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碩　士　論　文

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摘　　　要

1973 年，G.Shimura 利用 theta-函數來定義半模函式，並且寫出 Hecke 算子作用在半模函式上的一般項。他發現 Hecke 算子作用在半模函式上的特徵值與整模函式的特徵值有對應關係，這就是所謂的 Shimura 對應。

另一方面，eta-函數是一個 (1/2) weight 的模函式在 Shimura 的定義之下。本篇論文當中主要探討用 eta-函數所定義出來的半模函式空間，這也許會使我們能夠證明出一些分析函數的同於式。歷史上這種由 eta-函數所定義出來的半模函式空間的研究開使於 Li Guo 和 Ken Ono 在“The partition function and the arithmetic of certain modular L-functions”中，並且證明了在某些例子中這種子空間是同構於一個整模函式的子空間，而這個整模函式的子空間是一些算子的不變子空間。我們現在把他們的結果更一般化，並且算出對應空間的維度。不過由於時間的關係，我們仍無法算出對於一般 Hecke 算子的 trace formula，也許在將來的日子裡會有機會把他算出來。

Half Integral Weight Modular Forms and Shimura correspondence

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ABSTRACT

In 1973, G.Shimura defined modular forms of half-integral weight by using theta-function. He showed that there are Hecke operators on half-integral weight modular forms, and he found that there is a correspondence between each eigenvalue for Hecke operator for integral weight modular form and half-integral weight modular form. And it is the so-called Shimura correspondence.

On the other hand, eta-function is a modular form of weight (1/2) in Shimura's sense. In this paper, we study the space of half-integral weight modular forms defined by eta-function, so that we may find some congruence of partition functions. Historically, these spaces were first studied by Li Guo and Ken Ono in their paper "The partition function and the arithmetic of certain modular L-functions". They proved that in some case these space are

isomorphic to space of integral-weight modular forms which is eigenspace of some operators. Now we make more general results, and we compute the dimensions in our cases. For the isomorphism, we try to prove it by using trace formula, but it is so complicated that we have not figured it out yet.

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目錄

Chapter 1

Introduction

In 1973, G.Shimura[4] laid the foundations of a theory of half integral weight modular forms with level M is always divisible by 4. We consider the space of cusp forms denoted by $S_{k/2}(M,\chi)$ mainly, where k is a positive odd number and χ is a Dirichlet character. Firstly, Shimura showed that there are Hecke operations T_{n^2} for every natural number n with $\gcd(n,M)=1$. Secondly, in the Main Theorem and its corollary, Shimura associated half integral weight modular forms with modular forms of integral weight. It is the so-called Shimura correspondence. In S.Niwa's paper [5], he proved that $S_{k/2}(M,\chi)$ is always isomorphic to $S_{k-1}(M/2)$.

Later, W. Kohnen in his paper[7] looked for a subspace which corresponds to the space of cusp forms of weight $2k$ on $\Gamma_0(M/4)$, where $M/4$ is square free. Elements contained in the subspace are cusp forms and with Fourier expansions of the form

$$\sum_{n\geq 1,(-1)^k n\equiv 0,1(4)} a(n)q^n,$$

and he denoted this subspace by $S^+_{k+1/2}(M,\chi)$. And we can introduce Hecke operators $T^+(n^2)$ on $S^+_{k+1/2}(M,\chi)$ for all n prime to M. He set up a theory of newforms similar to Atkin-Lehner-Li-Miyake. There is a canonically defined subspace $S^{\text{new}}_{k+1/2}(M,\chi)\subset S^+_{k+1/2}(M,\chi)$ and a canonical decomposition

$$S^+_{k+1/2}(M,\chi)=\bigoplus_{r,d\geq 1,rd|N} S^{\text{new}}_{k+1/2}(d,\chi)|U(r^2)$$

(where $U(r^2)$ is the operator replacing the nth Fourier coefficient of a modular form by its r^2nth one, and $N=M/4$), and $S^{\text{new}}_{k+1/2}(M,\chi)$ isomorphic to space of newforms $S^{\text{new}}_{2k}(N)\subset S_{2k}(N)$ (space of cusp forms with weight $2k$ over $\Gamma_0(N)$) as Hecke modules. In particular, we have a strong "multiplicity 1 theorem" for $S^{\text{new}}_{k+1/2}(M,\chi)$. See Remark 2.8.1.

In this thesis, we focus on a special subspace of $S_{k+1/2}(\Gamma_0(576))$, defined by

$$S_{r,s}(\Gamma_0(N)):=\{\eta(24\tau)^r f(24\tau): f(\tau)\in M_s(\Gamma_0(N))\},$$

where N is a positive integer. It is known that $\eta(24\tau)$ is a weight $\frac{1}{2}$ modular form on $\Gamma_0(576)$ with character χ_{12}. Y.Yang proved in [8] that this subspace is an invariant subspace of $S_{k+1/2}(\Gamma_0(576))$ under the action of the Hecke algebra when $N=1$. Then he discovered some new congruences of the the partition function $p(n)$ by applying Hecke operators on the subspace $\mathcal{S}_{r,s}$. A remarkable result of [8] is

$$p\left(\frac{m\ell^{2uK-1}n+1}{24}\right)\equiv 0 \mod m,$$

where $m \geq 13$ is a prime number, K is a positive integer determined by Hecke operators applying on the subspace $S_{r,s}(\Gamma_0(1))$, n is positive integer depend on the Hecke operator, and u is any positive integer.

Our main result is concerned with the space $\mathcal{S}_{r,s}(\Gamma_0(3))$. We compute some examples in Maple, and conjecture that $S_{r,s}(\Gamma_0(3)$ and $S_{r,s}(\Gamma_0(2)$ are also an invariant subspace of $S_{k/2}(\Gamma_0(576))$ for $r = 1, 5$ and 7, $s = 2, 4, 6, 8$, and some Hecke operators. Moreover, Yang conjecture that $S_{r,s}(\Gamma_0(1)$ isomorphic to a space of newforms of integral weight(see Proposition 3.2.1) as Hecke module when $r = 1, 5, 7, 11, 13, 17, 19, 23$. And here we also make a similar conjecture, but in our case, we can only find the correspondences for $r = 1, 5, 7$. For $r = 11, 13, 17, 19, 23$ we will check our conjecture fails by computing dimensions.

The invariance of $\mathcal{S}_{r,s}$ in our cases($N = 2, 3$) can most likely be proved by a way similar to the proof of theorem 2 in [8], but we have not work it out. One of the key points in the proof is the choice of

$$h(\tau) = \eta(\ell^2\tau)^{24-r} g(\tau/24) = \eta(\ell^2\tau)^{24-\tau} \eta(\tau)^r f(\tau),$$

where l is a prime number. When $N = 3$ we have to make some modification on h.

And now in this thesis, we observe some Hecke operator acting on the basis of $\mathcal{S}_{r,s}$ and claim invariance by checking Fourier coefficients(see example $4.1.1$).

Furthermore $S_{r,s}(\Gamma_0(2))$ and $S_{r,s}(\Gamma_0(3))$ isomorphic to a space of newforms of weight $2s+r-1$ on some congruence subgroups as Hecke modules.

Chapter 2

Standard Definition and Background

2.1 Notations

$\mathbb{Z}$: set of integers.
$\mathbb{H}$: upper half plane.
N : positive integer.
p : prime number.
$SL_2(\mathbb{Z})$: special linear group over $\mathbb{Z}$ of dimension 2.
R_Γ : fundamental domain of congruence subgroup Γ in $SL_2(\mathbb{Z})$.

2.2 Congruence subgroup $\Gamma_0(N)$

If N is any positive integer we define $\Gamma_0(N)$ to be the set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

It is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.
In particular, if p is a prime and let $S\tau = -1/\tau$ and $T\tau = \tau + 1$ be the generators of $\mathrm{SL}_2(\mathbb{Z})$, then for every V in $\mathrm{SL}_2(\mathbb{Z})$, but not in $\Gamma_0(p)$, there exists an element $P \in \Gamma_0(p)$ and an integer k, $0 \leq k < p$, such that

$$V = PST^k.$$

Let R be the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Then

$$R_{\Gamma_0(p)} = R \cup \bigcup_{k=0}^{p-1} ST^k(R),$$

where p is a prime.

Generally, we can also compute the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ and find the coset representations. Explicitly,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = 2N \prod_{p|N} (1 + 1/p).$$

and let $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $j = 1, 2$. The following three statement are equivalent:

1. The right cosets $\Gamma_0(N)\gamma_1$and $\Gamma_0(N)\gamma_1$ are equal.

2. $c_1 d_2 \equiv c_2 d_1 \mod N$.

3. There exist an integer r with $\gcd(r, N) = 1$ such that $c_1 \equiv rc_2$ and $d_1 \equiv rd_2 \mod N$.

Then we have

Theorem 2.2.1. *Let S be the set of pairs $(c, d) \in \mathbb{Z}^2$ with $\gcd(c, d, N) = 1$. Define an equivalence relation on S by $(c_1, d_1) \sim (c_2, d_2)$ if and only if $c_1 d_2 \equiv c_2 d_1 \mod N$. Then the coset representations of $\Gamma_0(N) \, \mathrm{SL}_2(\mathbb{Z})$ is*

$$\left\{ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z) : (c, d) \in S/\sim \right\},$$

where $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ means the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.2.2. *A set of inequivalent cusps for $\Gamma_0(N)$ is given by*

$$\left\{ \frac{a}{c} : c|N, a = 1, \ldots, \gcd(c, N/c), \gcd(a, c) = 1 \right\}.$$

Hence the number of inequivalent cusps is

$$\sum_{c|N} \phi(\gcd(c, N/c)),$$

where ϕ is the Euler totient function.

Theorem 2.2.3. **(a)** *The number v_2 of inequivalent elliptic points of order 2 for $\Gamma_0(N)$ is equal to the number of solutions of $x^2 + 1 = 0$ in $\mathbb{Z}/N\mathbb{Z}$. That is, when $4|N$, v_2=0, and when $4 \nmid N$,*

$$v_2 = \prod_{p|N, p \text{ odd}} \left(1 + \left(\frac{-1}{p} \right) \right),$$

where $\left(\frac{-1}{p} \right)$ is the Jacobi symbol.

(b) *The number v_3 of inequivalent elliptic points of order 3 for $\Gamma_0(N)$ is equal to the number of solutions of $x^2+x+1 = 0$ in $\mathbb{Z}/N\mathbb{Z}$. That is, when $9|N$, v_3=0, and when $9 \nmid N$,*

$$v_3 = \prod_{p|N} \left(1 + \left(\frac{-3}{p} \right) \right),$$

where $\left(\frac{-3}{p} \right)$ is the Jacobi symbol.

2.3 Atkin-Lehner involutions

Let N be a positive integer ≥ 2. Let n be a divisor of N such that $\gcd(n, N/n) = 1$. The elements in

$$\omega_n = \left\{ \frac{1}{\sqrt{n}} \begin{pmatrix} an & b \\ cN & dn \end{pmatrix}, adn^2 - bcN = n \right\}$$

are the *Atkin-Lehner involutions*, which normalize $\Gamma_0(N)$. The set of $\Gamma_0(N)$ union all possible Atkin-Lehner involutions is denoted by $\Gamma_0^*(N)$.

2.4 Modular forms of integral weight

Here we let k be a positive integer. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{GL}_2^+(\mathbb{R})$ (general linear group over $\mathbb{R}$ with positive determinant).

Definition 2.4.1. Let f be a meromorphic function on $\mathbb{H}$ and α as above. Then we define

$$f(\tau)|[\alpha]_k = \det(\alpha)^{k/2}(c\tau + d)^{-k} f(\alpha\tau) \quad (k \in \mathbb{N}).$$

Then we call a holomorphic function on $\mathbb{H}$ is a modular form of weight k with respect to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ if f satisfies:

1. $f(\tau)|[\alpha]_k = f(\tau)$, where $\alpha \in \Gamma$ and $\tau \in \mathbb{H}$,

2. f is holomorphic at every cusp of Γ.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, $f(\tau + 1) = f(\tau)|[\alpha]_k = f(\tau)$. Hence f has a Fourier expansion of the form

$$\sum_{n=0}^{\infty} a_n q^n,$$

where $q = e^{2\pi i \tau}$ and $a_n \in \mathbb{H}$. If $f(\tau)$ is vanish at all cusps, $a_0 = 0$ in the Fourier expansion, and we call such f cusp form.

We denote by

1. $M_k(\Gamma)$ the set of all holomorphic modular forms.

2. $S_k(\Gamma)$ the set of all cusp forms.

And there are dimension formulas:

Theorem 2.4.2. *Let Γ be a subgroup of finite index of $SL_2(\mathbb{Z})$. Assume that the genus of $X(\Gamma)$(the compactified modular curve $\Gamma \setminus \mathbb{H}^*$) is g. Let c be the number of inequivalent cusps of Γ, and $e_1, \ldots, e_r$ be the orders of inequivalent elliptic points. Let k be an even integer. We have*

$$\dim M_k(\Gamma) = \begin{cases} (k-1)(g-1) + \sum_{i=1}^{r} \left\lfloor \frac{k}{2}(1 - \frac{1}{e_i}) \right\rfloor + \frac{kc}{2}, & \text{if } s > 2, \\ g + c - 1, & \text{if } k = 2, \\ 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0. \end{cases}$$

and

$$\dim S_k(\Gamma) = \begin{cases} \dim M_k(\Gamma) - c, & \text{if } k > 2, \\ g, & \text{if } k = 2, \\ 0, & \text{if } k \le 0. \end{cases}$$

2.5 Hecke operators on integral weight modular forms

Hecke(1937) introduced a certain ring of operators acting on modular forms. The commutativity of this ring leads to Euler products associated with modular forms. Here we make a brief guide to Hecke operators.

