國立交通大學

應用數學系

碩 士 論 文

薛丁格算子前兩個固有值差距之下界估計

On the lower bounds of the first two eigenvalues in the Schrödinger operator

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摘　　　要

在這篇文章中，我們找到一個由一些微分不等式所組成的系統，這個系統的解可以使 Gradient estimate 成立。適當選取這個系統的解，可以幫助我們估計薛丁格算子前兩個固有值差距之下界。

On the lower bounds of the first two eigenvalues
in the Schrödinger operator

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Abstract

We find a system of differential inequalities under which the gradient estimate holds. Using appropriately chosen test functions, we find some lower bounds of the gap of the first two eigenvalues in the Schrödinger operator.

誌　　謝

這一段求學路上，感謝許義容老師的包容，對於學生不懂的地方，老師總是不厭其煩的細心講解，令學生獲益良多。除了認真教學以外，老師總是以笑容面對學生、以鼓勵代替責備，讓學生在遭遇挫折時，能很快的建立起信心、面對問題。很慶幸在這旅途上找到許義容老師這樣的指導教授，老師，謝謝您。另外，感謝交大、清大數學系的老師們在這段日子的教導，特別感謝王夏聲教授與黃明傑教授，不但抽空擔任口試委員，並且給予許多寶貴的建議以及想法，讓學生以不同的方式切入思考這個主題，使得這篇文章得以改進。

一路上，若不是家人的支持與鼓勵，在下可能也沒辦法全心全力放在讀書上，謝謝你們一路的提攜、拉拔。也要感謝りか的關心，りか犧牲了返台假期，時常陪同在下到醫院看診，由衷感謝りか的關心、照顧。最後，一定要感謝那些甘苦與共的朋友們，奈々、宇軒、俞碩、曉恩、佩錚，你們為我的研究生活增色不少、也帶給我不少有趣的回憶，衷心的感謝所有給予幫助的朋友、貴人，謝謝大家。

Contents

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and V be a smooth potential in Ω. The spectrum of the Dirichlet eigenvalue problem

$$\begin{cases} (\triangle - V)f + \lambda f = 0 & \text{in } \ \Omega\ , \\ f = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{1.1}$$

are discrete, and can be arranged in nondecreasing order as follows

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots .$$

It is an interesting problem to find a lower bound for the first gap $\lambda_2 - \lambda_1$ in terms of the geometrical invariants of Ω and the given potential function V.

In 1983, consider bounded convex domains Ω with convex potentials V, M. van den Berg conjectured that the lower bound is $3\frac{\pi^2}{d^2}$, where d is the diameter of Ω. In 1985, Bun Wong, Shing-Tung Yau and Stephen S.-T. Yau [1] showed that $\lambda \geq \frac{\pi^2}{4d^2}$, where $\lambda = \lambda_2 - \lambda_1$. Qihuang Yu and Jia-Qing Zhong [2] later proved that $\lambda \geq \frac{\pi^2}{d^2}$ by using the log-convexity of the first eigenfunction and choosing an appropriate test function. For some special potential function V and for some special domain Ω, the conjecture has been proved. In 1989, Mark S. Ashbaugh and Rafael Benguria [4] proved that $\lambda \geq 3\frac{\pi^2}{d^2}$ for one-dimensional case under the additional assumption that V is a symmetric "single-well" potential.

Another direction of estimate on the lower bound λ is considering the quantity of "global log-convexity." In [5], Shing-Tung Yau proved that

$$\lambda \geq \theta\frac{\pi^2}{d^2} + 2[\cos(\sqrt{\theta}\pi)]^2\alpha$$

where θ is any constant with $0 < \theta < \frac{1}{4}$, and $\alpha > 0$ is the infimum of the global log-convexity of the first eigenfunction,

$$\alpha = \inf_{\substack{x \in \Omega \\ \tau \in T_x\Omega \\ |\tau| = 1}} (-\log f_1)_{\tau\tau}.$$

Shing-Tung Yau [5] gave an interesting estimate on the lower bound of α. He showed that if V is strictly convex function such that

$$\inf_{\substack{x \in \Omega \\ \tau \in T_x\Omega \\ |\tau| = 1}} V_{\tau\tau} \ \geq \ c$$

for some positive constant c, then $\alpha \geq \sqrt{\frac{c}{2}} > 0$.

