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碩 士 論 文

探討量子形變的新途徑

A New Approach to Quantum Deformation

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摘要

q−形變在許多數學及物理的不同領域裡被廣泛討論。然而，所有已知的討論都是建立在保角變換$x \mapsto qx$之上。在本篇論文中，我們考慮另一種形式的形變，稱之為$\tilde{q}$−形變，它是建立在$x \mapsto x^q$的變換之上。換言之，這類形變發生在變數x的次方數上。我們也期待$\tilde{q}$−形變最終會是研究量子群理論的另一途徑。

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ABSTRACT

The q-deformation had been wide discussed in many different fields of mathematics and physics. However, all the discussions that we know of are simply based on the conformal mapping $x \mapsto qx$. Throughout the thesis, we consider deformation of another kind, says $\tilde{q}$-deformation, which is based on the mapping $x \mapsto x^q$. In other words, these kinds of deformations appear in the power of variable x. We expect $\tilde{q}$- deformation to be a new approach to the studying of Quantum Groups eventually.

誌　謝

轉眼間，來到交通大學已近兩個年頭，碩士階段也告尾聲。回首一路渾渾噩噩、跌跌撞撞、起起落落、暗濤洶湧的十數年求學生涯，雖不盡完美，能於此安然畫下句點，實是多年以前始料未及之事，亦感觸良多。當然，還必須感謝無數的人們在這一路上給予我多方的協助與關懷。

回想中學時候的求學過程，讀書的目的很狹隘，不外乎讓考試卷上的分數更接近滿分，不外乎拿到第一名時候的優越感，不外乎能唸第一志願的學校…。當誘因一個個消失的同時，書本也就變得不值一提。大學的生活，可謂多采多姿，也可謂五味雜陳，單看你從什麼樣的觀點與角度切入。如果從課業、成績這個面向切入，我的名字實在與用功、優秀這樣的字眼怎也沾不上邊。前兩年的功課，真是混到一種昏天黑地無以言喻的極致，能翹的課幾乎讓我翹光了，能當的主科也幾乎讓我當光了，還有每學期一張讓父母憂心的成績單，如果有個比誰『混』地徹頭徹尾的獎項，我想我應該能得『書卷獎』吧。優秀的學生拿著選課加簽單上修課程，而一路下修、重修的我，當然是拿著加簽單，在爆學分邊緣的壓力下，時時出入系主任辦公室接受主任的關心與輔導，從大三開始，一連三個學期都是如此。高微、代數、複變、微分方程、拓樸、機率統計在同一時間連續轟炸的感覺，我想很少數學系的學生有我這般得天獨厚的機遇吧，只能如此苦笑地安慰自己，再往臉上貼金，這是天將降大任於斯人也云云。然而，天不曾降大任於斯人，成敗與否卻往往操之在己。也許是帶著再當一科鐵定延畢的決心，也許是自己開始約略懂得唸書的要領，也許是自己同時進行著好幾科更高等深難的課程，許多人聞之色變的高等微積分，在我身上卻很意外地如魚得水，我只能這麼解釋，微積分我修過兩次，昨天複變講得更廣，前天拓樸教得更深，當然用功與否才是主因吧。最終還是順利地如期畢業了，也糊裡糊塗地考上了自己學校的研究所。然而，讀書之於我的意義何在？我依舊不甚了解。

也許對純數學興趣缺缺，也許我這隻老鳥老到每個老師的習性與底限都摸透了，研究所的生活，又是另一個昏天黑地的開始。說來慚愧，連開課老師都不時好心地向我耳提面命嘮叨幾句，勸我要好好唸書，別再浪費時間，同學也半開玩笑地提醒我，明天要考試別又睡過頭忘了來上課了，渾渾噩噩之中，第一學期也就這樣莫名其妙地過去了。寒假四十天，下定決心至少要讓大一以來一直呈現空白狀態的線性代數從零到有，畢竟一個不懂線性代數的數學系研究生在面子上是掛不住的，也再次報名了研究所考試做為自我能力的檢測。我很驚訝於我的投入與專注程度，整整四十天，每天從圖書館開館坐到閉館，一坐下便能馬上全神貫注，投入程度令我常常中餐不吃，也因此有生以來第一次體會到自己竟然可以和

線性代數如此親近。也許是自己夠用功，加上考運不錯，沒想到就這樣來到了交大。踏入交大的校園，是兩種奇妙而矛盾的感覺，畢竟原本只能在網站上才看得到的老師們，一個個栩栩如生地出現在你的生活圈周圍，不盡懷疑道，沒想到像我這樣稱不上優秀的人竟也有機會站在這裡？然而，一旦意識到我確實在站這裡的同時，交大在我心中的地位，似乎也就不再如想像中那般神話了。

來到這裡，目標很明確，我希望朝理論物理的方向走去，而且這所學校自由而興盛的學術風氣足以讓我有機會這麼做。一個沒修過任何物理課程的數學系學生，竟會想往這個方向走去？不管走到那裡，這樣的想法總是會招來無數異樣的眼光與備受質疑。如果要說原因，我想是源自於高中時候對於物理的興趣以及對宇宙論的憧憬與熱情，也源自於童年時候對萬事萬物單純的好奇之心。然而，大學聯考一翻兩瞪眼的結果，加上自己並不積極於轉換跑道之上，就這麼半推半就地在數學領域不斷打滾至今。不過，我從沒想過自己要做出什麼樣的成果或將來能否有所輝煌成就，我只是很單純地希望有人能用數學語言很粗略地告訴我，這個宇宙模模糊糊地看上去會是什麼樣子爾爾…。雖然，最終這樣的理想仍是遠比我想像之中的要來遙遠地太多太多，我還是願意將第二次的研究所生涯投注在其上。在課業上，簡值跌破了自己的眼鏡，不曾翹課，很少遲到，甚至懂得安排自己的時間，這些在過去根本不可能出現的事情，竟也一一發生在我身上。一旦開始下定決心唸書，便發現動力是如此之大，也發覺再多的時間永遠總嫌不夠，更驚覺過去自己真的蹉跎不少光陰。大學時候作業總是一再依靠著同組同學完成，一路摸魚打混、得過且過的我，竟然破天荒地自己獨立地把習題一題又一題地完成，連自己都直呼不可思議。這兩年來，親自做了好多好多的習題，也從中獲取到很多無法用分數來評價的意義。只要有心，很多事情我也能夠做得很好的。

能來到今天，有許多錯綜複雜、無以解釋的因緣際會，要感謝的人真的很多，無以一一道盡。首先，我要感謝我的父母，在求學的路上給予我經濟上及精神上很大的支持，使我沒有後顧之憂。其次，我必須感謝台中一中的美術教師蘇俊吉先生，在他幽默詼諧的課堂裡，從生死學的角度提到超空間的概念，並不厭其煩地鼓勵我們將來要以科學的角度去探索 26 維的宇宙方程式，也期許我們有朝一日與他分享，我想這是我對宇宙論激起熱情的一個開端。另一個很大的原因，是在大學的啟蒙時期，受到一本科普著作的感染很深，也幾乎影響了我往後整個思維模式與哲學觀。我必須感謝 Prof. Michio Kaku 及他的科普著作 Hyperspace，書中以理論物理學家的觀點，從更高、更廣闊的視野，以生動易懂卻深刻的比喻，帶著我漫游想像之外的宇宙，並粗略地介紹了理論物理的發展過程及相對應的數學工具，對我最終走向這條道路，幾乎是辦演著絕對性的角色。當然，能順利完成碩士論文，最要感謝的是我兩個指導教授，金周新先生與翁志文先生。因為金老師的專業知識背景及多年的研究經驗，給了我一條進入數學物理、理論物理的捷徑，因而過程中鮮少碰壁並獲益良多；再則金老師的耐心與包容，願意傾聽我

許多無意間提到的荒誕想法，並願意花時間與我討論紙上隨手寫下的不知道有沒有意義的數學式子，讓我在沒有壓力之下，能放縱思想、暢所欲言，渡過不少快樂的時光。因為翁老師對學生的深切的關懷及胸懷大度，在門生眾多的情況下，還願意擔任我的共同指導，為我擔下不少額外的心力與責任，並在遇到困難時候，不遺餘力地給予我多方的協助與心靈上的支持，才得以使生活上、課業上、系上運作上諸事順利，最終完成此學業。我還必須感謝原本在彰化師大準備要找的指導教授曾旭堯先生，是他鼓勵我前來交大唸書，並不時對我的生涯規劃提出諸多建議與關注。我也必須感謝彰化師大的蕭守仁教授及劉承楷教授，願意隨時與我長談，傾聽我的想法並給予我許多寶貴意見。雖然，最終我走上一條在他們想像中很不一樣的道路，不過，那也許也是最貼近我想法與感覺中的路。當然，我還必須感謝交大應數系課程委員會的諸位老師，因為他們的認可，我才得以前往系外尋找論文指導教授，過程中麻煩了許多人，也得到了很多幫忙，方得以達此願，只有在此簡短謝過。我還要感謝博士班的黃皓文學長，在我課業繁忙的碩一下時候，願意時時與我討論，在功課上為我提點許多，才得以順利渡過。最後，我還要感謝目前就讀資工博一的陳毅睿同學，也是我過去大學的同班同學，系籃的好戰友，在 Latex 軟體的指令操作及使用技巧上不厭其煩地給予我許多幫助與解惑，論文初稿才得以順利產出。

如果將求學過程比喻成人生的縮影，那麼發生在我身上的許多事情，真如同一齣齣什麼都有的鬧劇，時而哭笑不得，時而揚眉吐氣，能寫的，不能寫的，高潮迭起，充滿戲劇性，幸虧在我身邊總有無以計數的貴人適時牽引、援助，如今才得以安然於此處。我想是我的個性過於自負、專斷、自我，縱使不想標新立異卻也顯得獨特立行，我想我必須學會在處事上更加圓融，考慮事情更加全面、周詳，使自己的稜角融入大環境之中，而非一味橫衝直撞。非常感謝曾經幫助過我、包容過我的無數人們，從我出生至今，否則我真不知如今會身在何處，大恩無以言謝，僅以此書聊表謝意。

巍巍蒼穹，無奇不有，浩浩宇宙，無所不包。人類是否也只是一頭擁有三維體態的『螞蟻』，爬行在一個多維時空的交大校園內呢？從這頭螞蟻的眼中看出去的世界，又是那座系館的一角或是那片草地的一部分呢？時常我不得不放縱思維，浸淫在這樣看似無意義的遐想世界裡，而後回到現實。然而，這樣的問題，縱使是世界第一流的大人物們，都不見得能給予我明確的答案。也許，我們都應放下身段，減少意識形態上的對立，以更低下的姿態、更廣闊的視野，重新檢視自己以及周遭的人、事、地、物；避免以自我為中心，獨斷地認定一些自以為是的事實，卻驕傲而不可一世地否定最寶貴的真實。如此，便能以更加謙遜而崇敬的態度，包容世間無奇不有的萬事萬物。

黃山齊　2010 年 7 月

Contents

Chapter 1

Introduction

Almost all functions can be represented explicitly or approximately by a finite sum or an infinite sum of the simplest algebraic functions x^n with some coefficients. More precisely, $\sum_n a_n x^n$. For the nature exponential function, the simplest transcendental function to our philosophy, one has the expression from Elementary Calculus:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

When the simplest algebraic function comes to the simplest transcentdental function and take integral from 0 to ∞, a fantastic formula comes into being:

$$n! = \int_0^{\infty} x^n e^{-x} dx.$$

For complex z except the non-positive integers, the complex generalization of the factorial is the well-known gamma function

$$\begin{aligned}\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}dt \\ &= \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)...(z+n)} \\ &:= (z-1)!\end{aligned}$$

Retrospect to the history, it is the intelligence of Euler [1] to wrote down the two expressions of the factorial function:

$$\frac{1\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}...$$

and

$$\int_0^1 (-logt)^n dt.$$

Later, Gauss and Legendre rewrote the product and integral expression in the modern form, respectively. Although Euler discussed some cases for n is rational in his formulas and refound the work of Wallis (1616-1703) for $n=\frac{1}{2}$:

$$\frac{\pi}{2} = (\frac{2\cdot 2}{1\cdot 3})\cdot(\frac{4\cdot 4}{3\cdot 5})\cdot(\frac{6\cdot 6}{5\cdot 7})\cdot(\frac{8\cdot 8}{7\cdot 9})...,$$

he does not appear to be considered in the factorial of a complex number, as Gauss did first. Nowadays, the gamma function play an important role for applications in engineering and many applied sciences for special functions, such as beta function, hypergeometric function, Bessel function, Legendre function..., are original from the simplest gamma function. In reality, each time when a topic concerning the special functions was discussed, the gamma function was mentioned. As a result, the gamma function is essential to the theoretical research of mathematics and physics, and worth studying.