For $N \in \mathbb{N}$, if $\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$ and $\Gamma_0(N)$ and $\alpha^{-1}\Gamma_0(N)\alpha$ are commensurable. By [1, sec.1.4], The double coset $\Gamma_0(N)\alpha\Gamma_0(N)$ is a finite union of right cosets:

$$\Gamma_0(N)\alpha\Gamma_0(N) = \bigcup_{i=1}^{h} \Gamma_0(N)\alpha_i,$$

where $\alpha_i \in \mathrm{GL}_2^+(\mathbb{Z})$ and $h = [\Gamma_0(N) : \alpha^{-1}\Gamma_0(N)\alpha]$. Then we define a linear operator $[\Gamma_0(N)\alpha\Gamma_0(N)]$ on $M_k(\Gamma_0(N))$ by

$$f|[\Gamma_0(N)\alpha\Gamma_0(N)]_k = \sum f|\alpha_i$$

In particular, for $n \in \mathbb{N}$ with $\gcd(n, N) = 1$ we denote by

$$T_n = n^{2k-1}[\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma_0(N)]_k$$

the Hecke operator of degree n.

Proposition 2.5.1. *Let the q-expansion of a modular form f is $\sum_{i \ge 0} a_i q^i$, for a prime p we have*

$$f|T_p = \sum_{i \ge 0} (a_{pi} + p^{k-1} a_{i/p}) q^i.$$

We can define an inner product called Peterson inner product on the vector space of cusp forms of weight k on $\Gamma_0(N)$. The precise formula is

$$\langle f, g \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int\int_D y^k f(\tau)\overline{g(\tau)} \frac{dxdy}{y^2},$$

where D is the fundamental domain of $\Gamma_0(N)$ and we write $\tau = x + iy$ for $\tau \in \mathbb{H}$.With respect to this inner product on the space of cusp forms, we can show that T_n is self-adjoint if n and N are relatively prime, and thus diagonalizable.

We call f a Hecke eigenform if f is non-vanishing modular form on $\Gamma_0(N)$ and a simultaneous eigenfunction for all Hecke operators.

When $N = 1$, then it can be proved that every Hecke operators commute with each other i.e. $T_m T_n = T_n T_m$. So we have a nice result from linear algebra that is there is a basis consisting entirely of Hecke eigenforms such that all the Hecke operators are simultaneously diagonalizable. The space of cusps forms of weight k on $\Gamma_0(1)$ is spanned by Hecke eigenforms. Then the Fourier coefficients a_i of f satisfying the following:

1. $a_1 \neq 0$.

2. if $a_1 = 1$, then the coefficients a_i are multiplicative.

Thus we may adjust any Hecke eigenform by a constant so that $a_1 = 1$. Such Hecke eigenform is called normalized. And it was shown that the L-function of a Hecke eigenform has a Euler product.

Theorem 2.5.2. *[1, Theorem1.4.4] If f is a simultaneous eigenform, then*

$$L(s,f) = \sum \frac{a_n}{n^s} = \prod_p (1 - a_p p^{-s} + p^{k-1+2s})^{-1}$$

For example, $S_{12}(\Gamma_0(1))$ is a one dimensional vector space spanned by a normalized Hecke eigenform $\Delta(\tau) = \eta^{24}(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$, where $\tau \in \mathbb{H}$ and $\tau(n)$ are Ramanujan's tau functiona. Here we obtain the Euler product formula

$$\sum \frac{\tau(n)}{n^s} = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}$$

of Ramanujan and Mordell.

But in the cases of $N > 1$, $S_k(\Gamma_0(N))$ may not have a basis consisting entirely of simultaneous eigenforms for all Hecke operators T_n. Here is an example.

Example 2.5.3. Consider the Hecke operator T_2 acts on $S_4(\Gamma_0(16))$. Then the Jordan form for T_2 is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, there does not exist a basis whose elements are all simultaneous eigenforms for all Hecke operator.

But if $f = \sum_{n=1}^{\infty} a_n q^n$ is a simultaneous eigenform for all T_n, then f still has the property that $T_n f = a_n f$ and its L-function has the Euler product

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p|N} (1 - a_p p^{-s})^{-1} \prod_{p\nmid N} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

The main reason for this is that some of the cusp forms in $S_k(\Gamma_0(N))$ actually have level smaller than N.

Lemma 2.5.4. *[9] Let M, $N \in \mathbb{N}$ and M dividing N. Then we have $S_t(\Gamma_0(M)) \in \mathcal{S}_t(\Gamma_0(N))$. And let $f \in S_t(\Gamma_0(M))$, then for any $h \mid (N/M)$, the function $f(h\tau) \in \mathcal{S}_t(\Gamma_0(N))$.*

Proof. By assumption Let $N = kM$ for some $k \in \mathbb{Z}$, and let $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. So

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} a & b \\ ckM & d \end{pmatrix} \in \Gamma_0(M),$$

thus we have $\mathcal{S}_t(\Gamma_0(M)) \in \mathcal{S}_t(\Gamma_0(N))$.

Let $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$. And note that,

$$h\gamma\tau = \frac{a(h\tau) + hb}{\frac{cN}{h}(h\tau) + d} = \begin{pmatrix} a & hb \\ \frac{cN}{h} & d \end{pmatrix} h\tau.$$

By assumption $h \mid (N/M)$, we have $\gamma' = \begin{pmatrix} a & hb \\ \frac{cN}{h} & d \end{pmatrix} \in \Gamma_0(M)$.
So

$$f(h\gamma\tau) = f(\gamma'(h\tau)) = (cN\tau + d)^t f(h\tau).$$

Then $f(h\tau)$ is a cusp form on $\Gamma_0(N)$.

□

To define such cusp forms precisely, Let $f(\tau) \in S_k(\Gamma_0(N))$ satisfies $f(\tau) = g(h\tau)$ for some simultaneous eigenform $g(\tau) \in \Gamma_0(M)$ with $M|N$ and $h|(N/M)$, then f is called an $oldform$. The space spaned by all oldforms are denoted by $S_k^{\text{old}}(\Gamma_0(N))$. And the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N))$ in $S_k(\Gamma_0(N))$ is call space of $newforms$, denoted by $S_k^{\text{new}}(\Gamma_0(N))$. In particular, The space $S_k^{\text{new}}(\Gamma_0(N))$ has a basis consisting of simultaneous eigenforms for all T_n with $\gcd(n, N) = 1$.

Now we introduce some theorems we will use:
Define the *degeneracy map* α_h as:

$$\alpha_h : S_k(\Gamma_0(M)) \longrightarrow S_k(\Gamma_0(N))$$

by

$$\alpha_h(f(\tau)) = f(h\tau),$$

where h is the divisors of N/M if N is divisible by M.

Proposition 2.5.5. *We have a decomposition*

$$S_k(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{d|N/M} \alpha_d(S_k^{\text{new}}(\Gamma_0(M))). \tag{2.1}$$

2.6 Modular forms of half integral weight

From now on we let k be an positive odd integer.

To define the modular forms of half integral weight, one may try to make a definition similar to modular forms of integral weight: Let γ be a discrete subgroup of $\text{GL}_2^+(\mathbb{R})$. Assume that f is a holomorphic (or meromorphic) function on $\mathbb{H}$, and it satisfies an appropriate condition at cusps. Then f is a *modular form of weight k/2* if

$$f(\gamma\tau) = (c\tau + d)^{k/2} f(\tau),$$

where $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.

Suppose we accept this definition. Then we have a statement:

Proposition 2.6.1. *[3] Let $\Gamma' \subset \text{SL}_2(\mathbb{Z})$ be any congruence subgroup. Let f be a modular form of weight $k/2$ satisfies the above definition. Then $f = 0$.*

Proof. Γ' is a congruence subgroup of $\text{SL}_2(\mathbb{Z})$, so we can assume that for some $N > 2$

$$\Gamma(N) = \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \subset \Gamma'.$$

Let

$$\alpha = \begin{pmatrix} N+1 & N \\ -N & 1-N \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix}.$$

and we compute

$$\alpha\beta = \begin{pmatrix} -N^2+N+1 & N \\ N^2-2N & 1-N \end{pmatrix}.$$

For any nonzero modular form of weight $k/2$, by the definition we require that

$$f(\alpha\tau) = (-N\tau + (1-N))^{k/2} f(\tau)$$

and

$$f(\beta\tau) = (-N\tau+1)^{k/2} f(\tau).$$

Therefore,

$$\begin{aligned} f(\alpha\beta\tau) &= (-N(\beta\tau) + (1-N))^{k/2} f(\beta\tau) \\ &= (-N(\beta\tau) + (1-N))^{k/2}(-N\tau+1)^{k/2} f(\tau) \\ &= \left(\frac{-N\tau}{-N\tau+1} + (1-N)\right)^{k/2} (-N\tau+1)^{k/2} f(\tau) \end{aligned}$$

By applying the definition to the matrix $\alpha\beta$ directly, we have

$$f(\alpha\beta\tau) = ((N^2-2N)\tau + (1-N))^{k/2} f(\tau)$$

This implies that

$$((N^2-2N)\tau + (1-N))^{k/2} = \left(\frac{-N\tau}{-N\tau+1} + (1-N)\right)^{k/2} (-N\tau+1)^{k/2}. \tag{2.2}$$

When k is even, this equality holds. We may assume that $k = 1$. Then since the two expressions in the radicals on the right are in lower half plane, the right side is the product of two complex number in the fourth quadrant(we take the branch of the square root having argument in $(-\pi/2, \pi/2]$). But the left side is in the first quadrant, since $(N^2-2N)\tau + (1-N) \in \mathbb{H}$.

Hence (2.2) is wrong by a factor of -1 for $k = 1$, and also for any odd k.

□

To see why this definition fails. Note that square root function is multivalued, so our choice of a branch of the square root necessary led to problems. We may handle this group by requiring that our modular forms act on a covering space of $\mathrm{GL}_2^+(\mathbb{R})$, where we allow all branches of the square root simultaneously.

Let an element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ act on $\mathbb{H} \bigcup \mathbb{R} \bigcup \{\infty\}$ by $\alpha(\tau) = (a\tau+b)/(c\tau+d)$. Let $\mathfrak{B}$ denote the set of all couples $(\alpha, \phi(\tau))$ formed by an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of GL_2^+ and a holomorphic function ϕ on $\mathbb{H}$ such that

$$\phi^2 = t \cdot \det(\alpha)^{-1/2}(c\tau+d),$$

where $t \in T^2 = \{\tau \in \mathbb{C} : |\tau| = 1\}$. Define the law of multiplication by

$$(\alpha, \phi(\tau))(\beta, \psi(\tau)) = (\alpha\beta, \phi(\beta(\tau))\psi(\tau)),$$

we can make $\mathfrak{B}$ a group.

Let $\xi = (\alpha, \phi) \in \mathfrak{B}$, we define the action of ξ on $\mathbb{C} \bigcup \{\infty\}$ to be the same as that of α. Furthermore, for a complex valued function $f(\tau)$ on $\mathbb{H}$ and an integer k, we define a function $f|[\xi]_k$ on $\mathbb{H}$ by

$$(f|[\xi]_k)(\tau) = f(\xi(\tau))\phi(\tau)^{-k}.$$

Note that $f|[\xi\eta]_k = (f|[\xi]_k)|[\eta]_k$.

Let the function P be the natural projection map defined as:

$$P : \mathfrak{B} \longrightarrow \mathrm{GL}_2^+(\mathbb{R}),$$

by

$$(\alpha, \phi) \mapsto \alpha.$$

And we denote by $L : \Gamma \longrightarrow \Delta$ the inverse map of P.

Let Δ be a *Fuchsian subgroup* of $\mathfrak{B}$ satisfying the following:

1. $P(\Delta)$, the projection of Δ onto $\mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, and this projection is one to one.

2. The fundamental domain $R_{P(\Delta)}$ is of finite measure with respect to the invariant measure $y^{-2}dxdy$.

3. If $-1 \in P(\Delta)$, then its preimage is $(-1, 1)$.

We call a meromorphic function $f(\tau)$ on $\mathbb{H}$ an *automorphic form of weight* $k/2$ *with respect to* Δ if the following conditions are satisfied:

1. $f|[\xi]_k = f$ for all $\xi \in \Delta$.

2. f is meromorphic at each cusp of $P(\Delta)$, where $P(\Delta)$ is the projection of Δ on $GL_2^+(\mathbb{R})$.

We denote by $G_k(\Delta)$ the vector space of all such f which are holomorphic on $\mathbb{H}$ and for which $c_n = 0$ if $n < 0$, and further by $S_k(\Delta)$ the subspace of $G_k(\Delta)$ consisting of all f for which $c_0 = 0$ if $r = 0$ at every cusp of $P(\Delta)$.

We should now specialize our discussion to the case where Δ is obtained from a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $\Delta_0(M)$ denote the image of $\Gamma_0(M)$ under L, for every positive integer M divisible by 4. Define an automorphic factor $j(\gamma, \tau)$ by

$$j(\gamma, \tau) = \frac{\theta(\gamma\tau)}{\theta(\tau)}, \quad for \quad \gamma \in \Gamma_0(4). \tag{2.3}$$

where

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{2\pi i \tau} \tag{2.4}$$

and τ is in on H.

Then we have by [3, p.148]

$$j(\gamma, \tau)^2 = \left(\frac{-1}{d}\right)(c\tau + d), \tag{2.5}$$

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$=$\gamma \in \Gamma_0(4)$. In Equation (2.5), $\left(\frac{-1}{d}\right)$ is the Jacobi symbol. Note that if d is negative, we set $\left(\frac{-1}{d}\right) = -\left(\frac{-1}{|d|}\right)$.

Here we put $\gamma^* = (\gamma, j(\gamma, \tau))$, then we consider a cusp form $f(\tau)$ satisfying

$$f(\tau)|[\gamma^*]_k = f(\gamma(\tau)) \cdot j(\gamma, \tau)^{-k} \tag{2.6}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$=$\gamma \in \Gamma_0(4)$. Let χ be a Dirichlet character modulo M. We denote by $G_k(M, \chi)$(resp.$S_k(M, \chi)$) the set of all elements f of $G_k(\Delta_0(M))$ satisfying

$$f|[\gamma^*]_k = \chi(d) \cdot f$$

for all $\gamma \in \Gamma_0(M)$.

