卜松過程。同時，我們也討論一些隨機微分方程的解，觀察它們是不是碎形布朗運動。

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ABSTRACT

From Föllmer, Wu, Yor(1999) we know when the Brownian motion with nonzero linear drift is again a Brownian motion. In this thesis, instead of Brownian motion we discuss the case of compensated Poisson processes with nonzero. So we can construct new compensated compound Poisson processes. We also discuss whether the solutions of some particular form of stochastic differential equations are fractional Brownian motions.

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目　　　　錄

CHAPTER 1

Introduction

We consider some generalization of Brownian bridge. We want to change the Brownian motion which is in a Brownian bridge to a compensated Poisson process or a compensated compound Poisson process. We also try to change the Brownian motion which is in a Brownian bridge to a fractional Brownian motion. Then we discuss about what are the properties of these processes. We consider the process

$$X_t = B_t - tB_1, \qquad \text{for } 0 \le t \le 1 \tag{1.1}$$

where B is a Brownian motion is a Brownian bridge from 0 to 0 on $[0,1]$ (see Shreve [**14**] Definition 4.7.4). Denote by $(\mathcal{F}_t^B)$ the filtration generated by B. The Brownian bridge $(X_t)_{0\le t\le 1}$ is not adapted to the filtration $(\mathcal{F}_t^B)_{0\le t\le 1}$. In the following we consider the Brownian Bridge which is adapted to the filtration $(\mathcal{F}_t^B)_{0\le t\le 1}$. Consider the stochastic differential equation

$$dX_t = dB_t + \frac{-X_t}{1-t}\,dt \tag{1.2}$$

with the initial value $X_0 = 0$. The solution $(X_t)_{0\le t<1}$ of (1.2) is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s, \qquad \text{for } 0 \le t < 1.$$

Then the process $(X_t)_{0\le t<1}$ is a Brownian bridge from 0 to 0 on $[0,1)$ and it has the same law of the Brownian bridge which is in (1.1) (see Shreve [**14**] Section 4.7). The process $(X_t)_{0\le t<1}$ is adapted to the filtration $(\mathcal{F}_t^B)_{0\le t<1}$. Now we consider two independent Brownian motions $(B_t)_{t\ge 0}$ and $(\tilde{B}_t)_{t\ge 0}$. The solution $(X_t)_{0\le t\le 1}$ of the stochastic differential equation

$$dX_t = dB_t + \frac{\tilde{B}_1 - X_t}{1-t}\,dt$$

with the initial value $X_0 = 0$ is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s + \tilde{B}_1\,t, \qquad \text{for } 0 \le t < 1. \tag{1.3}$$

The process $(X_t)_{0\le t<1}$ in (1.3) is a standard Brownian motion with respect to the filtration $(\mathcal{F}_t^X)_{0\le t<1}$ which is the filtration generated by $(X_t)_{0\le t<1}$ and the process $(X_t)_{0\le t<1}$ converges to the final value $\tilde{B}_1$ (cf., for example, Jeulin-Yor [**7**]). The following solution $(X_t)_{0\le t<1}$ of the stochastic differential equation

$$dX_t = dB_t + \frac{\tilde{B}_t - X_t}{1-t}\,dt$$

with the initial value $X_0 = 0$ is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s + (1-t)\int_0^t \frac{\tilde{B}_s}{(1-s)^2}\,ds, \qquad \text{for } 0 \le t < 1. \tag{1.4}$$

The process $(X_t)_{0\le t<1}$ in (1.4) converges to $\tilde{B}_1$ $\mathbb{P}$-a.s. as $t \to 1$ and $(X_t)_{0\le t<1}$ is no longer a Brownian motion (see Föllmer, H. [**5**]).

In Shreve [**14**] we see the introduction about compensated Poisson process and compensated compound Poisson process. We know their basic properties from Shreve [**14**]. In Chapter 2 we change the Brownian motion which is in a Brownian bridge to a compensated Poisson process. We will see some bridges with respect to the compensated Poisson process start from zero to fixed points and see a bridge between two independent compensated Poisson processes. In Chapter 3 we will construct a new compensated Poisson process and a new compensated compound Poisson process.

In the last chapter we introduce the fractional Brownian motion and its basic properties. The fractional Brownian motion was first introduced by Kolmogorov [**10**]. Mandelbrot and Van Ness [**11**] established the integral representation for fractional Brownian motion on the whole real line. By the approach of [**12**], we have the integral representation for fractional Brownian motion on a finite interval. These integral representations are all integrals of deterministic integrands with respect to the Brownian motion. Then we know that the fractional Brownian motion is adapted to the filtration which is generated by the Brownian motion. Gani, Heyde, Jagers, Kurtz [**6**] tell us a fractional Brownian motion is not a semimartingale, so we can't use Itô stochastic calculus which is defined for semimartingales to define the stochastic integral with respect to the fractional Brownian motion. In Section 4.2 we have the definition for the Wiener integral of a deterministic integrand with respect to the fractional Brownian motion for the Hurst index $H > \frac{1}{2}$ (see Gani, Heyde, Jagers, Kurtz [**6**]). From the definition we know that the Brownian motion which is in the integral representation for the fractional Brownian motion can be represented by an integral with respect to the fractional Brownian motion. Hence, we know that the Brownian motion and the fractional Brownian motion generate the same filtration. In Section 4.3 we will see a bridge with respect to the fractional Brownian motion starts from zero to a fixed point and a bridge between the fractional Brownian motion and a random variable.

CHAPTER 2

A Bridge with Respect to the Compensated Poisson Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this chapter we consider the properties of the stochastic process if the Brownian bridge is driven by a compensated Poisson process instead of the Brownian motion. First we would introduce some basic properties of Poisson process.

2.1. Poisson Process

Definition 1. A random variable τ is said to have exponential distribution if τ is a random variable with the probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

where λ is a positive constant. We also say that τ is an exponential random variable.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of independent exponential random variables, all with the same parameter λ. Let

$$S_n = \sum_{k=1}^{n} \tau_k,$$

i.e., $S_1 = \tau_1$, $S_2 = \tau_1 + \tau_2$, $\cdots$.

Definition 2. The Poisson Process (N_t) is defined by

$$N_t = \inf\{n-1 : S_n > t\} = \max\{n : S_n \leq t\}.$$

Moreover, we say that (N_t) is a Poisson process with intensity λ.

The Poisson process (N_t) is right-continuous in t and it has stationary independent increments, i.e., for $0 \leq t_0 < t_1 < \cdots < t_m$, the random variables

$$N_{t_1},\ N_{t_2} - N_{t_1}, \cdots, N_{t_m} - N_{t_{m-1}}$$

are stationary and independent. The mean and variance of N_t are given by

$$\mathbb{E}\left[N_t\right] = \lambda t \quad \text{and} \quad \mathrm{Var}\left(N_t\right) = \lambda t$$

respectively. The Poisson process is no more a martingale. We consider a martingale which has similar properties of Poisson process.

Definition 3. Let (N_t) be a Poisson process with intensity λ. The stochastic process defined by

$$M_t = N_t - \lambda t, \qquad t \geq 0,$$

is called the compensated Poisson process.

Denote by $(\mathcal{F}_t^N)$ and $(\mathcal{F}_t^M)$ the filtrations generated by (N_t) and (M_t), respectively. From the definition of the compensated Poisson process, we know that $\mathcal{F}_t^M = \mathcal{F}_t^N$, for all $t \geq 0$. The compensated Poisson process (M_t) with intensity λ is a martingale with respect to the filtration $(\mathcal{F}_t^N)$. In next two sections we will discuss about some models with respect to the compensated Poisson process.

2.2. A Bridge Starts from Zero to a Fixed Point

Consider the process

$$X_t = M_t - tM_1, \qquad \text{for } 0 \leq t \leq 1$$

which is a bridge with respect to the compensated Poisson process from 0 to 0 on $[0, 1]$. Because the term M_1 is in the difinition of X_t, for $0 \leq t \leq 1$, the bridge X_t is not adapted to the filtration $(\mathcal{F}_t^N)$. We shall later obtain a different process which is also from 0 to 0 but is adapted to the filtration $(\mathcal{F}_t^N)$.

We consider the stochastic differential equation

$$dX_t = dM_t + \frac{-X_t}{1-t}\,dt \tag{2.1}$$

with the initial value $X_0 = 0$. The equation can be solved by applying the Itô's formula to the function

$$f(t,x) = x\exp\left\{\int_0^t \frac{1}{1-s}\,ds\right\} = \frac{x}{1-t}\,.$$

We have

$$f_t(t,x) = \frac{x}{(1-t)^2}, \quad f_x(t,x) = \frac{1}{1-t}, \quad f_{xx}(t,x) = f_{tx}(t,x) = f_{xt}(t,x) = 0.$$

The Itô's formula implies

$$f(t,X_t) = \frac{X_t}{1-t} = \int_0^t \frac{X_s}{(1-s)^2}\,ds + \int_0^t \frac{1}{1-s}\,dX_s. \tag{2.2}$$

From (2.1), we obtain

$$\int_0^t \frac{X_t}{(1-s)^2}\,ds + \int_0^t \frac{1}{1-s}\,dX_s = \int_0^t \frac{1}{1-s}\,dM_s. \tag{2.3}$$

By (2.2) and (2.3), we have that the explicit formula of solution X_t, for $0 \le t < 1$ is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dM_s, \qquad \text{for } 0 \le t < 1. \tag{2.4}$$

Due to (2.4) we see that (X_t) is adapted to the filtration $(\mathcal{F}_t^N)$, for $0 \le t < 1$. From Shreve [**14**] we have the following theorem.

Theorem 4 (Theorem 11.4.5, Shreve [**14**])**.** Consider the jump process (X_t) given by

$$X_t = X_0 + \int_0^t \Gamma_s\,dB_s + \int_0^t \Theta_s\,ds + J_t,$$

where Γ, Θ are adapted processes, B is an adapted Brownian motion, and J is an adapted, right-continuous pure jump process with $J_0 = 0$ having finitely many jumps on finite interval. Assume the process (X_t) is a martingale, the integrand Φ is left-continuous and adapted, and satisfies

$$\mathbb{E}\left[\int_0^t \Phi_s^2\,\Gamma_s^2\,ds\right] < \infty, \qquad \text{for all } t \geq 0.$$

Then the stochastic integral $\displaystyle\int_0^t \Phi_s\,dX_s$ is a martingale.

Since the compensated Poisson process (M_t) is a martingale, the process $\displaystyle\int_0^t \frac{1}{1-s}\,dM_s$ is also a martingale. Then X_t has zero mean for all t. Next, we compute the value of the variance of X_t and use the mean and variance of X_t to see where the process (X_t) approaches when $t \to 1^-$.

