國立交通大學

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碩 士 論 文

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Spatial Entropy and Minimal Cycle of Corner Coloring

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摘　　要

這篇研究在角落著色的平面方塊的複雜性。在平面上對角落著色，角落有 p 種顏色選擇的單位方塊並肩排著，相鄰的邊必須要有一樣的顏色 在[12] 王浩猜測任意可以拼成全平面的磁磚集合就可以週期性的拼成全平面。

$P(B)$ 是所有可以由 B 生成的週期花樣。如果 $P(B)\neq\phi$ 那麼 B 就有一個最小週期生成的子集 B'，使得 $P(B')\neq\phi$ 並且 $P(B")=\phi$ 對於 $B"\subset B'$ 及 $B"\neq B'$。所有最小週期生成的集合 $C(2)$ 包含 17 個元素。所有最大非週期生成的集合 $N(2)$: 如果 $B\in N(2)$ 那麼 $P(B)=\phi$ 並且 $B_1\supset B$ 及 $B_1\neq B$ 意味著 $P(B_1)\neq\phi$

胡文貴學長和林松山老師證明過，在兩個顏色的角落著色時，王浩的猜測，藉由表現出對於任意的 $B\in N(2)$，$\sum(B)=\phi$。更精確的說，$\sum(B)\neq\phi$ 充要 B 有一個最小週期生成的子集

我主要的工作是決定兩個顏色時任意最小週期生成的聯集的熵是否大於零，剩下七個未能決定的情況。

Spatial Entropy and Minimal Cycle of Corner Coloring

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ABSTRACT

This investigation studies the complexity problems of plane square tiling with colored corners . In the corner coloring of a plane, unit squares with colored corners of p colors are arranged side by side such that the corners of the adjacent sides have the same colors.In [12],Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically.i.e., if $\Sigma(B)\neq\phi$ then $P(B)\neq\phi$.

$P(B)$ is the set of all periodic patterns on $\mathbb{Z}^2$ that can be generated by B. If $P(B)\neq\phi$, then B has a subset B' of minimal cycle generator such that $P(B')\neq\phi$ and $P(B'')=\phi$ for $B''\subset B'$.and $B''\neq B'$ The set of all minimal cycle generators $C(2)$ contains 17 elements.N(2) is the set of all maximal non-cycle generators: if $B\in N(2)$, then $P(B)=\phi$ and $B_1\supset B$ and $B_1\neq B$ implies $P(B_1)\neq\phi$.

W.G. Hu and S.S.Lin proved that Wang's conjecture in corner coloring holds when p=2 by showing that $\Sigma(B)=\phi$ for any $B\in N(2)$ More precisely, $\Sigma(B)\neq\phi$ if and only if B has a subset of minimal cycle generator.

My primary work is to decide whether the spatial entropy of any union of minimal cycle generators equals to zero or not when p=2 ,except 7 undetermined cases.

致　　　謝

首先要感謝我的指導教授,林松山教授,協助我完成本篇論文。在老師的教導之下，學到了不同思考問題的方向，瞭解到如何當一個好老師，以及做人處事之道。

接著感謝胡文貴學長。當我有疑惑時，學長就會義不容辭的幫助我解決問題，讓我有信心繼續完成我的論文。

還有處理邊著色問題的陳晉育同學，他獨到的想法，讓我在角落著色上也可以仿照他的方法處理問題。

最後，也是最重要的就是我的家人，包容我不能常常回家陪他們，是我精神上的支柱。

目　　錄

1. Introduction

The coloring of unit squares on $\mathbb{Z}^2$ has been studied for many years [6]. In 1961, in studying proving theorem by pattern recognition, Wang [12] started to study the square tiling of a plane. The unit squares with colored corners of p colors are arranged side by side such that the corners of the adjacent sides have the same colors ; the tiles cannot be rotated or reflected.

The 2×2 unit squares is denoted by $\mathbb{Z}_{2\times 2}$. Let $\mathcal{S}_p$ be a set of p (≥ 1) colors. The total set of all tiles is denoted by $\Sigma_{2\times 2}(p) \equiv \mathcal{S}_p^{\mathbb{Z}_{2\times 2}}$.A set $\mathcal{B}$ of corner-colored tiles, such that $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, is called a basic set. Let $\Sigma(\mathcal{B})$ be the set of all global patterns on $\mathbb{Z}^2$ that can be constructed from the corner-colored tiles in $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns on $\mathbb{Z}^2$ that can be constructed from the corner-colored tiles in $\mathcal{B}$. Clearly, $\mathcal{P}(\mathcal{B}) \subseteq \Sigma(\mathcal{B})$. The nonemptiness problem is to determine whether or not $\Sigma(\mathcal{B}) \neq \emptyset$. In [12], Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically, i.e.,

$$\text{if } \Sigma(\mathcal{B}) \neq \emptyset \quad \text{then } \mathcal{P}(\mathcal{B}) \neq \emptyset. \tag{1.1}$$

First,the minimal cycle generator is introduced in [13]. $\mathcal{B} \subset \Sigma^{w}_{2\times 2}(p)$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ whenever $\mathcal{B}' \subsetneqq \mathcal{B}$. Given $p \geq 2$, denote the set of all minimal cycle generators by $\mathcal{C}(p)$.

In this study, for $p = 2$, $\mathcal{C}(2)$ can be listed explicitly, $\mathcal{C}(2)$ has 17 members . Furthermore, under the symmetry group D_4 of $\mathbb{Z}_{2\times 2}$ and the permutation group S_p of colors of corners , $\mathcal{C}(2)$ can be classified into seven classes .

In corner coloring, the basic set of 44 tiles with six colors that can tile the plane aperiodically without any periodic patterns has been established [9]. And then (1.1) fails for $p = 6$. The method used to study Wang tiles (edge coloring) can also be applied to study corner coloring. For $p = 2$, no allowable pattern on $\mathbb{Z}_{5\times 4}$ can be generated from any maximal non-cycle generator. Hence, the nonemptiness problem is decidable for corner coloring.

For recent results on Wang tiles (colored edges) and colored corners with their applications to computer graphics, see Lagae and Dutré [8] and references therein.

We consider $p = 2$ and the rest of this paper is arranged as follows. Section 2 introduces notations ,minimal cycle generators of corner coloring and their serial numbers . Section 3 states the theorems and methods which is used to get the bounds of the spatial entropy .

2. Minimal cycle generators

Some notations must be introduced first. In this section, $\mathbb{Z}_{2\times 2}$ represents the square lattice with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Furthermore, for any $(i,j)\in\mathbb{Z}^2$, define $\mathbb{Z}_{2\times 2}(i,j)=\{(i,j),(i,j+1),(i+1,j),(i+1,j+1)\}$.

For given positive integers m and n, the rectangular lattice $\mathbb{Z}_{m\times n}$ is defined by

$$\mathbb{Z}_{m\times n}=\{(i,j)|0\leq i\leq m-1 \text{ and } 0\leq j\leq n-1\}\,.$$

Denote the set of p colors by $\mathcal{S}_p=\{0,1,\cdots,p-1\}$. The set of all global patterns on $\mathbb{Z}^2$ with colors in $\mathcal{S}_p$ is denoted by

$$\Sigma_p^2=\mathcal{S}_p^{\mathbb{Z}^2}=\left\{U|U:\mathbb{Z}^2\to\mathcal{S}_p\right\}.$$

The set of all local patterns on $\mathbb{Z}_{m\times n}$ is defined by

$$\Sigma_{m\times n}(p)=\{U|_{\mathbb{Z}_{m\times n}}:U\in\Sigma_p^2\}.$$

Now, for any given $\mathcal{B}\subset\Sigma_{2\times 2}(p)$, $\mathcal{B}$ is called a basic set. The set$\Sigma_{m\times n}(\mathcal{B})$ of all patterns on $\mathbb{Z}_{m\times n}$ generated by $\mathcal{B}$ is defined by

$$\Sigma_{m\times n}(\mathcal{B})=\left\{U\in\Sigma_{m\times n}(p):U\mid_{\mathbb{Z}_{2\times 2}(i,j)}\in\mathcal{B}\text{ for }0\leq i\leq m-2,0\leq j\leq n-2\right\},$$

and the set $\Sigma(\mathcal{B})$ of all global patterns on$\mathbb{Z}^2$ generated by $\mathcal{B}$ is defined by

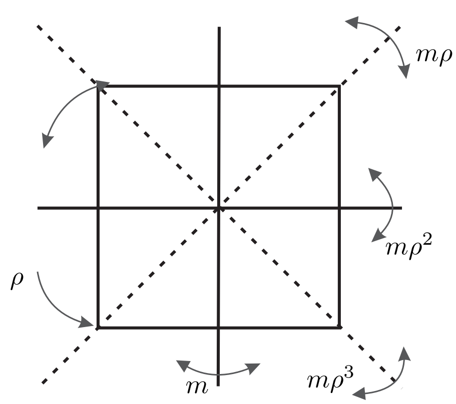
$$\Sigma(\mathcal{B})=\left\{U\in\Sigma_p^2:U\mid_{\mathbb{Z}_{2\times 2}(i,j)}\in\mathcal{B}\text{ for }i,j\in\mathbb{Z}\right\}.$$

Then, the set $\mathcal{P}(\mathcal{B})$ of all periodic patterns generated by $\mathcal{B}$ is defined by

$$\mathcal{P}(\mathcal{B})=\{U=(u_{i,j})\in\Sigma(\mathcal{B})\mid u_{i,j}=u_{i+n,j}=u_{i,j+k}\text{ for all }i,j\in\mathbb{Z}\text{ and for some }n,k\geq 1\}\,.$$

For simplicity, a periodic pattern is also called a cycle.