Then the half integral weight modular forms of weight $k/2$ over $\Gamma_0(M)$ denoted by $G_k(\Delta_0(M))$ is the complex vector space of all such f. If $f \in G_k(\Delta_0(M))$, we see that $f(\tau+1) = f(\tau)$, since $\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1\right) \in \Delta_0(M)$. Hence the Fourier expansion of f has the form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i \tau}$.

And then we can define a linear operator on $G_k(\Delta_0(M))$ for each prime p.

2.7 Hecke operator T_m on $G_{k/2}(M,\chi)$

Let $f(\tau) = \displaystyle\sum_{n=0}^{\infty} a_f(n) q^n$ in $G_{k/2}(\Gamma_0(M),\chi)$. Let m be a square of a positive integer, and

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, \quad \xi = (\alpha, m^{1/4}).$$

Suppose we have a disjoint union

$$\Delta_0(M)\xi\Delta_0(M) = \bigcup_{\nu=1}^{r} \Delta_0(M)\xi_\nu (\text{disjoint}), \quad \Gamma_N(M)\alpha\Gamma_0(M) = \bigcup_{\nu=1}^{r} \Gamma_0(M)\alpha_\nu.$$

We define a linear operator on $S_{k/2}(M,\chi)$ by

$$[\Delta_0(M)\xi\Delta_0(M)]_{k/2} : G_{k/2}(M,\chi) \longrightarrow G_{k/2}(M,\chi),$$

by

$$f \mapsto f|[\Delta_0(M)\xi\Delta_0(M)]_{k/2} = m^{(k/4)-1} \cdot \sum_{\nu=1}^{r} \chi(a_\nu) f|[\xi_\nu]_{k/2},$$

which is independent of choice of the representative of ξ_ν.

Definition 2.7.1. The Hecke operator T_m on $G_{k/2}(M,\chi)$ is given by

$$f|T_m = f|[\Delta_0(M)\xi\Delta_0(M)]_{k/2} = m^{(k/4)-1} \cdot \sum_{\nu=1}^{r} \chi(a_\nu) f|[\xi_\nu]_{k/2},$$

where $\xi = \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, m^{1/4}\right)$ and $P(\xi_\nu) = \begin{pmatrix} a_\nu & * \\ * & * \end{pmatrix}$.

If m is not a square and $\gcd(N,m) = 1$, then $[\Delta_0(M)\xi\Delta_0(M)]$ is a zero operator on $G_{k/2}(M,\chi)$. So we can consider T_m only for square m.

In [4] Shimura proved that

$$T_{p^2} : f(\tau) \to \sum_{n=0}^{\infty} \left(a_f(p^2 n) + \chi(p) \left(\frac{(-1)^\lambda n}{p}\right) p^{\lambda-1} a_f(n) + \chi(p^2) p^{2\lambda-1} a_f(n/p^2) \right) q^n. \tag{2.7}$$

and if n is not divisible by p^2, $a_f(n/p^2) = 0$.

2.8 Shimura correspondence

One can define certain liftings of cusp forms of $k/2$ on $\Gamma_0(M)$ to cusp forms of weight $2k$ on $\Gamma_0(M')$ for a certain M' depending on M; these liftings commute with the action of Hecke operators.
Suppose f is a common eigen-function of the operator T_{p^2} for all prime p, and let $f|T_{p^2} = w_p f$. Define a function F on $\mathbb{H}$ by

$$F(\tau) = \sum_{n=1}^{\infty} A_n q^n, \quad A_n \in \mathbb{C},$$
$$\sum_{n=1}^{\infty} A_n n^{-s} = \prod_p [1 - \omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}]^{-1},$$

if $k \geq 5$, F is a cusp form of weight $k-1$ over $\Gamma_0(N)$ with character χ^2.

In Shimura's original theorem, the determination of the level M' was a little complicated. However, it was been shown ([5]) that one can always take $M' = M/2$.

Next let us focus on the space $S^+_{k/2}(4N)$ (Kohnen space) of cusp forms of weight $k/2$ on $\Gamma_0(4N)$ ($N \in \mathbb{N}$ is square free). Recall that the space $S^+_{k/2}(4N)$ is the set consisting of elements with the Fourier series of the form

$$\sum_{n\geq 1, (-1)^k n \equiv 0,1(4)} a(n) q^n.$$

Let $f = \sum_{m\geq 1} c_m q^m$ be in $S^+_{k/2}(4N, \chi)$. Then define

$$\mathfrak{T}_{k/2,4N,\chi}(p)(f) = \sum_{m\geq 1, \varepsilon(-1)^k m\equiv 0,1(4)} \left(c_{p^2 m} + \chi(p) \left(\frac{\varepsilon(-1)^k m}{p} \right) p^{k-1} c_m + p^{k-2} c_{m/p^2} \right) q^m.$$

If we define Pertersson inner product by

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(4):\Gamma]} \int_{R_\Gamma} f(\tau)\overline{g(\tau)} y^{k/2-2} dxdy \quad (x = \mathrm{Re}\tau, y = \mathrm{Re}\tau),$$

then $\mathfrak{T}_{k/2,4N,\chi}(p)$ generate a commutative $\mathbb{C}$-algebra of hermitean operators, and the space $S^+_{k/2}(M,\chi)$ has an orthogonal basis consisting of common eigenfunctions of all operators $\mathfrak{T}_{k/2,4N,\chi}(p)$.

Next we define the space of oldforms in $S_{k'/2}(M,\chi)$ to be

$$\sum_{d|N, d<N} (S_{k/2}(d,\chi) + S_{k/2}(d,\chi)|U(N^2/d^2))$$

and the space of newforms $S^{\mathrm{new}}_{k/2}(M,\chi)$ to be the orthogonal complement of the space of oldforms in $S^+_{k/2}(M,\chi)$. And since the operator $u(\mathfrak{f})$ is an isomorphism between $S^+_{k/2}(M,\chi)$ and $S^+_{k/2}(M)$, it is enough to study $S^+_{k/2}(M)$, where $\mathfrak{f}$ is the conductor of χ, and denote by $u(\mathfrak{f})$ to be the restriction of $U(\mathfrak{f})$ to $S^+_{k/2}(M,\chi)$.

Let $\mathcal{H}_N$ be the Hecke algebra spanned by the elements $\Gamma_0(N) \left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) \Gamma_0(N)$, where $a, d > 0$, $a|d$ and $\gcd(d, 2N) = 1$. Define a linear map R from $\mathcal{H}_N$ to $\mathrm{End}_{\mathbb{C}}(S^+_{k/2}(4N))$ by

$$R\left(\Gamma_0(N)\left(\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\right)\Gamma_0(N)\right) = a(ad)^{(k-4)/2}\left[\Delta_0(4N)\left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2\end{pmatrix}, (ad)^{k/2}\right)\Delta_0(4N)\right]\Big|_{S^+_{k/2}(4N)}$$

Then R is a representation of $\mathcal{H}_N$[6]. Also, we have a representation

$$\tilde{R} : \mathcal{H}_N \longrightarrow \mathrm{End}_{\mathbb{C}}(S_{k-1}(N))$$

defined by

$$\tilde{R}\left(\Gamma_0(N)\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right)\Gamma_0(N)\right) = (ad)^{k-2}\left[\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(N)\right]_{k-1}$$

Since R and $\tilde{R}$ are semisimple and in [7] Kohnen showed that

$$\mathrm{tr}(R(\xi), S^+_{k/2}(4N)) = \mathrm{tr}(\tilde{R}(\xi), S_{k-1}(N)),$$

the representations R and $\tilde{R}$ are equivalent.

Recall that for every prime divisor p for N the operator $U(p)$ preserves $S^{\mathrm{new}}_{2k}(N)$ and that $U(p) = -p^{k-1}w^N_{p,2k}$ on $S^{\mathrm{new}}_{2k}(N)$, where $w^N_{p,2k}$ is Atkin-Lehner involution on $S_{2k}(N)$ defined by

$$w^N_{p,2k}(f(\tau)) = p^k(4N\tau + p\beta)^{-2k} f\left(\frac{p\tau+\alpha}{4N\tau+p\beta}\right) \qquad (\alpha,\beta\in\mathbb{Z}, p^2\beta - 4N\alpha = p).$$

And there are analogous results for newforms of half integral weight.

One can define an involution on $S_{k/2}(M,\chi)$ as follows. For each prime prime divisor p of N, we define an "Atkin-Lehner involution" $W(p)$ by

$$W(p) = \left(\begin{pmatrix} p & a \\ 4N & pb \end{pmatrix}, \left(\frac{-4}{p}\right)^{-k} p^{-k'/4}(4N\tau + pb)^{k/2}\right),$$

where a and b are integers with $p^2b - 4Na = p$. In particular, for each prime divisor p of N we put

$$w^N_{p,k/2} = p^{-k/4+1/2}U(p)W(p)$$

and define $\mathfrak{S}^{\pm,p}_{k/2}(4N)$ as the subspace of $S^+_{k/2}(4N)$ consisting of the forms whose nth Fourier coefficients vanish for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$; then we set $w^N_{p,k/2,\chi} = u(\mathfrak{f})^{-1}w^N_{p,k/2}u(\mathfrak{f})$ and $\mathfrak{S}^{\pm,p}_{k/2}(4N,\chi) = \mathfrak{S}^{\pm,p}_{k/2}(4N)|u(\mathfrak{f})$.

The operator $w^N_{p,k/2,\chi}$ is a hermitean involution on $S^+_{k/2}(4N,\chi)$ whose $(\pm)$-eigenspace is $\mathfrak{S}^{\pm,p}_{k/2}(4N,\chi)$. In particular, for each prime divisor of N we have an orthogonal decomposition

$$S^+_{k/2}(4N) = \mathfrak{S}^{+,p}_{k/2}(4N,\chi) \oplus \mathfrak{S}^{-,p}_{k/2}(4N,\chi).$$

If p does not divide $\mathfrak{f}$, then $w^N_{p,k/2,\chi}$ coincides with $\left(\frac{\mathfrak{f}}{p}\right)p^{-k/4+1/2}U(p)W(p)|S^+_{k/2}(4N,\chi)$, and $\mathfrak{S}^{\pm,p}_{k/2}(4N,\chi)$ coincide with the subspace of $S^+_{k/2}(4N,\chi)$ consisting of the forms whose Fourier coefficients vanish for n with $\left(\frac{(-1)^{k-1/2}\mathfrak{f}n}{p}\right) = \mp 1$.

The space $S^{\mathrm{new}}_{k/2}(4N,\chi)$ has an orthogonal basis of common eigenfunctions for all operators $\mathfrak{T}_{k/2,4N,\chi}(p)$ (p prime, p does not divide N), uniquely determined up to multiplication by non-zero complex numbers. These eigenfunctions are also eigenfunctions for the operators $U(p^2)$ (p prime, $p|N$), the corresponding eigenvalues being $\pm p^{k/2-3/2}$. If f is such an eigenfunction and λ_p is the eigenvalue by respect to $\mathfrak{T}_{k/2,4N,\chi}(p)$ (resp. $U(p^2)$), then there is an eigenfunction $F \in$

$S_{2k}^{\rm new}(N)$, uniquely determined up to multiplication with a nonzero complex number, which satisfies $T_{k'-1,N}(p)(F) = \lambda_p F$ (resp. $U(p^2)(F) = \lambda_p F$) for all prime p does not divide N (resp. $p|N$). The Fourier coefficients are related as follows: if $f = \sum_{n\geq 1} a_n q^n$ and $F = \sum_{n\geq 1} A_n q^n$, and if D is a fundamental discriminant with $\varepsilon(-1)^k D > 0$, then

$$L(s-k+1, \chi\left(\frac{D}{\cdot}\right)) \sum_{n\geq 1} a_{|D|n^2} n^{-s} = a(|D|) \sum_{n\geq 1} A_n n^{-s}.$$

Then we define a map $\varphi_{D,k/2-1/2,N,\chi}$ by

$$\sum_{n\geq 1} b_n q^n \mapsto \sum_{n\geq 1} \left(\sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{k/2-3/2} b\left(\frac{n^2}{d^2}|D|\right)\right) q^n$$

maps $S_{k/2}^{+}(4N,\chi)$ to $S_{k-1}(N)$, $S_{k/2}^{\rm new}(4N,\chi)$ to $S_{k-1}^{\rm new}(N)$ and for every prime divisor p of N, $\mathfrak{S}_{k/2}^{\pm}(4N,\chi) \bigcap S_{k/2}^{\rm new}(4N,\chi)$ to $S_{k-1}^{\pm}(N) \bigcap S_{k-1}^{\rm new}(N)$. It satisfies

$$\mathfrak{T}_{k/2,N,\chi}(p)\varphi_{D,k/2-1/2,N,\chi} = \varphi_{D,k/2-1/2,N,\chi} T_{k-1,N}(p)$$

for all prime p with $p \nmid N$ and $U(p^2)\varphi_{D,k/2-1/2,N,\chi} = \varphi_{D,k/2-1/2,N,\chi} U(p)$ for all prime p with $p|N$. There exist a linear combination of the $\varphi_{D,k/2-1/2,N,\chi}$ which maps $S_{k/2}^{\rm new}(4N,\chi)$(resp. $\mathfrak{S}_{k/2}^{\pm,p}(4N,\chi) \bigcap S_{k/2}^{\rm new}(4N,\chi)$)isomorphically onto $S_{k-1}^{\rm new}(N)$ (resp. $S_{k-1}^{\pm,p}(N) \bigcap S_{k-1}^{\rm new}(N)$).

Remark 2.8.1. We see that $S_{k/2}^{\rm new}(4N,\chi)$ and $S_{k-1}^{\rm new}(N)$ are isomorphic, and since strong multiplicity one theorem holds for $S_{k-1}^{\rm new}(N)$, also holds for $S_{k/2}^{\rm new}(4N,\chi)$.

But it is naturally to ask that does multiplicity one theorem hold for the set of all cusp forms of half-integral weight over $\Gamma_0(M)$?