Theorem 5. Consider the process $(X_t)_{0\leq t<1}$ which is given by (2.4). For $0 \leq t < 1$, we have that the variance of X_t is given by

$$\operatorname{Var}(X_t) = -\lambda t^2 + \lambda t. \tag{2.5}$$

Proof. For $0 \leq t < 1$,

$$\mathbb{E}\left[X_t^2\right] = (1-t)^2\,\mathbb{E}\left[\left(\int_0^t \frac{1}{1-s}\,dM_s\right)^2\right].$$

We will apply the Itô's formula to $\displaystyle\left(\int_0^t \frac{1}{1-s}\,dM_s\right)^2$, so that we can get the value of $\displaystyle\mathbb{E}\left[\left(\int_0^t \frac{1}{1-s}\,dM_s\right)^2\right]$. We set for $0 \leq t < 1$,

$$\begin{aligned} Y_t &= \int_0^t \frac{1}{1-s}\,dM_s \\ &= \int_0^t \frac{1}{1-s}\,dN_s - \lambda\int_0^t \frac{1}{1-s}\,ds. \end{aligned}$$

Note that the continuous part of Y_t, Y_t^c, is given by $dY_s^c = \dfrac{-\lambda}{1-s}\,ds$. Take $f(x) = x^2$ so that $f^{'}(x) = 2x$, $f^{''}(x) = 2$. The Itô's formula implies

$$f\left(Y_t\right) = f(Y_0) + \int_0^t f^{'}(Y_s)\,dY_s^c + \frac{1}{2}\int_0^t f^{''}(Y_s)\,dY_s^c\,dY_s^c + \sum_{0<s\leq t}\left[f(Y_s) - f(Y_{s-})\right].$$

Then we have

$$Y_t^2 = Y_0^2 + \int_0^t 2\,Y_s\,dY_s^c + \frac{1}{2}\int_0^t 2\,dY_s^c\,dY_s^c + \sum_{0<s\le t}\left[Y_s^2 - Y_{s-}^2\right]$$

$$= \int_0^t 2\,Y_s\left(\frac{-\lambda}{1-s}\right)ds + \sum_{0<s\le t}\left[Y_s^2 - Y_{s-}^2\right]. \tag{2.6}$$

Next, we take the expectation of both sides of (2.6) and use Fubini's theorem

$$\mathbb{E}\left[Y_t^2\right] = -2\lambda\int_0^t \frac{1}{1-s}\,\mathbb{E}\left[Y_s\right]\,ds + \mathbb{E}\left[\sum_{0<s\le t}\left(Y_s^2 - Y_{s-}^2\right)\right].$$

Since $\mathbb{E}\left[Y_s\right] = 0$, we obtain

$$\mathbb{E}\left[Y_t^2\right] = \mathbb{E}\left[\sum_{0<s\le t}\left(Y_s^2 - Y_{s-}^2\right)\right]. \tag{2.7}$$

The sum $\sum_{0<s\le t}\left(Y_s^2 - Y_{s-}^2\right)$ can be transformed to an integral with respect to the Poisson process (N_t) by using the fact

$$Y_s - Y_{s^-} = \frac{1}{1-s}\Delta N_s,$$

and

$$(Y_s - Y_{s-})^2 = \frac{1}{(1-s)^2}\Delta N_s.$$

Then we have that

$$\begin{aligned}
\sum_{0<s\le t}\left[Y_s^2 - Y_{s-}^2\right] &= \sum_{0<s\le t}\left[(Y_s - Y_{s-})^2 + 2\,(Y_s - Y_{s-})\,Y_{s-}\right] \\
&= \sum_{0<s\le t}\left[\frac{1}{(1-s)^2}\,\Delta N_s + 2\,\frac{1}{1-s}\,\Delta N_s\,Y_{s-}\right] \\
&= \int_0^t\left(\frac{1}{(1-s)^2} + \frac{2}{1-s}\,Y_{s-}\right)dN_s.
\end{aligned} \tag{2.8}$$

We change the form of the last integral so that we can get the mean of it easierly.

$$\begin{aligned}\int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,Y_{s^-}\right) dN_s = \int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,Y_{s^-}\right) dM_s \\ + \int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,Y_{s^-}\right) \lambda\, ds.\end{aligned} \tag{2.9}$$

Since Y_{s^-} is left continuous in s, for $0 \le s < 1$ and (M_t) is a martingale, the process $\int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,Y_{s^-}\right) dM_s$ is also a martingale. So we see that

$$\mathbb{E}\left[\int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,Y_{s^-}\right) dM_s\right] = 0. \tag{2.10}$$

Due to (2.8), (2.9), (2.10) and by Fubini's theorem, we have

$$\mathbb{E}\left[\sum_{0<s\le t}\left(Y_s^2 - Y_{s^-}^2\right)\right] = \int_0^t \left(\frac{1}{(1-s)^2}+\frac{2}{1-s}\,\mathbb{E}\left[Y_{s^-}\right]\right) \lambda\, ds. \tag{2.11}$$

Now we compute the value of the right side of (2.11). $\mathbb{E}\left[Y_{s^-}\right] = 0$ since $\mathbb{E}\left[Y_s\right] = 0$, for $0 \le s < 1$. From (2.7) and (2.11), we obtain

$$\begin{aligned}\mathbb{E}\left[X_t^2\right] &= (1-t)^2\,\mathbb{E}\left[Y_t^2\right] \\ &= (1-t)^2 \int_0^t \frac{1}{(1-s)^2}\,\lambda\, ds \\ &= -\lambda t^2 + \lambda t.\end{aligned}$$

Since $\mathbb{E}\left[X_t\right] = 0$, we know that $\mathrm{Var}(X_t) = -\lambda t^2 + \lambda t$. $\square$

We have known the mean and variance of X_t, then we can use these to see where the process $(X_t)_{0\le t<1}$ approaches as time approaches 1. Since $\mathbb{E}\left[X_t\right] = 0$ and $\mathrm{Var}(X_t) = -\lambda t^2 + \lambda t$ which converges to 0 as $t \to 1^-$, we obtain that $X_t \to 0$ $\mathbb{P}$-a.s. as $t \to 1^-$. The process $(X_t)_{0\le t<1}$ is a bridge from 0 to 0 on $[0,1]$ and it is adapted to the filtration $\left(\mathcal{F}_t^N\right)_{0\le t<1}$.

In the following we see a process with respect to the compensated Poisson process starting from 0 to b, for some constant b. Consider the stochastic differential equation

$$dX_t = dM_t + \frac{b - X_t}{1 - t}\, dt, \tag{2.12}$$

where (M_t) is a compensated Poisson process with intensity λ, b is a constant and $X_0 = 0$. We can solve the stochastic differential equation by the same method as before. Then we obtain the solution X_t, for $0 \leq t < 1$ which is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\, dM_s + bt, \qquad \text{for } 0 \leq t < 1. \tag{2.13}$$

Then we obtain $\mathbb{E}\left[X_t\right] = bt$. By **Theorem 5**, we have

$$\begin{aligned} \mathrm{Var}\left(X_t\right) &= \mathrm{Var}\left((1-t)\int_0^t \frac{1}{1-s}\, dM_s\right) \\ &= -\lambda t^2 + \lambda t. \end{aligned}$$

Since we have known that the first term of (2.13) converges to 0 $\mathbb{P}$-a.s as $t \to 1^-$, we have that $X_t \to b$ $\mathbb{P}$-a.s. as $t \to 1^-$. Hence, we see that the process (X_t) is a bridge from 0 to b and is adapted to the filtration $\left(\mathcal{F}_t^N\right)$. We have discussed about a bridge with respect to the compensated Poisson process from 0 to a fixed point. In next section, we will discuss about a bridge between two independent compensated Poisson process.

2.3. A Bridge between two Independent Compensated Poisson Processes

We consider the stochastic differential equation

$$dX_s = dM_s + \frac{\tilde{M}_s - X_s}{1-s}\, ds,$$

where M is a compensated Poisson process with intensity λ, $\tilde{M}$ is another compensated Poisson process which has the same intensity λ is independent of M and X starts from 0.

We can also use the same method as before to solve the stochastic differential equation and get the solution X_t, for $0 \leq t < 1$, is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dM_s + (1-t)\int_0^t \frac{\tilde{M}_s}{(1-s)^2}\,ds, \qquad \text{for } 0 \leq t < 1. \tag{2.14}$$

The process $(X_t)_{0\leq t<1}$ has another form by applying Itô product rule to the second term of (2.14)

$$\begin{aligned} X_t &= (1-t)\int_0^t \frac{1}{1-s}\,dM_s + (1-t)\left[\tilde{M}_t\frac{1}{1-t} - \int_0^t \frac{1}{1-s}\,d\tilde{M}_s\right] \\ &= (1-t)\int_0^t \frac{1}{1-s}\,dM_s + \tilde{M}_t - (1-t)\int_0^t \frac{1}{1-s}\,d\tilde{M}_s. \end{aligned} \tag{2.15}$$

We have known that the process which has the form as (2.4) converges to 0 $\mathbb{P}$-a.s. as $t \to 1^-$, so the first term and the third term in (2.15) both converge to 0 $\mathbb{P}$-a.s. as $t \to 1^-$. Then $X_t \to \tilde{M}_1$ $\mathbb{P}$-a.s. as $t \to 1^-$. The process $(X_t)_{0\leq t<1}$ is a bridge from 0 to $\tilde{M}_1$ on $[0,1]$.

Remark 6. Suppose that the process $(X_t)_{0\leq t<1}$ is given by

$$X_t = M_t + \int_0^t \frac{\tilde{M}_s - X_s}{1-s}\,ds. \tag{2.16}$$

The filtration $(\mathcal{F}_t^X)_{0\leq t<1}$ which is generated by $(X_t)_{0\leq t<1}$ contains the information about when $(X_t)_{0\leq t<1}$ jumps. Since the second term of (2.16) is continuous in t, $(X_t)_{0\leq t<1}$ jumps at the same time as $(M_t)_{0\leq t<1}$ jumps. So the filtration $(\mathcal{F}_t^X)_{0\leq t<1}$ contains the information about when $(M_t)_{0\leq t<1}$ jumps, i.e., $(M_t)_{0\leq t<1}$ is adapted to $(\mathcal{F}_t^X)_{0\leq t<1}$. From (2.16), we know that $(\tilde{M}_t)$ is also adapted to $(\mathcal{F}_t^X)_{0\leq t<1}$. Hence, we obtain the Doob-Meyer decomposition of $(X_t)_{0\leq t<1}$ in its own filtration $\left(\mathcal{F}_t^X\right)_{0\leq t<1}$. We may regard X as a semimartingale in its natural filtration $\left(\mathcal{F}_t^X\right)_{0\leq t<1}$.

Consider two independent compensated Poisson processes $(M_t)_{t\geq 0}$ and $(\tilde{M}_t)_{t\geq 0}$. The stochastic process X is given by

$$X_t = M_t + \int_0^t Z_s \, ds, \tag{2.17}$$

where M is a Poisson process and the drift Z depends linearly on X and $\tilde{M}$. We want to characterize those cases where X is again a compensated Poisson process. Since the second term of (2.17) is continuous in t, we have that (X_t) jumps at the same time as (M_t) jumps. So (M_t) is adapted to the filtration $\left(\mathcal{F}_t^X\right)$. Suppose (X_t) is a compensated Poisson process. Since X and M all start from 0, and they jump at the same time, X is just equal to M. So the process $\int_0^t Z_s \, ds$ is equal to zero. Recall that the process $(X_t)_{0\leq t<1}$ which is given by (2.16) is a semimartingale in its natural filtration $\left(\mathcal{F}_t^X\right)_{0\leq t<1}$, but it is not a compensated Poisson process since the second term of (2.16) is not equal to zero.