In this thesis, we first derive the following theorem for finding test functions of gradient estimate. Let v be the normalized ratio of the first two eigenfunctions f_1

and f_2,

$$v = \frac{2\frac{f_2}{f_1} - (M-m)}{M+m},$$

where $M = \max_{x\in\overline{\Omega}}(\frac{f_2}{f_1})$, $m = -\min_{x\in\overline{\Omega}}(\frac{f_2}{f_1})$ with $M \geq m$ and $a = \frac{M-m}{M+m}$.

Theorem 1.1. *If f is a function of v, and satisfies the following conditions:*

(a): $f > 0$
(b): $f'' + 2(\lambda - 2\alpha) \leq 0$
(c): $f[f'' + 2(\lambda - 2\alpha)] - \frac{1}{2}f'[f' + 2\lambda(v+a)] < 0$

then $|\nabla v|^2 \leq f(v)$ *in* $\overline{\Omega}$.

In 2008, Jun Ling [3] proved that $\lambda \geq \frac{\pi^2}{d^2} + 0.62\alpha$. More detailed, he proved that $\lambda \geq \frac{\pi^2}{d^2} + \alpha$ if $a = 0$ or $a \geq \frac{\pi^2\alpha}{4\lambda}$, and $\lambda \geq \frac{\pi^2}{d^2} + 0.62\alpha$ if $0 < a < \frac{\pi^2\alpha}{4\lambda}$. As an application of Theorem 1.1, we finally show that the lower bound of λ near $\frac{\pi^2}{d^2} + \alpha$ if a near 0.

2. Proof of the Main Theorem

Throughout this thesis, we assume that the the domain Ω is strictly convex, and the potential function V is strictly convex.

Let f_1 and f_2 be the first and second eigenfunctions of (1.1). It is well known that the first eigenfunction f_1 must be a positive function and the second eigenfunction changes sign since $\int f_1 f_2 = 0$. Since $f_1 > 0$, $u = f_2/f_1$ is a well-defined function, and smooth to the boundary of Ω [1].

Suppose

$$M = \max_{x\in\overline{\Omega}} u(x) \ ; \ -m = \min_{x\in\overline{\Omega}} u(x).$$

We may assume that $M \geq m$, otherwise, we can use $-f_2$ instead of f_2. Setting

$$\begin{aligned} v &= \left(u - \frac{M-m}{2}\right) \Bigg/ \left(M - \frac{M-m}{2}\right) \\ &= \frac{2u-(M-m)}{M+m}, \end{aligned} \tag{2.2}$$

then v is a smooth function on $\overline{\Omega}$ and $\max_{x\in\overline{\Omega}} v(x) = 1$; $\min_{x\in\overline{\Omega}} v(x) = -1$.

By computing, we have

$$
\begin{aligned}
\triangle v &= \sum_i v_{ii} = \frac{2}{M+m}\sum_i u_{ii} = \frac{2}{M+m}\sum_i \left(\frac{f_2}{f_1}\right)_{ii} \\
&= \frac{2}{M+m}\sum_i \left(\frac{(f_2)_{ii}f_1 - f_2(f_1)_{ii}}{f_1^2} - 2\frac{(f_1)_i}{f_1}\frac{(f_2)_i f_1 - f_2(f_1)_i}{f_1^2}\right) \\
&= \frac{2}{M+m}\left(\frac{f_1\triangle f_2 - f_2\triangle f_1}{f_1^2} + 2\sum_i(-\log f_1)_i u_i\right) \\
&= \frac{2}{M+m}\left(\frac{f_1(-\lambda_2 f_2 + V f_2) - f_2(-\lambda_1 f_1 + V f_1)}{f_1^2} + 2\nabla(-\log f_1)\nabla u\right) \\
&= -\lambda\frac{2u}{M+m} + 2\nabla(-\log f_1)\nabla v,
\end{aligned}
$$