The answer is not. Take $S_{13/2}(\Gamma_0(4))$ for an example. By [?] we know that $S_{13/2}(\Gamma_0(4))$ isomorphic to $S_{12}(\Gamma_0(2))$ as modules over the Hecke Algebra. Note that $\dim S_{12}(\Gamma_0(2)) = 2$. And then we compute the matrix representations for the Hecke operators T_p on $S_{12}(\Gamma_0(2))$ directly, we see that for $p = 3$, the Jordan form of the matrix representation is

$$\begin{pmatrix} 252 & 0 \\ 0 & 252 \end{pmatrix}.$$

It is diagonalizable, so there are two linearly independent eigenfunctions in $S_{12}(\Gamma_0(2))$ with same eigenvalue. Note that $S_{12}(\Gamma_0(2))$ is spanned by $\{\eta(\tau)^{24}, \eta(2\tau)^{24}\}$, so $S_{12}(\Gamma_0(2)) = S_{12}^{\rm old}(\Gamma_0(2))$. That is say multiplicity one theorem does not hold for $S_{12}(\Gamma_0(2))$, so does $S_{13/2}(\Gamma_0(4))$.

Chapter 3

Explicit formulas

In this chapter we compute the dimensions of some spaces corresponding to $S_{r,s}(\Gamma_0(3))$ which is defined in introduction. As we mentioned in the introduction, our conjecture holds for $r = 1, 5, 7$ and fails for $r = 11, 13, 17, 19, 23$. But there maybe some space of newforms corresponding to $S_{r,s}(\Gamma_0(3))$ for large r. From now on, we always let t be a positive even number.

3.1 Dimension of $S_t(\Gamma_0(2), \epsilon_1)$ and $S_t(\Gamma_0(3), \epsilon_2)$

Proposition 3.1.1. *Let* $S_t^{\mathrm{new}}(\Gamma_0(2), \epsilon_1)$ *denote the space of newforms of weight t(t is even) on* $\Gamma_0(2)$ *that is eigenfunction for* w_2 *with eigenvalues* ϵ_1*.Then the dimension of this space is*

$$\dim S_t^{\mathrm{new}}(\Gamma_0(2), \epsilon_1) = \begin{cases} \lfloor 3t/8 \rfloor - \lfloor t/3 \rfloor, & \text{if } \epsilon_1 = 1 \\ t - \lfloor 3t/8 \rfloor - \lfloor t/3 \rfloor - \lfloor t/4 \rfloor - 1, & \text{if } \epsilon_1 = -1. \end{cases} \tag{3.1}$$

Before proving this proposition, we shall prove a lemma first.

Lemma 3.1.2. *Let* $S_t(\Gamma_0(2), \epsilon_1)$ *denote the space of cusp forms of weight t(t is even) on* $\Gamma_0(2)$ *that is eigenfunctions for* w_2 *with eigenvalue* ϵ_1*.Then the dimension of this space is*

$$\dim(S_t(\Gamma_0(2), \epsilon_1)) = \begin{cases} \lfloor 3t/8 \rfloor + \lfloor t/4 \rfloor - t/2 & \text{if } \epsilon_1 = 1 \\ t/2 - \lfloor 3t/8 \rfloor - 1 & \text{if } \epsilon_1 = -1. \end{cases} \tag{3.2}$$

Proof. Since $\Gamma_0(2)$ is of genus 0, the group $\Gamma_0^+(2) = \Gamma_0(2) \cup w_2\Gamma_0(2)$ is of genus 0, where w_2 is the Atkin-Lehner involution on $\Gamma_0(2)$.
And the genus formula says

$$g = 1 + \frac{\mathrm{vol}(\Gamma_0^+(2) \setminus \mathbb{H})}{12\mathrm{vol}(SL_2(\mathbb{Z}) \setminus \mathbb{H})} - \frac{1}{2}\sum_{n=1}^{r}(1 - \frac{1}{e_n}) - \frac{c}{2}, \tag{3.3}$$

where $\mathrm{vol}(\Gamma \setminus \mathbb{H})$ is the volume of the fundamental domain $\Gamma \setminus \mathbb{H}$, r is the number of elliptic points, $e_1, \ldots, e_r$ is the order of inequivalent elliptic point, and c is number of cusps.
Since $[SL_2(\mathbb{Z}) : \Gamma_0(2)] = 3$, the fundamental domain of $\Gamma_0^+(2)$ is equal to $\frac{3}{2}$ of the fundamental domain of $SL_2(\mathbb{Z})$. Than we have

$$0 = 1 + \frac{1}{8} - \sum_{n=1}^{r}(1 - \frac{1}{e_n}) - \frac{1}{2}.$$

(Note that $\Gamma_0(2)$ has two inequivalent cusp are 0, $\frac{1}{2}$, but in $\Gamma_0^+(2)$, 0 and $\frac{1}{2}$ are equivalent, since $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \frac{1}{2} = 0$).
Thus we have

$$\sum_{n=1}^{r}(1-\frac{1}{e_n})=\frac{5}{4}.$$

So we can see that there are one elliptic point of order 2, and one elliptic point of order 4.
Then

$$\dim(\mathcal{S}_t(\Gamma_0(2)),+)=\lfloor 3t/8\rfloor+\lfloor t/4\rfloor-t/2.$$

Since $\dim(S_t(\Gamma_0(2))=\lfloor t/4\rfloor-1$, we have

$$\dim(\mathcal{S}_t(\Gamma_0(2),-))=t/2-\lfloor 3t/8\rfloor-1.$$

□

Now we can prove the proposition.

Proof of proposition 3.1.1. Let $f\in S_t(\Gamma_0(1))$ and w_2 the Atkin-Lehner involution on $\Gamma_0(2)$

$$w_2=\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} w_2(f(\tau)+2^t f(2\tau)) &= (f(\tau)+2^t f(2\tau)) \mid [w_2]_t \\ &= f(\tau)\mid [w_2]_t + 2^t f(2\tau)\mid [w_2]_t \\ &= \det(w_2)^{t/2}(-2\tau)^{-t} f\left(\frac{-1}{2\tau}\right) + 2^t \det(w_2)^{t/2}(-2\tau)^{-t} f\left(\frac{-2}{2\tau}\right) \\ &= \det(w_2)^{t/2}(-2\tau)^{-t} f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} 2\tau\right) + 2^t \det(w_2)^{t/2}(-2\tau)^{-t} f\left(\frac{-1}{\tau}\right) \\ &= f(\tau)+2^t f(2\tau). \end{aligned}$$

So the eigenvalue of w_2 respect to $f(\tau)+2^t f(2\tau)$ is 1. And we also have if $f\in S_t(\Gamma_0(1))$

$$\begin{cases} w_2(f(\tau)) = 2^t f(2\tau) \\ w_2(f(2\tau)) = 2^{-t} f(\tau). \end{cases} \tag{3.4}$$

Let $\{f_1(\tau), f_2(\tau),\ldots,f_n(\tau)\}$ be a basis for $S_t(\Gamma_0(1))$. Then $\{f_1(2\tau), f_2(2\tau),\ldots,f_n(2\tau)\}$ is a basis for $\alpha_2 S_t(\Gamma_0(1))$, where α_2 is the degeneracy map we defined in chapter 2. Let Q is a subspace of $S_t(\Gamma_0(2),+)$ defined as

$$Q=\text{span}\{f_1(\tau)+2^t f_1(2\tau),\ldots,f_n(\tau)+2^n f_n(2\tau)\}.$$

We claim that

$$S_t(\Gamma_0(2),+)=S_t^{\text{new}}(\Gamma_0(2),+)\oplus Q.$$

By Theorem2.1, we know that

$$S_t(\Gamma_0(2))=S_t(\Gamma_0(1))\oplus\alpha_2 S_t(\Gamma_0(1))\oplus S_t^{\text{new}}(\Gamma_0(2)).$$

Suppose we have a $g(\tau)\in S_t^{\text{new}}(\Gamma_0(2),+)\bigcap Q$. Since $Q\subset S_t(\Gamma_0(1))\oplus\alpha_2 S_t(\Gamma_0(1))$ and $S_t^{\text{new}}(\Gamma_0(2),+)\subset S_t^{\text{new}}(\Gamma_0(2))$, $g\in S_t(\Gamma_0(1))\oplus\alpha_2 S_t(\Gamma_0(1))\bigcap S_t^{\text{new}}(\Gamma_0(2))=\{0\}$.

Next given any $f\in S_t(\Gamma_0(2),+)$. Since $S_t(\Gamma_0(2),+)\subset S_t(\Gamma_0(2))$, we can write f as

$$f=\sum_{i=1}^{n}a_if_i(\tau)+\sum_{i=1}^{n}b_if_i(2\tau)+h(\tau),$$

where $h\in S_t^{\text{new}}(\Gamma_0(2))$. Because of $f\in S_t(\Gamma_0(2),+)$ and newforms are Hecke eigenforms of w_2, $w_2h=h$. Then By $w_2f=f$ and Equation(3.4), we have

$$\begin{aligned}
&2^t\sum_{i=1}^{n}a_if_i(2\tau)+2^{-t}\sum_{i=1}^{n}b_if_i(\tau)+h(\tau)=\sum_{i=1}^{n}a_if_i(\tau)+\sum_{i=1}^{n}b_if_i(2\tau)+h(\tau)\\
\Longrightarrow&\sum_{i=1}^{n}(2^ta_i-b_i)f_i(2\tau)=\sum_{i=1}^{n}(a_i-2^{-t}b_i)f_i(\tau)\\
\Longrightarrow&\sum_{i=1}^{n}(a_i-2^{-t}b_i)f_i(\tau)=0\\
\Longrightarrow&b_i=2^ta_i.
\end{aligned}$$

So $f\in S_t^{\text{new}}(\Gamma_0(2),+)\oplus Q$, and then $S_t(\Gamma_0(2),+)=S_t^{\text{new}}(\Gamma_0(2),+)\oplus Q$.

Therefore,

$$\begin{aligned}
\dim S_t^{\text{new}}(\Gamma_0(2),+)&=\dim S_t(\Gamma_0(2),+)-\dim S_t(\Gamma_0(1)\\
&=\lfloor 3t/8\rfloor+\lfloor t/4\rfloor-t/2-\lfloor t/3\rfloor-\lfloor t/4\rfloor+t/2\\
&=\lfloor 3t/8\rfloor-\lfloor t/3\rfloor
\end{aligned}$$

To compute $\dim S_t^{\text{new}}(\Gamma_0(2),-)$, we follow the same process as above but change $f(\tau)+2^tf(2\tau)$ to $f(\tau)-2^tf(2\tau)$. □

Similarly, we have formulas for dimension of $S_t(\Gamma_0(3),\epsilon_2)$ and $S_t^{\text{new}}(\Gamma_0(3),\epsilon_2)$.

Proposition 3.1.3. *Let $S_t(\Gamma_0(3),\epsilon_2)$ denote the space of cusp forms of weight t(t is even) on $\Gamma_0(3)$ that is eigenfunctions for w_3 with eigenvalue ϵ_2.Then the dimension of this space is*

$$\dim(S_t(\Gamma_0(3),\epsilon_2))=\begin{cases}\lfloor 3t/8\rfloor+\lfloor t/4\rfloor-t/2, & \text{if } \epsilon_2=1,\\ t/2-\lfloor 3t/8\rfloor-1, & \text{if } \epsilon_2=-1.\end{cases}$$

Moreover, let $S_t^{\text{new}}(\Gamma_0(3),\epsilon_2)$ denote the space of newforms of weight t(t is even) on $\Gamma_0(3)$ that is eigenfunction for w_3 with eigenvalues ϵ_2.Then the dimension of this space is

$$\dim S_t^{\text{new}}(\Gamma_0(3),\epsilon_2)=\begin{cases}\lfloor 5t/12\rfloor-\lfloor t/3\rfloor, & \text{if } \epsilon_2=1\\ t-\lfloor 5t/12\rfloor-2\lfloor t/4\rfloor-1, & \text{if } \epsilon_2=-1.\end{cases}$$

Proposition 3.1.4. *Dimension of $S_t^{\text{new}}(\Gamma_0(4),\epsilon_3)$ is*

$$\dim S_t^{\text{new}}(\Gamma_0(4),\epsilon_3)=\begin{cases}0 & \text{if } \epsilon_3=1\\ \lfloor k/3\rfloor-\lfloor k/4\rfloor & \text{if } \epsilon_3=-1.\end{cases}$$

where ϵ_3 is the eigenvalue of Atkin Lehner involution w_4.

3.2 Dimension of $S_t(\Gamma_0(6), \epsilon_1, \epsilon_2)$

Proposition 3.2.1. *Let $S_t^{\text{new}}(\Gamma_0(6), \epsilon_1, \epsilon_2)$ denote the space of newforms of weight t(t is even) on $\Gamma_0(6)$ that is eigenfunction for w_2, w_3 with eigenvalues ϵ_1, ϵ_2. Then the dimension of this space is*

$$\dim S_t^{\text{new}}(\Gamma_0(6), \epsilon_1, \epsilon_2) = \begin{cases} 2\lfloor t/4 \rfloor + \lfloor t/3 \rfloor - \lfloor 3t/8 \rfloor - \lfloor 5t/12 \rfloor, & \text{if } \epsilon_1 = 1, \quad \epsilon_2 = 1 \\ \lfloor 5t/12 \rfloor - \lfloor 3t/8 \rfloor, & \text{if } \epsilon_1 = 1, \quad \epsilon_2 = -1, \\ \lfloor t/3 \rfloor - \lfloor t/4 \rfloor + \lfloor 3t/8 \rfloor - \lfloor 5t/12 \rfloor, & \text{if } \epsilon_1 = -1, \quad \epsilon_2 = 1, \\ \lfloor t/4 \rfloor + \lfloor 3t/8 \rfloor + \lfloor 5t/12 \rfloor - t + 1, & \text{if } \epsilon_1 = -1, \quad \epsilon_2 = -1. \end{cases}$$

We prove some lemmas first.