Now, the symmetry of the unit square $\mathbb{Z}_{2\times 2}$ is introduced. The symmetry group of the rectangle $\mathbb{Z}_{2\times 2}$ is D_4, the dihedral group of order eight. The group D_4 is generated by the rotation ρ, through $\frac{\pi}{2}$, and the reflection m about the y-axis. Denote by $D_4=\{I,\rho,\rho^2,\rho^3,m,m\rho,m\rho^2,m\rho^3\}$.



Therefore, given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}(p)$ and any element $\tau \in D_4$, another basic set $(\mathcal{B})_\tau$ can be obtained by transforming the local patterns in $\mathcal{B}$ by τ.

Additionally, consider the permutation group S_p on $\mathcal{S}_p$. If $\eta \in S_p$ and $\eta(0) = i_0, \eta(1) = i_1, \cdots, \eta(p-1) = i_{p-1}$, we write $\eta = \begin{pmatrix} 0 & 1 & \cdots & p-1 \\ i_0 & i_1 & \cdots & i_{p-1} \end{pmatrix}$. For $\eta \in S_p$ and $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, another basic set $(\mathcal{B})_\eta$ can be obtained.

D_4 and S_p can be combined to define the equivalentclasses of basic sets, as follows: given $\mathcal{B} \subset \Sigma_{2\times 2}(p)$, define the class $[\mathcal{B}]$ of $\mathcal{B}$ by

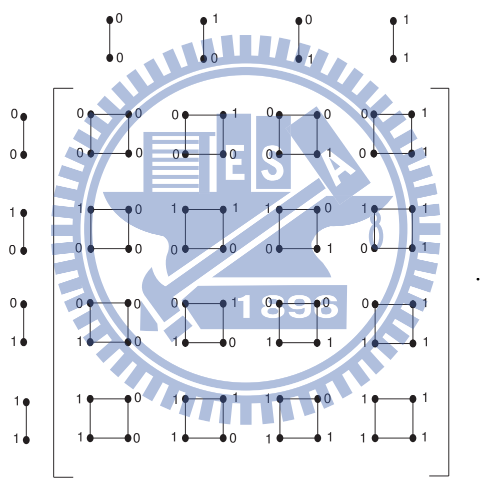
$$[\mathcal{B}] = \{\mathcal{B}' \subset \Sigma_{2\times 2}(p) : \mathcal{B}' = ((\mathcal{B})_\tau)_\eta, \tau \in D_4, \eta \in S_p\}.$$

Definition 2.1. For $\mathcal{B} \subset \Sigma_{2\times 2}(p)$,

(i) $\mathcal{B}$ is called a cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$.
(ii) $\mathcal{B}$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ for all $\mathcal{B}' \subsetneqq \mathcal{B}$.
(iii) $\mathcal{C}(p)$ is the set of all minimal cycle generators that are subsets of $\Sigma_{2\times 2}(p)$.

From now on, only the case $p = 2$ is considered: $\mathcal{S}_2 = \{0, 1\}$. In another work [1], the horizontal ordering matrix $\mathbf{X}_{2\times 2} = [x_{p,q}]_{4\times 4}$ for all local patterns in $\Sigma_{2\times 2}(2)$ is defined by

(2.1)



For convenience, the name of each local pattern in $\Sigma_{2\times 2}(2)$ is given as follows.

Definition 2.2. Denote by

$$\mathbf{X}_{2\times 2} = \begin{bmatrix} O & e_2 & e_4 & r \\ e_1 & t & I & \overline{e}_3 \\ e_3 & J & b & \overline{e}_1 \\ l & \overline{e}_4 & \overline{e}_2 & E \end{bmatrix}. \tag{2.2}$$

The following theorem groups all 17 minimal cycle generators into seven classes . Table A.1 presents the symmetries of minimal cycle generators in $\mathcal{C}_c(2)$

Theorem 2.3. (i) *$\mathcal{C}_c(2)$ contains 17 elements and is classified into seven classes of minimal cycle generators which are given by*
(1) $[\{O\}] = \{\{O\}, \{E\}\}$,

(2) $[\{b,t\}] = \{\{b,t\},\{l,r\}\}$,
(3) $[\{I,J\}] = \{\{I,J\}\}$,
(4) $[\{e_1,e_2,e_3,e_4\}] = \{\{e_1,e_2,e_3,e_4\},\{\overline{e}_1,\overline{e}_2,\overline{e}_3,\overline{e}_4\}\}$,
(5) $[\{e_1,e_2,\overline{e}_3,\overline{e}_4,b\}] = \{\{e_1,e_2,\overline{e}_3,\overline{e}_4,b\},\{e_3,e_4,\overline{e}_1,\overline{e}_2,t\},\{e_1,e_3,\overline{e}_2,\overline{e}_4,r\},$
$\{e_2,e_4,\overline{e}_1,\overline{e}_3,l\}\}$,
(6) $[\{e_1,e_4,J\}] = \{\{e_1,e_4,J\},\{e_2,e_3,I\},\{\overline{e}_2,\overline{e}_3,J\},\{\overline{e}_1,\overline{e}_4,I\}\}$,
(7) $[\{e_1,e_4,\overline{e}_1,\overline{e}_4\}] = \{\{e_1,e_4,\overline{e}_1,\overline{e}_4\},\{e_2,e_3,\overline{e}_2,\overline{e}_3\}\}$.

For convenience, the mainimal cycle generators mentioned above are numbered first by classes and then ordering.

Serial number	Original minimal cycle
(1.1)	{O}
(1.2)	{E}
(2.1)	{b,t}
(2.2)	{l,r}
(3)	{I,J}
(4.1)	$\{e_1,e_2,e_3,e_4\}$
(4.2)	$\{\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4}\}$
(5.1)	$\{e_1,e_2,\overline{e_3},\overline{e_4},b\}$
(5.2)	$\{e_3,e_4,\overline{e_1},\overline{e_2},t\}$
(5.3)	$\{e_1,e_3,\overline{e_2},\overline{e_4},r\}$
(5.4)	$\{e_2,e_4,\overline{e_1},\overline{e_3},l\}$
(6.1)	$\{e_1,e_4,J\}$
(6.2)	$\{e_2,e_3,I\}$
(6.3)	$\{\overline{e_2},\overline{e_3},J\}$
(6.4)	$\{\overline{e_1},\overline{e_4},I\}$
(7.1)	$\{e_1,e_4,\overline{e_1},\overline{e_4}\}$
(7.2)	$\{e_2,e_3,\overline{e_2},\overline{e_3}\}$

Table 2.1

Notations:

1 : $(a)_1$ which means any minimal cycle generator of (a) , for $1 \leq a \leq 7$

2 : $[a]_1$ which means there exists one minimal cycle generator of (a) , for $1 \leq a \leq 7$

3. Spatial entropy

Let $\Gamma_{m\times n}(\mathcal{B}) = card(\Sigma_{m\times n}(\mathcal{B}))$, be the number of distinct patterns in $\Sigma_{m\times n}(\mathcal{B})$.

The Spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is defined as

$$h(\mathcal{B}) \equiv \lim_{m\to\infty, n\to\infty} \frac{\log(\Gamma_{m\times n}(\mathcal{B}))}{m\times n}.$$

The spatial entropy , $h(\mathcal{B})$ is clearly independent of the choice of elements in $[\mathcal{B}]$: for any $\mathcal{B}' \in [\mathcal{B}]$.