$$
\triangle v = -\lambda(v+a) + 2\nabla v\cdot\nabla(-\log f_1), \tag{2.3}
$$

where

$$
a = \frac{M-m}{M+m}\ ;\ 0 \le a < 1. \tag{2.4}
$$

Since $\frac{\partial}{\partial n}f_1\mid_{\partial\Omega}\neq 0$, here n is the outward normal of $\partial\Omega$, using (2.3), v satisfies the Neumann boundary condition $\frac{\partial}{\partial n}v\mid_{\partial\Omega}= 0$.

Proof of Theorem 1.1. Define a function on $\overline{\Omega}$ by

$$
P(x) = |\nabla v|^2 - f(v) \tag{2.5}
$$

where f satisfies (a),(b) and (c).

Case 1: If $P(x)$ attains its maximum at $x_0\in\partial\Omega$, we can choose an orthonormal frame $l_1, l_2, \ldots, l_n$ around x_0 such that l_n is perpendicular to $\partial\Omega$ and pointing outward. Since $P(x_0)$ is the maximum of $P(x)$,

$$
\begin{aligned}
0 \le \frac{\partial P}{\partial x_n}(x_0) &= 2\sum_{i=1}^{n-1} v_i v_{in} + 2v_n v_{nn} - f'(v)v_n \\
&= 2\sum_{i=1}^{n-1} v_i v_{in},
\end{aligned} \tag{2.6}
$$

where the notation $\partial/\partial x_n$ is denote the restriction of l_n on $\partial\Omega$.

From the definition of the second fundamental form of $\partial\Omega$ in $\mathbb{R}^n$, we have

$$
v_{in} = -\sum_{j=1}^{n-1} h_{ij} v_j \tag{2.7}
$$

where (h_{ij}) is the second fundamental form and (h_{ij}) is positive definite. Putting (2.7) into (2.6), we obtain

$$0 \leq -2 \sum_{i,j=1}^{n-1} h_{ij} v_i v_j \leq 0. \tag{2.8}$$

This implies that $v_1 = v_2 = \cdots = v_{n-1} = 0$, and hence $\nabla v(x_0) = 0$. Therefore, we have $P(x) \leq P(x_0) = -f(x_0) < 0$ which implies $|\nabla v|^2 < f(v)$.

Case 2: If $P(x)$ attains its maximum at $x_0 \in \Omega$ and $\nabla v(x_0) = 0$, then we have the same conclusion as above.

Case 3: If $P(x)$ attains its maximum at $x_0 \in \Omega$ and $\nabla v(x_0) \neq 0$. First, we compute the Laplacian of $P(x)$ for $x \in \Omega$,

$$\begin{aligned}
\triangle P(x) &= \sum_j P_{jj} \\
&= 2\sum_{i,j} v_{ij}^2 + 2\sum_{i,j} v_i v_{ijj} - f''(v)\sum_j v_j^2 - f'(v)\sum_j v_{jj} \\
&= 2\sum_{i,j} v_{ij}^2 + 2\nabla v \cdot \nabla(\triangle v) - f''(v)|\nabla v|^2 - f'(v)\triangle v \\
&= 2\sum_{i,j} v_{ij}^2 + 2\nabla v \cdot \nabla[-\lambda(v+a) + 2\nabla v \cdot \nabla(-\log f_1)] \\
&\quad -f''(v)|\nabla v|^2 - f'(v)[-\lambda(v+a) + 2\nabla v \cdot \nabla(-\log f_1)] \\
&= 2\sum_{i,j} v_{ij}^2 + [-2\lambda - f''(v)]|\nabla v|^2 \underbrace{+4\sum_{i,j} v_j v_{ij}(-\log f_1)_i} \\
&\quad +4\sum_{i,j}(-\log f_1)_{ij} v_i v_j + \lambda f'(v)(v+a) \\
&\quad \underbrace{-2f'(v)\sum_i v_i(-\log f_1)_i} \\
&= 2\sum_{i,j} v_{ij}^2 + [-2\lambda - f''(v)]|\nabla v|^2 + 4\sum_{i,j}(-\log f_1)_{ij} v_i v_j \\
&\quad +\lambda f'(v)(v+a) \underbrace{+2\nabla P \cdot \nabla(-\log f_1)}
\end{aligned} \tag{2.9}$$