In the world of q (or more formal terminology "q-analogue"), roughly speaking, a q-analogue of a classical mathematical object (a constant, a variable, a function, an equation, an identity, an algebra, a theorem...) means a new mathematical object which was deformed from the classical with scales of functions of variable q in its substructures, and the q-deformation recover the classical form when q tends to 1^-. To our knowledge, the earliest q-deformation studied in detail is the basic hypergeometric series [2]

$$\begin{aligned}\varphi(\alpha,\beta,\gamma,q,x) &= 1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x \\ &\quad +\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2+...,\end{aligned}$$

which was first considered by E. Heine in 1846 and is the q-analogue of the classical hypergeometric series

$$F(\alpha,\beta,\gamma,x)=1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+...,$$

From the inspiration of the fact that $F(1,k,1,\frac{t}{k})$ for $k=\infty$ is e^t, Heine constructed two analogues

$$\sum_{m=0}^{\infty}\frac{q^{1+2+...+(m-1)}t^m}{(q-1)(q^2-1)...(q^m-1)}$$

and

$$\sum_{m=0}^{\infty}\frac{t^m}{(q-1)(q^2-1)...(q^m-1)}$$

by $\varphi(1,k,1,q,\frac{t}{q^k})$, $\varphi(1,k,1,q,-t)$ for $k=\infty$. In fact, the two series were discovered by Euler earlier and introduced by F. H. Jackson [3] after Heine as two q-analogue of e^x in 1903: (The notations used differ from Jackson slightly)

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!}$$

and

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

where

$$[n]_q := \frac{1-q^n}{1-q}$$

is the q-analogue of n, and

$$[n]_q! := [1]_q[2]_q[3]_q...[n]_q.$$

The q-gamma function $\Gamma_q(x)$ was introduced by J. Thomae [4], a student of Heine, in 1869 and later by F. H. Jackson [5] in 1904 as the infinite products (We use the notations nowadays instead of what the original authors used)

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(\frac{1}{1-q})^{x-1} = \frac{(q;q)_{x-1}}{(1-q)^{x-1}},$$

where $0<q<1$ and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) = (1-a)(1-qa)(1-q^2a)\cdots,$$

$$(a;q)_x := \frac{(a;q)_\infty}{(aq^x;q)_\infty}.$$

However, the q-integral representation of $\Gamma_q(x)$ was not quite right until it was recosidered by T. H. Koornwinder [6] in 1990:

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t,$$

where the q-integral is defined by

$$\int_0^x f(t) d_q t = \sum_{j=0}^{\infty} f(q^j x)(q^j x - q^{j+1} x).$$

The q-beta function, first introduce by J. Thomae [4], has a quite similar result to classical beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \frac{t^{x-1}}{(1-t)^{y-1}} d_q t$:

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} = \int_0^1 t^{x-1} \frac{(qt;q)_\infty}{(q^y t;q)_\infty} d_q t = \int_0^1 \frac{t^{x-1}}{(qt;q)_{y-1}} d_q t,$$

where $x, y > 0$. Other q-extensions of beta function $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt$ are original from the famous formula of Ramanujan [7]

$$\int_0^\infty \frac{(1+aqt)(1+aq^2t)\ldots}{(1+t)(1+qt)(1+q^2t)\ldots} t^{x-1} dt = \frac{\pi}{\sin \pi x} \prod_{n=1}^{\infty} \frac{(1-q^{n-x})(1-aq^n)}{(1-q^n)(1-aq^{n-x})}.$$

Some more details was discussed by R. Askey and G. E. Andrews in [8, 9]:

$$\begin{aligned}
\int_0^\infty t^{x-1} \frac{(-q^{x+y}t;q)_\infty}{(-t;q)_\infty} dt &:= \int_0^\infty \frac{t^{x-1}}{(-t;q)_{x+y}} dt \\
&= \frac{\Gamma_q(y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(1-x)} \\
&= (\frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)})(\frac{\Gamma_q(1)}{\Gamma_q(x)\Gamma_q(1-x)})(\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}) \\
&= \frac{B(x,1-x)}{B_q(x,1-x)} B_q(x,y)
\end{aligned}$$

$$\xrightarrow{\text{q}\to 1^-} B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt,$$

$$\begin{aligned}\int_0^\infty t^{x-1}e_q^{-t}dt &:= \int_0^\infty t^{x-1}\frac{1}{(-(1-q)t;q)_\infty}dt\\ &= \frac{\Gamma(x)\Gamma(1-x)}{\Gamma_q(1-x)}\\ &= (\frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)})(\frac{\Gamma_q(1)}{\Gamma_q(x)\Gamma_q(1-x)})\Gamma_q(x)\\ &= \frac{B(x,1-x)}{B_q(x,1-x)}\Gamma_q(x) \xrightarrow{\text{q}\to 1^-} \Gamma(x,y),\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty t^{x-1}\frac{(-q^{x+y}t;q)_\infty}{(-t;q)_\infty}d_qt &:= \int_0^\infty \frac{t^{x-1}}{(-t;q)_{x+y}}d_qt\\ &= \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\frac{(-q^x;q)_\infty(-q^{1-x};q)_\infty}{(-1;q)_\infty(-q;q)_\infty}\\ &= B_q(x,y)\frac{(-q^x;q)_\infty(-q^{1-x};q)_\infty}{(-1;q)_\infty(-q;q)_\infty}\\ &\xrightarrow{\text{q}\to 1^-} B(x,y),\end{aligned}$$

where

$$\int_{-\infty}^\infty f(t)d_qt := \sum_{j=-\infty}^\infty \int_{q^{j+1}}^{q^j} f(t)d_qt = \sum_{j=-\infty}^\infty f(q^j)(q^j-q^{j+1}).$$

Everytime when a subject of q-series was concerned, the famous Jacobi Tripe Product Indentity should not be missed:

$$(1-x)(qx;q)_\infty(q;q)_\infty(q/x;q)_\infty = \sum_{n=-\infty}^\infty (-1)^n q^{\frac{n(n-1)}{2}} x^n.$$

A simple proof, which conncets two identities of Euler

$$\prod_{n=0}^\infty (1+q^n x) = \sum_{m=0}^\infty \frac{q^{1+2+\ldots+(m-1)}x^m}{(1-q)(1-q^2)\ldots(1-q^m)},$$

$$\prod_{n=0}^{\infty}(\frac{1}{1-q^n x}) = \sum_{m=0}^{\infty}\frac{x^m}{(1-q)(1-q^2)...(1-q^m)}$$

with the Jacobi Triple Product, is due to G. E. Andrews [10]. An application that we know of was given by R. Askey [11], who generalized the well-known formula

$$cos\pi x = \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}-x)}$$

to q-analogue by considering the relations between q-gamma function, Jacobi Triple Product, properities of Theta functions and then obtained the nice formula :

$$\frac{(\Gamma_q(\frac{1}{2}))^2}{\Gamma_q(\frac{1}{2}+x)\Gamma_q(\frac{1}{2}-x)} = q^{\frac{-x^2}{2}} cos\pi x \prod_{n=1}^{\infty}(\frac{1+2r^{2n}cos2\pi x + r^{4n}}{1+2r^{2n}+r^{4n}}),$$

where $r = e^{\frac{2\pi^2}{logq}} \to 0^+$ as $q \to 1^-$. A more general version of Jacobi Triple Product is the the famous Ramanujan ${}_1\Psi_1$ summation [8]:

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n = \frac{(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(x;q)_\infty(b/ax;q)_\infty(b;q)_\infty(q/a;q)_\infty},$$

which is the generalization of the q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

If we replace a by $1/\alpha$, x by αt and set $\alpha = b = 0$ in the Ramanujan ${}_1\Psi_1$ summation, the Jacobi Triple Product Indentity appeared.

So far, we have reviewed the history of the development of q-analogues briefly. A notice, in brief, is that the deformations of the substructures of the q-formulas above are based on the conformal mapping $x \mapsto qx$. It is a

q times deformation in scale with variable x. An interesting subject appear to be open: what are the analogues when the deformations occur in the ‶power″of variable x? That is, what are the analogues when qx becomes x^q? This is what we want to discuss throughout this article. For simplicity, we use the terminology ‶$\tilde{q}$-analogue″or ‶$\tilde{q}$-deformation″to describe deformations of this kind from now on.

Chapter 2

$\tilde{q}$-Calculus for differentiation

No one would doubt about the formula "$\frac{d}{dx}(x^n) = nx^{n-1}$", the simplest differential rule in any Elementary Calculus books. The main objective in this chapter is to construct a $\tilde{q}$-analgoue of this formula with self-consistence. Some preknowledge are given in 2.1. The $\tilde{q}$-analogues of n and x^n are constructed in 2.2. Two $\tilde{q}$-analogues of n-factorial $n!$ and the corresponding Taylor's expansions of e^x are considered in 2.3.

2.1 $\tilde{q}$-differential operator

First of all, we define the $\tilde{q}$-differential operator [12, 13](or the $\tilde{q}$-derivative) with respect to coordinate x as follow:

$$\tilde{\Delta}_{(x;q)} f(x) := \frac{f(x) - f(x^q)}{x - x^q}$$

or more generally

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)} f(x) := \frac{f(x^{q^\alpha}) - f(x^{q^\beta})}{x^{q^\alpha} - x^{q^\beta}},$$

where the lower index q or (q^α, q^β) denotes a parameter and x is a variable or a function.

It is clear that the $\tilde{q}$-differential operator is a generalization of the classical derivative and the q-derivative

$$\Delta_q f(x) := \frac{f(x) - f(qx)}{x - qx},$$

$$\Delta_{(q^\alpha,q^\beta)} f(x) := \frac{f(q^\alpha x) - f(q^\beta x)}{q^\alpha x - q^\beta x}.$$

A simple exercise is to act $\tilde{\Delta}_{(x;q^\alpha,q^\beta)}$ on the function x^n, and one may easily obtain

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)} x^n = \frac{(x^{q^\alpha})^n - (x^{q^\beta})^n}{x^{q^\alpha} - x^{q^\beta}} = (\frac{x^{(q^\alpha-1)n} - x^{(q^\beta-1)n}}{x^{q^\alpha-1} - x^{q^\beta-1}})x^{n-1}$$

which is a analogue of classical $(x^n)' = nx^{n-1}$ and q-analogue

$$\Delta_{(q^\alpha,q^\beta)} x^n = (\frac{q^{n\alpha} - q^{n\beta}}{q^\alpha - q^\beta})x^{n-1}.$$

For arbitrary ω, we also have the analogue

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)} x^\omega = \frac{(x^{q^\alpha})^\omega - (x^{q^\beta})^\omega}{x^{q^\alpha} - x^{q^\beta}} = (\frac{x^{(q^\alpha-1)\omega} - x^{(q^\beta-1)\omega}}{x^{q^\alpha-1} - x^{q^\beta-1}})x^{\omega-1}$$

corresponding to classical and q. It is not surprising that the factor

$$\frac{x^{(q^\alpha-1)n} - x^{(q^\beta-1)n}}{x^{q^\alpha-1} - x^{q^\beta-1}}$$

or the factor

$$\frac{x^{(q^\alpha-1)\omega}-x^{(q^\beta-1)\omega}}{x^{q^\alpha-1}-x^{q^\beta-1}}$$

give us some idea to define the $\tilde{q}$-analogue of integer n(or any ω). However, they are not what we quite expected n or ω of $\tilde{q}$-analogue. In next section, we find a better expression than this one!