Definition 3.1. A K+1 multiple index

$$\mathcal{B}_K \equiv \beta_1\beta_2\cdots\beta_k\beta_1$$

is called a *diagonal cycle* if

$$\beta_{K+1} = \beta_1 \qquad and \qquad \beta_k \in \{1,4\}.$$

for each $1 \le k \le K+1$

Theorem 3.2. *Let $\beta_1\beta_2\cdots\beta_k\beta_1$ be a diagonal cycle . Then for any $m \ge 2$,*

$$h(\mathbb{A}_2) \ge \frac{1}{mK}\log\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1})$$

and

$$h(\mathbb{A}_2) \ge \frac{1}{mK}\log\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1})$$

In particular, if a diagonal cycle $\beta_1\beta_2\cdots\beta_K\beta_1$ exists and $m \ge 2$ such that

$$\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}) > 1,$$

or

$$\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}) > 1$$

then $h(\mathbb{A}_2) > 0$

This theorem is in [J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns generation and spatial entropy in two dimensional lattice models*, Asian J. Math. **11** (2007), 497–534.]

Theorem 3.3. *Given a basic set $\mathcal{B}$,if there are two adjacent sides of all tiles in $\mathcal{B}$,such that the corners of the sides have different colors . Then,if given the colored corners of the adjacent sides in $\Sigma_{m\times n}(\mathcal{B})$,the rest colors will be determined. Hence we can show the upper bound of $\Gamma_{m\times n}(\mathcal{B})$ is related to the (m+n)th power ,then we can get $h(\mathcal{B}) = 0$ by the definition of spatial entropy.*

[This theorem is first proved by J.Y. Chen in edge coloring.]

<u>Methods:</u>

1 :

$$h(\mathcal{B}) > 0 \begin{cases} P_1 : \text{Theorem 3.2} \\ P_2 : \text{If } h(\mathcal{A}) > 0 \text{ and } \mathcal{B} \supseteq \mathcal{A}, \text{ then } h(\mathcal{B}) > 0 \end{cases}$$

2 :

$$h(\mathcal{B})=0\begin{cases} O_1 : \text{Theorem 3.3} \\ O_{1.1} : \text{If the patterns in the horizontal(or vertical) way have only one} \\ \qquad \text{choice,then } \Gamma_{m\times n}(\mathcal{B}) \leq (|\mathcal{B}|)^n(\text{or } (|\mathcal{B}|)^m)\text{,hence } h(\mathcal{B})=0. \\ O_{1.2} : \text{If } \mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2\text{,the patterns in each set can't be attached together} \\ \qquad \text{and } h(\mathcal{B}_1)=h(\mathcal{B}_2)=0 \text{ ,then } h(\mathcal{B})=0. \\ O_2 : \text{If } h(\mathcal{A})=0 \text{ and } \mathcal{B}\subseteq\mathcal{A},\ \text{then } h(\mathcal{B})=0 \end{cases}$$

2-1	2 minimal cycle generators: $(\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	Spatial entropy
$(1.1) \cup (1.2)$	$\mathcal{B} = \{O, E\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (2.1)$	$\mathcal{B} = \{O, b, t\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (3)$	$\mathcal{B} = \{O, I, J\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (4.1)$	$\mathcal{B} = \{O, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;11}) = g$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (4.2)$	$\mathcal{B} = \{O, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (5.1)$	$\mathcal{B} = \{O, e_1, e_2, \overline{e_3}, \overline{e_4}, b\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (6.1)$	$\mathcal{B} = \{O, e_1, e_4, J\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (6.3)$	$\mathcal{B} = \{O, \overline{e_2}, \overline{e_3}, J\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (7.1)$	$\mathcal{B} = \{O, e_1, e_4, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(2.1) \cup (2.2)$	$\mathcal{B} = \{b, t, l, r\}$	$h(\mathcal{B}) = O_1$
$(2.1) \cup (3)$	$\mathcal{B} = \{b, t, I, J\}$	$h(\mathcal{B}) = O_1$
$\mathbf{(2)_1} \cup \mathbf{(4)_1}$	$\mathcal{B} = \{b, t, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;41} S_{2;14}) = 2$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (5.1)$	$\mathcal{B} = \{b, t, e_1, e_2, \overline{e_3}, \overline{e_4}\}$ $\Gamma_{m\times n}(\mathcal{B}) \leq 6 \cdot 3^{(n)}$	$h(\mathcal{B}) = O_{1,1}$
$(2.1) \cup (5.3)$	$\mathcal{B} = \{b, t, e_1, e_3, \overline{e_2}, \overline{e_4}, r\}$ $\Gamma_{m\times n}(\mathcal{B}) \leq 7 \cdot 2^{(m)}$	$h(\mathcal{B}) = O_{1,1}$
$(2.1) \cup (6.1)$	$\mathcal{B} = \{b, t, e_1, e_4, J\}$	$h(\mathcal{B}) = O_1$
$(2.1) \cup (7.1)$	$\mathcal{B} = \{b, t, e_1, e_4, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$\mathbf{(3)} \cup \mathbf{(4)_1}$	$\mathcal{B} = \{I, J, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;14} S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$\mathbf{(3)} \cup \mathbf{(5)_1}$	$\mathcal{B} = \{I, J, e_1, e_2, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{2;41} S_{2;14}) = g$	$h(\mathcal{B}) = P_1$
$(3) \cup (6.1)$	$\mathcal{B} = \{I, J, e_1, e_4\}$	$h(\mathcal{B}) = O_1$
$(3) \cup (7.1)$	$\mathcal{B} = \{I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(4.1) \cup (4.2)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(4.1) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_3}, \overline{e_4}, b\}$	$h(\mathcal{B}) = O_1$

(4.1) ∪ (6.1)	$\mathcal{B} = \{e_1, e_2, e_3, e_4, J\}$	$h(\mathcal{B}) = O_1$
(4.1) ∪ (6.3)	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_2}, \overline{e_3}, J\}$	$h(\mathcal{B}) = O_1$
(4.1) ∪ (7.1)	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_4}\}$ $\subset (4.1) \cup (4.2)$	$h(\mathcal{B}) = O_2$
(5.1) ∪ (5.2)	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, e_4, \overline{e_1}, \overline{e_2}, t\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
(5.1) ∪ (5.3)	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, \overline{e_2}, r\}$	$h(\mathcal{B}) = O_1$
(5.1) ∪ (6.1)	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J\}$	$h(\mathcal{B}) = O_1$
(5.1) ∪ (7.1)	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, \overline{e_1}\}$	$h(\mathcal{B}) = O_1$
(6.1) ∪ (6.2)	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, I\}$ $= (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
(6.1) ∪ (6.3)	$\mathcal{B} = \{e_1, e_4, J, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_1$
(6.1) ∪ (6.4)	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_4}, I\}$ $= (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
(6.3) ∪ (6.4)	$\mathcal{B} = \{\overline{e_2}, \overline{e_3}, J, \overline{e_1}, \overline{e_4}, I\}$ $= (\mathbf{3}) \cup (\mathbf{4.2})$	$h(\mathcal{B}) = P_2$
(6.1) ∪ (7.1)	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_4}\}$ $\subset (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
(6.1) ∪ (7.2)	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_1$
(7.1) ∪ (7.2)	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $= (4.1) \cup (4.2)$	$h(\mathcal{B}) = O_2$

Summary 2-1	2 minimal cycle generators	
Serial number	Basic set Method	$h(\mathcal{B}) = P_1$ or Undetermined
(1.1) ∪ (4.1)	$\mathcal{B} = \{O, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;11}) = g$	$h(\mathcal{B}) = P_1$
$(\mathbf{2})_\mathbf{1} \cup (\mathbf{4})_\mathbf{1}$	$\mathcal{B} = \{b, t, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;41} S_{2;14}) = 2$	$h(\mathcal{B}) = P_1$
$(\mathbf{3}) \cup (\mathbf{4})_\mathbf{1}$	$\mathcal{B} = \{I, J, e_1, e_2, e_3, e_4\}$ $\rho(S_{2;14} S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$(\mathbf{3}) \cup (\mathbf{5})_\mathbf{1}$	$\mathcal{B} = \{I, J, e_1, e_2, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{2;41} S_{2;14}) = g$	$h(\mathcal{B}) = P_1$