At x_0, we have $\nabla P(x_0) = 0$, i.e., $2\sum_i v_i v_{ij} - f'(v) v_j = 0$ for all j.

We can choose an orthonormal frame around x_0 such that

$$v_1(x_0) \neq 0\ ,\ \ v_i(x_0) = 0 \text{ for } 2 \leq i \leq n.$$

Hence at x_0, $2v_1v_{1j} = f'(v)v_j$ for all j, which implies,

$$v_{1j} = \frac{1}{2}f'(v)\frac{v_j}{v_1} = \begin{cases} \frac{1}{2}f'(v) & \text{if } j=1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Thus,

$$\begin{aligned} 0 &\geq \triangle P(x_0) \geq 2v_{11}^2 + [-2\lambda - f''(v)]|\nabla v|^2 \\ &\quad +4\sum_{i,j}(-\log f_1)_{ij}v_iv_j + \lambda f'(v)(v+a) \\ &= \frac{1}{2}[f'(v)]^2 + [-2\lambda - f''(v)]v_1^2 + 4(-\log f_1)_{11}v_1^2 + \lambda f'(v)(v+a) \\ &\geq \frac{1}{2}[f'(v)]^2 + [4\alpha - 2\lambda - f''(v)]v_1^2 + \lambda f'(v)(v+a) \\ &= \frac{1}{2}[f'(v)]^2 + [4\alpha - 2\lambda - f''(v)][|\nabla v|^2 - f(v)] \\ &\quad +f(v)[4\alpha - 2\lambda - f''(v)] + \lambda f'(v)(v+a). \end{aligned} \tag{2.11}$$

Rewrite (2.11), we have

$$\begin{aligned} f(v)[f''(v) + 2(\lambda - 2\alpha)] &- \frac{1}{2}f'(v)[f'(v) + 2\lambda(v+a)] \\ &\geq -[f''(v) + 2(\lambda - 2\alpha)]P(x_0). \end{aligned} \tag{2.12}$$

The inequality (2.12) holds for any arbitrary $P(x) = |\nabla v|^2 - f(v)$, there is no restriction on f. Now, let $\sigma = \max_{x\in\overline{\Omega}}[|\nabla v|^2 - f(v)]$. Suppose $|\nabla v|^2 > f(v)$ somewhere, then $\sigma > 0$ and $\widetilde{f} = f + \sigma$ satisfies (2.12) for $\widetilde{P} = |\nabla v|^2 - \widetilde{f}(v)$ and there exists a point $x_0 \in \overline{\Omega}$ such that $\max_{x\in\overline{\Omega}} \widetilde{P}(x) = \widetilde{P}(x_0) = 0$. Hence, at x_0

$$(f+\sigma)[f'' + 2(\lambda - 2\alpha)] - \frac{1}{2}f'[f' + 2\lambda(v+a)] \geq 0,$$

$$\sigma[f'' + 2(\lambda - 2\alpha)] + f[f'' + 2(\lambda - 2\alpha)] - \frac{1}{2}f'[f' + 2\lambda(v+a)] \geq 0,$$

which is impossible. Thus $\sigma \leq 0$ and $|\nabla v|^2 \leq f(v)$ for all $x \in \overline{\Omega}$.