One shall not be curious to know the two differential rules in any Elementary Calculus books:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

and

$$f(g(x))' = f'(g(x))g'(x),$$

the Product Rule and the Chain Rule. In $\tilde{q}$-Calculus, our analogues to the Product Rule has the following two kinds:
The first kind is

$$\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)}(f(x)g(x)) &= \frac{f(x^{q^\alpha})g(x^{q^\alpha})-f(x^{q^\beta})g(x^{q^\beta})}{x^{q^\alpha}-x^{q^\beta}} \\
&= \frac{1}{x^{q^\alpha}-x^{q^\beta}}\{f(x^{q^\alpha})g(x^{q^\alpha})-f(x^{q^\beta})g(x^{q^\alpha})\}+ \\
&\quad \frac{1}{x^{q^\alpha}-x^{q^\beta}}\{f(x^{q^\beta})g(x^{q^\alpha})-f(x^{q^\beta})g(x^{q^\beta})\} \\
&= (\frac{f(x^{q^\alpha})-f(x^{q^\beta})}{x^{q^\alpha}-x^{q^\beta}})g(x^{q^\alpha})+f(x^{q^\beta})(\frac{g(x^{q^\alpha})-g(x^{q^\beta})}{x^{q^\alpha}-x^{q^\beta}}) \\
&= \tilde{\Delta}_{(x;q^\alpha,q^\beta)}f(x)\cdot g(x^{q^\alpha})+f(x^{q^\beta})\cdot\tilde{\Delta}_{(x;q^\alpha,q^\beta)}g(x).
\end{aligned}$$

If we replace the term $f(x^{q^\beta})g(x^{q^\alpha})$ by $f(x^{q^\alpha})g(x^{q^\beta})$ in the second line of the above formula, we obtain the second kind:

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)}(f(x)g(x)) = f(x^{q^\alpha}) \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)}g(x) + \tilde{\Delta}_{(x;q^\alpha,q^\beta)}f(x) \cdot g(x^{q^\beta}).$$

A similar argument is to the $\tilde{q}$-quotient rule:

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)}\frac{f(x)}{g(x)} = \frac{\tilde{\Delta}_{(x;q^\alpha,q^\beta)}f(x) \cdot g(x^{q^\beta}) - f(x^{q^\beta}) \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)}g(x)}{g(x^{q^\alpha})g(x^{q^\beta})}$$

and

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)}\frac{f(x)}{g(x)} = \frac{\tilde{\Delta}_{(x;q^\alpha,q^\beta)}f(x) \cdot g(x^{q^\alpha}) - f(x^{q^\alpha}) \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)}g(x)}{g(x^{q^\alpha})g(x^{q^\beta})}.$$

For the $\tilde{q}$-chain rule, by computing directly, we have

$$\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)}f(g(x)) &= \frac{f(g(x^{q^\alpha})) - f(g(x^{q^\beta}))}{x^{q^\alpha} - x^{q^\beta}} \\
&= \frac{f(g(x^{q^\alpha})) - f(g(x^{q^\beta}))}{g(x^{q^\alpha}) - g(x^{q^\beta})} \cdot \frac{g(x^{q^\alpha}) - g(x^{q^\beta})}{x^{q^\alpha} - x^{q^\beta}} \\
&= \frac{f(g^{q'^\omega}(x)) - f(g^{q'^\mu}(x))}{g^{q'^\omega}(x) - g^{q'^\mu}(x)} \cdot \frac{g(x^{q^\alpha}) - g(x^{q^\beta})}{x^{q^\alpha} - x^{q^\beta}} \\
&= \tilde{\Delta}_{(g(x);q'^\omega,q'^\mu)}f(g(x)) \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)}g(x)
\end{aligned}$$

if there are q', ω, μ such that

$$g^{q'^\omega}(x) = g(x^{q^\alpha})$$

and

$$g^{q'^\mu}(x) = g(x^{q^\beta}).$$

This formula, of course, is not regorous! However, it is enough for our argument in later chapters.

2.2 x^n: A new approach

As we state in the beginning of Chapter 1, most functions are original from the simple function x^n. When we act the $\tilde{q}$-differential operator on x^n, however, there is no more surprising than we expected. Of course, it is also indifferent to any other functions we were familiar with. This inspired us to try an absurd way: Why don't we change these functions? More specifically, why don't we change x^n?

A simple and conceivable generalization for x^n is of the form $x^{1+q+q^2+\ldots+q^{n-1}}$ or more general

$$x^{q^{(n-1)\alpha}+q^{(n-2)\alpha+\beta}+\ldots+q^{\alpha+(n-2)\beta}+q^{(n-1)\beta}}.$$

For simplicity, we use the notaions $[n]_q := \frac{1-q^n}{1-q}$, $[n]_{(q^\alpha,q^\beta)} := \frac{q^{n\alpha}-q^{n\beta}}{q^\alpha-q^\alpha}$ to rewrite them by $x^{[n]_q}$, $x^{[n]_{(q^\alpha,q^\beta)}}$, respectively. Then we have the following results when we act $\tilde{q}$-differential operator:

$$\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)}x^{[n]_{(q^\alpha,q^\beta)}} &= \tilde{\Delta}_{(x;q^\alpha,q^\beta)}x^{q^{(n-1)\alpha}+q^{(n-2)\alpha+\beta}+\ldots+q^{\alpha+(n-2)\beta}+q^{(n-1)\beta}} \\
&= \frac{x^{q^{n\alpha}+q^{(n-1)\alpha+\beta}+\ldots+q^{\alpha+(n-1)\beta}} - x^{q^{(n-1)\alpha+\beta}+\ldots+q^{\alpha+(n-1)\beta}+q^{n\beta}}}{x^{q^\alpha}-x^{q^\beta}} \\
&= \frac{(x^{q^{n\alpha}}-x^{q^{n\beta}})\cdot x^{(q^{\alpha+\beta})(q^{(n-2)\alpha}+q^{(n-3)\alpha+\beta}+\ldots+q^{\alpha+(n-3)\beta}+q^{(n-2)\beta})}}{x^{q^\alpha}-x^{q^\beta}} \\
&= (\frac{x^{q^{n\alpha}}-x^{q^{n\beta}}}{x^{q^\alpha}-x^{q^\beta}})\cdot x^{(q^{\alpha+\beta})[n-1]_{(q^\alpha,q^\beta)}} \\
&= [\![n]\!]_{(x;q^\alpha,q^\beta)}\cdot x^{(q^{\alpha+\beta})[n-1]_{(q^\alpha,q^\beta)}},
\end{aligned}$$

where

$$[\![n]\!]_{(x;q^\alpha,q^\beta)} := \frac{x^{q^{n\alpha}}-x^{q^{n\beta}}}{x^{q^\alpha}-x^{q^\beta}} = \frac{x^{(q^{n\alpha}-1)}-x^{(q^{n\beta}-1)}}{x^{(q^\alpha-1)}-x^{(q^\beta-1)}}.$$

Take $\alpha = 0$ and $\beta = 1$, it reduces to

$$\begin{aligned}
\tilde{\Delta}_{(x;q)} x^{[n]_q} &= \tilde{\Delta}_{(x;q)} x^{1+q+q^2+\ldots+q^{n-1}} \\
&= (\frac{1-x^{q^n-1}}{1-x^{q-1}}) x^{q(1+q+q^2+\ldots+q^{n-2})} \\
&= [\![n]\!]_{(x;q)} x^{q[n-1]_q},
\end{aligned}$$

where

$$[\![n]\!]_{(x;q)} := \frac{1-x^{q^n-1}}{1-x^{q-1}} = \frac{1-(x^{q-1})^{[n]_q}}{1-x^{q-1}}.$$

For any ω, it is intuitive to generalize x^ω to

$$x^{[\omega]_q} = x^{(\frac{1-q^\omega}{1-q})}$$

or more generally

$$x^{[\omega]_{(q^\alpha,q^\beta)}} = x^{(\frac{q^{\omega\alpha}-q^{\omega\beta}}{q^\alpha-q^\beta})},$$

and

$$\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)} x^{[\omega]_{(q^\alpha,q^\beta)}} &= \tilde{\Delta}_{(x;q^\alpha,q^\beta)} x^{(\frac{q^{\omega\alpha}-q^{\omega\beta}}{q^\alpha-q^\beta})} \\
&= \frac{x^{q^\alpha(\frac{q^{\omega\alpha}-q^{\omega\beta}}{q^\alpha-q^\beta})} - x^{q^\beta(\frac{q^{\omega\alpha}-q^{\omega\beta}}{q^\alpha-q^\beta})}}{x^{q^\alpha}-x^{q^\beta}} \\
&= \frac{x^{(\frac{q^{(\omega+1)\alpha}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})} - x^{\frac{q^{\omega\alpha+\beta}-q^{(\omega+1)\beta}}{q^\alpha-q^\beta}}}{x^{q^\alpha}-x^{q^\beta}} \\
&= \frac{x^{q^{\omega\alpha}} x^{(\frac{q^{(\omega+1)\alpha}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta}-q^{\omega\alpha})} - x^{q^{\omega\beta}} x^{(\frac{q^{\omega\alpha+\beta}-q^{(\omega+1)\beta}}{q^\alpha-q^\beta}-q^{\omega\beta})}}{x^{q^\alpha}-x^{q^\beta}} \\
&= \frac{x^{q^{\omega\alpha}} x^{(\frac{q^{\omega\alpha+\beta}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})} - x^{q^{\omega\beta}(\frac{q^{\omega\alpha+\beta}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})}}{x^{q^\alpha}-x^{q^\beta}} \\
&= (\frac{x^{q^{\omega\alpha}}-x^{q^{\omega\beta}}}{x^{q^\alpha}-x^{q^\beta}}) \cdot x^{q^{\alpha+\beta}(\frac{q^{(\omega-1)\alpha}-q^{(\omega-1)\beta}}{q^\alpha-q^\beta})} \\
&= [\![\omega]\!]_{(x;q^\alpha,q^\beta)} \cdot x^{q^{\alpha+\beta}[\omega-1]_{(q^\alpha,q^\beta)}},
\end{aligned}$$

where

$$[\![\omega]\!]_{(x;q^\alpha,q^\beta)} := \frac{x^{q^{\omega\alpha}} - x^{q^{\omega\beta}}}{x^{q^\alpha} - x^{q^\beta}} = \frac{x^{(q^{\omega\alpha}-1)} - x^{(q^{\omega\beta}-1)}}{x^{(q^\alpha-1)} - x^{(q^\beta-1)}}.$$

Similarly,

$$\tilde{\Delta}_{(x;q)} x^{[\omega]_q} = \tilde{\Delta}_{(x;q)} x^{(\frac{1-q^\omega}{1-q})} = (\frac{1-x^{q^\omega-1}}{1-x^{q-1}}) x^{q(\frac{1-q^{\omega-1}}{1-q})} = [\![\omega]\!]_{(x;q)} x^{q[\omega-1]_q},$$

where

$$[\![\omega]\!]_{(x;q)} := \frac{1-x^{q^\omega-1}}{1-x^{q-1}} = \frac{1-(x^{q-1})^{[\omega]_q}}{1-x^{q-1}}.$$

Our objective in this section is nearly achieved:

$$\tilde{\Delta}_q(x^{[\omega]}) = [\![\omega]\!] \cdot x^{q[\omega-1]}. \tag{2.2.1}$$

The $\tilde{q}$-differential operator $\tilde{\Delta}_q$ is consistent with $[\![\omega]\!]$ and $x^{[\omega]}$, the $\tilde{q}$-analogue of ω and x^ω, respectively. Here we omit all the indices for clearity and use (2.2.1) to denote the two cases for single q and double q^α, q^β. Note also that the factor $[\![\omega]\!]$ is a function of x and q, not a "constant" anymore.