3-1	3 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	Spatial entropy
$(1.1) \cup (2.1) \cup (3)$	$\mathcal{B} = \{O, b, t, I, J\}$	$h(\mathcal{B}) = O_1$
$\mathbf{(1)_1} \cup \mathbf{(2)_1} \cup \mathbf{(5)_1}$		$h(\mathcal{B}) = P$
$(1.1) \cup (2.1) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, t\}$ $\rho(S_{2;11}S_{2;14}S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.2) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, l, r\}$ $\rho(S_{2;11}S_{2;11}S_{2;14}S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, b, t\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 3$	$h(\mathcal{B}) = P_1$
$(1.2) \cup (2.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, E, b, t\}$	$h(\mathcal{B}) = O_1$
$\mathbf{(1)_1} \cup \mathbf{(2)_1} \cup \mathbf{(7)_1}$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, O, b, t\}$ $\rho(S_{5;14}S_{5;44}S_{5;41}S_{5;11}) = 3.73$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, I\}$	$h(\mathcal{B}) = O_1$
$(1.2) \cup (3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, E, I\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (3) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, O, I, J\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (4.1) \cup (5.1)$	$\mathcal{B} \supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.2) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (4.1) \cup (6.1)$	$\mathcal{B} \supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.2) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(1.2) \cup (4.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, E, e_2, e_3\}$	$h(\mathcal{B}) = O_1$
$(1.2) \cup (4.2) \cup (6.1)$	$\mathcal{B} \supset [(1.1) \cup (4.1)]_1$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.1) \cup (7.1)$	$\mathcal{B} \supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.2) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, O, \overline{e_2}, \overline{e_3}\}$ $\subset (1.1) \cup (4.2) \cup (6.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, e_2, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}) = 4.08$	$h(\mathcal{B}) = P_1$
$(1.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, E, e_2, \overline{e_3}, \overline{e_4}, b\}$	$h(\mathcal{B}) = O_1$
$\mathbf{(1)_1} \cup \mathbf{(5)_1} \cup \mathbf{(7)_1}$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, e_4, \overline{e_1}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 3.24$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, O, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(1.2) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, E, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, O, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.2) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, E, e_2, e_3, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_1$

$(\mathbf{2})_\mathbf{1} \cup (\mathbf{3}) \cup (\mathbf{6})_\mathbf{1}$	$\mathcal{B} = \{e_1, e_4, J, b, t, I\}$ $\rho(S_{3;14}S_{3;41}S_{3;14}S_{3;41}S_{3;11}) = 3$	$h(\mathcal{B}) = P_1$
$(\mathbf{2})_\mathbf{1} \cup (\mathbf{3}) \cup (\mathbf{7})_\mathbf{1}$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, b, t, I, J\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{3}) \cup (\mathbf{6.1})$	$h(\mathcal{B}) = P_2$
$(\mathbf{2})_\mathbf{1} \cup (\mathbf{5})_\mathbf{1} \cup (\mathbf{6})_\mathbf{1}$		$h(\mathcal{B}) = P$
$(2.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, t, e_4, J\}$ $\rho(S_{7;14}S_{7;41}S_{7;14}S_{7;41}S_{7;11}) = 3.41$	$h(\mathcal{B}) = P_1$
$(2.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, l, r, e_4, J\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 2$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, t, e_4, \overline{e_1}\}$	Undetermined
$(2.2) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, l, r, e_4, \overline{e_1}\}$ $\rho(S_{4;11}) = 1.46$	$h(\mathcal{B}) = P_1$
$(\mathbf{2})_\mathbf{1} \cup (\mathbf{6})_\mathbf{1} \cup (\mathbf{7})_\mathbf{1}$		$h(\mathcal{B}) = P$
$(2.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, b, t, \overline{e_1}, \overline{e_4}\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 2.61$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, b, t, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(3) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, I, \overline{e_1}, \overline{e_4}\}$ $= (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(3) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, I, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(4.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_3}, \overline{e_4}, b\}$	Undetermined
$(4.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_2, b\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 1.58$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, e_4, \overline{e_1}\}$ $\boxplus$	$h(\mathcal{B}) = O_1$
$(4.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_1}, \overline{e_4}\}$	Undetermined
$(4.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $= (6.1) \cup (7.2)$	$h(\mathcal{B}) = O_2$
$(4.2) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_2, e_3\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 3.4$	$h(\mathcal{B}) = P_1$
$(5.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, \overline{e_1}\}$	Undetermined
$(5.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, e_3, \overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 1.58$	$h(\mathcal{B}) = P_1$

Summary 3-1	3 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	$h(\mathcal{B}) = P_1$ or Undetermined
$\mathbf{(1)_1} \cup \mathbf{(2)_1} \cup \mathbf{(5)_1}$		$h(\mathcal{B}) = P$
$(1.1) \cup (2.1) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, t\}$ $\rho(S_{2;11}S_{2;14}S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.2) \cup (5.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, l, r\}$ $\rho(S_{2;11}S_{2;11}S_{2;14}S_{2;41}) = 2$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, b, t\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 3$	$h(\mathcal{B}) = P_1$
$\mathbf{(1)_1} \cup \mathbf{(2)_1} \cup \mathbf{(7)_1}$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, O, b, t\}$ $\rho(S_{5;14}S_{5;44}S_{5;41}S_{5;11}) = 3.73$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, e_2, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}) = 4.08$	$h(\mathcal{B}) = P_1$
$\mathbf{(1)_1} \cup \mathbf{(5)_1} \cup \mathbf{(7)_1}$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, e_4, \overline{e_1}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 3.24$	$h(\mathcal{B}) = P_1$
$\mathbf{(2)_1} \cup \mathbf{(3)} \cup \mathbf{(6)_1}$	$\mathcal{B} = \{e_1, e_4, J, b, t, I\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 3$	$h(\mathcal{B}) = P_1$
$\mathbf{(2)_1} \cup \mathbf{(5)_1} \cup \mathbf{(6)_1}$		$h(\mathcal{B}) = P$
$(2.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, t, e_4, J\}$ $\rho(S_{7;14}S_{7;41}S_{7;14}S_{7;41}S_{7;11}) = 3.41$	$h(\mathcal{B}) = P_1$
$(2.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, l, r, e_4, J\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 2$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, t, e_4, \overline{e_1}\}$	Undetermined
$(2.2) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, l, r, e_4, \overline{e_1}\}$ $\rho(S_{4;11}) = 1.46$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, b, t, \overline{e_1}, \overline{e_4}\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 2.61$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_3}, \overline{e_4}, b\}$	Undetermined
$(4.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_2, b\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 1.58$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, \overline{e_1}, \overline{e_4}\}$	Undetermined
$(4.2) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_2, e_3\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 3.4$	$h(\mathcal{B}) = P_1$
$(5.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, \overline{e_1}\}$	Undetermined
$(5.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, e_3, \overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 1.58$	$h(\mathcal{B}) = P_1$