Lemma 3.2.2. *Let $S_t(\Gamma_0(6), \epsilon_1, \epsilon_2)$ denote the space of cusp forms of weight t(t is even) on $\Gamma_0(6)$ that is eigenfunctions for w_2, w_3 with eigenvalues ϵ_1, ϵ_2. Then the dimension of this space is*

$$\dim S_t(\Gamma_0(6), \epsilon_1, \epsilon_2) \begin{cases} 3\lfloor t/4 \rfloor - t/2, & \text{if } \epsilon_1 = 1, \quad \epsilon_2 = 1, \\ t/2 - \lfloor t/4 \rfloor - 1, & \text{if } \epsilon_1 = 1, \quad \epsilon_2 = -1, \\ t/2 - \lfloor t/4 \rfloor - 1, & \text{if } \epsilon_1 = -1, \quad \epsilon_2 = 1, \\ t/2 - \lfloor t/4 \rfloor - 1, & \text{if } \epsilon_1 = -1, \quad \epsilon_2 = -1. \end{cases}$$

Proof. $\Gamma_0(6)$ is genus 0, so $\Gamma_0^+(6) = \Gamma_0(6) \cup w_2\Gamma_0(6) \cup w_3\Gamma_0(6) \cup w_6\Gamma_0(6)$ is genus 0. $\{0, \frac{1}{2}, \frac{1}{3}, \infty\}$ are inequivalent cusps of $\Gamma_0(6)$, but in $\Gamma_0^+(6)$, we have only one inequivalent cusp, since

$$w_2(0) = \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix} 0 = \frac{1}{2}.$$

and also $w_3(0) = \frac{1}{3}$, $w_6(0) = \infty$. The fundamental domain of $\Gamma_0^+(6)$ is equal to 3 of fundamental domain of $SL_2(\mathbb{Z})$, since the index [$\mathrm{SL}_2(\mathbb{Z}, \Gamma_0(6))$]=12. And we have

$$0 = 1 + \frac{3}{12} - \frac{1}{2}\sum_{n=1}^{r}(1 - \frac{1}{e_r}) - \frac{1}{2},$$

where r is the number of inequivalent elliptic points, and e_i is the order of each elliptic point. then

$$\sum_{n=1}^{r}(1 - \frac{1}{e_r}) = \frac{3}{2}.$$

so we can see that $\Gamma_0^+(6)$ has three elliptic points of order 2.
And then by dimension formula of modular forms, we have $\dim(S_t(\Gamma_0(6), +, +)) = 3\lfloor t/4 \rfloor - t/2$.
For the others, we first compute the inequivalent cusps and elliptic points by similar method above. Define that

1. $(\Gamma_0(6), +, -) = \Gamma_0(6) \cup w_2\Gamma_0(6)$.

2. $(\Gamma_0(6), -, +) = \Gamma_0(6) \cup w_3\Gamma_0(6)$.

3. $(\Gamma_0(6), -, -) = \Gamma_0(6) \cup w_6\Gamma_0(6)$

And we see that all of $(\Gamma_0(6), +, -)$,$(\Gamma_0(6), -, +)$, and $(\Gamma_0(6), -, -)$ have 2 inequivalent elliptic points of order 2 and 2 inequivalent cusps. So $\dim(S_t(\Gamma_0(6), +, +)) = \dim(\mathcal{S}_t(\Gamma_0(6), +, -)) = \dim(\mathcal{S}_t(\Gamma_0(6), -, -)) = \frac{1}{3}(\dim(\mathcal{S}_t(\Gamma_0(6))) - \dim(\mathcal{S}_t(\Gamma_0(6), +, +))) = t/2 - \lfloor t/4 \rfloor - 1$.

∎

Lemma 3.2.3. *Let $f\in S_t(\Gamma_0(2))$, then $g_i(\tau):=f(\tau)+(-1)^i3^{\frac{t}{2}}f(3\tau)\in\mathcal{S}_t(\Gamma_0(6))$. Moreover the eigenvalue of Atkin-Lehner operator w_3 with respect to g_i is $(-1)^i$, where i=$\{1,2\}$.*

Proof. By previous lemma, we see that $g_i(\tau)\in\mathcal{S}_t(\Gamma_0(6))$.
Let $w_3=\begin{pmatrix}3&1\\6&3\end{pmatrix}$.
Note that $w_3(f(\tau))=f(\tau)\mid[w_3]_t$,
and $f\left(\dfrac{3\tau+1}{6\tau+3}\right)=f(\gamma3\tau)=(2\tau+3)^tf(3\tau)$, where $\gamma=\begin{pmatrix}1&1\\2&3\end{pmatrix}$. So we have

$$w_3(f(\tau))=(\det(w_3))^{t/2}(6\tau+3)^{-t}f\left(\frac{3\tau+1}{6\tau+3}\right)=3^{t/2}f(3\tau), \tag{3.5}$$

and

$$w_3(f(3\tau))=(\det(w_3))^{t/2}(6\tau+3)^{-t}f\left(3\left(\frac{3\tau+1}{6\tau+3}\right)\right)=3^{-t/2}f(\tau). \tag{3.6}$$

Therefore, $w_3(g_i(\tau))=g_i(\tau)\mid_{[w_3]_t}=(-1)^ig_i$.

□

Similarly, we can prove following:

Lemma 3.2.4. *Let $f\in S_t(\Gamma_0(3))$, then $h_i(\tau):=f(\tau)+(-1)^i2^{\frac{t}{2}}f(2\tau)\in\mathcal{S}_t(\Gamma_0(6))$. Moreover the eigenvalue of Atkin-Lehner operator w_2 with respect to h_i is $(-1)^i$, where i=$\{1,2\}$.*

And now we can prove the proposition.

Proof of Proposition3.2.1. We prove the case $\epsilon_1=\epsilon_2=1$, and the others are same. By Equation 2.1 we have an unique decomposition of $S_t(\Gamma_0(6))$

$$\begin{aligned}S_t(\Gamma_0(6))=&S_t(\Gamma_0(1))\oplus\alpha_2S_t(\Gamma_0(1))\alpha_3S_t(\Gamma_0(1))\alpha_6S_t(\Gamma_0(1))\\&\bigoplus S_t^{\text{new}}(\Gamma_0(2))\oplus\alpha_3S_t^{\text{new}}(\Gamma_0(2))\\&\bigoplus S_t^{\text{new}}(\Gamma_0(3))\oplus\alpha_2S_t^{\text{new}}(\Gamma_0(3))\\&\bigoplus S_t^{\text{new}}(\Gamma_0(6))\end{aligned}$$

If $f(\tau)\in S_t(\Gamma_0(1))$ and let $w_2=\begin{pmatrix}2&1\\6&4\end{pmatrix}$ be the atkin-lehner involution on $\Gamma_0(6)$, then we have

$$\begin{aligned}w_2f(\tau)&=2^{t/2}(6\tau+4)^{-t}f\left(\frac{2\tau+1}{6\tau+4}\right)\\&=2^{t/2}(6\tau+4)^{-t}f\left(\begin{pmatrix}1&1\\3&4\end{pmatrix}2\tau\right)\\&=2^{t/2}f(2\tau).\end{aligned}$$

Let $w_3 = \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}$ be the atkin-lehner involution on $\Gamma_0(6)$, then we have

$$\begin{aligned} w_3 f(\tau) &= 3^{t/2}(6\tau+3)^{-t} f\left(\frac{3\tau+1}{6\tau+3}\right) \\ &= 3^{t/2}(6\tau+3)^{-t} f\left(\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} 3\tau\right) \\ &= 3^{t/2} f(3\tau). \end{aligned}$$

If $f(2\tau) \in \alpha_2 S_t(\Gamma_0(1))$, then we have

$$\begin{aligned} w_2 f(2\tau) &= 2^{t/2}(6\tau+4)^{-t} f\left(\frac{4\tau+2}{6\tau+4}\right) \\ &= 2^{t/2}(6\tau+4)^{-t} f\left(\frac{2\tau+1}{3\tau+2}\right) \\ &= 2^{t/2}(6\tau+4)^{-t} f\left(\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \tau\right) \\ &= 2^{-t/2} f(\tau), \end{aligned}$$

and

$$\begin{aligned} w_3 f(2\tau) &= 3^{t/2}(6\tau+3)^{-t} f\left(\frac{6\tau+2}{6\tau+3}\right) \\ &= 3^{t/2}(6\tau+3)^{-t} f\left(\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} 6\tau\right) \\ &= 3^{t/2} f(6\tau). \end{aligned}$$

If $f(3\tau) \in \alpha_3 S_t(\Gamma_0(1))$, then we have

$$\begin{aligned} w_2 f(3\tau) &= 2^{t/2}(6\tau+4)^{-t} f\left(\frac{6\tau+3}{6\tau+4}\right) \\ &= 2^{t/2}(6\tau+4)^{-t} f\left(\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} 6\tau\right) \\ &= 2^{t/2} f(6\tau), \end{aligned}$$

and

$$\begin{aligned} w_3 f(3\tau) &= 3^{t/2}(6\tau+3)^{-t} f\left(\frac{9\tau+3}{6\tau+3}\right) \\ &= 3^{t/2}(6\tau+3)^{-t} f\left(\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \tau\right) \\ &= 3^{-t/2} f(\tau). \end{aligned}$$

If $f(6\tau)\in\alpha_6 S_t(\Gamma_0(1))$, then we have

$$\begin{aligned}
w_2 f(6\tau) &= 2^{t/2}(6\tau+4)^{-t} f\left(\frac{12\tau+6}{6\tau+4}\right)\\
&= 2^{t/2}(6\tau+4)^{-t} f\left(\frac{6\tau+3}{3\tau+2}\right)\\
&= 2^{t/2}(6\tau+4)^{-t} f\left(\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix} 3\tau\right)\\
&= 2^{-t/2} f(3\tau),
\end{aligned}$$

and

$$\begin{aligned}
w_3 f(6\tau) &= 3^{t/2}(6\tau+3)^{-t} f\left(\frac{18\tau+6}{6\tau+3}\right)\\
&= 3^{t/2}(6\tau+3)^{-t} f\left(\frac{6\tau+2}{2\tau+1}\right)\\
&= 3^{t/2}(6\tau+3)^{-t} f\left(\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix} 2\tau\right)\\
&= 3^{-t/2} f(2\tau).
\end{aligned}$$

Let $g = af(\tau)+bf(2\tau)+cf(3\tau)+df(6\tau)$, where a,b,c,d are scalars such that $w_2 g = g$, and $w_3 g = g$. Then we have a linear system

$$\begin{pmatrix} 2^{t/2} & -1 & 0 & 0\\ 0 & 0 & 2^{t/2} & -1\\ 3^{t/2} & 0 & -1 & 0\\ 0 & 3^{t/2} & 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{3.7}$$

Note that this system has only one solution that is $(a,b,c,d)=(1,2^{t/2},3^{t/2},6^{t/2})$. If $f(\tau)\in S_t^{\mathrm{new}}(\Gamma_2,+)$, then by Lemma 3.2.4 and Lemma 3.2.3 we know that

$$w_3 f(\tau) = 3^{t/2} f(3\tau)$$

and

$$w_3 f(3\tau) = 3^{-t/2} f(\tau)$$

If $f(\tau)\in S_t^{\mathrm{new}}(\Gamma_3,+)$, then by Lemma 3.2.4 and Lemma 3.2.3 we know that

$$w_2 f(\tau) = 2^{t/2} f(2\tau)$$

and

$$w_2 f(2\tau) = 2^{-t/2} f(\tau)$$

Now we let

$$\begin{aligned}
V_1 &= \mathrm{span}\left\{f_i(\tau)+3^{t/2}f_i(3\tau) : f_i \text{ is basis of } S_t(\Gamma_0(2))\right\}\\
V_2 &= \mathrm{span}\left\{g_i(\tau)+2^{t/2}g_i(2\tau) : g_i \text{ is basis of } S_t(\Gamma_0(3))\right\}\\
V_3 &= \mathrm{span}\left\{h_i(\tau)+2^{t/2}h_i(2\tau)+3^{t/2}h_i(3\tau)+6^{t/2}h_i(6\tau) : h_i \text{ is basis of } S_t(\Gamma_0(1))\right\}.
\end{aligned}$$

By considering the decomposition of $S_t(\Gamma_0(6))$ and the calculations above we have

$$S_t(\Gamma_0(6),+,+) = S_t^{\text{new}}(\Gamma_0(6),+,+) \oplus V_1 \oplus V_2 \oplus V_3.$$

So

$$\dim S_t^{\text{new}}(\Gamma_0(6),+,+) = \dim S_t(\Gamma_0(6),+,+) - \dim V_1 - \dim V_2 - \dim V_3 \quad = 2\lfloor t/4\rfloor + \lfloor t/3\rfloor - \lfloor 3t/8\rfloor - \lfloor 5t/12\rfloor.$$

□

Proposition 3.2.5. *Dimension of $S_t^{\text{new}}(\Gamma_0(12),\epsilon_2,\epsilon_3)$ is:*

$$\dim S_t^{\text{new}}(\Gamma_0(12),\epsilon_2,\epsilon_3) = \begin{cases} 0 & \text{if } \epsilon_2 = 1, \epsilon_3 = 1, \\ \lfloor 5k/12\rfloor - \lfloor k/3\rfloor & \text{if } \epsilon_2 = 1, \epsilon_3 = -1, \\ 0 & \text{if } \epsilon_2 = -1, \epsilon_3 = 1, \\ 2\lfloor k/4\rfloor - \lfloor 5k/12\rfloor & \text{if } \epsilon_2 = -1, \epsilon_3 = -1, \end{cases}$$

where ϵ_3 is the eigenvalue of the atkin-lehner involution w_4.

Theorem 3.2.6. [2] *Let $f = \sum_{n=1}^{\infty} a_n q^n \in M_t(\Gamma_0(N),\phi)$, where ϕ be a Dirichlet character with conductor c. Let χ be a primitive Dirichlet character modulo m. Then*

$$f \otimes \chi = \sum_{n\geq 0} \chi(n) a_n q^n. \tag{3.8}$$

and $f \otimes \chi \in M_t(\Gamma_0(M),\phi\chi^2)$, where M is the least common multiple of N, cm, m^2. If f is a cusp form, so is $f \otimes \chi$.