Another interesting attempt is to construct the $\tilde{q}$-analogue of "$\frac{d}{dx}(\frac{1}{x^n}) = -\frac{1}{x^{n+1}}$". We consider the relation between $\tilde{\Delta}_q$ and $(\frac{1}{x})^{[n]}$, and use the $\tilde{q}$-chain rule and (2.2.1) to justify our following argument:

$$\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)}(\frac{1}{x})^{[n]_{(q^\alpha,q^\beta)}} &= \tilde{\Delta}_{(\frac{1}{x};q^\alpha,q^\beta)}(\frac{1}{x})^{q^{\alpha+\beta}[n]_{(q^\alpha,q^\beta)}} \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)}(\frac{1}{x}) \\
&= [\![n]\!]_{(\frac{1}{x};q^\alpha,q^\beta)} \cdot (\frac{1}{x})^{q^{\alpha+\beta}[n-1]_{(q^\alpha,q^\beta)}} \cdot (\frac{\frac{1}{x^{q^\alpha}} - \frac{1}{x^{q^\beta}}}{x^{q^\alpha} - x^{q^\alpha}}) \\
&= (\frac{(\frac{1}{x})^{q^{n\alpha}} - (\frac{1}{x})^{q^{n\beta}}}{(\frac{1}{x})^{q^\alpha} - (\frac{1}{x})^{q^\beta}}) \cdot \\
&\quad (\frac{1}{x})^{q^{(n-1)\alpha+\beta}+\ldots+q^{\alpha+(n-1)\beta}} \cdot (-1)(\frac{1}{x})^{q^\alpha+q^\beta}
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{x^{q^{n\beta}} - x^{q^{n\alpha}}}{x^{q^\beta} - x^{q^\alpha}}) \cdot (\frac{1}{x})^{q^{n\alpha}+q^{n\beta}-q^\alpha-q^\beta} \cdot \\
&\quad (\frac{1}{x})^{q^{(n-1)\alpha+\beta}+\ldots+q^{\alpha+(n-1)\beta}} \cdot (-1)(\frac{1}{x})^{q^\alpha+q^\beta} \\
&= -(\frac{x^{q^{n\alpha}} - x^{q^{n\beta}}}{x^{q^\alpha} - x^{q^\beta}}) \cdot (\frac{1}{x})^{q^{n\alpha}+q^{(n-1)\alpha+\beta}+\ldots+q^{\alpha+(n-1)\beta}+q^{n\beta}} \\
&= -\llbracket n \rrbracket_{(x;q^\alpha,q^\beta)} \cdot (\frac{1}{x})^{[n+1]_{(q^\alpha,q^\beta)}},
\end{aligned}
$$

or another equivalent representation

$$\tilde{\Delta}_{(x;q^\alpha,q^\beta)} x^{-[n]_{(q^\alpha,q^\beta)}} = -\llbracket n \rrbracket_{(x;q^\alpha,q^\beta)} \cdot x^{-[n+1]_{(q^\alpha,q^\beta)}}.$$

For single q case, it is easier to check that

$$\tilde{\Delta}_{(x;q)} x^{-[n]_q} = -\llbracket n \rrbracket_{(x;q)} x^{-[n+1]_q}$$

if we take $\alpha = 0$ and $\beta = 1$ in the previous argument. For any ω, we have

$$
\begin{aligned}
\tilde{\Delta}_{(x;q^\alpha,q^\beta)} (\frac{1}{x})^{[\omega]_{(q^\alpha,q^\beta)}} &= \tilde{\Delta}_{(\frac{1}{x};q^\alpha,q^\beta)} (\frac{1}{x})^{[\omega]_{(q^\alpha,q^\beta)}} \cdot \tilde{\Delta}_{(x;q^\alpha,q^\beta)} (\frac{1}{x}) \\
&= \llbracket \omega \rrbracket_{(\frac{1}{x};q^\alpha,q^\beta)} \cdot (\frac{1}{x})^{q^{\alpha+\beta}[\omega-1]_{(q^\alpha,q^\beta)}} \cdot (-1)(\frac{1}{x})^{q^\alpha+q^\beta} \\
&= (\frac{(\frac{1}{x})^{q^{\omega\alpha}} - (\frac{1}{x})^{q^{\omega\beta}}}{(\frac{1}{x})^{q^\alpha} - (\frac{1}{x})^{q^\alpha}}) \cdot (\frac{1}{x})^{(\frac{q^{\omega\alpha+\beta}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})} \cdot (-1)(\frac{1}{x})^{q^\alpha+q^\beta} \\
&= (\frac{x^{q^{\omega\alpha}} - x^{q^{\omega\beta}}}{x^{q^\alpha} - x^{q^\alpha}}) \cdot (\frac{1}{x})^{q^{\omega\alpha}+q^{\omega\beta}-q^\alpha-q^\beta} \cdot \\
&\quad (\frac{1}{x})^{(\frac{q^{\omega\alpha+\beta}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})} \cdot (-1)(\frac{1}{x})^{q^\alpha+q^\beta} \\
&= -(\frac{x^{q^{\omega\alpha}} - x^{q^{\omega\beta}}}{x^{q^\alpha} - x^{q^\alpha}}) \cdot (\frac{1}{x})^{q^{\omega\alpha+}+q^{\omega\beta}+(\frac{q^{\omega\alpha+\beta}-q^{\alpha+\omega\beta}}{q^\alpha-q^\beta})} \\
&= -(\frac{x^{q^{\omega\alpha}} - x^{q^{\omega\beta}}}{x^{q^\alpha} - x^{q^\alpha}}) \cdot (\frac{1}{x})^{(\frac{q^{(\omega+1)\alpha}-q^{(\omega+1)\beta}}{q^\alpha-q^\beta})} \\
&= -\llbracket \omega \rrbracket_{(x;q^\alpha,q^\beta)} \cdot (\frac{1}{x})^{[\omega+1]_{(q^\alpha,q^\beta)}},
\end{aligned}
$$

or another equivalent representaion

$$\tilde{\Delta}_{(x;q^{\alpha},q^{\beta})}x^{-[\omega]_{(q^{\alpha},q^{\beta})}} = -[\![\omega]\!]_{(x;q^{\alpha},q^{\beta})} \cdot x^{-[\omega+1]_{(q^{\alpha},q^{\beta})}},$$

and the single q case is

$$\tilde{\Delta}_{(x;q)}x^{-[\omega]_q} = -[\![\omega]\!]_{(x;q)}x^{-[\omega+1]_q}.$$

Now, we obtain another self-consistent relation between $\tilde{\Delta}_q$, $-[\![\omega]\!]$, and $x^{-[\omega]}$:

$$\tilde{\Delta}_q(x^{-[\omega]}) = -[\![\omega]\!] \cdot x^{-[\omega+1]} \tag{2.2.2}$$

Notice also that the algebraic structure of (2.2.2) is quite different from (2.2.1) in nature since an additional factor "q "appeared in the power of x in right-hand side of (2.2.1), but it does't in (2.2.2) instead.

Actually, (2.2.2) inspired us to construct another version of $\tilde{q}$-analogue of "$\frac{d}{dx}(x^{\omega}) = \omega x^{\omega-1}$ "with no additional "q "in the power of x:

$$\tilde{\Delta}_q(x^{-[-\omega]}) = -[\![-\omega]\!] \cdot x^{-[-\omega+1]}. \tag{2.2.3}$$

This formula seems to be more artifical than (2.2.1) for the first look. However, they are two reciprocal representaions of classical $\frac{d}{dx}(x^{\omega}) = \omega x^{\omega-1}$ inherently. We will give some reasons in 2.3 to convince the readers of our opinions.

2.3 $\tilde{\mathrm{E}}_q^x$, $\tilde{\varepsilon}_q^x$: Incomplete discussions for Taylor's expansion approach

As a result of $\frac{d^n}{dx^n}x^n = n!$ and $\Delta_q^n x^n = [n]_q[n-1]_q[n-2]_q...[2]_q[1]_q = [n]_q!$, we have some idea to extend the n-factorial to $\tilde{q}$-analogue. One way, but not quite right, is that

$$[\![n]\!]^+_{(x;q)}! := [\![n]\!]_{(x;q)}[\![n-1]\!]_{(x^q;q)}[\![n-2]\!]_{(x^{q^2};q)}...[\![2]\!]_{(x^{q^{n-2}};q)}[\![1]\!]_{(x^{q^{n-1}};q)}, \quad (2.3.1)$$

where

$$[\![n-i]\!]_{(x^{q^i};q)} := \frac{1-x^{q^i(q^{n-i}-1)}}{1-x^{q^i(q-1)}}.$$

The idea is direct from the consequences:

$$\tilde{\Delta}_{(x;q)}x^{1+q+q^2+...+q^{n-1}} = (\frac{1-x^{q^n-1}}{1-x^{q-1}})x^{q(1+q+q^2+...+q^{n-2})},$$

$$\tilde{\Delta}_{(x^q;q)}x^{q(1+q+q^2+...+q^{n-2})} = (\frac{1-x^{q(q^{n-1}-1)}}{1-x^{q(q-1)}})x^{q^2(1+q+q^2+...+q^{n-3})},$$

$$\tilde{\Delta}_{(x^{q^2};q)}x^{q^2(1+q+q^2+...+q^{n-3})} = (\frac{1-x^{q^2(q^{n-2}-1)}}{1-x^{q^2(q-1)}})x^{q^3(1+q+q^2+...+q^{n-4})},$$

$$\vdots$$

$$\tilde{\Delta}_{(x^{q^i};q)}x^{q^i(1+q+q^2+...+q^{n-i-1})} = (\frac{1-x^{q^i(q^{n-i}-1)}}{1-x^{q^i(q-1)}})x^{q^{i+1}(1+q+q^2+...+q^{n-i-2})},$$

$$\vdots$$

It is intuitive to define an e^x of $\tilde{q}$-analogue as the Taylor's expansion

$$\tilde{\mathrm{E}}_q^x := 1 + \sum_{n=1}^{\infty}\frac{1}{[\![n]\!]^+_{(x;q)}!}x^{1+q+q^2+...+q^{n-1}}. \quad (2.3.2)$$

Then we have the following consequences by the above differential laws and $\tilde{q}$-product rule of first kind from section 2.1:

$$\begin{aligned}
\tilde{\Delta}_{(x;q)}\tilde{\mathrm{E}}_q^x &= \tilde{\Delta}_{(x;q)}(1+\sum_{n=1}^{\infty}\frac{1}{[\![n]\!]_{(x;q)}^{+}!}x^{1+q+q^2+\ldots+q^{n-1}}) \\
&= \sum_{n=1}^{\infty}\tilde{\Delta}_{(x;q)}(\frac{1}{[\![n]\!]_{(x;q)}^{+}!}x^{1+q+q^2+\ldots+q^{n-1}}) \\
&= \sum_{n=1}^{\infty}(\frac{1}{[\![n]\!]_{(x;q)}^{+}!}\tilde{\Delta}_{(x;q)}x^{1+q+q^2+\ldots+q^{n-1}}) \\
&\quad +\sum_{n=1}^{\infty}(x^{q(1+q+q^2+\ldots+q^{n-1})}\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]_{(x;q)}^{+}!}) \\
&= 1+\sum_{n=2}^{\infty}\frac{[\![n]\!]_{(x;q)}}{[\![n]\!]_{(x;q)}^{+}!}x^{q(1+q+q^2+\ldots+q^{n-2})} \\
&\quad +\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]_{(x;q)}^{+}!})x^{q(1+q+q^2+\ldots+q^{n-1})} \\
&= 1+\sum_{n=2}^{\infty}\frac{1}{[\![n-1]\!]_{(x^q;q)}^{+}!}x^{q(1+q+q^2+\ldots+q^{n-2})} \\
&\quad +\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]_{(x;q)}^{+}!})x^{q(1+q+q^2+\ldots+q^{n-1})} \\
&= 1+\sum_{n=1}^{\infty}\frac{1}{[\![n]\!]_{(x^q;q)}^{+}!}x^{q(1+q+q^2+\ldots+q^{n-1})} \\
&\quad +\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]_{(x;q)}^{+}!})x^{q(1+q+q^2+\ldots+q^{n-1})} \\
&= \tilde{\mathrm{E}}_q^{x^q}+\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]_{(x;q)}^{+}!})x^{q(1+q+q^2+\ldots+q^{n-1})}.
\end{aligned}$$