3-2	3 minimal cycle generators: $(\ )_1 \cup (\ )_2$	
Serial number	Basic set Method	Spatial entropy
$(1.1)\cup(2.1)\cup(2.2)$	$\mathcal{B}=\{O,b,t,l,r\}$	$h(\mathcal{B})=O_1$
$(1.1)\cup(1.2)\cup(2.1)$	$\mathcal{B}=\{O,E,b,t\}$	$h(\mathcal{B})=O_1$
$(1.1)\cup(1.2)\cup(3)$	$\mathcal{B}=\{O,E,I,J\}$	$h(\mathcal{B})=O_1$
$(\mathbf{1})_\mathbf{1}\cup(\mathbf{5})_\mathbf{2}$		$h(\mathcal{B})=P$
$(1.1)\cup(5.1)\cup(5.2)$	$\mathcal{B}=\{O,e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,\overline{e_1},\overline{e_2},t\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(5.1)\cup(5.3)$	$\mathcal{B}=\{O,e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,\overline{e_2},r\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=3.54$	$h(\mathcal{B})=P_1$
$(\mathbf{1})_\mathbf{2}\cup(\mathbf{5})_\mathbf{1}$	$\mathcal{B}=\{O,E,e_1,e_2,\overline{e_3},\overline{e_4},b\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}S_{4;11})=2$	$h(\mathcal{B})=P_1$
$(1.1)\cup(6.1)\cup(6.2)$	$\mathcal{B}=\{O,e_1,e_4,J,e_2,e_3,I\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{O,e_1,e_4,J,\overline{e_2},\overline{e_3}\}$	$h(\mathcal{B})=O_1$
$(1.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{O,e_1,e_4,J,\overline{e_1},\overline{e_4},I\}$ $=(1.1)\cup(3)\cup(7.1)$	$h(\mathcal{B})=O_2$
$(1.1)\cup(6.3)\cup(6.4)$	$\mathcal{B}=\{O,\overline{e_2},\overline{e_3},J,\overline{e_1},\overline{e_4},I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.2})$	$h(\mathcal{B})=P_2$
$(1.1)\cup(1.2)\cup(6.1)$	$\mathcal{B}=\{O,E,e_1,e_4,J\}$	$h(\mathcal{B})=O_1$
$(\mathbf{1})_\mathbf{1}\cup(\mathbf{7})_\mathbf{2}$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},e_2,e_3,\overline{e_2},\overline{e_3}\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(1.2)\cup(7.1)$	$\mathcal{B}=\{O,E,e_1,e_4,\overline{e_1},\overline{e_4}\}$	$h(\mathcal{B})=O_1$
$(2.1)\cup(2.2)\cup(3)$	$\mathcal{B}=\{b,t,l,r,I,J\}$	$h(\mathcal{B})=O_1$
$(\mathbf{2})_\mathbf{1}\cup(\mathbf{5})_\mathbf{2}$		$h(\mathcal{B})=P$
$(2.1)\cup(5.1)\cup(5.2)$	$\mathcal{B}=\{b,t,e_1,e_2,\overline{e_3},\overline{e_4},e_3,e_4,\overline{e_1},\overline{e_2}\}$ $\supset(\mathbf{2.1})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(2.1)\cup(5.1)\cup(5.3)$	$\mathcal{B}=\{b,t,e_1,e_2,\overline{e_3},\overline{e_4},e_3,\overline{e_2},r\}$ $\supset[(2.2)\cup(5.1)\cup(7.1)]_1$	$h(\mathcal{B})=P_2$
$(2.1)\cup(5.3)\cup(5.4)$	$\mathcal{B}=\{b,t,e_1,e_3,\overline{e_2},\overline{e_4},r,e_2,e_4,\overline{e_1},\overline{e_3},l\}$ $\supset(\mathbf{2.1})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(2.1)\cup(2.2)\cup(5.1)$	$\mathcal{B}=\{b,t,l,r,e_1,e_2,\overline{e_3},\overline{e_4}\}$ $\Gamma_{m\times n}(\mathcal{B})\leq 8\cdot 3^{(n)}$	$h(\mathcal{B})=O_{1,1}$
$(\mathbf{2})_\mathbf{1}\cup(\mathbf{6})_\mathbf{2}$		$h(\mathcal{B})=P$
$(2.1)\cup(6.1)\cup(6.2)$	$\mathcal{B}=\{b,t,e_1,e_4,J,e_2,e_3,I\}$ $\supset(\mathbf{2.1})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(2.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{b,t,e_1,e_4,J,\overline{e_2},\overline{e_3}\}$ $\rho(S_{5;14}S_{5;41})=1.73$	$h(\mathcal{B})=P_1$
$(2.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{b,t,e_1,e_4,J,\overline{e_1},\overline{e_4},I\}$ $=(\mathbf{2.1})\cup(\mathbf{3})\cup(\mathbf{7.1})$	$h(\mathcal{B})=P_2$
$(2.1)\cup(2.2)\cup(6.1)$	$\mathcal{B}=\{b,t,l,r,e_1,e_4,J\}$	$h(\mathcal{B})=O_1$

$(\mathbf{2})_\mathbf{1} \cup (\mathbf{7})_\mathbf{2}$	$\mathcal{B} = \{b, t, e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(2.1) \cup (2.2) \cup (7.1)$	$\mathcal{B} = \{b, t, l, r, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ 	$h(\mathcal{B}) = O_1$
$(3) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{I, J, e_1, e_4, e_2, e_3\}$ $= (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(3) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{I, J, e_1, e_4, \overline{e_2}, \overline{e_3}\}$ $= \{I, J, e_1, e_4\} \cup \{I, J, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_{1,2}$
$(3) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ 	$h(\mathcal{B}) = O_1$
$(\mathbf{3}) \cup (\mathbf{7})_\mathbf{2}$	$\mathcal{B} = \{I, J, e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(\mathbf{4})_\mathbf{1} \cup (\mathbf{5})_\mathbf{2}$		$h(\mathcal{B}) = P$
$(4.1) \cup (5.1) \cup (5.2)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_3}, \overline{e_4}, b, \overline{e_1}, \overline{e_2}, t\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(4.1) \cup (5.1) \cup (5.3)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_3}, \overline{e_4}, b, \overline{e_2}, r\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 3$	$h(\mathcal{B}) = P_1$
$(\mathbf{4})_\mathbf{2} \cup (\mathbf{5})_\mathbf{1}$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{2;11}S_{2;14}S_{2;41}) = g$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, J, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(4.1) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, J, \overline{e_2}, \overline{e_3}\}$ 	$h(\mathcal{B}) = O_1$
$(4.1) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, J, \overline{e_1}, \overline{e_4}, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(4.1) \cup (6.3) \cup (6.4)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_2}, \overline{e_3}, J, \overline{e_1}, \overline{e_4}, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(\mathbf{4})_\mathbf{2} \cup (\mathbf{6})_\mathbf{1}$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, J\}$ $= (4.2) \cup (6.1) \cup (7.2)$	$h(\mathcal{B}) = P_2$
$(4.1) \cup (7.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_4}, \overline{e_2}, \overline{e_3}\}$ $= (4.1) \cup (4.2)$	$h(\mathcal{B}) = O_2$
$(4.1) \cup (4.2) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\}$ $= (4.1) \cup (4.2)$	$h(\mathcal{B}) = O_2$
$(5.1) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, e_3, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(5.1) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, \overline{e_2}\}$ 	$h(\mathcal{B}) = O_1$
$(5.1) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, \overline{e_1}, I\}$ $\rho(S_{2;14}S_{2;41}) = g$	$h(\mathcal{B}) = P_1$
$(\mathbf{5})_\mathbf{2} \cup (\mathbf{6})_\mathbf{1}$		$h(\mathcal{B}) = P$
$(5.1) \cup (5.2) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, \overline{e_3}, \overline{e_4}, b, e_3, \overline{e_1}, \overline{e_2}, t\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(5.1) \cup (5.3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, \overline{e_3}, \overline{e_4}, b, e_3, \overline{e_2}, r\}$ $\rho(S_{5;14}S_{5;41}S_{5;11}S_{5;11}) = 4.41$	$h(\mathcal{B}) = P_1$
$(5.2) \cup (5.3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_3, \overline{e_1}, \overline{e_2}, t, \overline{e_4}, r\}$ $\rho(S_{7;14}S_{7;41}S_{7;11}S_{7;11}) = 2.30$	$h(\mathcal{B}) = P_1$
$(\mathbf{5})_\mathbf{1} \cup (\mathbf{7})_\mathbf{2}$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, \overline{e_1}, \overline{e_2}, e_3\}$ $= (\mathbf{4.1}) \cup (\mathbf{4.2}) \cup (\mathbf{5.1})$	$h(\mathcal{B}) = P_2$
$(5.1) \cup (5.2) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, \overline{e_3}, b, e_3, \overline{e_2}, t\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(5.1) \cup (5.3) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, \overline{e_3}, b, e_3, \overline{e_2}, r\}$ $\supset (\mathbf{4.1}) \cup (\mathbf{5.1}) \cup (\mathbf{5.3})$	$h(\mathcal{B}) = P_2$
$(5.1) \cup (5.4) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, \overline{e_3}, b, l\}$ $= (5.1) \cup (5.4)$	$h(\mathcal{B}) = O_2$

$(\mathbf{6})_{\mathbf{1}} \cup (\mathbf{7})_{\mathbf{2}}$	$\mathcal{B} = \{e_1, e_4, J, \overline{e_1}, \overline{e_4}, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $= (4.2) \cup (6.1) \cup (7.2)$	$h(\mathcal{B}) = P_2$
$(6.1) \cup (6.2) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, J, e_2, e_3, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(6.1) \cup (6.3) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, J, \overline{e_2}, \overline{e_3}\}$ $= (4.2) \cup (6.1)$	$h(\mathcal{B}) = O_2$
$(6.1) \cup (6.4) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, I, J\}$ $= (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(6.2) \cup (6.3) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, e_3, I, \overline{e_2}, \overline{e_3}, J\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$