□

Remark. The result holds if Ω is a smooth strictly convex domain in a Riemannian manifold with nonnegative Ricci curvature.

These conditions $(a), (b)$ and (c) can simplify if we let $w = v + a$.

Corollary 2.1. *If φ is a function of $w = v + a$, $w \in [-1+a, 1+a]$, φ satisfies the following conditions:*

(a′): $\varphi > 0$
(b′): $\varphi'' + 2(\lambda - 2\alpha) \leq 0$

(c′): $\varphi[\varphi'' + 2(\lambda - 2\alpha)] - \frac{1}{2}\varphi'[\varphi' + 2\lambda w] < 0$

then $|\nabla w|^2 \le \varphi(w)$ *in* $\overline{\Omega}$.

3. Lower bounds

In this section, we use Theorem 1.1 and its corollary to derive some interesting lower bound. The following Theorem was proved by Shing-Tung Yau [5], and the proof is similarly to Theorem 1.1.

Theorem 3.1. $\lambda \ge 2\alpha$.

Proof. We repeat the process of Theorem 1.1. Let $P(x) = |\nabla v|^2$ and $P(x_0) = \max P$.

Case 1: If $x_0 \in \partial\Omega$ or $x_0 \in \Omega$ with $\nabla v(x_0) = 0$. then $|\nabla v|^2(x) = P(x) \le P(x_0) = |\nabla v|^2(x_0) = 0$. This implies that $v_1 = v_2 = \cdots = v_n = 0$ for $x \in \overline{\Omega}$. Thus v is a constant which is impossible.

Case 2: If $x_0 \in \Omega$ and $\nabla v(x_0) \neq 0$. From (2.12) we have

$$\begin{aligned} &0 \ge -2(\lambda - 2\alpha)P(x_0), \\ &0 \le 2(\lambda - 2\alpha), \\ &\lambda \ge 2\alpha. \end{aligned}$$

□

Bun Wong, Shing-Tung Yau and Stephen S.-T. Yau [1] use gradient estimate to show that $\lambda \ge \pi^2/4d^2$. Here we verify this result by choosing an appropriate test function that satisfies the assumption(a′), (b′) and(c′).

Theorem 3.2.

$$\lambda \ge \left[\frac{\pi}{2} + \arcsin\left(\frac{1-a}{1+a}\right)\right]^2 \Bigg/ d^2.$$

Proof. If we choose $\varphi(w)$ with following form [2]

$$\varphi(w) = \lambda(b^2 - w^2)$$

for $b > 1 + a$, $w \in [-1 + a, 1 + a]$ and $w = v + a$. Then φ satisfies (a′), (b′) and (c′). Thus $|\nabla w|^2 \le \lambda(b^2 - w^2)$ in $\overline{\Omega}$. Hence,

$$\sqrt{\lambda} \ge \frac{|\nabla w|}{\sqrt{b^2 - w^2}}. \tag{3.13}$$

Suppose that x_1 and $x_2 \in \overline{\Omega}$ such that $w(x_1) = -1 + a, w(x_2) = 1 + a$, and let $\mathcal{L}$ be the line segment between x_1 and x_2. $\mathcal{L}$ lies on $\overline{\Omega}$ completely, because it is convex. We integrate both sides of (3.13) along $\mathcal{L}$ from x_1 to x_2 and obtain

$$\sqrt{\lambda}d \ge \int_{\mathcal{L}} \frac{|\nabla w|}{\sqrt{b^2 - w^2}} dl \ge \arcsin\left(\frac{1+a}{b}\right) - \arcsin\left(\frac{-1+a}{b}\right). \tag{3.14}$$

Letting $b \longrightarrow 1+a$, we have

$$\sqrt{\lambda} d \geq \frac{\pi}{2} - \arcsin\left(\frac{-1+a}{1+a}\right). \tag{3.15}$$

□

Remark. If $a = 0$, then $\lambda \geq \pi^2/d^2$ [2]. Furthermore, since

$$I(a) = \left[\frac{\pi}{2} + \arcsin\left(\frac{1-a}{1+a}\right)\right]^2 \Big/ d^2$$

is a continuous function of a, the lower bound near π^2/d^2 as a near 0.