Corollary 3.2.7. *Let $f \in \mathcal{S}_t(\Gamma_0(N))$, and χ_{-4} be a primitive Dirichlet character modulo 4. Then $f \otimes \chi_{-4} \in \mathcal{S}_t(\Gamma_0(16N'),\chi_{-4}^2)$, where $N' = \operatorname{lcm}(16,N)/16$. Similarly, Let χ_{12} be Dirichlet primitive character modulo 12, then $f \otimes \chi_{12} \in \mathcal{S}_t(\Gamma_0(144N'),\chi_{12}^2)$, where $N' = \operatorname{lcm}(144,N)/144$.*

Conjecture 3.2.8. *Let $r \in \{1,5,7\}$, and s be a non-negative even integer.*
Define

$$S_{r,s}(\Gamma_0(3)) = \{\eta(24\tau)^r f(24\tau) : f \in M_s(\Gamma_0(3))\}. \tag{3.9}$$

Then $S_{r,s}(\Gamma_0(3))$ is an invariant subspace of $S_{\frac{r}{2}+s}(\Gamma_0(576),\chi_{12})$ under the action of the Hecke algebra. That is for all primes $\ell \neq 2,3$ and all $f \in S_{r,s}(\Gamma_0(3))$, we have $f \mid T_{\ell^2} \in S_{r,s}(\Gamma_0(3))$.
Furthermore, let $S_t^{\text{new}}(\Gamma_0(6),\epsilon_1,\epsilon_2)$ denote the space of newforms of weight t on $\Gamma_0(6)$ that are eigenfunctions for w_2 and w_3 with eigenvalues ϵ_1 and ϵ_2, respectively.
Let also $S_t^{\text{new}}(\Gamma_0(2),\epsilon_1)$ denote the space of newforms of weight t on $\Gamma_0(2)$ that are eigenfunctions for w_2 with eigenvalue ϵ_1.
Then they have

$$S_{r,s}(\Gamma_0(3)) \cong V \oplus W, \tag{3.10}$$

where

$$V = S_{r+2s-1}^{\text{new}}(\Gamma_0(6), -\left(\frac{2}{r}\right), -\left(\frac{3}{r}\right)) \otimes \chi_{12},$$

and

$$W = \{S_{r+2s-1}^{\text{new}}(\Gamma_0(2),\left(\frac{2}{r}\right)) \oplus S_{r+2s-1}^{\text{new}}(\Gamma_0(6),\left(\frac{2}{r}\right),\left(\frac{3}{r}\right)) \oplus S_{r+2s-1}^{\text{new}}(\Gamma_0(6),\left(\frac{2}{r}\right),-\left(\frac{3}{r}\right))\} \otimes \chi_{-4}.$$

For convenience, we let

$$\begin{cases} W_1 = S^{\text{new}}_{r+2s-1}(\Gamma_0(2), \left(\frac{2}{r}\right)), \\ W_2 = S^{\text{new}}_{r+2s-1}(\Gamma_0(6), \left(\frac{2}{r}\right), \left(\frac{3}{r}\right)), \\ W_3 = S^{\text{new}}_{r+2s-1}(\Gamma_0(6), \left(\frac{2}{r}\right), -\left(\frac{3}{r}\right)). \end{cases}$$

And moreover, we also have a conjecture for $S_{r,s}(\Gamma_0(2))$. Let $r \in \{1, 5, 7\}$

$$S_{r,s}(\Gamma_0(2)) \cong S^{\text{new}}_{r+2s-1}(\Gamma_0(6), -\left(\frac{2}{r}\right), -\left(\frac{3}{r}\right)) \otimes \chi_{12} \bigoplus S^{\text{new}}_{r+2s-1}(\Gamma_0(12), \epsilon_3, -\left(\frac{3}{r}\right)) \otimes \chi_{12}, \tag{3.11}$$

where ϵ_3 is the eigenvalue of Atkin-Lehner involution w_4 on $\Gamma_0(12)$ and is always negative with respect to $S^{\text{new}}_t(\Gamma_0(12))$.

Chapter 4

Result

4.1 $S_{r,s}(\Gamma_0(3))$

Example 4.1.1. Consider $S_{1,2}(\Gamma_0(3)) = \{\eta(24\tau)f(24\tau) : f(\tau) \in M_2(\Gamma_0(3))\}$ in $S_{2+1/2}(\Gamma_0(576), \chi_{12})$. Let $G_2(\tau)$ be the Eisentein series, then $f = \dfrac{3G_2(3\tau) - G_2(\tau)}{2}$ is a modular form on $\Gamma_0(3)$ of weight 2. And by the dimension formula for the modular forms, we know that $\dim M_2(\Gamma_0(3)) = 1$, so $\{f\}$ is a basis. Thus $\{\eta(24\tau)f(24\tau)\}$ is a basis for $S_{1,2}(\Gamma_0(3))$, now we apply Hecke operators T_{p^2} on this basis.

For example, let $p = 5$, then the q-expansion of $\eta(24\tau)f(24\tau)|T_{5^2}$ is

$$6q + 66q^{25} + 138q^{49} - 216q^{73} + 216q^{97} - 138q^{121} - 648q^{145} + 150q^{169} + 432q^{193} - 864q^{217} \ldots.$$

and $6\eta(24\tau)f(24\tau) - \eta(24\tau)f(24\tau)|T_{5^2} = O(q^{241})$ means that the initial segment of q-expansions of $6\eta(24\tau)f(24\tau)$ and $\eta(24\tau)f(24\tau)|T_{5^2}$ agree more than $\frac{5}{2}[SL_2(\mathbb{Z}) : \Gamma_0(576)]/12$ terms, thus $\eta(24\tau)f(24\tau)|T_{5^2} = 6\eta(24\tau)f(24\tau)$. It is similar for other prime numbers, and we have the following table.

	5	7	11	13	17	19
$S_{1,2}(\Gamma_0(3))$	6	16	-12	38	-126	-20

On the other hand, since $\left(\frac{2}{r}\right) = 1$ and $\left(\frac{3}{r}\right) = 1$, by Proposition 3.1.1 and Proposition 3.2.1 we know that

$$\begin{cases} \dim V = \dim S_4(\Gamma_0(6), -, -) = 0 \\ \dim W_1 = \dim S_4(\Gamma_0(2), +) = 0 \\ \dim W_2 = \dim S_4(\Gamma_0(6), +, +) = 1 \\ \dim W_3 = \dim S_4(\Gamma_0(6), +, -) = 0. \end{cases}$$

So $V \oplus W$ is $S_4^{\text{new}}(\Gamma_0(6), +, +) \otimes \chi_{-4}$. We compute the basis of space of newforms by Sage, and the q-expansion of the basis of $\mathcal{S}_4^{\text{new}}(\Gamma_0(6), +, +) \otimes \chi_{-4}$ is

$$q + 3q^3 + 6q^5 + 16q^7 + 9q^9 - 12q^{11} + 38q^{13} + 32q^{14} + 18q^{15} - 126q^{17} - 20q^{19} + O(q^{20}).$$

And next case are concerned with $r = 5, 7$ and $s = 2$.

Example 4.1.2. $S_{r,2}(\Gamma_0(3))$ For $r = 5, 7$.

a. For $S_{5,2}(\Gamma_0(3))$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{5+4-1}(\Gamma_0(6),+,+) = 0, \\ \dim W_1 = \dim S^{\text{new}}_{5+4-1}(\Gamma_0(2),-) = 0, \\ \dim W_2 = \dim S^{\text{new}}_{5+4-1}(\Gamma_0(6),-,-) = 1, \\ \dim W_3 = \dim S^{\text{new}}_{5+4-1}(\Gamma_0(6),-,+) = 0. \end{cases}$$

So from our conjecture

$$S_{5,2}(\Gamma_0(3)) \cong S^{\text{new}}_{5+4-1}(\Gamma_0(6),-,-) \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,2}$(by Maple)

	5	7	11	13	17	19
$S_{5,2}(\Gamma_0(3))$	-144	1576	-7332	-3802	-6606	-24860

(By Sage.)The normalized newform in $S^{\text{new}}_{5+4-1}(\Gamma_0(6),-,-)$ is

$$q + 8q^2 + 27q^3 + 64q^4 - 114q^5 + 216q^6 - 1576q^7 + 512q^8 + \dots,$$

then we twist this newform by χ_{-4}:

$$q - 27q^3 - 114q^5 + 1576q^7 + 729q^9 - 7332q^{11} - 3802q^{13} + 3078q^{15} - 6606q^{17} - 24860q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_8(\Gamma_0(48))$, and its Atkin-Lehner eigenvalues for w_2 and w_3 are -1 and 1(the eigenvalue of w_3 change to -1 since we twist by χ_{-4}), respectively.

b. For $S_{5,2}(\Gamma_0(3))$.

Next we deal with the case $r = 7$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{7+4-1}(\Gamma_0(6),+,+) = 0, \\ \dim W_1 = \dim S^{\text{new}}_{7+4-1}(\Gamma_0(2),+) = 0, \\ \dim W_2 = \dim S^{\text{new}}_{7+4-1}(\Gamma_0(6),+,-) = 1, \\ \dim W_3 = \dim S^{\text{new}}_{7+4-1}(\Gamma_0(6),-,+) = 0. \end{cases}$$

From our conjecture

$$S_{7,2}(\Gamma_0(3)) \cong S^{\text{new}}_{7+4-1}(\Gamma_0(6),+,-) \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{7,2}(\Gamma_0(3))$

	5	7	11	13	17	19
$S_{7,2}(\Gamma_0(3))$	2694	3544	-29580	-44818	-101934	895084

(By Sage.)The normalized newform in $S^{\text{new}}_{7+4-1}(\Gamma_0(6),+,-)$ is

$$q - 16q^2 + 81q^3 + 256q^4 + 2694q^5 - 1296q^6 - 3544q^7 - 4096q^8 + \dots,$$

then we twist this newform by χ_{-4}:

$$q - 81q^3 + 2694q^5 + 3544q^7 + 6561q^9 - 29580q^{11} - 44818q^{13} + 218214q^{15} - 101934q^{17} + 895084q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_{10}(\Gamma_0(48))$, and its Atkin-Lehner eigenvalues for w_2 and w_3 are -1 and 1(the eigenvalue of w_3 change to 1 since we twist by χ_{-4}), respectively.

For the cases $r = 11, 13, 17, 19, 23$. Our conjecture never holds by checking the dimensions of $S_{r,2}(\Gamma_0(3))$ and $V \oplus W$. By Proposition 3.1.1 and Proposition 3.2.1, we have

$$\dim V \oplus W = \begin{cases} 3 & \text{if } r = 11, \\ 2 & \text{if } r = 13, \\ 2 & \text{if } r = 17, \\ 3 & \text{if } r = 19, \\ 3 & \text{if } r = 23. \end{cases}$$

Example 4.1.3. For $r = 1, 5, 7$ and $s = 4$. Note that

$$\dim \mathcal{S}_{r,4}(\Gamma_0(3)) = 2.$$

a. For $S_{1,4}(\Gamma_0(3))$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{1+8-1}(\Gamma_0(6), -, -) = 1, \\ \dim W_1 = \dim S^{\text{new}}_{1+8-1}(\Gamma_0(2), +) = 1, \\ \dim W_2 = \dim S^{\text{new}}_{1+8-1}(\Gamma_0(6), +, +) = 0, \\ \dim W_3 = \dim S^{\text{new}}_{1+8-1}(\Gamma_0(6), +, -) = 0. \end{cases}$$

So from our conjecture

$$S_{5,2}(\Gamma_0(3)) \cong S^{\text{new}}_{1+8-1}(\Gamma_0(6), -, -) \otimes \chi_{12} \bigoplus S^{\text{new}}_{8}(\Gamma_0(2), +) \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,2}$(by Maple)

	5	7	11	13	17	19
$\mathcal{S}_{1,4}$	144 -210	1576 -1016	7332 -1092	-3802 1382	6606 14706	-24860 39904

(By Sage.)The normalized newform in $S^{\text{new}}_{1+8-1}(\Gamma_0(6), -, -)$ is

$$q + 8q^2 + 27q^3 + 64q^4 - 144q^5 + 216q^6 - 1576q^7 + 512q^8 + \ldots,$$

then we twist this newform by χ_{12}:

$$q + 114q^5 + 1576q^7 + 7332q^{11} - 3802q^{13} + 6606q^{17} - 24860q^{19} + O(q^20)$$

We can find this normalized newform in $S_8(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_2 and w_3 are -1 and -1, respectively.

(By Sage.)The normalized newform in $S^{\text{new}}_{1+8-1}(\Gamma_0(2), +)$ is

$$q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 + \ldots.$$

then we twist this newform by χ_{-4}:

$$q + 12q^3 - 210q^5 - 1016q^7 - 2043q^9 - 1092q^{11} + 1382q^{13} + 2520q^{15} + 14706q^{17} + 39940q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_8(\Gamma_0(16))$, and its Atkin-Lehner eigenvalues for w_2 is -1

b. For $S_{5,4}(\Gamma_0(3))$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{5+8-1}(\Gamma_0(6),+,+) = 1, \\ \dim W_1 = \dim S^{\text{new}}_{5+8-1}(\Gamma_0(2),-) = 0, \\ \dim W_2 = \dim S^{\text{new}}_{5+8-1}(\Gamma_0(6),-,-) = 1, \\ \dim W_3 = \dim S^{\text{new}}_{5+8-1}(\Gamma_0(6),-,+) = 0. \end{cases}$$

From our conjecture

$$S_{5,4}(\Gamma_0(3)) \cong S^{\text{new}}_{5+8-1}(\Gamma_0(6),+,+) \otimes \chi_{12} \bigoplus S^{\text{new}}_{5+8-1}(\Gamma_0(6),+,-) \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,4}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{5,4}$	-5766 3630	-72464 -32936	-408948 758748	-2482858 1367558	-5422914 8290386	-15166100 10867300

(By Sage.)The normalized newform in $S^{\text{new}}_{5+8-1}(\Gamma_0(6),+,+)$ is

$$q - 32q^2 - 243q^3 + 1024q^4 + 5766q^5 + 7776q^6 + 72464q^7 - 32768q^8 + \dots.$$

then we twist this newform by χ_{12}:

$$q - 5766q^5 - 72464q^7 - 408948q^{11} + 1367558q^{13} - 5422914q^{17} - 15166100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.