Similarly,

$$\tilde{\Delta}_{(x^q;q)}\tilde{\mathrm{E}}_q^{x^q}=\tilde{\mathrm{E}}_q^{x^{q^2}}+\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x^q;q)}\frac{1}{[\![n]\!]_{(x^q;q)}^{+}!})x^{q^2(1+q+q^2+\ldots+q^{n-1})},$$

$$\tilde{\Delta}_{(x^{q^2};q)}\tilde{\mathrm{E}}_q^{x^{q^2}} = \tilde{\mathrm{E}}_q^{x^{q^3}} + \sum_{n=1}^{\infty}(\tilde{\Delta}_{(x^{q^2};q)}\frac{1}{[\![n]\!]^+_{(x^{q^2};q)}!})x^{q^3(1+q+q^2+...+q^{n-1})},$$

$$\vdots$$

$$\tilde{\Delta}_{(x^{q^i};q)}\tilde{\mathrm{E}}_q^{x^{q^i}} = \tilde{\mathrm{E}}_q^{x^{q^{i+1}}} + \sum_{n=1}^{\infty}(\tilde{\Delta}_{(x^{q^i};q)}\frac{1}{[\![n]\!]^+_{(x^{q^i};q)}!})x^{q^{i+1}(1+q+q^2+...+q^{n-1})},$$

$$\vdots$$

Another $n!$ of $\tilde{q}$-analogue is from the consequences of (2.2.3):

$$\tilde{\Delta}_{(x;q)}(x^{-[-n]_q}) = -[\![-n]\!]_{(x;q)} \cdot x^{-[-n+1]_q}$$

$$\tilde{\Delta}_{(x;q)}(x^{-[-n+1]_q}) = -[\![-n+1]\!]_{(x;q)} \cdot x^{-[-n+2]_q}$$

$$\vdots$$

$$\tilde{\Delta}_{(x;q)}(x^{-[-n+i]_q}) = -[\![-n+i]\!]_{(x;q)} \cdot x^{-[-n+(i+1)]_q}$$

$$\vdots$$

and we define

$$[\![n]\!]^-_{(x;q)}! := (-1)^n[\![-n]\!]_{(x;q)}[\![-(n-1)]\!]_{(x;q)}[\![-(n-2)]\!]_{(x;q)}...[\![-2]\!]_{(x;q)}[\![-1]\!]_{(x;q)}. \tag{2.3.3}$$

Then the corresponding Taylor's expansion for e^x of $\tilde{q}$-analogue is

$$\tilde{\varepsilon}_q^x := 1 + \sum_{n=1}^{\infty}\frac{1}{[\![n]\!]^-_{(x;q)}!}x^{-[-n]_{(x;q)}} = 1 + \sum_{n=1}^{\infty}\frac{1}{[\![n]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+...+\frac{1}{q^n}}, \tag{2.3.4}$$

where

$$-[-n]_q = -(\frac{1-q^{-n}}{1-q}) = \frac{1}{q^n}(\frac{1-q^n}{1-q}) = \frac{1}{q} + \frac{1}{q^2} + ... + \frac{1}{q^n}.$$

The relation between the $\tilde{q}$-differential operator and (2.3.4) is

$$
\begin{aligned}
\tilde{\Delta}_{(x;q)}\tilde{\varepsilon}_q^x &= \tilde{\Delta}_{(x;q)}(1+\sum_{n=1}^{\infty}\frac{1}{[\![n]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^n}}) \\
&= \sum_{n=1}^{\infty}\tilde{\Delta}_{(x;q)}(\frac{1}{[\![n]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^n}}) \\
&= \sum_{n=1}^{\infty}(\frac{1}{[\![n]\!]^-_{(x;q)}!}\cdot\tilde{\Delta}_{(x;q)}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^n}}) \\
&\quad +\sum_{n=1}^{\infty}(x^{q(\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^n})}\cdot\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]^-_{(x;q)}!}) \\
&= 1+\sum_{n=2}^{\infty}\frac{-[\![-n]\!]_{(x;q)}}{[\![n]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}} \\
&\quad +\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]^-_{(x;q)}!})x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}} \\
&= 1+\sum_{n=2}^{\infty}\frac{1}{[\![n-1]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}} \\
&\quad +\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]^-_{(x;q)}!})x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}} \\
&= 1+\sum_{n=1}^{\infty}\frac{1}{[\![n]\!]^-_{(x;q)}!}x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n}}} \\
&\quad +\sum_{n=1}^{\infty}+(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]^-_{(x;q)}!})x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}} \\
&= \tilde{\varepsilon}_q^x+\sum_{n=1}^{\infty}(\tilde{\Delta}_{(x;q)}\frac{1}{[\![n]\!]^-_{(x;q)}!})x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n-1}}}.
\end{aligned}
$$

Now, we obtain two different relations between $\tilde{\mathrm{E}}_q^x$, $\tilde{\varepsilon}_q^x$, and $\tilde{\Delta}_{(x,q)}$:

$$\tilde{\Delta}_{(x;q)}\tilde{\mathrm{E}}_q^x \approx \tilde{\mathrm{E}}_q^{x^q} \tag{2.3.5}$$

and

$$\tilde{\Delta}_{(x;q)}\tilde{\varepsilon}_q^x \approx \varepsilon_q^x \tag{2.3.6}$$

if we ignore the mixed terms. Indeed, the analogues of "$\frac{d}{dx}(e^x) = e^x$"in $\tilde{q}$-Calculus is not completely made attempt on since the terms $\tilde{\Delta}_{(xq^i,q)}\frac{1}{[\![n]\!]^+_{(xq^i,q)}!}$ for $\tilde{\mathrm{E}}_q^x$, and the terms $\tilde{\Delta}_{(x,q)}\frac{1}{[\![n]\!]^-_{(x,q)}!}$ for $\tilde{\varepsilon}_q^x$ are still preserved. However, the corresponding terms in classical and q-Calculus would be cancelled for that n, $[n]_q$ are constant and $(\frac{1}{n!})' = 0$, $\Delta_q\frac{1}{[n]_q!} = 0$. As a result, these expressions are not nice enough to our philosophy. We will discuss two nicer forms in 3.2.

Chapter 3

$\tilde{q}$-infinite products and $\tilde{q}$-infinite sums

e^x, the simplest and well-known transcendental function, is given by the following two products $(1+\frac{x}{n})^n$ and $\frac{1}{(1-\frac{x}{n})^n}$ when n tends to infinity. When we consider the q-analogue, the corresponding infinite products of E_q^x and e_q^x are $\prod_{n=0}^{\infty}(1+\frac{q^n x}{[n]})$ and $\prod_{n=0}^{\infty}\frac{1}{(1-\frac{q^n x}{[n]})}$, respectively. An immediate attempt made is to find the corresponding $\tilde{q}$-infinite products for e^x of $\tilde{q}$-analogue .

A short review, related q-infinite products to q-infinite sums for E_q^x and e_q^x, is given in 3.1. $\tilde{E}_q^x$, $\tilde{e}_q^x$, infinite product representations for e^x of $\tilde{q}$-analogue, and the coresponding nice properties are discussed in 3.2. Two recurrences for ‟sum part ″of $\tilde{E}_q^x$ and $\tilde{e}_q^x$ with some incomplete discussions are considered in 3.3. Unsolved problems are listed in 3.4.

3.1 E_q^x, e_q^x: A rough review

First of all, we introduce two identities from [14], the Gauss's binomial formula

$$\prod_{j=0}^{n-1}(1+q^j x) = \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array} \right] q^{1+2+...+(n-1)} x^j,$$

where

$$\left[\begin{array}{c} n \\ j \end{array} \right] := \frac{[n]!}{[j]![n-j]!},$$

and the Heine's binomial formula

$$\prod_{j=0}^{n-1} \frac{1}{(1-q^j x)} = 1 + \sum_{j=1}^{\infty} \frac{[n][n+1]...[n+j-1]}{[j]!} x^j,$$

which is the q-analogue of Taylor's expansion of $\frac{1}{(1-x)^n}$ in elementary Calculus.

It is not difficult to verify that the Gauss's and Heine's binomial formulas become two different q-analogue of e^x when replacing x by $x/[n]$ and letting $n \to \infty$:

$$\begin{aligned} \prod_{j=0}^{n-1}(1+q^j \frac{x}{[n]}) &= \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array} \right] q^{1+2+...+(n-1)} (\frac{x}{[n]})^j \\ &= \sum_{j=0}^{n} \frac{1}{[j]!} \frac{[n][n-1]...[n-j+1]}{[n]^j} q^{1+2+...+(n-1)} x^j \\ &= \sum_{j=0}^{n} \frac{1}{[j]!} \{(\frac{[n-1]}{[n]})(\frac{[n-2]}{[n]})...(\frac{[n-j+1]}{[n]})\} q^{1+2+...+(n-1)} x^j, \end{aligned}$$

which implies that

$$\prod_{n=0}^{\infty}(1+(1-q)q^n x) = \sum_{j=0}^{\infty} \frac{1}{[j]!} q^{1+2+...+(n-1)} x^j := E_q^x, \qquad (3.1.1)$$

and

$$\prod_{j=0}^{n-1}\frac{1}{(1-q^j\frac{x}{[n]})}=1+\sum_{j=1}^{\infty}\frac{[n][n+1]...[n+j-1]}{[j]!}(\frac{x}{[n]})^j$$
$$=1+\sum_{j=1}^{\infty}\frac{1}{[j]!}\{(\frac{[n+1]}{[n]})(\frac{[n+2]}{[n]})...(\frac{[n+j-1]}{[n]})\}x^j,$$

which implies that

$$\prod_{n=0}^{\infty}\frac{1}{(1-(1-q)q^n x)}=\sum_{j=0}^{\infty}\frac{1}{[j]!}x^j:=e_q^x. \tag{3.1.2}$$

On the other hand, E_q^x, e_q^x also appear when we replace x by $(1-q)x$ in the two identities of Euler

$$\prod_{n=0}^{\infty}(1+q^n x)=\sum_{m=0}^{\infty}q^{1+2+...+(m-1)}\frac{x^m}{(1-q)(1-q^2)...(1-q^m)},$$

$$\prod_{n=0}^{\infty}(\frac{1}{1-q^n x})=\sum_{m=0}^{\infty}\frac{x^m}{(1-q)(1-q^2)...(1-q^m)}.$$

Interestingly, the two identities, relate infinite products to infinite sums, were discovered by Euler who lived before Gauss and Heine. However, they would be derivated from Gauss's and Heine's binomial formulas when n tends to infinity.