Jun Ling[3] gave several lower bounds for different a. He showed that $\lambda \geq \frac{\pi^2}{d^2} + \alpha$ if $a = 0$ or $a \geq \frac{\delta\pi^2}{4}$ where $\delta = \frac{\alpha}{\lambda}$. And $\lambda \geq \frac{\pi^2}{d^2} + 0.62\alpha$ for $0 < a < \frac{\delta\pi^2}{4}$. However, using the same argument as above, we find that an estimate of the lower bound which is a continuous function of a, and show that the lower bound is almost $\frac{\pi^2}{d^2} + \alpha$ when a near 0.

Theorem 3.3.

$$\lambda \geq \frac{1}{d^2} \frac{\left(\int_{\arcsin[(-1+a)/(1+a)]}^{\pi/2} 1 dt\right)^3}{\int_{\arcsin[(-1+a)/(1+a)]}^{\pi/2} z(t) dt}$$

where $\quad z(t) = \delta\left(1 + 2t\tan t + \frac{t^2}{(\cos t)^2} + \frac{-\pi^2/4}{(\cos t)^2}\right) + 1$ *and* $\delta = \frac{\alpha}{\lambda}$

In particular, if $a = 0$ we have $\lambda \geq \frac{\pi^2}{d^2} + \alpha$.

Proof. Let $t = \arcsin\left(\frac{w}{b}\right)$ where $w = v + a$, $b > 1 + a$ and let z be a function of t such that $z(t) > 0$ for $t \in [\arcsin\left(\frac{-1+a}{b}\right), \arcsin\left(\frac{1+a}{b}\right)]$. Consider the test function with following form:

$$\varphi(w) = \lambda b^2 \cos^2 t \; z(t) \tag{3.16}$$

By direct computation, we have

$$\frac{d}{dw}\varphi(w) = -2\lambda b \sin t \; z + \lambda b \cos t \; z' \tag{3.17}$$

$$\frac{d^2}{dw^2}\varphi(w) = -2\lambda z - 3\lambda \frac{\sin t}{\cos t} z' + \lambda z'' \tag{3.18}$$

Putting (3.16)–(3.18) into (c′), one can get

$$\begin{aligned} &\lambda b^2 \cos^2 t \; z \left[\lambda z'' - 3\lambda \frac{\sin t}{\cos t} z' - 2\lambda z + 2(\lambda - 2\alpha)\right] - \\ &\frac{1}{2}[\lambda b \cos t \; z' - 2\lambda b \sin t \; z][\lambda b \cos t \; z' - 2\lambda b \sin t \; z + 2\lambda w] < 0 \end{aligned} \tag{3.19}$$

Dividing both sides of the above inequality by $2b^2\lambda^2 z > 0$, we have

$$\begin{aligned} &\frac{1}{2}\cos^2 t\ z'' - \sin t \cos t\ z' - z + 1 - 2\delta\cos^2 t + \\ &\frac{z'}{4z}\cos t[2\sin t\ z - \cos t\ z' - 2\sin t] < 0 \end{aligned} \tag{3.20}$$

where $\delta = \alpha/\lambda$.