(By Sage.)The normalized newform in $S^{\text{new}}_{5+8-1}(\Gamma_0(6),-,-)$ is

$$q + 32q^2 + 243q^3 + 1024q^4 + 3630q^5 + 7776q^6 + 32936q^7 + 32768q^8 + \dots,$$

then we twist this newform by χ_{-4}:

$$q - 243q^3 + 3630q^5 - 32936q^7 + 59049q^9 + 758748q^{11} - 2482858q^{13} + 8290386q^{17} + 10867300q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.

c. For $S_{7,4}(\Gamma_0(3))$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{7+8-1}(\Gamma_0(6),-,+) = 1, \\ \dim W_1 = \dim S^{\text{new}}_{7+8-1}(\Gamma_0(2),+) = 1, \\ \dim W_2 = \dim S^{\text{new}}_{7+8-1}(\Gamma_0(6),+,-) = 0, \\ \dim W_3 = \dim S^{\text{new}}_{7+8-1}(\Gamma_0(6),+,+) = 0. \end{cases}$$

From our conjecture

$$S_{7,4}(\Gamma_0(3)) \cong S^{\text{new}}_{7+8-1}(\Gamma_0(6),-,+) \otimes \chi_{12} \bigoplus S^{\text{new}}_{7+8-1}(\Gamma_0(2),+) \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{7,4}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{7,4}$	-54654 3990	-176336 433432	-1619772 6612420	-24028978 -10878466	60569298 154665054	-190034876 243131740

(By Sage.)The normalized newform in $S^{\text{new}}_{7+8-1}(\Gamma_0(6),-,+)$ is

$$q+64q^2-729q^3+4096q^4+54654q^5-46656q^6+176336q^7+262144q^8+\dots.$$

then we twist this newform by χ_{12}:

$$q-54654q^5-176336q^7+6612420q^{11}-24028978q^{13}+154665054q^{17}-190034876q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.
(By Sage.)The normalized newform in $S^{\text{new}}_{7+8-1}(\Gamma_0(2),+)$ is

$$q-64q^2-1836q^3+4096q^4+3990q^5+117504q^6-433432q^7-262144q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q+1836q^3+3990q^5+433432q^7+1776573q^9-1619772q^{11}-10878466q^{13}+60569298q^{17}+243131740q^{19}+O(q^{20}),$$

We can find this normalized newform in $S_{12}(\Gamma_0(16))$.

Example 4.1.4. For $r=1,5,7$ and $s=8$. $\dim S_{r,6}(\Gamma_0(3))=3$.

a. For $\mathcal{S}_{1,8}$.
Note that,

$$\begin{cases}\dim V=\dim S^{\text{new}}_{1+12-1}(\Gamma_0(6),-,-)=1,\\ \dim W_1=\dim S^{\text{new}}_{1+12-1}(\Gamma_0(2),+)=0,\\ \dim W_2=\dim S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,+)=1,\\ \dim W_3=\dim S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,-)=1.\end{cases}$$

From our conjecture

$$S_{1,6}(\Gamma_0(3))\cong S^{\text{new}}_{1+12-1}(\Gamma_0(6),-,+)\otimes\chi_{12}\bigoplus\{S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,+)\oplus S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,-)\}\otimes\chi_{-4}$$

And we compute eigenvalues for T_p^2, $p=5,7,11,13,17,19$ on $S_{1,6}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{1,6}(\Gamma_0(3))$	-3630	-72464	531420	1332566	5422914	-2901404
	5766	50008	-758748	-2482858	-8290386	10867300
	-11730	-32936	408948	1367558	-5109678	-15166100

(By Sage.)The normalized newform in $S^{\text{new}}_{1+12-1}(\Gamma_0(6),-,-)$ is

$$q+32q^2+243q^3+1024q^4+3630q^5+7776q^6+32936q^7+32768q^8+\dots.$$

then we twist this newform by χ_{12}:

$$q-3630q^5-32936q^7-758748q^{11}-2482858q^{13}-8290386q^{17}+10867300q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.
(By Sage.)The normalized newform in$S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,+)$ is

$$q-32q^2-243q^3+1024q^4+5766q^5+7776q^6+72464q^7-32768q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q+243q^3+5766q^5-72464q^7+59049q^9+408948q^{11}+1367558q^{13}+1401138q^{15}+5422914q^{17}-15166100q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(48))$.

(By Sage.)The normalized newform in $S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,-)$ is

$$q-32q^2+243q^3+1024q^4-11730q^5-7776q^6-50008q^7-32768q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q-11730q^5+50008q^7+59049q^9+531420q^{11}+1332566q^{13}+2850390q^{15}-5109678q^{17}-2901404q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(48))$.

b. For $\mathcal{S}_{5,6}$.

Note that,

$$\begin{cases}\dim V=\dim S^{\text{new}}_{5+12-1}(\Gamma_0(6),+,+)=1,\\ \dim W_1=\dim S^{\text{new}}_{5+12-1}(\Gamma_0(2),-)=0,\\ \dim W_2=\dim S^{\text{new}}_{5+12-1}(\Gamma_0(6),-,-)=1,\\ \dim W_3=\dim S^{\text{new}}_{5+12-1}(\Gamma_0(6),-,+)=1.\end{cases}$$

From our conjecture

$$S_{5,6}(\Gamma_0(3))\cong S^{\text{new}}_{5+12-1}(\Gamma_0(6),+,+)\otimes\chi_{12}\bigoplus\{S^{\text{new}}_{5+12-1}(\Gamma_0(6),-,-)\oplus S^{\text{new}}_{5+12-1}(\Gamma_0(6),-,+)\}\otimes\chi_{-4}$$

And we compute eigenvalues for T_p^2, $p=5,7,11,13,17,19$ on $S_{5,6}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{5,6}(\Gamma_0(3))$	-114810	-762104	103451700	-104365834	-3173671566	5895116260
	77646	3034528	110255052	56047862	997689762	-2163188180
	314490	-2025056	-48011172	285130118	1930104414	-4934015444

(By Sage.)The normalized newform in$S^{\text{new}}_{5+12-1}(\Gamma_0(6),+,+)$ is

$$q-128q^2-2187q^3+16384q^4-314490q^5+279936q^6+2025056q^7-2097152q^8+\dots,$$

then we twist this newform by χ_{12}:

$$q+314490q^5-2025056q^7+110255052q^{11}+56047862q^{13}+1930104414q^{17}-2163188180q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{16}(\Gamma_0(144))$.

(By Sage.)The normalized newform in$S^{\text{new}}_{5+12-1}(\Gamma_0(6),-,-)$ is

$$q+128q^2+2187q^3+16384q^4+77646q^5+279936q^6+762104q^7+2097152q^8+\dots$$

then we twist this newform by χ_{-4}:

$$q-2187q^3+77646q^5-762104q^7+4782969q^9-48011172q^{11}+285130118q^{13}-169811802q^{15}$$
$$-3173671566q^{17}+5895116260q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{16}(\Gamma_0(48))$.
(By Sage.)The normalized newform in$S^{\text{new}}_{5+12-1}(\Gamma_0(6),+,-)$ is

$$q+128q^2-2187q^3+16384q^4-114810q^5-279936q^6-3034528q^7+2097152q^8+\ldots,$$

then we twist this newform by χ_{-4}:

$$q+2187q^3-114810q^5+3034528q^7+4782969q^9+103451700q^{11}-104365834q^{13}-251089470q^{15}+997689762q^{17}-4934015444q^{19}+O(q^{20}).$$

We can find this normalized newform in $S_{16}(\Gamma_0(48))$.

c. For $S_{7,6}$.
Note that,

$$\begin{cases}\dim V=\dim S^{\text{new}}_{7+12-1}(\Gamma_0(6),-,+)=1,\\ \dim W_1=\dim S^{\text{new}}_{7+12-1}(\Gamma_0(2),+)=0,\\ \dim W_2=\dim S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,-)=1,\\ \dim W_3=\dim S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,+)=1.\end{cases}$$

From our conjecture

$$S_{5,6}(\Gamma_0(3))\cong S^{\text{new}}_{7+12-1}(\Gamma_0(6),-,+)\otimes\chi_{12}\bigoplus\{S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,-)\oplus S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,+)\}\otimes\chi_{-4}$$

And we compute eigenvalues for T_p^2, $p=5,7,11,13,17,19$ on $S_{7,6}(\Gamma_0(3))$

	5	7	11	13	17	19
$S_{7,6}(\Gamma_0(3))$	645150	-24959264	-1159304460	-5425661314	-35551782594	-5778498836
	199650	-3974432	500068668	2801062862	-5466992958	64354589764
	-72186	8640184	125556420	4227195518	32979662226	53889877060

(By Sage.)The normalized newform in$S^{\text{new}}_{7+12-1}(\Gamma_0(6),-,+)$ is

$$q+256q^2-6561q^3+65536q^4-199650q^5-1679616q^6+24959264q^7+16777216q^8+\ldots,$$

then we twist this newform by χ_{12}:

$$q+199650q^5-24959264q^7+125556420q^{11}+4227195518q^{13}-35551782594q^{17}+64354589764q^{19}+O(q^{20}).$$

We can find this normalized newform in $S_{18}(\Gamma_0(144))$.
(By Sage.)The normalized newform in$S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,-)$ is

$$q-256q^2+6561q^3+65536q^4-72186q^5-1679616q^6-8640184q^7-16777216q^8+\ldots$$

then we twist this newform by χ_{-4}:

$$q-6561q^3-72186q^5+8640184q^7+43046721q^9-1159304460q^{11}+2801062862q^{13}+473612346q^{15}+32979662226q^{17}-5778498836q^{19}+O(q^{20}).$$

We can find this normalized newform in $S_{18}(\Gamma_0(48))$.

(By Sage.)The normalized newform in$S^{\text{new}}_{7+12-1}(\Gamma_0(6),+,+)$ is

$$q-256q^2-6561q^3+65536q^4+645150q^5+1679616q^6+3974432q^7-16777216q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q+6561q^3+645150q^5-3974432q^7+43046721q^9+500068668q^{11}-5425661314q^{13}+4232829150q^{15}-5466992958q^{17}+53889877060q^{19}+O(q^{20}),$$

We can find this normalized newform in $S_{18}(\Gamma_0(48))$.

For the cases $r=11,13,17,19,23$. Our conjecture never holds by checking the dimensions of $S_{r,2}(\Gamma_0(3))$ and $V\oplus W$. By Proposition 3.1.1 and Proposition 3.2.1, we have

$$\dim V\oplus W=\begin{cases}3 & \text{if } r=11,\\ 2 & \text{if } r=13,\\ 2 & \text{if } r=17,\\ 3 & \text{if } r=19,\\ 3 & \text{if } r=23.\end{cases}$$

Example 4.1.5. For $r=1,5,7$ and $s=8$. $\dim S_{r,8}(\Gamma_0(3))=3$.

a. For $\mathcal{S}_{1,8}$.

Note that,

$$\begin{cases}\dim V=\dim S^{\text{new}}_{1+16-1}(\Gamma_0(6),-,-)=1,\\ \dim W_1=\dim S^{\text{new}}_{1+16-1}(\Gamma_0(2),+)=1,\\ \dim W_2=\dim S^{\text{new}}_{1+16-1}(\Gamma_0(6),+,+)=1,\\ \dim W_3=\dim S^{\text{new}}_{1+16-1}(\Gamma_0(6),+,-)=0.\end{cases}$$

From our conjecture

$$S_{1,8}(\Gamma_0(3))\cong S^{\text{new}}_{1+16-1}(\Gamma_0(6),-,-)\otimes\chi_{12}\bigoplus\{S^{\text{new}}_{1+12-1}(\Gamma_0(2),+)\oplus S^{\text{new}}_{1+12-1}(\Gamma_0(6),+,+)\}\otimes\chi_{-4}$$

And we compute eigenvalues for T_p^2, $p=5,7,11,13,17,19$ on $S_{1,6}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{1,6}(\Gamma_0(3))$	-314490	-2025056	95889948	56047862	-1355814414	5895116260
	90510	-762104	48011172	-59782138	-1930104414	-2163188180
	-77646	-56	-110255052	285130118	3173671566	-3783593180

(By Sage.)The normalized newform in$S^{\text{new}}_{1+12-1}(\Gamma_0(6),-,-)$ is

$$q+32q^2+243q^3+1024q^4+3630q^5+7776q^6+32936q^7+32768q^8+\dots.$$

then we twist this newform by χ_{12}:

$$q-3630q^5-32936q^7-758748q^{11}-2482858q^{13}-8290386q^{17}+10867300q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.