A simple extension of Euler's identities is to find the infinite sums corresponding to $\prod_{n=0}^{\infty}(1+x^{q^n})$ and $\prod_{n=0}^{\infty}(\frac{1}{1-x^{q^n}})$. Computing the first product finite times, one shall obtain a formula of the form:

$$\prod_{j=0}^{n-1}(1+x^{q^j})=1+\sum_{j=1}^{n}\left[\!\!\left[\begin{matrix} n \\ j \end{matrix}\right]\!\!\right]_{(x,q)}x^{1+q+q^2+...+q^{j-1}},$$

where the "coefficients" $\left[\!\left[\begin{matrix} n \\ j \end{matrix} \right]\!\right]_{(x,q)}$ are functions of variable x with parameter q and satisfy the $\tilde{q}$-Pascal rule, manely,

$$\left[\!\left[\begin{matrix} n \\ j \end{matrix} \right]\!\right]_{(x,q)} = x^{q^{n-1}-q^{j-1}} \left[\!\left[\begin{matrix} n-1 \\ j-1 \end{matrix} \right]\!\right]_{(x,q)} + \left[\!\left[\begin{matrix} n-1 \\ j \end{matrix} \right]\!\right]_{(x,q)},$$

and $\left[\!\left[\begin{matrix} n \\ 0 \end{matrix} \right]\!\right]_{(x,q)} := 1$. However, it seems hard to express $\left[\!\left[\begin{matrix} n \\ j \end{matrix} \right]\!\right]_{(x,q)}$ by an explicit formula like the classical $\left(\begin{matrix} n \\ j \end{matrix} \right) = \frac{n!}{j!(n-j)!}$. For this reason, our destination, finding a $\tilde{q}$-analogue of Gauss's (or Heine's) binomial formula and then letting n tends to infinite to obtain a $\tilde{q}$-analogue of Euler identity, is failed due to our poor knowledge. In other words, to find the e^x of $\tilde{q}$-analogues seems to be impossible if we just rely on the wisdom of the predecessors.

3.2 $\tilde{E}_q^x$, $\tilde{e}_q^x$: Infinite product approach

The product representations for e^x of $\tilde{q}$-analogue that we found are of the following two forms:

$$\tilde{E}_q^x := \prod_{n=0}^{\infty} (1 + (1 - x^{q^n(q-1)})x^{q^n}) \tag{3.2.1}$$

and

$$\tilde{e}_q^x := \prod_{n=0}^{\infty} \frac{1}{(1 - (1 - x^{q^n(q-1)})x^{q^n})}. \tag{3.2.2}$$

It is clear that (3.2.1) and (3.2.2) are $\tilde{q}$-generalizations of "product part" corresponding to (3.1.1) and (3.1.2), respectively. For convenience, we use the following equivalent representaions from now on:

$$\tilde{E}_q^x = (1 + (x - x^q))(1 + (x^q - x^{q^2}))(1 + (x^{q^2} - x^{q^3}))..., \tag{3.2.3}$$

$$\tilde{e}_q^x = \frac{1}{(1-(x-x^q))(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))...}. \tag{3.2.4}$$

It is easy to check that $\tilde{E}_q^x$ and $\tilde{e}_q^x$ inherit the nice differential property of classical, that is, $(e^x)' = e^x$:

$$\begin{aligned}
&\tilde{\Delta}_{(x;q)}\tilde{E}_q^x \\
&= \frac{(1+(x-x^q))(1+(x^q-x^{q^2}))... - (1+(x^q-x^{q^2}))(1+(x^{q^2}-x^{q^3}))...}{x-x^q} \\
&= \frac{(1+(x-x^q)-1)}{x-x^q}\{(1+(x^q-x^{q^2}))(1+(x^{q^2}-x^{q^3}))(1+(x^{q^3}-x^{q^4}))...\} \\
&= (1+(x^q-x^{q^2}))(1+(x^{q^2}-x^{q^3}))(1+(x^{q^3}-x^{q^4}))... \\
&= \tilde{E}_q^{x^q}
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\Delta}_{(x;q)}\tilde{e}_q^x \\
&= \frac{\frac{1}{(1-(x-x^q))(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))...} - \frac{1}{(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))(1-(x^{q^3}-x^{q^4}))...}}{x-x^q} \\
&= (\frac{\frac{1}{1-(x-x^q)}-1}{x-x^q})\{\frac{1}{(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))(1-(x^{q^3}-x^{q^4}))...}\} \\
&= \frac{1}{(1-(x-x^q))(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))...} \\
&= \tilde{e}_q^x.
\end{aligned}$$

For the $\tilde{q}$-analogue of e^{-x}, we define by the following:

$$\begin{aligned}
\frac{1}{\tilde{e}_q^x} &:= \prod_{n=0}^{\infty}(1-(1-x^{q^n(q-1)})x^{q^n}) \\
&= (1-(x-x^q))(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))...
\end{aligned}$$

and

$$\frac{1}{\tilde{E}_q^x} := \prod_{n=0}^{\infty} \frac{1}{(1+(1-x^{q^n(q-1)})x^{q^n})}$$
$$= \frac{1}{(1+(x-x^q))(1+(x^q-x^{q^2}))(1+(x^{q^2}-x^{q^3}))\ldots}.$$

Then the similar arguments are that:

$$\tilde{\Delta}_{(x;q)} \frac{1}{\tilde{e}_q^x} = -\frac{1}{\tilde{e}_q^{x^q}}, \qquad \tilde{\Delta}_{(x;q)} \frac{1}{\tilde{E}_q^x} = -\frac{1}{\tilde{E}_q^x}.$$

It is intuitive to define the $\tilde{q}$-analogue of $\sinh x$ and $\cosh x$ by the following:

$$\widetilde{Sinh}_q x := \frac{1}{2}(\tilde{E}_q^x - \frac{1}{\tilde{e}_q^x}), \qquad \widetilde{Cosh}_q x := \frac{1}{2}(\tilde{E}_q^x + \frac{1}{\tilde{e}_q^x}),$$

$$\widetilde{sinh}_q x := \frac{1}{2}(\tilde{e}_q^x - \frac{1}{\tilde{E}_q^x}), \qquad \widetilde{cosh}_q x := \frac{1}{2}(\tilde{e}_q^x + \frac{1}{\tilde{E}_q^x}).$$

Then we have similar arguments as classical by applying the $\tilde{q}$-differential rules above:

$$\tilde{\Delta}_{(x;q)} \widetilde{Sinh}_q x = \widetilde{Cosh}_q x^q, \qquad \tilde{\Delta}_{(x;q)} \widetilde{Cosh}_q x = \widetilde{Sinh}_q x^q,$$

$$\tilde{\Delta}_{(x;q)} \widetilde{sinh}_q x = \widetilde{cosh}_q x, \qquad \tilde{\Delta}_{(x;q)} \widetilde{cosh}_q x = \widetilde{sinh}_q x.$$

A nice formula also be found by direct computation,

$$(\widetilde{Cosh}_q x)(\widetilde{cosh}_q x) - (\widetilde{Sinh}_q x)(\widetilde{sinh}_q x) = 1.$$

For $\sin x$ and $\cos x$, unfortunately, we have no nice formulas like $\sinh x$ and $\cosh x$. More precisely, the representaions

$$\widetilde{Sin}_q x := \frac{1}{2i}(\tilde{E}_q^{ix} - \frac{1}{\tilde{e}_q^{ix}}), \qquad \widetilde{Cos}_q x := \frac{1}{2}(\tilde{E}_q^{ix} + \frac{1}{\tilde{e}_q^{ix}}),$$

$$\widetilde{sin}_q x := \frac{1}{2i}(\tilde{e}_q^{ix} - \frac{1}{\tilde{E}_q^{ix}}), \widetilde{cos}_q x := \frac{1}{2}(\tilde{e}_q^{ix} + \frac{1}{\tilde{E}_q^{ix}})$$

seem to have no nice properties that we have is last paragraph. We use another way to define $\tilde{q}$-analogue of $\sin x$ and $\cos x$ as follow:

$$\widetilde{Sin}_q x := \frac{1}{2i}\{\prod_{n=0}^{\infty}(1 + i(1 - x^{q^n(q-1)})x^{q^n}) - \prod_{n=0}^{\infty}(1 - i(1 - x^{q^n(q-1)})x^{q^n})\},$$

$$\widetilde{Cos}_q x := \frac{1}{2}\{\prod_{n=0}^{\infty}(1 + i(1 - x^{q^n(q-1)})x^{q^n}) + \prod_{n=0}^{\infty}(1 - i(1 - x^{q^n(q-1)})x^{q^n})\},$$

$$\widetilde{sin}_q x := \frac{1}{2i}\{\prod_{n=0}^{\infty}\frac{1}{(1 - i(1 - x^{q^n(q-1)})x^{q^n})} - \prod_{n=0}^{\infty}\frac{1}{(1 + i(1 - x^{q^n(q-1)})x^{q^n})}\},$$

$$\widetilde{cos}_q x := \frac{1}{2}\{\prod_{n=0}^{\infty}\frac{1}{(1 - i(1 - x^{q^n(q-1)})x^{q^n})} + \prod_{n=0}^{\infty}\frac{1}{(1 + i(1 - x^{q^n(q-1)})x^{q^n})}\}.$$

It is easy to verify that

$$\tilde{\Delta}_{(x;q)}\prod_{n=0}^{\infty}(1 + i(1 - x^{q^n(q-1)})x^{q^n}) = i\prod_{n=1}^{\infty}(1 + i(1 - x^{q^n(q-1)})x^{q^n}),$$

$$\tilde{\Delta}_{(x;q)}\prod_{n=0}^{\infty}(1 - i(1 - x^{q^n(q-1)})x^{q^n}) = -i\prod_{n=1}^{\infty}(1 - i(1 - x^{q^n(q-1)})x^{q^n}),$$

$$\tilde{\Delta}_{(x;q)}\prod_{n=0}^{\infty}(\frac{1}{1 - i(1 - x^{q^n(q-1)})x^{q^n}}) = i\prod_{n=0}^{\infty}(\frac{1}{1 - i(1 - x^{q^n(q-1)})x^{q^n}}),$$

$$\tilde{\Delta}_{(x;q)}\prod_{n=0}^{\infty}(\frac{1}{1 + i(1 - x^{q^n(q-1)})x^{q^n}}) = -i\prod_{n=0}^{\infty}(\frac{1}{1 + i(1 - x^{q^n(q-1)})x^{q^n}}),$$

and then the properties satisfy:

$$\tilde{\Delta}_{(x;q)}\widetilde{Sin}_q x = \widetilde{Cos}_q x^q, \qquad \tilde{\Delta}_{(x;q)}\widetilde{Cos}_q x = -\widetilde{Sin}_q x^q,$$

$$\tilde{\Delta}_{(x;q)}\widetilde{sin}_q x = \widetilde{cos}_q x, \qquad \tilde{\Delta}_{(x;q)}\widetilde{cos}_q x = -\widetilde{sin}_q x.$$

A nice formula also be found by direct computation,

$$(\widetilde{Cos}_q x)(\widetilde{cos}_q x) + (\widetilde{Sin}_q x)(\widetilde{sin}_q x) = 1.$$

3.3 $\tilde{E}_q^x$, $\tilde{e}_q^x$: Incomplete discussions for infinite sum approach

An interesting problem which proceeds with 3.2 is to find the infinite sums of $\tilde{E}_q^x$, $\tilde{e}_q^x$. It appears that there are some relations between (3.2.1), (3.2.2) and (2.3.2), (2.3.4) since they are inherent in some algebric structures for similarity. More precisely,

$$\tilde{\Delta}_{(x;q)}\tilde{E}_q^x = \tilde{E}_q^{x^q}, \qquad \tilde{\Delta}_{(x;q)}\tilde{\mathrm{E}}_q^x \approx \tilde{\mathrm{E}}_q^{x^q},$$

and

$$\tilde{\Delta}_{(x;q)}\tilde{e}_q^x = \tilde{e}_q^x, \qquad \tilde{\Delta}_{(x;q)}\tilde{\varepsilon}_q^x \approx \varepsilon_q^x.$$