The integral with respect to time in (2.17) makes X and M jump simultaneously. In next chapter, we will transform the second term of (2.17) to an integral with respect to the compensated Poisson process which is independent of M. Then the process M and the integral together decide when the process X jumps.

CHAPTER 3

Construction of a New Compensated Compound Poisson Process

We have discussed about whether X is a compensated Poisson process in the model

$$X_t = M_t + \int_0^t Z_s\, ds.$$

We will transform the above integral to an integral with respect to the compensated Poisson process which is independent of M and characterize those cases where X is again a compensated Poisson process. Since the jump size of Poisson process is equal to 1, we will extend the discussion to the compensated compound Poisson process which has random jump sizes.

3.1. Construction of a New Compensated Poisson Process

We consider the process X which is given by

$$X_t = M_t + \int_0^t f(s)\, d\tilde{M}_s, \tag{3.1}$$

where M is a compensated Poisson process with intensity λ, $\tilde{M}$ is another compensated Poisson process with intensity $\tilde{\lambda}$ is independent of M and f is a deterministic differentiable function. We want to know the form of the moment generating function of X_t, for $t \geq 0$, so that we can see in which cases the process X given by (3.1) is again a compensated Poisson process. The moment generating function of X_t, for $t \geq 0$ is given by

$$\varphi_{X_t}(u) = \mathbb{E}\left[\exp\left\{uM_t\right\}\right] \cdot \mathbb{E}\left[\exp\left\{u \int_0^t f(s)\, d\tilde{M}_s\right\}\right].$$

In Shreve [**14**] we have known that the moment generating function for the compensated Poisson process M_t is given by

$$\mathbb{E}\left[\exp\left\{uM_t\right\}\right] = \exp\left\{\lambda t\left(e^u - u - 1\right)\right\}.$$

So we only need to focus on the expectation

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\, d\tilde{M}_s\right\}\right].$$

Lemma 1. Consider the process

$$\int_0^t f(s)\, d\tilde{M}_s,$$

where $\tilde{M}$ is a compensated Poisson process with intensity $\tilde{\lambda}$ and f is a nonrandom differentiable function. Then it's moment generating function is given by

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\, d\tilde{M}_s\right\}\right] = \exp\left\{\tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right) ds - u\int_0^t f(s)\, ds\right)\right\}.$$

Proof. We will apply the Itô's formula to

$$\exp\left\{u\int_0^t f(s)\, d\tilde{M}_s - \tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right) ds - u\int_0^t f(s)\, ds\right)\right\},$$

so that we can know it is a martingale. We set for $t \geq 0$,

$$Y_t = u\int_0^t f(s)\, d\tilde{M}_s - \tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right) ds - u\int_0^t f(s)\, ds\right)$$

and

$$Z_t = \exp\left\{Y_t\right\}.$$

Note that the continuous part of Y_s, Y_s^c, is given by

$$dY_s^c = \tilde{\lambda}\left(-e^{uf(s)} + 1\right) ds.$$

Take $f(x) = e^x$ so that $f^{'}(x) = e^x$, $f^{''}(x) = e^x$. The Itô's formula implies

$$Z_t = Z_0 + \int_0^t Z_s\, dY_s^c + \frac{1}{2}\int_0^t Z_s\, dY_s^c\, dY_s^c + \sum_{0<s\leq t}\left[Z_s - Z_{s-}\right]. \tag{3.2}$$

Since Y_s^c is the continuous part of Y_s, we can change the integrand Z_s which is in the second term of (3.2) to Z_{s^-}. When the Poisson process $\tilde{N}$ jumps at time s, $Z_s = Z_{s^-} \times e^{uf(s)}$. When $(\tilde{N}_t)$ does not jump at time s, $Z_s = Z_{s^-}$. So we have

$$Z_s - Z_{s^-} = Z_{s^-}\left(e^{uf(s)} - 1\right)\Delta\tilde{N}_s.$$

Then

$$\begin{aligned} Z_t &= 1 + \tilde{\lambda}\int_0^t Z_{s^-}\left(-e^{uf(s)} + 1\right)ds + \sum_{0<s\le t}\left[Z_{s^-}\left(e^{uf(s)} - 1\right)\Delta\tilde{N}_s\right] \\ &= 1 + \int_0^t Z_{s^-}\left(e^{uf(s)} - 1\right)d\left(\tilde{M}_s - \tilde{N}_s\right) + \int_0^t Z_{s^-}\left(e^{uf(s)} - 1\right)d\tilde{N}_s \\ &= 1 + \int_0^t Z_{s^-}\left(e^{uf(s)} - 1\right)d\tilde{M}_s. \end{aligned}$$

Since M is a martingale and $Z_{s^-}\left(e^{uf(s)} - 1\right)$ is left continuous in s, the above integral is also a martingale. So the process (Z_t) is a martingale and we have $\mathbb{E}\left[Z_t\right] = 1$, i.e.,

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\tilde{M}_s - \tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right)ds - u\int_0^t f(s)\,ds\right)\right\}\right] = 1.$$

Hence, we obtain

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\tilde{M}_s\right\}\right] = \exp\left\{\tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right)ds - u\int_0^t f(s)\,ds\right)\right\}.$$

□

Moreover, we have that the moment generating function of X_t is given by

$$\begin{aligned} \varphi_{X_t}(u) &= \exp\left\{\lambda t\left(e^u - u - 1\right)\right\}\cdot\exp\left\{\tilde{\lambda}\left(\int_0^t \left(e^{uf(s)} - 1\right)ds - u\int_0^t f(s)\,ds\right)\right\} \\ &= \exp\left\{\lambda t\left(e^u - u - 1\right) + \tilde{\lambda}\int_0^t \left(e^{uf(s)} - 1\right)ds - \tilde{\lambda}u\int_0^t f(s)\,ds\right\}. \end{aligned} \tag{3.3}$$

Next, we use (3.3) to see in which cases the process X is a compensated Poisson process. If $f \equiv 0$, then it is obvious that $X_t = M_t$. In the following proposition we set $f \neq 0$.

Proposition 2. Let the stochastic process (X_t) satisfy (3.1). Then (X_t) is a compensated Poisson process if and only if $f \equiv 1$. Moreover, (X_t) has the intensity $\lambda + \tilde{\lambda}$.

PROOF. " $\Longrightarrow$ " : The moment generating function for compensated Poisson process must be as the form $\left(\exp\left\{\hat{\lambda} t\left(e^u - u - 1\right)\right\}\right)$, for some constant $\hat{\lambda}$. Suppose that (X_t) is a compensated Poisson process with intensity $\hat{\lambda}$. We let the moment generating function of X_t equal to $\left(\exp\left\{\hat{\lambda} t\left(e^u - u - 1\right)\right\}\right)$, i.e.

$$\exp\left\{\lambda t\left(e^u - u - 1\right) + \tilde{\lambda}\int_0^t \left(e^{uf(s)} - 1\right) ds - \tilde{\lambda} u \int_0^t f(s)\, ds\right\} = \exp\left\{\hat{\lambda} t\left(e^u - u - 1\right)\right\}.$$

Then we get the equation

$$\lambda t\left(e^u - u - 1\right) + \tilde{\lambda}\int_0^t \left(e^{uf(s)} - 1\right) ds - \tilde{\lambda} u \int_0^t f(s)\, ds = \hat{\lambda} t\left(e^u - u - 1\right). \tag{3.4}$$

We differentiate with respect to t on both sides of (3.4)

$$\lambda\left(e^u - u - 1\right) + \tilde{\lambda}\left(e^{uf(t)} - 1\right) - \tilde{\lambda} u f(t) = \hat{\lambda}\left(e^u - u - 1\right).$$

We differentiate with respect to t again

$$\tilde{\lambda} u e^{uf(t)} f'(t) - \tilde{\lambda} u f'(t) = 0\,.$$

Then we get

$$\tilde{\lambda} u f'(t)\left(e^{uf(t)} - 1\right) = 0\,.$$

This implies that $f'(t) = 0$ or $e^{uf(t)} - 1 = 0$. So we have that $f(t)$ is a constant. Set $f \equiv C$, where C is a positive constant. From (3.1), we have

$$X_t = M_t + C\tilde{M}_t\,.$$

The moment generating function of X_t is given by

$$\varphi_X(u) = \exp\left\{\lambda t\left(e^u - u - 1\right) + \tilde{\lambda} t\left(e^{uC} - uC - C\right)\right\}.$$

Suppose the following equation holds

$$\exp\left\{\lambda t\left(e^{u}-u-1\right)+\tilde{\lambda} t\left(e^{uC}-uC-C\right)\right\}=\exp\left\{\hat{\lambda} t\left(e^{u}-u-1\right)\right\}.$$

Then we have

$$\lambda t\left(e^{u}-u-1\right)+\tilde{\lambda} t\left(e^{uC}-uC-C\right)=\hat{\lambda} t\left(e^{u}-u-1\right).$$

We differentiate with respect to u twice, then we get

$$\lambda e^{u}+\tilde{\lambda} C^{2} e^{uC}=\hat{\lambda} e^{u}.$$

We multiply e^{-u} on both sides of the above formula

$$\lambda+\tilde{\lambda} C^{2} e^{u(C-1)}=\hat{\lambda}.$$

Then

$$e^{u(C-1)}=\frac{\hat{\lambda}-\lambda}{\tilde{\lambda} C^{2}}.$$

Hence, we obtain $C=1$ and then $\hat{\lambda}=\lambda+\tilde{\lambda}$.