(By Sage.)The normalized newform in$S^{\mathrm{new}}_{1+12-1}(\Gamma_0(6),+,+)$ is

$$q-32q^2-243q^3+1024q^4+5766q^5+7776q^6+72464q^7-32768q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q+243q^3+5766q^5-72464q^7+59049q^9+408948q^{11}+1367558q^{13}+1401138q^{15}+5422914q^{17}-15166100q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(48))$.
(By Sage.)The normalized newform in$S^{\mathrm{new}}_{1+12-1}(\Gamma_0(6),+,-)$ is

$$q-32q^2+243q^3+1024q^4-11730q^5-7776q^6-50008q^7-32768q^8+\dots,$$

then we twist this newform by χ_{-4}:

$$q-11730q^5+50008q^7+59049q^9+531420q^{11}+1332566q^{13}+2850390q^{15}-5109678q^{17}-2901404q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(48))$.

b. For $\mathcal{S}_{5,8}$.
Note that,

$$\begin{cases}\dim V=\dim S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),+,+)=1,\\ \dim W_1=\dim S^{\mathrm{new}}_{5+16-1}(\Gamma_0(2),-)=0,\\ \dim W_2=\dim S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),-,-)=1,\\ \dim W_3=\dim S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),-,+)=1.\end{cases}$$

From our conjecture

$$S_{5,8}(\Gamma_0(3))\cong S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),+,+)\otimes\chi_{12}\bigoplus\{S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),-,-)\oplus S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),-,+)\}\otimes\chi_{-4}$$

And we compute eigenvalues for T_p^2, $p=5,7,11,13,17,19$ on $S_{5,6}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{5,8}(\Gamma_0(3))$	-114810 77646 314490	-762104 3034528 -2025056	103451700 110255052 -48011172	-104365834 56047862 285130118	-3173671566 997689762 1930104414	5895116260 -2163188180 -4934015444

(By Sage.)The normalized newform in$S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),+,+)$ is

$$q-128q^2-2187q^3+16384q^4-314490q^5+279936q^6+2025056q^7-2097152q^8+\dots,$$

then we twist this newform by χ_{12}:

$$q+314490q^5-2025056q^7+110255052q^{11}+56047862q^{13}+1930104414q^{17}-2163188180q^{19}+O(q^{20})$$

We can find this normalized newform in $S_{20}(\Gamma_0(144))$.
(By Sage.)The normalized newform in$S^{\mathrm{new}}_{5+16-1}(\Gamma_0(6),-,-)$ is

$$q+128q^2+2187q^3+16384q^4+77646q^5+279936q^6+762104q^7+2097152q^8+\dots$$

then we twist this newform by χ_{-4}:

$$q - 2187q^3 + 77646q^5 - 762104q^7 + 4782969q^9 - 48011172q^{11} + 285130118q^{13} - 169811802q^{15} - 3173671566q^{17} + 5895116260q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{20}(\Gamma_0(48))$.

(By Sage.)The normalized newform in$S^{\text{new}}_{5+16-1}(\Gamma_0(6), +, -)$ is

$$q + 128q^2 - 2187q^3 + 16384q^4 - 114810q^5 - 279936q^6 - 3034528q^7 + 2097152q^8 + \dots,$$

then we twist this newform by χ_{-4}:

$$q + 2187q^3 - 114810q^5 + 3034528q^7 + 4782969q^9 + 103451700q^{11} - 104365834q^{13} - 251089470q^{15} + 997689762q^{17} - 4934015444q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_{20}(\Gamma_0(48))$.

c. For $S_{7,8}$.

Note that,

$$\begin{cases} \dim V = \dim S^{\text{new}}_{7+16-1}(\Gamma_0(6), -, +) = 1, \\ \dim W_1 = \dim S^{\text{new}}_{7+16-1}(\Gamma_0(2), +) = 0, \\ \dim W_2 = \dim S^{\text{new}}_{7+16-1}(\Gamma_0(6), +, -) = 1, \\ \dim W_3 = \dim S^{\text{new}}_{7+16-1}(\Gamma_0(6), +, +) = 1. \end{cases}$$

From our conjecture

$$S_{7,8}(\Gamma_0(3)) \cong S^{\text{new}}_{7+16-1}(\Gamma_0(6), -, +) \otimes \chi_{12} \bigoplus \{S^{\text{new}}_{7+16-1}(\Gamma_0(6), +, -) \oplus S^{\text{new}}_{7+16-1}(\Gamma_0(6), +, +)\} \otimes \chi_{-4}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{7,8}(\Gamma_0(3))$

	5	7	11	13	17	19
$\mathcal{S}_{7,8}(\Gamma_0(3))$	645150	-24959264	-1159304460	-5425661314	-35551782594	-5778498836
	199650	-3974432	500068668	2801062862	-5466992958	64354589764
	-72186	8640184	125556420	4227195518	32979662226	53889877060

(By Sage.)The normalized newform in$S^{\text{new}}_{7+16-1}(\Gamma_0(6), -, +)$ is

$$q + 256q^2 - 6561q^3 + 65536q^4 - 199650q^5 - 1679616q^6 + 24959264q^7 + 16777216q^8 + \dots,$$

then we twist this newform by χ_{12}:

$$q + 199650q^5 - 24959264q^7 + 125556420q^{11} + 4227195518q^{13} - 35551782594q^{17} + 64354589764q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_{22}(\Gamma_0(144))$.

(By Sage.)The normalized newform in$S^{\text{new}}_{7+16-1}(\Gamma_0(6), +, -)$ is

$$q - 256q^2 + 6561q^3 + 65536q^4 - 72186q^5 - 1679616q^6 - 8640184q^7 - 16777216q^8 + \dots$$

then we twist this newform by χ_{-4}:

$$q - 6561q^3 - 72186q^5 + 8640184q^7 + 43046721q^9 - 1159304460q^{11} + 2801062862q^{13} + 473612346q^{15} + 32979662226q^{17} - 5778498836q^{19} + O(q^{20}).$$

We can find this normalized newform in $S_{22}(\Gamma_0(48))$.

(By Sage.)The normalized newform in$S^{\mathrm{new}}_{7+16-1}(\Gamma_0(6), +, +)$ is

$$q - 256q^2 - 6561q^3 + 65536q^4 + 645150q^5 + 1679616q^6 + 3974432q^7 - 16777216q^8 + \dots,$$

then we twist this newform by χ_{-4}:

$$q + 6561q^3 + 645150q^5 - 3974432q^7 + 43046721q^9 + 500068668q^{11} - 5425661314q^{13} + 4232829150q^{15} - 5466992958q^{17} + 53889877060q^{19} + O(q^{20}),$$

We can find this normalized newform in $S_{22}(\Gamma_0(48))$.

For the cases $r = 11, 13, 17, 19, 23$. Our conjecture never holds by checking the dimensions of $S_{r,2}(\Gamma_0(3))$ and $V \oplus W$. By Proposition 3.1.1 and Proposition 3.2.1, we have

$$\dim V \oplus W = \begin{cases} 3 & \text{if } r = 11, \\ 2 & \text{if } r = 13, \\ 2 & \text{if } r = 17, \\ 3 & \text{if } r = 19, \\ 3 & \text{if } r = 23. \end{cases}$$

4.2 $S_{r,s}(\Gamma_0(2))$

We take

$$\begin{cases} V = S^{\mathrm{new}}_t(\Gamma_0(6), -\left(\frac{2}{r}\right), -\left(\frac{3}{r}\right)) \otimes \chi_{12}, \\ U = S^{\mathrm{new}}_t(\Gamma_0(6), \epsilon_3, -\left(\frac{3}{r}\right)) \otimes \chi_{12}, \end{cases}$$

for convenience.

Example 4.2.1. For $r = 1, 5, 7$ and $s = 2$.

a. For $S_{1,2}(\Gamma_0(2))$.

Note that,

$$\begin{cases} \dim V = \dim S^{\mathrm{new}}_4(\Gamma_0(6), -, -) = 0, \\ \dim U = \dim S^{\mathrm{new}}_4(\Gamma_0(12), -, -) = 1. \end{cases}$$

So from our conjecture

$$S_{1,2}(\Gamma_0(2)) \cong S^{\mathrm{new}}_{1+2-1}(\Gamma_0(12), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,2}$(by Maple)

	5	7	11	13	17	19
$S_{1,2}(\Gamma_0(2))$	18	-8	36	-10	-18	100

(By Sage.)The normalized newform in $S^{\text{new}}_{1+4-1}(\Gamma_0(12), -, -)$ is

$$q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 36q^{11} - 10q^{13} + \dots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_4(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

b. For $S_{5,2}(\Gamma_0(2))$.

Note that,

$$\begin{cases} \dim U_1 = \dim S^{\text{new}}_6(\Gamma_0(6), +, +) = 0, \\ \dim U_2 = \dim S^{\text{new}}_6(\Gamma_0(12), -, +) = 1. \end{cases}$$

So from our conjecture

$$S_{5,2}(\Gamma_0(2)) \cong S^{\text{new}}_{5+2-1}(\Gamma_0(12), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,2}$(by Maple)

	5	7	11	13	17	19
$S_{5,2}(\Gamma_0(2))$	378	832	-2484	14870	22302	16300

(By Sage.)The normalized newform in $S^{\text{new}}_{1+4-1}(\Gamma_0(12), -, -)$ is

$$q - 27q^3 - 378q^5 - 832q^7 + 729q^9 - 2484q^{11} + 14870q^{13} + \dots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_4(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

c. For $S_{7,2}(\Gamma_0(2))$.

Note that,

$$\begin{cases} \dim U_1 = \dim S^{\text{new}}_8(\Gamma_0(6), -, +) = 0, \\ \dim U_2 = \dim S^{\text{new}}_8(\Gamma_0(12), -, +) = 1. \end{cases}$$

So from our conjecture

$$S_{7,2}(\Gamma_0(2)) \cong S^{\text{new}}_{7+2-1}(\Gamma_0(12), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{7,2}$(by Maple)

	5	7	11	13	17	19
$S_{7,2}(\Gamma_0(2))$	-990	-8576	70596	-2530	200574	695620

(By Sage.)The normalized newform in $S^{\text{new}}_{7+4-1}(\Gamma_0(12), -, -)$ is

$$q - 81q^3 + 990q^5 + 8576q^7 + 6561q^9 + 70596q^{11} - 2530q^{13} + \dots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{10}(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

For the cases $r = 11, 13, 17, 19, 23$. Our conjecture never holds by checking the dimensions of $S_{r,2}(\Gamma_0(2))$ and $U_1 \oplus U_2$. By Proposition 3.2.1 and Proposition 3.2.5, we have

$$\dim U_1 \oplus U_2 = \begin{cases} 3 & \text{if } r = 11, \\ 2 & \text{if } r = 13, \\ 2 & \text{if } r = 17, \\ 3 & \text{if } r = 19, \\ 3 & \text{if } r = 23. \end{cases}$$

Example 4.2.2. For $r = 1, 5, 7$, $s = 6$

a. For $S_{1,6}(\Gamma_0(2))$.

Note that,

$$\begin{cases} \dim U_1 = \dim S_{12}^{\text{new}}(\Gamma_0(6), -, -) = 1, \\ \dim U_2 = \dim S_{12}^{\text{new}}(\Gamma_0(12), -, -) = 1. \end{cases}$$

So from our conjecture

$$S_{1,12}(\Gamma_0(2)) \cong S_t^{\text{new}}(\Gamma_0(6), -, -) \otimes \chi_{12} \bigoplus S_{1+12-1}^{\text{new}}(\Gamma_0(12), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{1,6}(\Gamma_0(2))$(by Maple)

	5	7	11	13	17	19
$S_{1,6}(\Gamma_0(2))$	-3630 -2862	-9128 -32936	668196 -758748	2052950 -2482858	-1604178 -8290386	10867300

(By Sage.)The normalized newform in $S_{1+12-1}^{\text{new}}(\Gamma_0(6), -, -)$ is

$$q + 32q^2 + 243q^3 + 1024q^4 + 3630q^5 + 7776q^6 + 32936q^7 + 32768q^8 + \ldots$$

then we twist this newform by χ_{12}:

$$q - 3630q^5 - 32936q^7 - 758748q^{11} - 2482858q^{13} - 8290386q^{17} + 10867300q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{12}(\Gamma_0(144))$.

(By Sage.)The normalized newform in $S_{1+12-1}^{\text{new}}(\Gamma_0(12), -, -)$ is

$$q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 36q^{11} - 10q^{13} + \ldots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_6(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

b. For $S_{5,2}(\Gamma_0(2))$.

Note that,

$$\begin{cases} \dim U_1 = \dim S_t^{\text{new}}(\Gamma_0(6), +, +) = 0, \\ \dim U_2 = \dim S_t^{\text{new}}(\Gamma_0(12), -, +) = 1. \end{cases}$$

So from our conjecture

$$S_{5,2}(\Gamma_0(2)) \cong S^{\text{new}}_{5+2-1}(\Gamma_0(6), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{5,2}$(by Maple)

	5	7	11	13	17	19
$S_{5,2}(\Gamma_0(2))$	378	832	-2484	14870	22302	16300

(By Sage.)The normalized newform in $S^{\text{new}}_{1+4-1}(\Gamma_0(12), -, -)$ is

$$q - 27q^3 - 378q^5 - 832q^7 + 729q^9 - 2484q^{11} + 14870q^{13} + \dots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_4(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

c. For $S_{7,2}(\Gamma_0(2))$.
Note that,

$$\begin{cases} \dim U_1 = \dim S_t^{\text{new}}(\Gamma_0(6), -, +) = 0, \\ \dim U_2 = \dim S_t^{\text{new}}(\Gamma_0(12), -, +) = 1. \end{cases}$$

So from our conjecture

$$S_{7,2}(\Gamma_0(2)) \cong S^{\text{new}}_{7+2-1}(\Gamma_0(6), -, -) \otimes \chi_{12}$$

And we compute eigenvalues for T_p^2, $p = 5, 7, 11, 13, 17, 19$ on $S_{7,2}$(by Maple)

	5	7	11	13	17	19
$S_{7,2}(\Gamma_0(2))$	-990	-8576	70596	-2530	200574	695620

(By Sage.)The normalized newform in $S^{\text{new}}_{7+4-1}(\Gamma_0(12), -, -)$ is

$$q - 81q^3 + 990q^5 + 8576q^7 + 6561q^9 + 70596q^{11} - 2530q^{13} + \dots$$

then we twist this newform by χ_{12}:

$$q + 18q^5 - 8q^7 + 36q^{11} - 10q^{13} - 18q^{17} + 100q^{19} + O(q^{20})$$

We can find this normalized newform in $S_{10}(\Gamma_0(144))$, and its Atkin-Lehner eigenvalues for w_4 and w_3 are -1 and -1, respectively.

For the cases $r = 11, 13, 17, 19, 23$. Our conjecture never holds by checking the dimensions of $S_{r,2}(\Gamma_0(2))$ and $U_1 \oplus U_2$. By Proposition 3.2.1 and Proposition 3.2.5, we have

$$\dim U_1 \oplus U_2 = \begin{cases} 3 & \text{if } r = 11, \\ 2 & \text{if } r = 13, \\ 2 & \text{if } r = 17, \\ 3 & \text{if } r = 19, \\ 3 & \text{if } r = 23. \end{cases}$$

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