Summary 3-2	3 minimal cycle generators: $(\ )_1 \cup (\ )_2$	
Serial number	Basic set Method	$h(\mathcal{B}) = P_1$ or Undetermined
$(1.1) \cup (5.1) \cup (5.3)$	$\mathcal{B} = \{O, e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, \overline{e_2}, r\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 3.54$	$h(\mathcal{B}) = P_1$
$(\mathbf{1})_{\mathbf{2}} \cup (\mathbf{5})_{\mathbf{1}}$	$\mathcal{B} = \{O, E, e_1, e_2, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}S_{4;11}) = 2$	$h(\mathcal{B}) = P_1$
$(2.1) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{b, t, e_1, e_4, J, \overline{e_2}, \overline{e_3}\}$ $\rho(S_{5;14}S_{5;41}) = 1.73$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (5.1) \cup (5.3)$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_3}, \overline{e_4}, b, \overline{e_2}, r\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}S_{6;11}) = 3$	$h(\mathcal{B}) = P_1$
$(\mathbf{4})_{\mathbf{2}} \cup (\mathbf{5})_{\mathbf{1}}$	$\mathcal{B} = \{e_1, e_2, e_3, e_4, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, b\}$ $\rho(S_{2;11}S_{2;14}S_{2;41}) = g$	$h(\mathcal{B}) = P_1$
$(5.1) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_4, J, \overline{e_1}, I\}$ $\rho(S_{2;14}S_{2;41}) = g$	$h(\mathcal{B}) = P_1$
$(5.1) \cup (5.3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_2, \overline{e_3}, \overline{e_4}, b, e_3, \overline{e_2}, r\}$ $\rho(S_{5;14}S_{5;41}S_{5;11}S_{5;11}) = 4.41$	$h(\mathcal{B}) = P_1$
$(5.2) \cup (5.3) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, e_3, \overline{e_1}, \overline{e_2}, t, \overline{e_4}, r\}$ $\rho(S_{7;14}S_{7;41}S_{7;11}S_{7;11}) = 2.30$	$h(\mathcal{B}) = P_1$

3-3	3 minimal cycle generators: $(\ )_3$	
Serial number	Basic set Method	Spatial entropy
$(5.1) \cup (5.2) \cup (5.3)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, e_4, \overline{e_1}, \overline{e_2}, t, r\}$ $\supset (\mathbf{2.1}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$
$(6.1) \cup (6.2) \cup (6.3)$	$\mathcal{B} = \{e_1, e_4, J, e_2, e_3, I, \overline{e_2}, \overline{e_3}, \overline{e_1}, \overline{e_4}\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.1})$	$h(\mathcal{B}) = P_2$

4-1	4 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	Spatial entropy
$(1.1)\cup(3)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_4,J,O,I,\overline{e_1},\overline{e_4}\}$ $=(1.1)\cup(3)\cup(7.1)$	$h(\mathcal{B})=O_2$
$(1.1)\cup(3)\cup(6.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_4,J,O,I,e_2,e_3,\overline{e_2},\overline{e_3}\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.2)\cup(3)\cup(6.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_4,J,E,I,e_2,e_3,\overline{e_2},\overline{e_3}\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.1)\cup(5.1)\cup(6.1)$	$\mathcal{B}\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.1)\cup(5.1)\cup(6.3)$	$\mathcal{B}\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.2)\cup(5.1)\cup(6.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_1},\overline{e_2},e_4,J\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=17.41$	$h(\mathcal{B})=P_1$
$(1.1)\cup(4.2)\cup(5.1)\cup(6.3)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_1},\overline{e_2},J\}$	Undetermined
$(1.1)\cup(4.1)\cup(5.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_3,e_4,\overline{e_1}\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.2)\cup(5.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_1},\overline{e_2},e_4\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=3.24$	$h(\mathcal{B})=P_1$
$(1.1)\cup(4.1)\cup(6.1)\cup(7.1)$	$\mathcal{B}\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.1)\cup(6.1)\cup(7.2)$	$\mathcal{B}\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.2)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_4,J,O,\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4}\}$ $=(1.1)\cup(4.2)\cup(6.1)$	$h(\mathcal{B})=O_2$
$(1.1)\cup(4.2)\cup(6.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_4,J,O,\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4},e_2,e_3\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.2)\cup(4.1)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_4,J,E,e_2,e_3,\overline{e_1},\overline{e_4}\}$	Undetermined
$(1.1)\cup(5.1)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,\overline{e_1}\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11})=12.98$	$h(\mathcal{B})=P_1$
$(1.1)\cup(5.1)\cup(6.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,e_3,\overline{e_2}\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(5.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_2},J,e_4,\overline{e_1}\}$ $\supset(1.1)\cup(5.1)\cup(6.1)\cup(7.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(5.1)\cup(6.3)\cup(7.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_2},J,e_3\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=3.24$	$h(\mathcal{B})=P_1$
$(4.1)\cup(5.1)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,\overline{e_1}\}$	Undetermined
$(4.1)\cup(5.1)\cup(6.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,\overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=1.58$	$h(\mathcal{B})=P_1$
$(4.1)\cup(5.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,\overline{e_2},J,\overline{e_1}\}$ $\supset(4.1)\cup(5.1)\cup(6.1)\cup(7.2)$	$h(\mathcal{B})=P_2$

Summary 4-1	4 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	$h(\mathcal{B}) = P_1$ or Undetermined
$(1.1) \cup (4.2) \cup (5.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}, e_4, J\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 17.41$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (4.2) \cup (5.1) \cup (6.3)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}, J\}$	Undetermined
$(1.1) \cup (4.2) \cup (5.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}, e_4\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 3.24$	$h(\mathcal{B}) = P_1$
$(1.2) \cup (4.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_4, J, E, e_2, e_3, \overline{e_1}, \overline{e_4}\}$	Undetermined
$(1.1) \cup (5.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, e_4, J, \overline{e_1}\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11}) = 12.98$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (5.1) \cup (6.3) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_2}, J, e_3\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 3.24$	$h(\mathcal{B}) = P_1$
$(4.1) \cup (5.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, e_4, J, \overline{e_1}\}$	Undetermined
$(4.1) \cup (5.1) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, e_3, e_4, J, \overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11}) = 1.58$	$h(\mathcal{B}) = P_1$

4-2	4 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_2$	
Serial number	Basic set Method	Spatial entropy
$(1.1) \cup (2.1) \cup (2.2) \cup (3)$	$\mathcal{B} = \{O, b, t, l, r, I, J\}$	$h(\mathcal{B}) = O_1$
$(1.1) \cup (1.2) \cup (2.1) \cup (3)$	$\mathcal{B} = \{O, E, b, t, I, J\}$	$h(\mathcal{B}) = O_1$
$(\mathbf{1_1}) \cup (\mathbf{2})_\mathbf{1} \cup (\mathbf{6})_\mathbf{2}$		$h(\mathcal{B}) = P$
$(1.1) \cup (2.1) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{O, b, t, e_1, e_4, J, e_2, e_3, I\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (2.1) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{O, b, t, e_1, e_4, J, \overline{e_2}, \overline{e_3}\}$ $\rho(S_{5;14}S_{5;41}) = 2.14$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.1) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{O, b, t, e_1, e_4, J, \overline{e_1}, \overline{e_4}, I\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 10$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (2.1) \cup (6.3) \cup (6.4)$	$\mathcal{B} = \{O, b, t, \overline{e_2}, \overline{e_3}, J, \overline{e_1}, \overline{e_4}, I\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.2})$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (2.1) \cup (2.2) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, O, b, t, l, r\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 3.14$	$h(\mathcal{B}) = P_1$
$(1.2) \cup (2.1) \cup (2.2) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, E, b, t, l, r\}$	$h(\mathcal{B}) = O_1$
$(\mathbf{1_2}) \cup (\mathbf{2})_\mathbf{1} \cup (\mathbf{6})_\mathbf{1}$		$h(\mathcal{B}) = P$
$(1.1) \cup (1.2) \cup (2.1) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, b, t, O, E\}$ $\rho(S_{4;14}S_{4;41}S_{4;11}) = 3$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (1.2) \cup (2.2) \cup (6.1)$	$\mathcal{B} = \{e_1, e_4, J, l, r, O, E\}$ $\rho(S_{3;14}S_{3;41}S_{3;11}S_{3;11}) = 2$	$h(\mathcal{B}) = P_1$
$(1.1) \cup (3) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{O, I, J, e_1, e_4, \overline{e_2}, \overline{e_3}\}$ $= \{O, I, J, e_1, e_4\} \cup \{I, J, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_{1;2}$
$(1.1) \cup (3) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{O, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ $= (1.1) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (3) \cup (6.3) \cup (6.4)$	$\mathcal{B} = \{O, I, J, \overline{e_2}, \overline{e_3}, \overline{e_1}, \overline{e_4}\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.2})$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (3) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4\}$	$h(\mathcal{B}) = O_1$
$(\mathbf{1})_\mathbf{1} \cup (\mathbf{3}) \cup (\mathbf{7})_\mathbf{2}$	$\mathcal{B} \supset [\mathbf{1}]_\mathbf{1} \cup [\mathbf{7}]_\mathbf{2}$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (3) \cup (7.1)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$	$h(\mathcal{B}) = O_1$