" $\Longleftarrow$ " : Since $f \equiv 1$, we have $X_t = M_t + \tilde{M}_t$. We will show that the law of X agrees with the law of a compensated Poisson process which has intensity $\lambda+\tilde{\lambda}$. Denote by $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$ the filtration generated by M and $\tilde{M}$. Let

$$Z_t=\exp\left\{uM_t-\lambda t\left(e^{u}-u-1\right)\right\}.$$

By the proof of **Lemma 1**, we have that the process (Z_t) is a martingale with respect to $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$. Let

$$\tilde{Z}_t=\exp\left\{u\tilde{M}_t-\tilde{\lambda} t\left(e^{u}-u-1\right)\right\}.$$

$(\tilde{Z}_t)$ is also a martingale with respect to $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$. The two processes (Z_t) and $(\tilde{Z}_t)$ are independent and they are all martingales with respect to the filtration $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$. From (3.3), we know that the moment generating function of X_t is

$$\varphi_{X_t}(u)=\exp\left\{\left(\lambda+\tilde{\lambda}\right)t\left(e^{u}-u-1\right)\right\}.$$

For fixed $u \in \mathbb{R}$, the process $V_t^{(u)}$ is defined by

$$\begin{aligned} V_t^{(u)} &= \exp\left\{uX_t - \left(\lambda + \tilde{\lambda}\right) t\,(e^u - u - 1)\right\} \\ &= Z_t\,\tilde{Z}_t\,. \end{aligned}$$

We will show that $\left(V_t^{(u)}\right)$ is a martingale with respect to the filtration $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$. For $0 < s < t$,

$$\begin{aligned} \mathbb{E}\left[V_t^{(u)} \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] &= \mathbb{E}\left[Z_t\,\tilde{Z}_t \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] \\ &= \mathbb{E}\left[(Z_t - Z_s)\left(\tilde{Z}_t - \tilde{Z}_s\right) + Z_s\,\tilde{Z}_t + Z_t\,\tilde{Z}_s - Z_s\,\tilde{Z}_s \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] \\ &= \mathbb{E}\left[(Z_t - Z_s)\left(\tilde{Z}_t - \tilde{Z}_s\right) \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] + \mathbb{E}\left[Z_s\,\tilde{Z}_t \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] \\ &\quad + \mathbb{E}\left[Z_t\,\tilde{Z}_s \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] - \mathbb{E}\left[Z_s\,\tilde{Z}_s \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right]. \end{aligned}$$

Since $Z_t - Z_s$, $\tilde{Z}_t - \tilde{Z}_s$ are independent of $\mathcal{F}_s^{M,\tilde{M}}$ and Z_s, $\tilde{Z}_s$ are adapted to $\mathcal{F}_s^{M,\tilde{M}}$, we have

$$\begin{aligned} \mathbb{E}\left[V_t^{(u)} \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] &= \mathbb{E}\left[(Z_t - Z_s)\left(\tilde{Z}_t - \tilde{Z}_s\right)\right] + Z_s\mathbb{E}\left[\tilde{Z}_t \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] + \tilde{Z}_s\mathbb{E}\left[Z_t \,\middle|\, \mathcal{F}_s^{M,\tilde{M}}\right] - Z_s\,\tilde{Z}_s \\ &= \mathbb{E}\left[(Z_t - Z_s)\right] \cdot \mathbb{E}\left[\left(\tilde{Z}_t - \tilde{Z}_s\right)\right] + Z_s\,\tilde{Z}_s + Z_s\,\tilde{Z}_s - Z_s\,\tilde{Z}_s \\ &= Z_s\,\tilde{Z}_s = V_s^{(u)}. \end{aligned}$$

So the process $\left(V_t^{(u)}\right)$ is a martingale with respect to $\left(\mathcal{F}_t^{M,\tilde{M}}\right)$. For fixed $u_2 \in \mathbb{R}$ and $0 < t_1 < t_2$,

$$V_{t_1}^{(u_2)} = \mathbb{E}\left[V_{t_2}^{(u_2)} \,\middle|\, \mathcal{F}_{t_1}^{M,\tilde{M}}\right].$$

Now fixed $u_1 \in \mathbb{R}$. Since $\dfrac{V_{t_1}^{(u_1)}}{V_{t_1}^{(u_2)}}$ is adapted to $\mathcal{F}_{t_1}^{M,\tilde{M}}$, we have

$$\begin{aligned} V_{t_1}^{(u_1)} &= \mathbb{E}\left[\frac{V_{t_1}^{(u_1)}V_{t_2}^{(u_2)}}{V_{t_1}^{(u_2)}} \,\middle|\, \mathcal{F}_{t_1}^{M,\tilde{M}}\right] \\ &= \mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\} \,\middle|\, \mathcal{F}_{t_1}^{M,\tilde{M}}\right] \cdot \exp\left\{-\left(\lambda + \tilde{\lambda}\right) t_1 \left(e^{u_1} - u_1 - 1\right)\right\} \\ &\quad \cdot \exp\left\{-\left(\lambda + \tilde{\lambda}\right)\left(t_2 - t_1\right)\left(e^{u_2} - u_2 - 1\right)\right\}. \end{aligned}$$

Now we use the martingale property of $V_t^{(u_1)}$ and we take expectation of both sides of the above formula

$$\begin{aligned} 1 &= V_0^{(u_1)} \\ &= \mathbb{E}\left[V_{t_1}^{(u_1)}\right] \\ &= \mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\}\right] \\ &\quad \cdot \exp\left\{-\left(\lambda + \tilde{\lambda}\right) t_1 \left(e^{u_1} - u_1 - 1\right)\right\} \cdot \exp\left\{-\left(\lambda + \tilde{\lambda}\right)\left(t_2 - t_1\right)\left(e^{u_2} - u_2 - 1\right)\right\}. \end{aligned}$$

So we obtain

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\}\right] \\ &= \exp\left\{\left(\lambda + \tilde{\lambda}\right) t_1 \left(e^{u_1} - u_1 - 1\right)\right\} \cdot \exp\left\{\left(\lambda + \tilde{\lambda}\right)\left(t_2 - t_1\right)\left(e^{u_2} - u_2 - 1\right)\right\}. \end{aligned}$$

Since the above joint moment generating function factors into the product of moment generating functions, X_{t_1} and $X_{t_2} - X_{t_1}$ must be independent. We also know that the moment generating function of $X_{t_2} - X_{t_1}$ is

$$\varphi_{X_{t_2} - X_{t_1}}(u) = \exp\left\{\left(\lambda + \tilde{\lambda}\right)\left(t_2 - t_1\right)\left(e^{u} - u - 1\right)\right\}.$$

Next, we computer the joint moment generating function of the random variables X_{t_1}, $X_{t_2}, \cdots, X_{t_n}$, for $0 < t_1 < t_2 < \cdots < t_n$, so that we can know whether X is a compensated

Poisson process. For $0 < t_1 < t_2 < \cdots < t_n$, the joint moment generating function of the random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is given by

$$\varphi_{X_{t_1},X_{t_2},\cdots,X_{t_n}}(u_1, u_2, \cdots, u_n)$$

$$= \mathbb{E}\left[\exp\left\{u_n X_{t_n} + u_{n-1}X_{t_{n-1}} + \cdots + u_1 X_{t_1}\right\}\right]$$

$$= \mathbb{E}\left[\exp\left\{u_n\left(X_{t_n} - X_{t_{n-1}}\right) + (u_{n-1}+u_n)\left(X_{t_{n-1}} - X_{t_{n-2}}\right) + \cdots + (u_1 + u_2 + \cdots u_n)\, X_{t_1}\right\}\right]$$

$$= \mathbb{E}\left[\exp\left\{u_n\left(X_{t_n} - X_{t_{n-1}}\right)\right\}\right] \cdot \mathbb{E}\left[\exp\left\{(u_{n-1}+u_n)\left(X_{t_{n-1}} - X_{t_{n-2}}\right)\right\}\right] \cdot$$

$$\cdots \mathbb{E}\left[\exp\left\{(u_1 + u_2 + \cdots u_n)\, X_{t_1}\right\}\right].$$

We have known the form of the moment generating function of increments of X, then we obtain

$$\begin{aligned}
&\varphi_{X_{t_1},X_{t_2},\cdots,X_{t_n}}(u_1, u_2, \cdots, u_n)\\
&= \exp\left\{\left(\lambda + \tilde{\lambda}\right)(t_n - t_{n-1})\left(e^{u_n} - u_n - 1\right)\right\}\\
&\quad \cdot \exp\left\{\left(\lambda + \tilde{\lambda}\right)(t_{n-1} - t_{n-2})\left(e^{(u_{n-1}+u_n)} + (u_{n-1}+u_n) - 1\right)\right\} \cdot\\
&\quad \cdots \exp\left\{\left(\lambda + \tilde{\lambda}\right) t_1 \left(e^{(u_1+u_2+\cdots+u_n)} - (u_1 + u_2 + \cdots + u_n) - 1\right)\right\}.
\end{aligned}$$

This is the moment generating function for a compensated Poisson process with intensity $\lambda + \tilde{\lambda}$. This completes the proof. □

3.2. Compensated Compound Poisson Process

Let (N_t) be a Poisson process with intensity λ and let $Y_1, Y_2, \ldots$ be a sequence of independent, identically distributed random variables with mean β, where $\beta = E[Y_i]$. The random variables $Y_1, Y_2, \ldots$ are independent of the Poisson process (N_t).

Definition 3. The stochastic process (Q_t) defined by

$$Q_t = \sum_{i=1}^{N_t} Y_i\,, \qquad t \geq 0$$

is called the compound Poisson process.

The compound Poisson process (Q_t) jumps at the same time as the Poisson process (N_t) jumps. The jump sizes of the compound Poisson process are random. The compensated compound Poisson process $(Q_t - \beta\lambda t)$ is a martingale. In this chapter, we only regard the compound Poisson process which has finitely many possible jump sizes on finite interval. The following theorem says that a compound Poisson process can be regarded as a sum of independent Poisson processes each has fixed jump-size.

Theorem 4 (Shreve [**14**] Theorem 11.3.3.)**.** Let $y_1,\, y_2, \ldots, y_M$ be a finite set of nonzero numbers and let $p(y_1),\, p(y_2), \ldots, p(y_M)$ be positive numbers that sum to 1. Let $Y_1,\, Y_2, \ldots$ be a sequence of independent, identically distributed random variables with $\mathbb{P}\left(Y_i = y_m\right) = p(y_m),\ m = 1, \ldots, M$. Let (N_t) be a Poisson process with intensity λ and define the compound Poisson process

$$Q_t = \sum_{i=1}^{N_t} Y_i\,.$$

For $m = 1, \ldots, M$, let $N_t^{(m)}$ denote the number of jumps in Q of size y_m in $[0, t]$. Then

$$N_t = \sum_{m=1}^{M} N_t^{(m)} \quad \text{and} \quad Q_t = \sum_{m=1}^{M} y_m N_t^{(m)},$$

where the process $N^{(1)}, \ldots, N^{(M)}$ are independent Poisson processes and each $N^{(m)}$ has intensity $\lambda p(y_m)$.

The theorem tells us the fact that a compound Poisson process can be represented by some independent Poisson processes each has fixed jump-size. We will use this theorem to construct a new compensated compound Poisson process.

3.3. Construction of a New Compensated Compound Poisson Process

We consider two independent compound Poisson process which have some conditions as follows. Let $y_1,\, y_2, \ldots, y_M$ be a finite set of nonzero numbers and let $p(y_1),\ p(y_2),$

$\ldots, p(y_M)$ be positive numbers whose summation is identical to 1. Let $Y_1, Y_2, \ldots$ be a sequence of independent, identically distributed random variables with $\mathbb{P}(Y_i = y_m) = p(y_m)$, $m = 1, \ldots, M$ and $E[Y_i] = \beta$. Let (N_t) be a Poisson process with intensity λ and define the compound Poisson process

$$Q_t = \sum_{i=1}^{N_t} Y_i\,. \tag{3.5}$$

For $m = 1, \ldots, M$, let $N_t^{(m)}$ denote the number of jumps in Q of size y_m in $[0, t]$. Then we have

$$Q_t = \sum_{m=1}^{M} y_m N_t^{(m)},$$

where the process $N^{(1)}, \ldots, N^{(M)}$ are independent Poisson process, and each $N^{(m)}$ has intensity $\lambda p(y_m)$.

Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{\tilde{M}}$ be another finite set of nonzero numbers and let $\tilde{p}(\tilde{y}_1)$, $\tilde{p}(\tilde{y}_2)$, $\ldots, \tilde{p}(\tilde{y}_{\tilde{M}})$ be positive numbers that sum to 1. Let $\tilde{Y}_1, \tilde{Y}_2, \ldots$ be another sequence of independent, identically distributed random variables with $\mathbb{P}\left(\tilde{Y}_i = \tilde{y}_m\right) = \tilde{p}(\tilde{y}_m)$, $m = 1, \ldots, \tilde{M}$, $E[\tilde{Y}_i] = \tilde{\beta}$ and $\tilde{Y}_1, \tilde{Y}_2, \ldots$ are independent of the sequence $Y_1, Y_2, \ldots$. Let $(\tilde{N}_t)$ be a Poisson process with intensity $\tilde{\lambda}$ and it is independent of (N_t). Define another compound Poisson process

$$\tilde{Q}_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i\,. \tag{3.6}$$

For $m = 1, \ldots, \tilde{M}$, let $\tilde{N}_t^{(m)}$ denote the number of jumps in $\tilde{Q}$ of size $\tilde{y}_m$ in $[0, t]$. Then we have

$$\tilde{Q}_t = \sum_{m=1}^{\tilde{M}} \tilde{y}_m \tilde{N}_t^{(m)},$$

where the process $\tilde{N}^{(1)}, \ldots, \tilde{N}^{(\tilde{M})}$ are independent Poisson process, and each $\tilde{N}^{(m)}$ has intensity $\tilde{\lambda}\tilde{p}(\tilde{y}_m)$.

Consider the stochastic process X which is given by

$$X_t = (Q_t - \beta\lambda t) + \int_0^t f(s)\, d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda} s\right), \tag{3.7}$$

where f is a nonrandom differentiable function and $f \neq 0$. We use the similar method in Section 3.1 to see that in which cases X is again a compound Poisson process. First, we want to know the form of the moment generating function of X_t, for $t \geq 0$. The moment generating function of X_t, for $t \geq 0$ is given by

$$\varphi_{X_t}(u) = \mathbb{E}\left[\exp\left\{u\left(Q_t - \beta\lambda t\right)\right\}\right] \cdot \mathbb{E}\left[\exp\left\{u\int_0^t f(s)\, d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}s\right)\right\}\right]. \tag{3.8}$$

The following theorem tells us the form of the moment generating function for a compound Poisson process, so that we can get the form of the moment generating function of X_t.

Theorem 5 (Shreve [**14**] Section 11.3.2)**.** The moment generating function for the compound Poisson process (Q_t) defined as (3.5) is given by

$$\varphi_{Q_t}(u) = \exp\left\{\lambda t\left(\varphi_{Y_1}(u) - 1\right)\right\}.$$

By the above theorem, we know that the moment generating function of $(Q_t - \beta\lambda t)$ is given by

$$\varphi_{(Q_t-\beta\lambda t)}(u) = \exp\left\{\lambda t\sum_{m=1}^{M} p(y_m)\left(e^{uy_m} - 1\right) - u\beta\lambda t\right\}. \tag{3.9}$$

We remain to obtain the form of the moment generating function of $\int_0^t f(s)\, d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}s\right)$, so that we can get the form of the moment generating function of X_t.

Theorem 6. Consider the process

$$\int_0^t f(s)\, d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}s\right),$$

where $(\tilde{Q}_t)$ is given by (3.6) with intensity $\tilde{\lambda}$, $\tilde{\beta} = \mathbb{E}[\tilde{Y}_i]$, and f is a nonrandom differentiable function. Then its moment generating function is given by

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}s\right)\right\}\right] = \exp\left\{\tilde{\lambda}\sum_{m=1}^{\tilde{M}}\left[\tilde{p}(\tilde{y}_m)\int_0^t\left(e^{uf(s)\tilde{y}_m}-1\right)ds\right]\right\}\cdot\exp\left\{-u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\,ds\right\}.$$

PROOF.

$$\begin{aligned}\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}s\right)\right\}\right] \\ = \mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\tilde{Q}_s\right\}\right]\cdot\exp\left\{-u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\,ds\right\}.\end{aligned} \tag{3.10}$$

We focus on the first term of (3.10). Using the fact taht $\tilde{N}^{(1)},\ldots,\tilde{N}^{(\tilde{M})}$ are independent Poisson processes, we know that

$$\begin{aligned}\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\tilde{Q}_s\right\}\right] &= \mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\left(\sum_{m=1}^{\tilde{M}}\tilde{y}_m\tilde{N}_s^{(m)}\right)\right\}\right] \\ &= \prod_{m=1}^{\tilde{M}}\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\tilde{y}_m\,d\tilde{N}_s^{(m)}\right\}\right].\end{aligned} \tag{3.11}$$

From **Theorem 1**, we have that for $1 \le m \le \tilde{M}$,

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\tilde{y}_m\,d\tilde{N}_s^{(m)}\right\}\right] = \exp\left\{\tilde{\lambda}\tilde{p}(\tilde{y}_m)\int_0^t\left(e^{uf(s)\tilde{y}_m}-1\right)ds\right\}. \tag{3.12}$$

Due to (3.10), (3.11) and (3.12), we have

$$\mathbb{E}\left[\exp\left\{u\int_0^t f(s)\,d\left(\tilde{Q}_s - \tilde{\beta}\tilde{\lambda}t\right)\right\}\right] = \exp\left\{\tilde{\lambda}\sum_{m=1}^{\tilde{M}}\left[\tilde{p}(\tilde{y}_m)\int_0^t\left(e^{uf(s)\tilde{y}_m}-1\right)ds\right]\right\}\cdot\exp\left\{-u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\,ds\right\}.$$

This completes the proof. □

From (3.8), (3.9) and **Theorem 6**, we obtain that the moment generating function of X_t is given by

$$\varphi_{X_t}(u) = \exp\left\{\lambda t \sum_{m=1}^{M} p(y_m)\left(e^{uy_m} - 1\right) - u\beta\lambda t + \tilde{\lambda}\sum_{m=1}^{\tilde{M}}\left[\tilde{p}(\tilde{y}_m)\int_0^t \left(e^{uf(s)\tilde{y}_m} - 1\right) ds\right]\right.$$
$$\left. -u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\, ds\right\}. \tag{3.13}$$

Next, we want to see in which cases the process X is a compensated compound Poisson process.

Proposition 7. Let the stochastic process (X_t) satisfy (3.7). Then (X_t) is a compensated compound Poisson process if and only if $f \equiv C$. Moreover, (X_t) has intensity $\left(\lambda + \tilde{\lambda}\right)$.

PROOF. " $\Longrightarrow$ " : Suppose that (X_t) is a compensated compound Poisson process with intensity $\hat{\lambda}$. We let the moment generating function of X_t be equal to

$$\exp\left\{\hat{\lambda} t \sum_{m=1}^{\hat{M}} \hat{p}(\hat{y}_m)\left(e^{u\hat{y}_m} - 1\right) - u\hat{\beta}\hat{\lambda} t\right\}, \tag{3.14}$$

for some $\hat{M}$, $\hat{\beta}$, $\hat{y}_m$ and $\hat{p}(\hat{y}_m)$, for $1 \le m \le \hat{M}$. Let $\hat{Y}_i$ denote the size of the ith jump for X, for $i \ge 1$. Then $\hat{Y}_1, \hat{Y}_2, \ldots$ are independent and from (3.14) we know that the distribution of finitely many jump sizes of X is given by $\mathbb{P}\left(\hat{Y}_i = \hat{y}_m\right) = \hat{p}(\hat{y}_m)$, for $1 \le m \le \hat{M}$. The mean of $\hat{Y}_i$ is equal to $\hat{\beta}$. Suppose that the following equation holds

$$\exp\left\{\lambda t \sum_{m=1}^{M} p(y_m)\left(e^{uy_m} - 1\right) - u\beta\lambda t + \tilde{\lambda}\sum_{m=1}^{\tilde{M}}\left[\tilde{p}(\tilde{y}_m)\int_0^t \left(e^{uf(s)\tilde{y}_m} - 1\right) ds\right] - u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\, ds\right\}$$
$$= \exp\left\{\hat{\lambda} t \sum_{m=1}^{\hat{M}} \hat{p}(\hat{y}_m)\left(e^{u\hat{y}_m} - 1\right) - u\hat{\beta}\hat{\lambda} t\right\}.$$

Then we have the equation

$$\lambda t \sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda t + \tilde{\lambda}\sum_{m=1}^{\tilde{M}}\left[\tilde{p}(\tilde{y}_m)\int_0^t \left(e^{uf(s)\tilde{y}_m}-1\right)ds\right] - u\tilde{\beta}\tilde{\lambda}\int_0^t f(s)\,ds$$

$$= \hat{\lambda} t \sum_{m=1}^{\hat{M}} \hat{p}(\hat{y}_m)\left(e^{u\hat{y}_m}-1\right) - u\hat{\beta}\hat{\lambda} t.$$

We differentiate with respect to t on both sides of the above equation

$$\lambda \sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda + \tilde{\lambda}\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\left(e^{uf(t)\tilde{y}_m}-1\right) - u\tilde{\beta}\tilde{\lambda}f(t)$$

$$= \hat{\lambda}\sum_{m=1}^{\hat{M}} \hat{p}(\hat{y}_m)\left(e^{u\hat{y}_m}-1\right) - u\hat{\beta}\hat{\lambda}\,.$$

If we differentiate with respect to t again, then we obtain

$$u\tilde{\lambda}f^{'}(t)\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\tilde{y}_m\left(e^{uf(t)\tilde{y}_m}\right) - u\tilde{\beta}\tilde{\lambda}f^{'}(t) = 0\,.$$

Then

$$u\tilde{\lambda}f^{'}(t)\left(\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\tilde{y}_m\left(e^{uf(t)\tilde{y}_m}\right) - \tilde{\beta}\right) = 0\,.$$

This implies that $f^{'}(t) = 0$ or

$$\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\tilde{y}_m\left(e^{uf(t)\tilde{y}_m}\right) - \tilde{\beta} = 0\,. \tag{3.15}$$

We differentiate with respect to t on both sides of (3.15), then we have

$$uf^{'}(t)\sum_{m=1}^{\tilde{M}}\ \tilde{p}(\tilde{y}_m)(\tilde{y}_m)^2\left(e^{uf(t)\tilde{y}_m}\right) = 0\,.$$

So we know that f must be a constant. Set $f \equiv C$, where C is a constant. From (3.7), we have

$$X_t = (Q_t - \beta\lambda t) + C\left(\tilde{Q}_t - \tilde{\beta}\tilde{\lambda}t\right).$$

The moment generating function of X_t is given by

$$\begin{aligned}\varphi_{X_t}(u) &= \exp\left\{\lambda t\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda t + \tilde{\lambda} t\sum_{m=1}^{\tilde{M}} \tilde{p}(\tilde{y}_m)\left(e^{uC\tilde{y}_m}-1\right) - u\tilde{\beta}\tilde{\lambda}Ct\right\}\\ &= \exp\left\{\left(\lambda+\tilde{\lambda}\right)t\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uy_m}-1\right) + \sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uC\tilde{y}_m}-1\right)\right]\right.\\ &\qquad \left. -u\left(\lambda+\tilde{\lambda}\right)t\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}} + \frac{\tilde{\lambda}C\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}.\end{aligned}$$