From (3.20), we find that if $z(t)$ satisfies the following conditions:

(i): $z(t) > 0$ for $t \in \left[\arcsin\left(\frac{-1+a}{b}\right), \arcsin\left(\frac{1+a}{b}\right)\right]$
(ii): $-2\lambda z - 3\lambda\frac{\sin t}{\cos t}z' + \lambda z'' + 2(\lambda - 2\alpha) \le 0$
(iii): $\frac{1}{2}\cos^2 t\ z'' - \sin t\cos t\ z' - z + 1 - 2\delta\cos^2 t \le 0$
(iv): $z'[2\sin t\ z - \cos t\ z' - 2\sin t] \le 0$

where the left hand side of (iii) and (iv) are not all zero at the same t, then $\varphi(w) = \lambda b^2\cos^2 t\ z(t)$ satisfies (a′), (b′) and (c′). Furthermore, from the corollary we know $|\nabla w|^2 \le \varphi(w)$. Now, we must find a function $z(t)$ that satisfies these conditions. To simplify this problem, we let $z(t) = \frac{\delta}{\cos^2 t}y(t) + 1$ and solve $y(t)$ that satisfies:

(i′): $y(t) > \frac{-\cos^2 t}{\delta}$
(ii′): $\frac{1}{2}y'' + \frac{1}{2}\tan t\ y' - 2\cos^2 t \le 0$
(iii′): $\frac{1}{2}y'' + \tan t\ y' - 2\cos^2 t \le 0$
(iv′): $y'[2\sin t\ y + \cos t\ y'] \ge 0$

where the left hand side of (iii′) and (iv′) are not all zero at the same t. Solve the differential equation (iii′), we have

$$y_0(t) = \cos^2 t + 2t\sin t\cos t + t^2 + C_1(t + \sin t\cos t) + C_2 \tag{3.21}$$

where C_1 and C_2 are constants. Put (3.21) into (ii′), we have

$$(2t + C_1)\sin t\cos t \ge 0 \tag{3.22}$$

which implies $C_1 = 0$. If we take $C_2 = \frac{-\pi^2}{4}$, then $y_0(t)$ satisfies (i′), (ii′) and (iii′). Now we have to modify $y_0(t)$ to get an appropriate test function $\varphi(w)$. Let $y_\epsilon(t) = y_0(t) + \epsilon\cos^2 t$, where $\epsilon > 0$ is a constant. Then we can find that $y_\epsilon(t)$ satisfies all conditions for sufficient small ϵ. Thus

$$\varphi_\epsilon(w) = \lambda b^2\cos^2 t\left(\frac{\delta}{\cos^2 t}y_\epsilon(t) + 1\right)$$

is an appropriate test function that satisfies $|\nabla w|^2 \le \varphi_\epsilon(w)$. Letting $\epsilon \to 0$, we have

$$|\nabla w|^2 \le \varphi(w),\ \text{where}\ \varphi(w) = \lambda b^2\cos^2 t\left(\frac{\delta}{\cos^2 t}y_0(t) + 1\right).$$

To estimate the lower bound of λ, we integrate both sides of $|\nabla w|^2 \leq \varphi(w)$ as in Theorem 3.2, we obtain

$$\begin{aligned} \sqrt{\lambda}d &\geq \int_{w=-1+a}^{w=1+a} \frac{1}{b\cos t}\frac{1}{\sqrt{z(t)}}dw \\ &= \int_{\arcsin[(-1+a)/b]}^{\arcsin[(1+a)/b]} \frac{1}{\sqrt{z(t)}}dt \\ &\geq \left(\int_{\arcsin[(-1+a)/b]}^{\arcsin[(1+a)/b]} 1dt\right)^{\frac{3}{2}} \Big/ \left(\int_{\arcsin[(-1+a)/b]}^{\arcsin[(1+a)/b]} z(t)dt\right)^{\frac{1}{2}} \end{aligned}$$

for any arbitrary constnat $b > 1 + a$. Theorem follows as we let $b \longrightarrow 1 + a$. □

Remark. The lower bound

$$\widetilde{I}(a) = \frac{1}{d^2}\frac{\left(\int_{\arcsin[(-1+a)/(1+a)]}^{\pi/2} 1dt\right)^3}{\int_{\arcsin[(-1+a)/(1+a)]}^{\pi/2} z(t)dt}$$

is a continuous function of $a \in [0, 1)$. Thus the lower bound of λ in Theorem 3.3 near $\frac{\pi^2}{d^2} + \alpha$ as a near 0.

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