$(1.1)\cup(4.2)\cup(6.3)\cup(6.4)$	$\mathcal{B}=\{O,\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4},e_1,e_4,J\}$ ⊞	$h(\mathcal{B})=O_1$
$(1.1)\cup(4.2)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{O,\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4},e_1,e_4,J,I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.2})$	$h(\mathcal{B})=P_2$
$(1.1)\cup(4.2)\cup(6.3)\cup(6.4)$	$\mathcal{B}=\{O,\overline{e_1},\overline{e_2},\overline{e_3},\overline{e_4},I,J\}$ $\supset(\mathbf{3})\cup(\mathbf{4.2})$	$h(\mathcal{B})=P_2$
$(\mathbf{1_1})\cup(\mathbf{5})_\mathbf{1}\cup(\mathbf{6})_\mathbf{2}$		$h(\mathcal{B})=P$
$(1.1)\cup(5.1)\cup(6.1)\cup(6.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,e_3,I\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(5.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,\overline{e_2}\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11})=4.08$	$h(\mathcal{B})=P_1$
$(1.1)\cup(5.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,\overline{e_1},I\}$ $\rho(S_{2;14}S_{2;41})=g$	$h(\mathcal{B})=P_1$
$(1.1)\cup(5.1)\cup(6.3)\cup(6.4)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,\overline{e_2},J,\overline{e_3},I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.2})$	$h(\mathcal{B})=P_2$
$(1.1)\cup(6.1)\cup(6.2)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},J,e_2,e_3,I\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(6.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},J,\overline{e_2},\overline{e_3}\}$ $=(1.1)\cup(4.2)\cup(6.1)$	$h(\mathcal{B})=O_2$
$(1.1)\cup(6.1)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},J,I\}$ $=(1.1)\cup(3)\cup(7.1)$	$h(\mathcal{B})=O_2$
$(1.1)\cup(6.2)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},e_2,e_3,I,\overline{e_2},\overline{e_3},J\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(6.2)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},e_2,e_3,I\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(1.1)\cup(6.2)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{O,e_1,e_4,\overline{e_1},\overline{e_4},\overline{e_2},\overline{e_3},J,I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.2})$	$h(\mathcal{B})=P_2$
$(1.1)\cup(1.2)\cup(6.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_4,J,O,E,\overline{e_1},\overline{e_4}\}$ ⊞	$h(\mathcal{B})=O_1$
$(1.1)\cup(6.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_4,J,O,E,e_2,e_3,\overline{e_2},\overline{e_3}\}$ $\supset(1.1)\cup(4.1)$	$h(\mathcal{B})=P_2$
$(\mathbf{2})_\mathbf{2}\cup(\mathbf{5})_\mathbf{1}\cup(\mathbf{7})_\mathbf{1}$		$h(\mathcal{B})=P$
$(2.1)\cup(2.2)\cup(5.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,t,l,r,e_4,\overline{e_1}\}$ $\rho(S_{4;11})=g$	$h(\mathcal{B})=P_1$
$(2.1)\cup(2.2)\cup(5.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,t,l,r,e_3,\overline{e_2}\}$ $\rho(S_{4;11})=g$	$h(\mathcal{B})=P_1$
$(3)\cup(6.1)\cup(6.2)\cup(7.1)$	$\mathcal{B}=\{I,J,e_1,e_4,\overline{e_1},\overline{e_4},e_2,e_3\}$ $\supset(\mathbf{3})\cup[\mathbf{4}]_\mathbf{1}$	$h(\mathcal{B})=P_2$
$(3)\cup(6.1)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{I,J,e_1,e_4,\overline{e_1},\overline{e_4}\}$ $=(3)\cup(7.1)$	$h(\mathcal{B})=O_2$
$(3)\cup(6.2)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{I,J,e_1,e_4,\overline{e_1},\overline{e_4},e_2,e_3,\overline{e_2},\overline{e_3}\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(\mathbf{4})_\mathbf{1}\cup(\mathbf{5})_\mathbf{1}\cup(\mathbf{6})_\mathbf{2}$		$h(\mathcal{B})=P$
$(4.1)\cup(5.1)\cup(6.1)\cup(6.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(4.1)\cup(5.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,\overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=1.58$	$h(\mathcal{B})=P_1$
$(4.1)\cup(5.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,\overline{e_1},I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(4.1)\cup(5.1)\cup(6.3)\cup(6.4)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,\overline{e_2},J,\overline{e_1},I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$

$(4.1)\cup(6.1)\cup(6.2)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,e_3,e_4,\overline{e_1},\overline{e_4},J,I\}$ $=(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(4.1)\cup(6.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,e_3,e_4,\overline{e_1},\overline{e_4},J,\overline{e_2},\overline{e_3}\}$ $=(4.2)\cup(6.1)\cup(7.2)$	$h(\mathcal{B})=P_2$
$(4.1)\cup(6.3)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,e_3,e_4,\overline{e_1},\overline{e_4},J,\overline{e_2},\overline{e_3}\}$ $=(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(4.1)\cup(6.3)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,e_3,e_4,\overline{e_1},\overline{e_4},I\}$ $=(4.1)\cup(6.4)$	$h(\mathcal{B})=O_2$
$(\mathbf{5})_\mathbf{1}\cup(\mathbf{6})_\mathbf{2}\cup(\mathbf{7})_\mathbf{1}$		$h(\mathcal{B})=P$
$(5.1)\cup(6.1)\cup(6.2)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_4,\overline{e_1},J,e_3,I\}$ $\supset(\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(5.1)\cup(6.1)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_4,\overline{e_1},J,\overline{e_2}\}$ $=(5.1)\cup(6.3)\cup(7.1)$	$h(\mathcal{B})=P_2$
$(5.1)\cup(6.3)\cup(6.4)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_4,\overline{e_1},J,I\}$ $\supset(\mathbf{3})\cup(\mathbf{5.1})$	$h(\mathcal{B})=P_2$
$(5.1)\cup(6.2)\cup(6.3)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_4,\overline{e_1},e_3,I,\overline{e_2},J\}$ $=((\mathbf{3})\cup(\mathbf{4.1})$	$h(\mathcal{B})=P_2$
$(\mathbf{5})_\mathbf{2}\cup(\mathbf{6})_\mathbf{1}\cup(\mathbf{7})_\mathbf{1}$	$\mathcal{B}\supset[\mathbf{5}]_\mathbf{2}\cup[\mathbf{6}]_\mathbf{1}$	$h(\mathcal{B})=P_2$

Summary 4-2	4 minimal cycle generators: $(\ )_1\cup(\ )_1\cup(\ )_2$	
Serial number	Basic set Method	$h(\mathcal{B})=P_1$ or Undetermined
$(1.1)\cup(2.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{O,b,t,e_1,e_4,J,\overline{e_2},\overline{e_3}\}$ $\rho(S_{5;14}S_{5;41})=2.14$	$h(\mathcal{B})=P_1$
$(1.1)\cup(2.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{O,b,t,e_1,e_4,J,\overline{e_1},\overline{e_4},I\}$ $\rho(S_{4;14}S_{4;41}S_{4;11})=10$	$h(\mathcal{B})=P_1$
$(1.1)\cup(2.1)\cup(2.2)\cup(6.1)$	$\mathcal{B}=\{e_1,e_4,J,O,b,t,l,r\}$ $\rho(S_{4;14}S_{4;41}S_{4;11})=3.14$	$h(\mathcal{B})=P_1$
$(1.1)\cup(1.2)\cup(2.1)\cup(6.1)$	$\mathcal{B}=\{e_1,e_4,J,b,t,O,E\}$ $\rho(S_{4;14}S_{4;41}S_{4;11})=3$	$h(\mathcal{B})=P_1$
$(1.1)\cup(1.2)\cup(2.2)\cup(6.1)$	$\mathcal{B}=\{e_1,e_4,J,l,r,O,E\}$ $\rho(S_{3;14}S_{3;41}S_{3;11}S_{3;11})=2$	$h(\mathcal{B})=P_1$
$(1.1)\cup(5.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,\overline{e_2}\}$ $\rho(S_{6;14}S_{6;41}S_{6;11}S_{6;11})=4.08$	$h(\mathcal{B})=P_1$
$(1.1)\cup(5.1)\cup(6.1)\cup(6.4)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,O,e_4,J,\overline{e_1},I\}$ $\rho(S_{2;14}S_{2;41})=g$	$h(\mathcal{B})=P_1$
$(2.1)\cup(2.2)\cup(5.1)\cup(7.1)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,t,l,r,e_4,\overline{e_1}\}$ $\rho(S_{4;11})=g$	$h(\mathcal{B})=P_1$
$(2.1)\cup(2.2)\cup(5.1)\cup(7.2)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,t,l,r,e_3,\overline{e_2}\}$ $\rho(S_{4;11})=g$	$h(\mathcal{B})=P_1$
$(4.1)\cup(5.1)\cup(6.1)\cup(6.3)$	$\mathcal{B}=\{e_1,e_2,\overline{e_3},\overline{e_4},b,e_3,e_4,J,\overline{e_2}\}$ $\rho(S_{6;14}S_{6;44}S_{6;41}S_{6;11})=1.58$	$h(\mathcal{B})=P_1$