This implies that X is a compensated Poisson process with intensity $\left(\lambda+\tilde{\lambda}\right)$ and the distribution for finitely many jump sizes of X is given by

$$\mathbb{P}\left(\hat{Y}_i = y_m\right) = \frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}, \qquad \text{for } 1\le m\le M$$

and

$$\mathbb{P}\left(\hat{Y}_i = C\tilde{y}_n\right) = \frac{\tilde{\lambda}\tilde{p}(\tilde{y}_n)}{\lambda+\tilde{\lambda}}, \qquad \text{for } 1\le n\le \tilde{M}\,.$$

We also know that the mean of $\hat{Y}_i$, for $i\ge 1$ is given by

$$\mathbb{E}[\hat{Y}_i] = \frac{\lambda\beta}{\lambda+\tilde{\lambda}} + \frac{\tilde{\lambda}C\tilde{\beta}}{\lambda+\tilde{\lambda}}\,.$$

" $\Longleftarrow$ " : If $f\equiv C$, then we have $X_t = (Q_t-\beta\lambda t) + C\left(\tilde{Q}_t - \tilde{\beta}\tilde{\lambda}t\right)$. We will show that the law of X agrees with the law of a compensated compound Poisson process which has intensity $\lambda+\tilde{\lambda}$. Set

$$Z_t = \exp\left\{u\left(Q_t-\beta\lambda t\right) - \left(\lambda t\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda t\right)\right\}.$$

We will show that (Z_t) is a martingale. Since $\beta = \sum_{m=1}^{M} y_m p(y_m)$, we obtain

$$Q_t - \beta\lambda t = \sum_{m=1}^{M} y_m\left(N_t^{(m)} - \lambda p(y_m)t\right).$$

Then we have

$$Z_t = \exp\left\{u\sum_{m=1}^{M} y_m\left(N_t^{(m)} - \lambda p(y_m)t\right) - \left(\lambda t\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda t\right)\right\}.$$

Set

$$Y_t = u\sum_{m=1}^{M} y_m\left(N_t^{(m)} - \lambda p(y_m)t\right) - \left(\lambda t\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right) - u\beta\lambda t\right).$$

Note that

$$dY_s^c = \left(-\lambda\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right)\right)ds.$$

Take $f(x)=e^x$ so that $f^{'}(x)=e^x$, $f^{''}(x)=e^x$. The Itô's formula implies

$$Z_t = Z_0 + \int_0^t Z_s\,dY_s^c + \frac{1}{2}\int_0^t Z_s\,dY_s^c\,dY_s^c + \sum_{0<s\le t}\left[Z_s - Z_{s^-}\right]. \tag{3.16}$$

Since Y_s^c is the continuous part of Y_s, we can change the integrand Z_s which is in the second term of (3.16) to Z_{s^-}. When the compound Poisson process Q jumps at time s, the jump size of Q at time s must be equal to one of $y_1, y_2, \cdots, y_M$. If the jump size of Q at time s is equal to y_m, for some m, then we have $Z_s = Z_{s^-}\times e^{uy_m}$. If Q does not jump at time s, then $Z_s = Z_{s^-}$. So we have

$$Z_s - Z_{s^-} = \sum_{m=1}^{M} Z_{s^-}\left(e^{uy_m}-1\right)\Delta N_s^{(m)}.$$

Then

$$Z_t = 1 + \int_0^t Z_{s^-}\left(-\lambda\sum_{m=1}^{M} p(y_m)\left(e^{uy_m}-1\right)\right)ds + \sum_{0<s\le t}\sum_{m=1}^{M} Z_{s^-}\left(e^{uy_m}-1\right)\Delta N_s^{(m)}.$$

Set $M_t^{(m)} = N_t^{(m)} - \lambda\,p(y_m)\,t$, for $1\le m\le M$. Then $M^{(m)}$ is a compensated Poisson process with intensity $\lambda p(y_m)$ and $M^{(m)}$ is a martingale. We have

$$\begin{aligned} Z_t &= 1 + \sum_{m=1}^{M}\int_0^t Z_{s^-}\left(e^{uy_m}-1\right)\,d\left(M_s^{(m)} - N_s^{(m)}\right) + \sum_{m=1}^{M}\sum_{0<s\le t} Z_{s^-}\left(e^{uy_m}-1\right)\Delta N_s^{(m)} \\ &= 1 + \sum_{m=1}^{M}\int_0^t Z_{s^-}\left(e^{uy_m}-1\right)\,d\left(M_s^{(m)} - N_s^{(m)}\right) + \sum_{m=1}^{M}\int_0^t Z_{s^-}\left(e^{uy_m}-1\right)dN_s^{(m)} \\ &= 1 + \sum_{m=1}^{M}\int_0^t Z_{s^-}\left(e^{uy_m}-1\right)\,dM_s^{(m)}. \end{aligned}$$

Since $M^{(m)}$ is a martingale and $Z_{s^-}\left(e^{uy_m}-1\right)$ is left continuous in s, $\int_0^t Z_{s^-}\left(e^{uy_m}-1\right)dM_s^{(m)}$ is also a martingale. The sum of finitely many martingales is a martingale, so the process (Z_t) is a martingale. Denote by $\left(\mathcal{F}_t^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right)$ the filtration generated by $N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}$. The process (Z_t) is a martingale with respect to $\left(\mathcal{F}_t^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right)$. Set

$$\tilde{Z}_t=\exp\left\{uC\left(\tilde{Q}_t-\tilde{\beta}\tilde{\lambda}t\right)-\left(\tilde{\lambda}t\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\left(e^{uC\tilde{y}_m}-1\right)-uC\tilde{\beta}\tilde{\lambda}t\right)\right\}.$$

By the similar method, we have that $\tilde{Z}_t$ is also a martingale with respect to $\left(\mathcal{F}_t^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right)$. The two processes (Z_t) and $(\tilde{Z}_t)$ are independent and they are all martingales with respect to the filtration $\left(\mathcal{F}_t^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right)$. From (3.13), we know that the moment generating function of X_t is given by

$$\begin{aligned}\varphi_{X_t}(u)&=\exp\left\{\lambda t\sum_{m=1}^{M}p(y_m)\left(e^{uy_m}-1\right)-u\beta\lambda t+\tilde{\lambda}t\sum_{m=1}^{\tilde{M}}\tilde{p}(\tilde{y}_m)\left(e^{uC\tilde{y}_m}-1\right)-uC\tilde{\beta}\tilde{\lambda}t\right\}\\&=\exp\left\{\left(\lambda+\tilde{\lambda}\right)t\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uy_m}-1\right)+\sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uC\tilde{y}_m}-1\right)\right]\right.\\&\qquad\left.-u\left(\lambda+\tilde{\lambda}\right)t\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}}+\frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}.\end{aligned}$$

For the sake of simplicity, we let

$$\begin{aligned}\eta_t(u)=\exp&\left\{\left(\lambda+\tilde{\lambda}\right)t\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uy_m}-1\right)+\sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{uC\tilde{y}_m}-1\right)\right]\right.\\&\qquad\left.-u\left(\lambda+\tilde{\lambda}\right)t\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}}+\frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}.\end{aligned}$$

For fixed $u\in\mathbb{R}$, the process $V_t^{(u)}$ is defined by

$$V_t^{(u)}=\frac{\exp\left\{uX_t\right\}}{\eta_t(u)}=Z_t\,\tilde{Z}_t.$$

Since $Z_t - Z_s$, $\tilde{Z}_t - \tilde{Z}_s$ are independent of $\mathcal{F}_s^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}$ and Z_s, $\tilde{Z}_s$ are adapted to $\mathcal{F}_s^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}$, we can use the same method as the proof in **Proposition 2** to show that $V_t^{(u)} = Z_t\,\tilde{Z}_t$ is a martingale with respect to $\left(\mathcal{F}_t^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right)$. For fixed $u_2 \in \mathbb{R}$ and $0 < t_1 < t_2$, $V_{t_1}^{(u_2)} = \mathbb{E}\left[V_{t_2}^{(u_2)} \,\middle|\, \mathcal{F}_{t_1}^{M,\tilde{M}}\right]$. Now fixed $u_1 \in \mathbb{R}$. Since $\dfrac{V_{t_1}^{(u_1)}}{V_{t_1}^{(u_2)}}$ is adapted to $\mathcal{F}_{t_1}^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}$, we have

$$
\begin{aligned}
V_{t_1}^{(u_1)} &= \mathbb{E}\left[\frac{V_{t_1}^{(u_1)}\,V_{t_2}^{(u_2)}}{V_{t_1}^{(u_2)}} \,\middle|\, \mathcal{F}_{t_1}^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right] \\
&= \mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\} \,\middle|\, \mathcal{F}_{t_1}^{N^{(1)},\cdots,N^{(M)},\tilde{N}^{(1)},\cdots,\tilde{N}^{(\tilde{M})}}\right] \cdot \eta_{t_1}^{-1}(u_1)\cdot \eta_{t_2-t_1}^{-1}(u_2).
\end{aligned}
$$

Now we use the martingale property of $V_t^{(u_1)}$. We take expectation of both sides of the above formula

$$
\begin{aligned}
1 &= \mathbb{E}\left[V_{t_1}^{(u_1)}\right] \\
&= \mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\}\right] \cdot \eta_{t_1}^{-1}(u_1)\cdot \eta_{t_2-t_1}^{-1}(u_2).
\end{aligned}
$$

So we obtain

$$
\mathbb{E}\left[\exp\left\{u_1 X_{t_1} + u_2\left(X_{t_2} - X_{t_1}\right)\right\}\right] = \eta_{t_1}(u_1)\cdot \eta_{t_2-t_1}(u_2).
$$

Since the above joint moment generating function factors into the product of moment generating functions, X_{t_1} and $X_{t_2} - X_{t_1}$ must be independent. We also know that the moment generating function of $X_{t_2} - X_{t_1}$ is

$$
\begin{aligned}
\varphi_{X_{t_2}-X_{t_1}}(u) = \exp\Bigg\{&\left(\lambda+\tilde{\lambda}\right)(t_2-t_1)\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{u_2 y_m}-1\right) + \sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{u_2 C\tilde{y}_m}-1\right)\right] \\
&-u_2\left(\lambda+\tilde{\lambda}\right)(t_2-t_1)\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}} + \frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\Bigg\}.
\end{aligned}
$$