For this reason, an irregular attempt is to set

$$\begin{aligned} \tilde{E}_q^x &= (1 + (x - x^q))(1 + (x^q - x^{q^2}))(1 + (x^{q^2} - x^{q^3}))\ldots \\ &= 1 + \sum_{m=1}^{\infty} a_m(x;q) x^{1+q+q^2+\ldots+q^{m-1}}, \end{aligned} \tag{3.3.1}$$

and $a_0(x;q) := 1$, and then to find the recurrence between the coefficients $a_m(x;q)$. Our argument is the following:

$$\begin{aligned}
\tilde{E}_q^x &= (1+(x-x^q))\tilde{E}_q^{x^q} \\
&= (1+(x-x^q))\{1+\sum_{m=1}^{\infty} a_m(x^q;q)x^{q+q^2+\ldots+q^m}\} \\
&= 1+\sum_{m=1}^{\infty} a_m(x^q;q)x^{q+q^2+\ldots+q^m} \\
&\quad +(1-x^{q-1})\{x+\sum_{m=1}^{\infty} a_m(x^q;q)x^{1+q+q^2+\ldots+q^m}\} \\
&= 1+\sum_{m=1}^{\infty}\{a_m(x^q;q)x^{q^m-1}\}x^{1+q+\ldots+q^{m-1}} \\
&\quad +(1-x^{q-1})\{x+\sum_{m=2}^{\infty} a_{m-1}(x^q;q)x^{1+q+\ldots+q^{m-1}}\} \\
&= 1+\sum_{m=1}^{\infty}\{a_m(x^q;q)x^{q^m-1}\}x^{1+q+\ldots+q^{m-1}} \\
&\quad +(1-x^{q-1})\sum_{m=1}^{\infty} a_{m-1}(x^q;q)x^{1+q+\ldots+q^{m-1}} \\
&= 1+\sum_{m=1}^{\infty}\{a_m(x^q;q)x^{q^m-1}+(1-x^{q-1})a_{m-1}(x^q;q)\}x^{1+q+\ldots+q^{m-1}}
\end{aligned}$$

Comparing with (3.3.1), we obtain a recurrence

$$a_m(x;q) = \begin{cases} a_m(x^q;q)x^{q^m-1}+(1-x^{q-1})a_{m-1}(x^q;q), & m \geq 1 \\ 1, & m=0 \end{cases} \tag{3.3.2}$$

A similar attempt is to set

$$\begin{aligned}
\tilde{e}_q^x &= \frac{1}{(1-(x-x^q))(1-(x^q-x^{q^2}))(1-(x^{q^2}-x^{q^3}))\ldots} \\
&= 1+\sum_{m=1}^{\infty} c_m(x;q)x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^m}},
\end{aligned} \tag{3.3.3}$$

and $c_0(x;q) := 1$. Then

$$\begin{aligned}
\tilde{e}_q^{x^q} &= (1-(x-x^q))\tilde{e}_q^x \\
&= (1-x+x^q)\{1+\sum_{m=1}^{\infty} c_m(x;q)x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^m}}\} \\
&= 1+\sum_{m=1}^{\infty} c_m(x;q)x^{\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^m}} \\
&\quad -(1-x^{q-1})\{x+\sum_{m=1}^{\infty} c_m(x;q)x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^m}}\} \\
&= 1+\sum_{m=1}^{\infty}\{c_m(x;q)x^{\frac{1}{q^m}-1}\}x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{m-1}}} \\
&\quad -(1-x^{q-1})\{x+\sum_{m=2}^{\infty} c_{m-1}(x;q)x^{1+\frac{1}{q}+\ldots+\frac{1}{q^{m-1}}}\} \\
&= 1+\sum_{m=1}^{\infty}\{c_m(x;q)x^{q^{-m}-1}\}x^{1+\frac{1}{q}+\ldots+\frac{1}{q^{m-1}}} \\
&\quad -(1-x^{q-1})\{\sum_{m=1}^{\infty} c_{m-1}(x;q)x^{1+\frac{1}{q}+\ldots+\frac{1}{q^{m-1}}}\} \\
&= 1+\sum_{m=1}^{\infty}\{c_m(x;q)x^{q^{-m}-1}-(1-x^{q-1})c_{m-1}(x;q)\}x^{1+\frac{1}{q}+\ldots+\frac{1}{q^{m-1}}}
\end{aligned}$$

Also,

$$\tilde{e}_q^{x^q} = 1+\sum_{m=1}^{\infty} c_m(x^q;q)x^{1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{m-1}}}$$

from (3.3.3), we obtain a recurrence

$$c_m(x^q;q) = \begin{cases} c_m(x;q)x^{q^{-m}-1}-(1-x^{q-1})c_{m-1}(x;q), & m \geq 1 \\ 1, & m = 0 \end{cases} \qquad (3.3.4)$$

Though we have obtained two recurrences of the coefficients $a_m(x;q)$ and $c_m(x;q)$, unfortunately, we still have no idea to express $a_m(x,q)$ and $c_m(x,q)$

by explicit formulas. An illogic and ambiguous view is to omit some q from $a_m(x^q;q)$ in (3.3.2) and $c_m(x^q;q)$ in (3.3.4), then we rewrite the two recurrences by the following:

$$a_m^{'}(x;q)=\begin{cases} a_m^{'}(x;q)x^{q^m-1}+(1-x^{q-1})a_{m-1}^{'}(x^q;q), & m\geq 1\\ 1, & m=0\end{cases}$$

$$c_m^{'}(x;q)=\begin{cases} c_m^{'}(x;q)x^{q^{-m}-1}-(1-x^{q-1})c_{m-1}^{'}(x;q), & m\geq 1\\ 1, & m=0\end{cases}$$

Apparently, the two recurrences are solvable though they arise form two unreasonalbe process. To solve these recurrences, we obtain the following consequences directly:

$$\begin{aligned} a_m^{'}(x;q) &= \left(\frac{1-x^{q-1}}{1-x^{q^m-1}}\right)\cdot\left(\frac{1-x^{q(q-1)}}{1-x^{q(q^{m-1}-1)}}\right)\ldots\left(\frac{1-x^{q^{m-2}(q-1)}}{1-x^{q^{m-2}(q^2-1)}}\right)\cdot\left(\frac{1-x^{q^{m-1}(q-1)}}{1-x^{q^{m-1}(q-1)}}\right)\\ &= \frac{1}{[\![m]\!]_{(x;q)}}\cdot\frac{1}{[\![m-1]\!]_{(x^q;q)}}\ldots\frac{1}{[\![2]\!]_{(x^{q^{m-2}};q)}}\cdot\frac{1}{[\![1]\!]_{(x^{q^{m-1}};q)}}\\ &= \frac{1}{[\![m]\!]^{+}_{(x;q)}!}\end{aligned}$$

and

$$\begin{aligned} c_m^{'}(x;q) &= (-1)^m\left(\frac{1-x^{q-1}}{1-x^{q^{-m}-1}}\right)\cdot\left(\frac{1-x^{q-1}}{1-x^{q^{-(m-1)}-1}}\right)\ldots\left(\frac{1-x^{q-1}}{1-x^{q^{-2}-1}}\right)\cdot\left(\frac{1-x^{q-1}}{1-x^{q^{-1}-1}}\right)\\ &= (-1)^m\cdot\frac{1}{[\![-m]\!]_{(x;q)}}\cdot\frac{1}{[\![-(m-1)]\!]_{(x;q)}}\ldots\frac{1}{[\![-2]\!]_{(x;q)}}\cdot\frac{1}{[\![-1]\!]_{(x;q)}}\\ &= \frac{1}{[\![m]\!]^{-}_{(x;q)}!}.\end{aligned}$$

Interestingly, $\tilde{\mathrm{E}}_q^x$ and $\tilde{\varepsilon}_q^x$ in (2.3.2) and (2.3.4) are refound by two artificial recurrences since the coefficients $a_m^{'}$ and $c_m^{'}$ are the reciprocal of (2.3.1) and (2.3.3) unexpectedly. Is it just a coincidence? No, it is just lack of a logical explanation for these "coincidence" to our philosophy. It seems to be some deep relations between $\tilde{E}_q^x$, $\tilde{e}_q^x$ and $\tilde{\mathrm{E}}_q^x$, $\tilde{\varepsilon}_q^x$ and there must be a simple and concise rule behind the algebraic structures which dictate $\tilde{E}_q^x$, $\tilde{e}_q^x$ and $\tilde{\mathrm{E}}_q^x$, $\tilde{\varepsilon}_q^x$ in our belief.

3.4 Problems, still problems

A numerous mathematical structures of self-consistence were established in the world of q-analgoue. For instances, the Jacobi Triple Prodcut was constructed simply by two identities of Euler which are the original versions for E_q^x and e_q^x. The q-gamma function has a elegant self-consistent structure with q-integral and q-infinite product. The similar nice structure is inherited by the q-beta function and the q-hypergeometric function as well. Much more than these, the Jacobi Triple Product, the q-gamma function, together with theta functions [15, 16] were unified successfully into a nice formula [11]

$$(\cos \pi x)_q = \frac{(\Gamma_q(\frac{1}{2}))^2}{\Gamma_q(\frac{1}{2}+x)\Gamma_q(\frac{1}{2}-x)} = q^{\frac{-x^2}{2}} cos\pi x \prod_{n=1}^{\infty}(\frac{1+2r^{2n}cos2\pi x + r^{4n}}{1+2r^{2n}+r^{4n}}) \tag{3.4.1}$$

with $r = e^{\frac{2\pi^2}{log q}} \to 0^+$ as $q \to 1^-$, which is the q-analogue of the classical formula

$$\cos \pi x = \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}-x)}.$$

An immediate attempt made is to find the q-analogue of another formula

$$\frac{\sin \pi x}{\pi} = \frac{1}{\Gamma(x)\Gamma(1-x)}.$$

As a result of $\pi = \Gamma(\frac{1}{2})^2$, it is intuitive to set $\pi_q = \Gamma_q(\frac{1}{2})^2$ and then the result is quite similar to Askey's: (Here we leave all the details in the Appendix.)

$$(\sin \pi x)_q = \frac{(\Gamma_q(\frac{1}{2}))^2}{\Gamma_q(x)\Gamma_q(1-x)} = q^{\frac{-1}{2}(x-\frac{1}{2})^2} sin\pi x \prod_{n=1}^{\infty}(\frac{1-2r^{2n}cos2\pi x + r^{4n}}{1+2r^{2n}+r^{4n}}) \tag{3.4.2}$$

with $r = e^{\frac{2\pi^2}{logq}} \to 0^+$ as $q \to 1^-$. Interestingly, (3.4.2) also appeares when we replace x by $x - \frac{1}{2}$ or $\frac{1}{2} - x$ in (3.4.1). More precisely,

$$(\sin \pi x)_q = (\cos \pi(x - \frac{1}{2}))_q = (\cos \pi(\frac{1}{2} - x))_q,$$

which preserves the properities of the classical trigonometric functions:

$$\sin \pi x = cos(\frac{\pi}{2} - \pi x) = cos(-(\frac{\pi}{2} - \pi x)).$$

Unfortunately, to find the $\tilde{q}$-analogue of (3.4.1) and (3.4.2) is still a challenge to our poor knowledge. A suitable defination of $\tilde{q}$-integral for the $\tilde{\Gamma}_q$(x) is still unknown to us even though a candidate of the integrand

$$t^{[x-1]_{(x;q)}} \cdot \frac{1}{e_q^{tq}} \tag{3.4.3}$$

was given before. The two identities of Euler could not be extended to $\tilde{q}$-analogues to this day and a complete Jacobi Triple Product Identity of $\tilde{q}$-analogue is still an imagination. In addition, to construct a self-consistent theory for the Theta functions of $\tilde{q}$-analogue would be another important problem.