4-3	4 minimal cycle generators: $(\ )_2 \cup (\ )_2$	
Serial number	Basic set Method	Spatial entropy
$(1)_2 \cup (2)_2$	$\mathcal{B} = \{O, E, b, t, l, r\}$	$h(\mathcal{B}) = O_1$
$(\mathbf{1})_\mathbf{2} \cup (\mathbf{5})_\mathbf{2}$	$\mathcal{B} \supset [\mathbf{1}]_\mathbf{1} \cup [\mathbf{5}]_\mathbf{2}$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{O, E, e_1, e_4, J, e_2, e_3, I\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{O, E, e_1, e_4, J, \overline{e_2}, \overline{e_3}\}$ $= \{O, e_1, e_4, J\} \cup \{E, J, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_{1,2}$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{O, E, e_1, e_4, J, \overline{e_1}, \overline{e_4}, I\}$ $= (1.1) \cup (1.2) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(\mathbf{4})_\mathbf{2} \cup (\mathbf{7})_\mathbf{2}$	$\mathcal{B} = (4)_2$	$h(\mathcal{B}) = O_2$
4-4	4 minimal cycle generators: $(\ )_1 \cup (\ )_3$	
$(\ )_\mathbf{1} \cup (\ )_\mathbf{3}$	$\mathcal{B} \supset (\ )_3$	$h(\mathcal{B}) = P_2$
4-5	4 minimal cycle generators: $(\ )_4$	
$(\ )_\mathbf{4}$	$\mathcal{B} \supset (\ )_3$	$h(\mathcal{B}) = P_2$

5-1	5 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1 \cup (\ )_1 \cup (\ )_1$	
Serial number	Basic set Method	Spatial entropy
$(1.1) \cup (4.2) \cup (5.1) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}, e_4, J\}$ $\supset (1.1) \cup (4.2) \cup (5.1) \cup (6.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.2) \cup (5.1) \cup (6.3) \cup (7.2)$	$\mathcal{B} = \{e_1, e_2, \overline{e_3}, \overline{e_4}, b, O, \overline{e_1}, \overline{e_2}, J, e_3\}$ $\supset [(1.1) \cup (4.2) \cup (5.1) \cup (7.1)]_1$	$h(\mathcal{B}) = P_2$
5-2	5 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_1 \cup (\ )_2$	
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (7.1)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ $= (1.1) \cup (1.2) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (7.2)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (3) \cup (6.1) \cup (6.2) \cup (7.1)$	$\mathcal{B} = \{O, I, J, e_1, e_4, e_2, e_3, \overline{e_1}, \overline{e_4}\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (3) \cup (6.1) \cup (6.3) \cup (7.1)$	$\mathcal{B} = \{O, I, J, e_1, e_4, \overline{e_2}, \overline{e_3}, \overline{e_1}, \overline{e_4}\}$ $\supset (\mathbf{3}) \cup (\mathbf{4.2})$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (3) \cup (6.1) \cup (6.4) \cup (7.1)$	$\mathcal{B} = \{O, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ $\supset (1.1) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (3) \cup (6.1) \cup (6.4) \cup (7.2)$	$\mathcal{B} = \{O, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}, e_2, e_3, \overline{e_2}, \overline{e_3}\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (4.2) \cup (6.1) \cup (6.3) \cup (7.1)$	$\mathcal{B} = \{O, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_1, e_4, J\}$ $= (1.1) \cup (4.2) \cup (6.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (4.2) \cup (6.1) \cup (6.4) \cup (7.1)$	$\mathcal{B} = \{O, \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, e_1, e_4, J, I\}$ $= (\mathbf{3}) \cup (\mathbf{4.2})$	$h(\mathcal{B}) = P_2$
5-3	5 minimal cycle generators: $(\ )_1 \cup (\ )_2 \cup (\ )_2$	
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (6.2)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, e_2, e_3\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (6.3)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, \overline{e_2}, \overline{e_3}\}$ $= \{O, I, J, e_1, e_4\} \cup \{E, I, J, \overline{e_2}, \overline{e_3}\}$	$h(\mathcal{B}) = O_{1,2}$
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (6.4)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ $= (1.1) \cup (1.2) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.2) \cup (7.1)$	$\mathcal{B} = \{O, E, e_1, e_4, \overline{e_1}, \overline{e_4}, J, e_2, e_3, I\}$ $\supset (1.1) \cup (4.1)$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.3) \cup (7.1)$	$\mathcal{B} = \{O, E, e_1, e_4, \overline{e_1}, \overline{e_4}, J, \overline{e_2}, \overline{e_3}\}$ $\mathcal{B} \supset [(1.1) \cup (4.1)]_1$	$h(\mathcal{B}) = P_2$
$(1.1) \cup (1.2) \cup (6.1) \cup (6.4) \cup (7.1)$	$\mathcal{B} = \{O, E, e_1, e_4, \overline{e_1}, \overline{e_4}, J, I\}$ $= (1.1) \cup (1.2) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$
6	6 minimal cycle generators: $(\ )_1 \cup (\ )_1 \cup (\ )_2 \cup (\ )_2$	
$(1.1) \cup (1.2) \cup (3) \cup (6.1) \cup (6.4) \cup (7.1)$	$\mathcal{B} = \{O, E, I, J, e_1, e_4, \overline{e_1}, \overline{e_4}\}$ $= (1.1) \cup (1.2) \cup (3) \cup (7.1)$	$h(\mathcal{B}) = O_2$

Appendix A

A.1. The symmetries of D_4 and S_2 of 17 minimal cycle-generators in $\mathcal{C}_c(2)$ are listed in Table A.1.

Minimal cycle-generator	ρ	ρ^2	ρ^3	m	$m\rho$	$m\rho^2$	$m\rho^3$	$0 \leftrightarrow 1$
(1) 1×1 $\{O\}$	•	•	•	•	•	•	•	(2)
(2) 1×1 $\{E\}$	•	•	•	•	•	•	•	(1)
(3) 1×2 $\{t, b\}$	(4)	•	(4)	•	(4)	•	(4)	•
(4) 2×1 $\{l, r\}$	(3)	•	(3)	•	(3)	•	(3)	•
(5) 2×2 $\{I, J\}$	•	•	•	•	•	•	•	•
(6) 2×2 $\{e_1, e_2, e_3, e_4\}$	•	•	•	•	•	•	•	(7)
(7) 2×2 $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4\}$	•	•	•	•	•	•	•	(6)
(8) 3×2 $\{r, e_1, e_3, \overline{e}_2, \overline{e}_4\}$	(10)	(9)	(11)	(9)	(10)	•	(11)	(9)
(9) 3×2 $\{l, e_2, e_4, \overline{e}_1, \overline{e}_3\}$	(11)	(8)	(10)	(8)	(11)	•	(10)	(8)
(10) 2×3 $\{t, e_3, e_4, \overline{e}_1, \overline{e}_2\}$	(9)	(11)	(8)	•	(8)	(11)	(9)	(11)
(11) 2×3 $\{b, e_1, e_2, \overline{e}_3, \overline{e}_4\}$	(8)	(10)	(9)	•	(9)	(10)	(8)	(10)
(12) 3×3 $\{e_1, e_4, J\}$	(13)	•	(13)	(13)	•	(13)	•	(15)
(13) 3×3 $\{e_2, e_3, I\}$	(12)	•	(12)	(12)	•	(12)	•	(14)
(14) 3×3 $\{\overline{e}_2, \overline{e}_3, J\}$	(15)	•	(15)	(15)	•	(15)	•	(13)
(15) 3×3 $\{\overline{e}_1, \overline{e}_4, I\}$	(14)	•	(14)	(14)	•	(14)	•	(12)
(16) 4×4 $\{e_2, e_3, \overline{e}_2, \overline{e}_3\}$	(17)	•	(17)	(17)	•	(17)	•	•
(17) 4×4 $\{e_1, e_4, \overline{e}_1, \overline{e}_4\}$	(16)	•	(16)	(16)	•	(16)	•	•

Table A.1

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