For $0 < t_1 < t_2 < \cdots < t_n$, the joint moment generating function of the random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is given by

$$
\begin{aligned}
&\varphi_{X_{t_1},X_{t_2},\cdots,X_{t_n}}(u_1,u_2,\cdots,u_n)\\
&= \mathbb{E}\left[\exp\left\{u_n\left(X_{t_n}-X_{t_{n-1}}\right)\right\}\right]\cdot\mathbb{E}\left[\exp\left\{(u_{n-1}+u_n)\left(X_{t_{n-1}}-X_{t_{n-2}}\right)\right\}\right]\cdot\\
&\quad\cdots\mathbb{E}\left[\exp\left\{(u_1+u_2+\cdots+u_n)X_{t_1}\right\}\right]\\
&= \exp\left\{\left(\lambda+\tilde{\lambda}\right)(t_n-t_{n-1})\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{u_n y_m}-1\right)+\sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{u_n C\tilde{y}_m}-1\right)\right]\right.\\
&\qquad\left.-u_n\left(\lambda+\tilde{\lambda}\right)(t_n-t_{n-1})\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}}+\frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}\\
&\quad\cdot\exp\left\{\left(\lambda+\tilde{\lambda}\right)(t_{n-1}-t_{n-2})\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{(u_{n-1}+u_n)y_m}-1\right)\right.\right.\\
&\qquad\qquad\left.+\sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{(u_{n-1}+u_n)C\tilde{y}_m}-1\right)\right]\\
&\qquad\qquad\left.-(u_{n-1}+u_n)\left(\lambda+\tilde{\lambda}\right)(t_{n-1}-t_{n-2})\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}}+\frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}\\
&\quad\cdots\exp\left\{\left(\lambda+\tilde{\lambda}\right)t_1\left[\sum_{m=1}^{M}\left(\frac{\lambda p(y_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{(u_1+\cdots+u_n)y_m}-1\right)+\sum_{m=1}^{\tilde{M}}\left(\frac{\tilde{\lambda}\tilde{p}(\tilde{y}_m)}{\lambda+\tilde{\lambda}}\right)\left(e^{(u_1+\cdots+u_n)C\tilde{y}_m}-1\right)\right]\right.\\
&\qquad\left.-(u_1+\cdots+u_n)\left(\lambda+\tilde{\lambda}\right)t_1\left(\frac{\lambda\beta}{\lambda+\tilde{\lambda}}+\frac{C\tilde{\lambda}\tilde{\beta}}{\lambda+\tilde{\lambda}}\right)\right\}.
\end{aligned}
$$

This is the moment generating function for a compensated compound Poisson process with intensity $\lambda+\tilde{\lambda}$. This completes the proof. □

CHAPTER 4

A Bridge with Respect to the Fractional Brownian Motion

In this chapter, we introduce the fractional Brownian motion and some properties of this process. We change the Brownian motion which is in the Brownian bridge to a fractional Brownian motion and check if the new process converges.

4.1. Fractional Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The process (X_t) is a Gaussian process if for all $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ are jointly normally distributed. The jointly normally distribution of the random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ is determined by the means and covariances of these random variables. So the law of a Gaussian process is entirely determined by the mean function $\mathbb{E}[X_t]$ and the covariance function $\mathrm{Cov}(X_t, X_s)$, for $t, s \geq 0$.

Definition 1. A fractional Brownian motion $(B_t^{(H)})_{t \geq 0}$ with Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process with the covariance function

$$\mathbb{E}\left[B_t^{(H)} B_s^{(H)}\right] = R_H(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{4.1}$$

The fractional Brownian motion was first introduced by Kolmogorov in [**10**] and studied by Mandelbrot and Van Ness in [**11**], where a stochastic integral representation of this process in terms of a standard Brownian motion was established. By the above definition we know that a fractional Brownian motion has the following properties.

(1) Self-similarity: From (4.1) we know that the process $\{a^{-H} B_{at}^{(H)}, t \geq 0\}$ and $\{B_t^{(H)}, t \geq 0\}$ have the same law, for any $a > 0$.

(2) Stationary increments: From (4.1) we have that the increments of the fractional Brownian motion in $[s,\, t]$ has a normal distribution with zero mean and variance

$$\mathbb{E}\left[\left(B_t^{(H)} - B_s^{(H)}\right)^2\right] = |t-s|^{2H}.$$

So $B_{t+s}^{(H)} - B_s^{(H)}$ has the same law of $B_t^{(H)}$, for $s,\, t \geq 0$.

For $H = \dfrac{1}{2}$, the covariance function is $R_{\frac{1}{2}}(t,\, s) = \min(s,\, t)$, then the process $B^{(\frac{1}{2})}$ is a standard Brownian motion. However, for $H \neq \dfrac{1}{2}$, the increments of the process are not independent. Now we discuss the integral representations for the fractional Brownian motion. The integral representation for fractional Brownian motion on the whole real line which is given by

$$B_t^{(H)} = \frac{1}{C_H}\int_{\mathbb{R}}\left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\, dB_s\right], \tag{4.2}$$

where B is a Brownian motion, $H \in (0,\, 1)$ and

$$C_H = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}}$$

is obtained by Mandelbrot and Van Ness in [**11**]. For $s \in \mathbb{R}$, $t \geq 0$ the function $f_t(s) = ((t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}$ satisfies $\displaystyle\int_{\mathbb{R}} f_t^2(s)\, ds < \infty$, so the stochastic integral on the right side of (4.2) is well defined. The following integral representation for fractional Brownian motion is over a finite interval. By [**12**], for $H > \dfrac{1}{2}$, the fractional Brownian motion can be represented as

$$B_t^{(H)} = \int_0^t K_H^{(1)}(t,\, s)\, dB_s, \qquad \text{for } t \geq 0 \tag{4.3}$$

where (B_t) is a standard Brownian motion and

$$K_H^{(1)}(t,\, s) = C_H^{(1)}\, s^{\frac{1}{2}-H}\int_s^t |u-s|^{H-\frac{3}{2}}\, u^{H-\frac{1}{2}}\, du,$$

where $C_H^{(1)} = \left[\dfrac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$ and $t > s$. For $H < \dfrac{1}{2}$, the integral representation on the finite interval is different from the integral representation for $H > \dfrac{1}{2}$. By [**12**], for

$H < \dfrac{1}{2}$, the fractional Brownian motion can be represented as

$$B_t^{(H)} = \int_0^t K_H^{(2)}(t,\, s)\, dB_s, \qquad \text{for } t \geq 0,$$

where (B_t) is a standard Brownian motion and

$$K_H^{(2)}(t,\, s) = C_H^{(2)} \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}}\, du \right]$$

where $C_H^{(2)} = \left[\dfrac{2H}{(1-2H)\beta(1-2H,\, H+\frac{1}{2})} \right]^{\frac{1}{2}}$ and $t > s$. In Section 1.8 of [**6**], we know that the fractional brownian motion is not a semimartingale, for $H \neq \dfrac{1}{2}$. So we can not use Itô stochastic calculus which is defined for semimartingales to define the stochastic integral with respect to the fractional brownian motion. In next section the definition of the integral of deterministic processes with respect to a fractional brownian motion will be introduced.

4.2. Wiener Integrals for the Fractional Brownian Motion for $H > \dfrac{1}{2}$

The stochastic integrals of deterministic processes with respect to a Gaussian process are called Wiener integrals. Let $(B_t^{(H)})_{t\geq 0}$ be a fractional brownian motion with Hurst index $H > \dfrac{1}{2}$ on the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$. Fix a time interval $[0,\, T]$. For $0 = t_0 < t_1 < \cdots < t_n = T$ the stochastic integral of a step function of the form

$$\varphi_t = \sum_{i=1}^{n} a_i I_{(t_{i-1},\, t_i]}(t)$$

is naturally defined by

$$\int_0^T \varphi_t\, dB_t^{(H)} = \sum_{i=1}^{n} a_i \left(B_{t_i}^{(H)} - B_{t_{i-1}}^{(H)} \right).$$

The integral can be extended to general functions by using the convergence in $L^2(\Omega)$. Denote by $\mathcal{H}$ the closure of $(L^2([0,\, T]),\, <,>_H)$ with respect to the scalar product defined as

$$\langle f,\, g\rangle_H = H(2H-1)\int_0^T\int_0^T f(r)\, g(u)\, |r-u|^{2H-2}\, du\, dr. \tag{4.4}$$

Then $\mathcal{H}$ is a Hilbert space. Section 2.1 of [**6**] tells us the mapping $\varphi \longrightarrow \int_0^T \varphi_t\, dB_t^{(H)}$, where φ is a step function on $[0,\, T]$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian subspace of $L^2(\Omega)$ which is spanned by the random variables $\{B_t^{(H)}\,;\, t \in [0,\, T]\}$. Section 2.1 of [**6**] also tells us the definition of the Wiener integral of the deterministic function with respect to the fractional Brownian motion.

Definition 2. For $H > \frac{1}{2}$ and $\psi \in \mathcal{H}$, the Wiener integral of the deterministic function ψ with respect to the fractional Brownian motion $B^{(H)}$ is defined as

$$\int_0^T \psi_s\, dB_s^{(H)} = \int_0^T (K_H^{(1)*}\psi)(s)\, dB_s, \tag{4.5}$$

where (B_t) is a standard Brownian motion and

$$(K_H^{(1)*}\psi)(s) = \int_s^T \psi_t\, \frac{\partial K_H^{(1)}}{\partial t}(t,\, s)\, dt$$

which is a square-integrable function.

The integral on the right side of (4.5) is well defined for $\psi \in \mathcal{H}$ and we get the representation of the Wiener integral of the deterministic function with respect to the fractional brownian motion in terms of an integral with respect to the Brownian motion. If $\psi = (K_H^{(1)*})^{-1} I_{[0,\,t]}$, then we have

$$B_t = \int_0^t \left((K_H^{(1)*})^{-1} I_{[0,\,t]} \right)(s)\, dB_s^{(H)}. \tag{4.6}$$

From (4.3) and (4.6), we know that $B^{(H)}$ and B generate the same filtration. Due to the isometry property of the mapping $\psi \longrightarrow \int_0^T \psi_t\, dB_t^{(H)}$, where $\psi \in \mathcal{H}$, we have

$$\left\| \int_0^T \psi_s\, dB_s^{(H)} \right\|^2_{L^2(\Omega)} = \|\psi\|^2_{\mathcal{H}},$$

i.e.,

$$\mathbb{E}\left[\left(\int_0^T \psi_t \, dB_t^{(H)}\right)^2\right] = H(2H-1)\int_0^T\int_0^T \psi_r\,\psi_u\,|r-u|^{2H-2}\,du\,dr. \tag{4.7}$$

The left side of (4.7) is the variance of the Wiener integral of the deterministic function ψ with respect to the fractional brownian motion $B^{(H)}$. In next section we use the variance of the Wiener integral to know where the integral converges.

4.3. A Bridge with Respect to the Fractional Brownian Motion

Let $(B_t^{(H)})$ be a fractional brownian motion with Hurst index $H > \frac{1}{2}$. Consider the stochastic differential equation

$$dX_t = dB_t^{(H)} + \frac{b - X_t}{1-t}\,dt,$$

with the initial value $X_0 = 0$ and some constant b. The solution $(X_t)_{0\le t\le 1}$ is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s^{(H)} + bt. \tag{4.8}$$

Since the fractional brownian motion $B^{(H)}$ is a centered Gaussian process, the process $(1-t)\int_0^t \frac{1}{1-s}\,dB_s^{(H)}$ is also a centered Gaussian process. Then we have

$$\mathbb{E}\left[(1-t)\int_0^t \frac{1}{1-s}\,dB_s^{(H)}\right] = 0.$$

We will use the variance of the first term of (4.8) to see where the process (X_t) approaches as $t \to 1$.

Theorem 3. Suppose that the process $(X_t)_{0\le t<1}$ satisfies (4.8). Then we have that $X_t \to b$ $\mathbb{P}$-a.s. as $t \to 1$.

Proof. From the formula (4.7), we get

$$\mathbb{E}\left[\left((1-t)\int_0^t \frac{1}{1-s}\,dB_s^{(H)}\right)^2\right] = (1-t)^2 H(2H-1)\int_0^t\int_0^t |r-u|^{2H-2}\frac{1}{1-r}\,\frac{1}{1-u}\,du\,dr. \tag{4.9}$$

By Fubini's theorem, we obtain

$$\begin{aligned}(1-t)^2 H(2H-1)\int_0^t\int_0^t |r-u|^{2H-2}\frac{1}{1-r}\,\frac{1}{1-u}\,du\,dr \\ = 2H\,(2H-1)\,(1-t)^2\int_0^t\int_0^r (r-u)^{2H-2}\frac{1}{1-r}\,\frac{1}{1-u}\,du\,dr.\end{aligned} \tag{4.10}$$

Since $0 \le u \le r \le t \le 1$, we know that $\dfrac{1}{1-u} \le \dfrac{1}{1-r}$. Then we have

$$\begin{aligned}&2H\,(2H-1)\,(1-t)^2\int_0^t\int_0^r (r-u)^{2H-2}\frac{1}{1-r}\,\frac{1}{1-u}\,du\,dr \\ &\le 2H(2H-1)(1-t)^2\int_0^t\int_0^r (r-u)^{2H-2}\frac{1}{(1-r)^2}\,du\,dr \\ &= 2H(1-t)^2\int_0^t r^{2H-1}\frac{1}{(1-r)^2}\,dr.\end{aligned}$$

Since $r \le 1$, we get

$$\begin{aligned}2H(1-t)^2\int_0^t r^{2H-1}\frac{1}{(1-r)^2}\,dr &\le 2H(1-t)^2\int_0^t \frac{1}{(1-r)^2}\,dr \\ &= 2Ht(1-t).\end{aligned}$$

Then we obtain

$$\mathbb{E}\left[\left((1-t)\int_0^t \frac{1}{1-s}\,dB_s^{(H)}\right)^2\right] \le 2Ht(1-t)$$

which converges to 0 as $t \to 1$. So we have that the process $(1-t)\displaystyle\int_0^t \frac{1}{1-s}\,dB_s^{(H)}$ converges to 0 $\mathbb{P}$-a.s. as $t \to 1$. Finally, we know that the process $(X_t)_{0\le t\le 1}$ converges to b $\mathbb{P}$-a.s. as $t \to 1$. □

The process $(X_t)_{0\le t\le 1}$ which satisfies (4.8) is a bridge with respect to the fractional Brownian motion from 0 to fixed point b. Next, we have a bridge with respect to the fractional Brownian motion from 0 to a random variable. Let Y be a random variable. Consider the stochastic differential equation

$$dX_t = dB_t^{(H)} + \frac{Y - X_t}{1-t}\,dt$$

with the initial value $X_0 = 0$. The solution $(X_t)_{0\leq t\leq 1}$ is given by

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\, dB_s^{(H)} + t\, Y. \tag{4.11}$$

By the proof of **Theorem 3**, we have that the process $(X_t)_{0\leq t\leq 1}$ satisfying (4.11) converges to Y $\mathbb{P}$-a.s. as $t \to 1$.

In the following we want to know whether the process $(X_t)_{0\leq t\leq 1}$ is a fractional Brownian motion if we let Y be a standard normally distributed random variable with $\mathbb{E}[Y] = 0$ and $\mathrm{Var}(Y) = 1$. From (4.11) we know $(X_t)_{0\leq t\leq 1}$ is a centered Gaussian process. Now we see whether $\mathbb{E}[X_t^2]$ is equal to t^{2H}. If $\mathbb{E}[X_t^2] \neq t^{2H}$, then $(X_t)_{0\leq t\leq 1}$ is not a fractional Brownian motion. The variance of X_t is given by

$$\begin{aligned}\mathbb{E}\left[X_t^2\right] = (1-t)^2\,\mathbb{E}\left[\left(\int_0^t \frac{1}{1-s}\, dB_s^{(H)}\right)^2\right] + t^2\,\mathbb{E}\left[Y^2\right] \\ + 2t(1-t)\,\mathbb{E}\left[Y\left(\int_0^t \frac{1}{1-s}\, dB_s^{(H)}\right)\right].\end{aligned}$$

$B^{(H)}$ and Y are independent and $Y \sim N(0,1)$, then we have

$$\begin{aligned}\mathbb{E}\left[X_t^2\right] &= (1-t)^2\,\mathbb{E}\left[\left(\int_0^t \frac{1}{1-s}\, dB_s^{(H)}\right)^2\right] + t^2 \\ &\quad + 2t(1-t)\,\mathbb{E}[Y]\cdot\mathbb{E}\left[\int_0^t \frac{1}{1-s}\, dB_s^{(H)}\right] \\ &= (1-t)^2\,\mathbb{E}\left[\left(\int_0^t \frac{1}{1-s}\, dB_s^{(H)}\right)^2\right] + t^2\end{aligned}$$

From (4.9) and (4.10), we have that

$$E\left[X_t^2\right] = t^2 + 2H\,(2H-1)\,(1-t)^2\int_0^t\int_0^r (r-u)^{2H-2}\frac{1}{1-r}\,\frac{1}{1-u}\,du\,dr. \tag{4.12}$$

By Taylor's formula, we have $\dfrac{1}{1-u} = \displaystyle\sum_{k=0}^{\infty} u^k$. Then we obtain

$$\int_0^r (r-u)^{2H-2}\,\frac{1}{1-u}\,du = \sum_{k=0}^{\infty}\int_0^r (r-u)^{2H-2}\,u^k\,du. \tag{4.13}$$

The first term of (4.13) is

$$\int_0^r (r-u)^{2H-2}\,du = \frac{1}{2H-1}\,r^{2H-1}.$$

The second term of (4.13) is

$$\begin{aligned}\int_0^r (r-u)^{2H-2}\,u\,du &= \int_0^r \frac{1}{2H-1}\,(r-u)^{2H-1}\,du \\ &= \frac{1}{(2H-1)(2H)}\,r^{2H}.\end{aligned}$$

For all $k \in \mathbb{Z}$, for $n = 1$, we have that

$$\begin{aligned}\int_0^r (r-u)^{2H+k}\,u^n\,du &= \int_0^r (r-u)^{2H+k}\,u\,du \\ &= \int_0^r \frac{1}{2H+k+1}\,(r-u)^{2H+k+1}\,du \\ &= \frac{1}{(2H+k+1)(2H+k+2)}\,r^{2H+k+2}.\end{aligned}$$

Suppose that for all $k \in \mathbb{Z}$, for $n = m-1$, the following equation holds

$$\int_0^r (r-u)^{2H+k}\,u^{m-1}\,du = \frac{(m-1)!}{(2H+k+1)(2H+k+2)\cdots(2H+k+m)}\,r^{2H+k+m}. \quad (4.14)$$

For $n = m$, we have

$$\begin{aligned}\int_0^r (r-u)^{2H+k}\,u^m\,du &= \int_0^r \frac{1}{2H+k+1}\,(r-u)^{2H+k+1}\,m\,u^{m-1}\,du \\ &= \frac{m}{2H+k+1}\int_0^r (r-u)^{2H+k+1}\,u^{m-1}\,du.\end{aligned}$$

Let $\tilde{k} = k+1$. Due to (4.14), we obtain

$$\begin{aligned}\int_0^r (r-u)^{2H+k}\,u^m\,du &= \frac{m}{2H+\tilde{k}}\int_0^r (r-u)^{2H+\tilde{k}}\,u^{m-1}\,du \\ &= \frac{m!}{(2H+\tilde{k})(2H+\tilde{k}+1)\cdots(2H+\tilde{k}+m)}\,r^{2H+\tilde{k}+m} \\ &= \frac{m!}{(2H+k+1)(2H+k+2)\cdots(2H+k+m+1)}\,r^{2H+k+m+1}\end{aligned}$$

By induction on n, the result

$$\int_0^r (r-u)^{2H+k}\, u^n\, du = \frac{n!}{(2H+k+1)(2H+k+2)\cdots(2H+k+n+1)}\, r^{2H+k+n+1}$$

holds for all $n \in \mathbb{N}$. Then we get

$$\int_0^r (r-u)^{2H-2}\, \frac{1}{1-u}\, du = \sum_{k=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)}\, r^{2H+k-1}.$$

Then we obtain

$$\begin{aligned}
&\int_0^t \int_0^r (r-u)^{2H-2} \frac{1}{1-r}\, \frac{1}{1-u}\, du\, dr \\
&= \int_0^t \frac{1}{1-r} \left(\sum_{k=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)}\, r^{2H+k-1} \right) dr \\
&= \sum_{k=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \int_0^t \frac{1}{1-r}\, r^{2H+k-1} dr.
\end{aligned}$$

Since $\dfrac{1}{1-r} = \sum_{j=0}^{\infty} r^j$, we get

$$\begin{aligned}
&\int_0^t \int_0^r (r-u)^{2H-2} \frac{1}{1-r}\, \frac{1}{1-u}\, du\, dr \\
&= \sum_{k=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \int_0^t \sum_{j=0}^{\infty} r^{2H+k+j-1}\, dr \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \int_0^t r^{2H+k+j-1}\, dr \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \cdot \frac{1}{2H+k+j}\, t^{2H+k+j}.
\end{aligned}$$

From (4.12), we have that the variance of X_t is given by

$$\begin{aligned}
&\mathbb{E}\left[X_t^2\right] \\
&= t^2 + (2H)(2H-1)(1-t)^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \cdot \frac{1}{2H+k+j}\, t^{2H+k+j}.
\end{aligned} \tag{4.15}$$

Suppose the following equation holds

$$t^2 + (2H)(2H-1)(1-t)^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \cdot \frac{1}{2H+k+j}\, t^{2H+k+j} = t^{2H}.$$

Then

$$(2H)(2H-1)(1-t)^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \cdot \frac{1}{2H+k+j}\, t^{k+j} = 1 - t^{2-2H}.$$

Let $s = 1-t$, then we have that

$$\begin{aligned} & (2H)(2H-1)s^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{k!}{(2H-1)(2H)\cdots(2H+k-1)} \cdot \frac{1}{2H+k+j}\, (1-s)^{k+j} \\ = \;& 1-(1-s)^{2-2H}. \end{aligned} \tag{4.16}$$

Since

$$\begin{aligned} 1-(1-s)^{2-2H} &= 1 - \sum_{m=0}^{\infty} \binom{2-2H}{m} (-s)^m \\ &= (2-2H)s + \sum_{m=2}^{\infty} \binom{2-2H}{m} (-s)^m. \end{aligned}$$

But from (4.16), the exponent of s in every term on the left side of (4.16) is larger than two. Then we know that $\mathbb{E}\left[X_t^2\right] \neq t^{2H}$, for $0 < t \leq 1$. Thus $(X_t)_{0 \leq t \leq 1}$ is not a fractional Brownian motion.

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