A weak version for Jacobi Triple Product of $\tilde{q}$-analogue is to replace x by x^{q-1} in the origial formula

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}} x^n = (1-x)(qx;q)_\infty (q;q)_\infty (q/x;q)_\infty.$$

Then we obtain

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(n-1)}{2}q-\frac{n(n-3)}{2}} = (1-x)(x^q;x^{q-1})_\infty (x^{q-1};x^{q-1})_\infty (x^{q-2};x^{q-1})_\infty, \tag{3.4.4}$$

where $\frac{n(n-1)}{2}q - \frac{n(n-2)}{2} \approx n$ for $q \approx 1$. In reality, it is not appropriate to involve the terminology "$\tilde{q}$-analogue"with (3.4.4) since it is not beyond the structures of q-series eventually, of course, helpless to our destination. An constructive formula to our expectation is at least of the form

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{1+q+q^2+\ldots+q^{n-1}}$$

which would be equal to some products involving

$$(1-t)(1-t^q)(1-t^{q^2})(1-t^{q^3})\ldots,$$

or whatever.

In conclusion, we have discussed "sum part "and "product part "of $\tilde{q}$-infinite sums and $\tilde{q}$-infinite products independently throughout this chapter. However, we still have no idea to find a transformation formula which connected $\tilde{q}$-sums with $\tilde{q}$-products simultaneously. More precisely, a explicit formula (or a weak version) for $\tilde{q}$-binomial theorem. This problem is desperate for settling and seems to be more fundamental than all of the above. Eventually, problems are still problems.

Chapter 4

$\Gamma(x)$, Euler Constant γ, and $\zeta(2)$: A new approach

Many different expressions can be given for the Euler constant

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n).$$

One way to obtain the magic constant is to evaluate the value of the logarithmic derivative for the reciprocal of Gamma function at $x = 1$ [17]. More precisely,

$$\frac{d}{dx}\log\frac{1}{\Gamma(x)}\,|_{x=1} = -\frac{\Gamma'(1)}{\Gamma(1)} = \gamma.$$

Differentiating $\log\frac{1}{\Gamma(x)}$ twice, the result is also worth studying:

$$\frac{\Gamma''(x)}{\Gamma(x)} = (\frac{\Gamma'(x)}{\Gamma(x)})^2 + \sum_{n=0}^{\infty}\frac{1}{(n+x)^2} = (\frac{\Gamma'(x)}{\Gamma(x)})^2 + \zeta(2,x)$$

and then

$$\frac{\Gamma''(1)}{\Gamma(1)} = \gamma^2 + \sum_{n=1}^{\infty}\frac{1}{n^2} = \gamma^2 + \frac{\pi^2}{6}.$$

In this Chapter, we desired to generalize the Euler Constant γ and Riemann zeta 2 function $\zeta(2)$ of classical to q- and $\tilde{q}$- analogue.

4.1 γ_q and $\zeta_q(2)$

For q-gamma function, one has the infinite product

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(\frac{1}{1-q})^{x-1},$$

or another equivalent representaion

$$\Gamma_q(x) = \lim_{n\to\infty} \frac{[1][2]...[n]}{[x+1][x+2]...[x+n]}[n]^x,$$

where we use the notation $[\cdot]$ to omit the lower index q of $[\cdot]_q$ for simplicity.

Our propositon are the following two identities:

$$-\frac{\Gamma_q'(1)}{\Gamma_q(1)} = \lim_{n\to\infty}(\frac{1}{\langle 1\rangle} + \frac{1}{\langle 2\rangle} + \frac{1}{\langle 3\rangle} + ... + \frac{1}{\langle n\rangle} - \log[n]) := \gamma_q \tag{4.1.1}$$

and

$$\frac{\Gamma_q''(1)}{\Gamma_q(1)} = \gamma_q^2 + \sum_{n=1}^{\infty} \frac{1}{q^n\langle n\rangle^2}, \tag{4.1.2}$$

where

$$\frac{1}{\langle n\rangle} := q^n \log(\frac{1}{q})^{\frac{1}{1-q}} \frac{1}{[n]}.$$

Note that the factor

$$(\frac{1}{q})^{\frac{1}{1-q}}$$

is a q-analogue of nature exponential e since it tends to e as q tends to 1^- and hence

$$\sum_{n=1}^{\infty} \frac{1}{q^n \langle n \rangle^2}$$

is a q-analogue of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

To verify the proposition is not difficult, the only pre-knowledge is just the Elementary Calculus.

$$\gamma_q^{(n)} := \frac{1}{\langle 1 \rangle} + \frac{1}{\langle 2 \rangle} + ... + \frac{1}{\langle n \rangle} - \log[n]$$

$$\Longrightarrow [n]^{-x} = e^{-x\log[n]} = e^{-x(\frac{1}{\langle 1 \rangle}+\frac{1}{\langle 2 \rangle}+...+\frac{1}{\langle n \rangle}-\gamma_q^{(n)})} = e^{\frac{-x}{\langle 1 \rangle}} e^{\frac{-x}{\langle 2 \rangle}} ... e^{\frac{-x}{\langle n \rangle}} e^{x\gamma_q^{(n)}}$$

$$\Longrightarrow \frac{1}{\Gamma_q(x)} = \lim_{n\to\infty} [x](\frac{[x+1]}{[1]}\frac{[x+2]}{[2]}...\frac{[x+n]}{[n]})[n]^{-x}$$

$$= [x] \lim_{n\to\infty} e^{\gamma_q^{(n)}x}(\frac{[x+1]}{[1]}e^{\frac{-x}{\langle 1 \rangle}}\frac{[x+2]}{[2]}e^{\frac{-x}{\langle 2 \rangle}}...\frac{[x+n]}{[n]}e^{\frac{-x}{\langle n \rangle}})$$

$$= [x]e^{\gamma_q x}\prod_{n=1}^{\infty}(\frac{[x+n]}{[n]}e^{\frac{-x}{\langle n \rangle}})$$

$$\Longrightarrow \log\frac{1}{\Gamma_q(x)} = \log[x] + \gamma_q x + \sum_{n=1}^{\infty}(\log[x+n] - \log[n] - \frac{x}{\langle n \rangle})$$

$$\because \frac{d}{dx}\log[x] = q^x \log(\frac{1}{q})^{\frac{1}{1-q}}\frac{1}{[x]} = \frac{1}{\langle x \rangle},$$

$$\frac{d}{dx}\log[x+n] = q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}}\frac{1}{[x+n]} = \frac{1}{\langle x+n \rangle}$$

$$\Longrightarrow \frac{d}{dx}\log\frac{1}{\Gamma_q(x)} = -\frac{\Gamma_q'(x)}{\Gamma_q(x)} = \gamma_q + \frac{1}{\langle x \rangle} + \sum_{n=1}^{\infty}(\frac{1}{\langle x+n \rangle} - \frac{1}{\langle n \rangle})$$

$$\Longrightarrow -\frac{\Gamma_q'(1)}{\Gamma_q(1)} = \gamma_q + \frac{1}{\langle 1\rangle} + \sum_{n=1}^{\infty}(\frac{1}{\langle n+1\rangle} - \frac{1}{\langle n\rangle})$$
$$= \gamma_q + \lim_{n\to\infty}\frac{1}{\langle n+1\rangle} = \gamma_q.$$

and

$$\frac{d^2}{dx^2}\log\frac{1}{\Gamma_q(x)} = \frac{d}{dx}\{\gamma_q + \frac{1}{\langle x\rangle} + \sum_{n=1}^{\infty}(\frac{1}{\langle x+n\rangle} - \frac{1}{\langle n\rangle})\}$$
$$= \frac{d}{dx}(\sum_{n=0}^{\infty}\frac{1}{\langle x+n\rangle})$$
$$= \sum_{n=0}^{\infty}\frac{d}{dx}(q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}}\frac{1}{[x+n]})$$
$$= -\sum_{n=0}^{\infty}\frac{1}{q^{n+x}}(\frac{q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}}}{[x+n]})^2$$
$$= -\sum_{n=0}^{\infty}\frac{1}{q^{n+x}\langle x+n\rangle^2}$$

Since

$$\frac{d^2}{dx^2}\log\frac{1}{\Gamma_q(x)} = -(\frac{\Gamma_q(x)\Gamma_q''(x) - \Gamma_q'(x)^2}{\Gamma_q^2(x)}),$$

$$\frac{\Gamma_q''(x)}{\Gamma_q(x)} = (\frac{\Gamma_q'(x)}{\Gamma_q(x)})^2 - \frac{d^2}{dx^2}\log\frac{1}{\Gamma_q(x)} = (\frac{\Gamma_q'(x)}{\Gamma_q(x)})^2 + \sum_{n=0}^{\infty}\frac{1}{q^{n+x}\langle x+n\rangle^2}.$$

Hence,

$$\frac{\Gamma_q''(1)}{\Gamma_q(1)} = \gamma_q^2 + \sum_{n=1}^{\infty}\frac{1}{q^n\langle n\rangle^2}.$$

4.2 $\tilde{\gamma}_{(t,q)}$ and $\tilde{\zeta}_{(t,q)}(2)$

A simple $\tilde{q}$-extension of Gamma function is of the form:

$$\tilde{\Gamma}_{(t,q)}(x) := \lim_{n\to\infty}\frac{[\![1]\!][\![2]\!]...[\![n]\!]}{[\![x]\!][\![x+1]\!]...[\![x+n]\!]}[\![n]\!]^x,$$

where

$$\llbracket w \rrbracket := \llbracket w \rrbracket_{(t,q)} = \frac{1-t^{q^{\omega}-1}}{1-t^{q-1}}.$$

And it is easy to check $\tilde{\Gamma}_{(q,t)}(\mathrm{x})$ satisfies the functional equation:

$$\begin{aligned}
\tilde{\Gamma}_{(t,q)}(x+1) &= \lim_{n\to\infty} \frac{\llbracket 1 \rrbracket \llbracket 2 \rrbracket ... \llbracket n \rrbracket}{\llbracket x+1 \rrbracket \llbracket x+2 \rrbracket ... \llbracket x+n+1 \rrbracket} \llbracket n \rrbracket^{x+1} \\
&= \llbracket x \rrbracket \lim_{n\to\infty} \frac{\llbracket 1 \rrbracket \llbracket 2 \rrbracket ... \llbracket n \rrbracket}{\llbracket x \rrbracket \llbracket x+1 \rrbracket ... \llbracket x+n \rrbracket} \llbracket n \rrbracket^{x} \left(\frac{\llbracket n \rrbracket}{\llbracket x+n+1 \rrbracket}\right) \\
&= \llbracket x \rrbracket \tilde{\Gamma}_{(t,q)}(x) \lim_{n\to\infty} \frac{\llbracket n \rrbracket}{\llbracket x+n+1 \rrbracket} \\
&= \llbracket x \rrbracket \tilde{\Gamma}_{(t,q)}(x),
\end{aligned}$$

which is an analogue of $\Gamma(x+1) = x!$.

Our propositon are the following two identities:

$$-\frac{\tilde{\Gamma}'_{(t,q)}(1)}{\tilde{\Gamma}_{(t,q)}(1)} = \lim_{n\to\infty}\left(\frac{1}{\langle\langle 1\rangle\rangle} + \frac{1}{\langle\langle 2\rangle\rangle} + \frac{1}{\langle\langle 3\rangle\rangle} + ... + \frac{1}{\langle\langle n\rangle\rangle} - \log \llbracket n \rrbracket\right) := \tilde{\gamma}_{(t,q)} \quad (4.2.1)$$

and

$$\frac{\tilde{\Gamma}''_{(t,q)}(1)}{\tilde{\Gamma}_{(t,q)}(1)} = \tilde{\gamma}^2_{(t,q)} + \sum_{n=1}^{\infty} \frac{1}{\langle\langle n\rangle\rangle^2} + \xi(t,q), \quad (4.2.2)$$

where

$$\frac{1}{\langle\langle n\rangle\rangle} := \left(q^n \log\left(\frac{1}{q}\right)^{\frac{1}{1-q}}\right)\left(t^{q^n-1} \log\left(\frac{1}{t^{q-1}}\right)^{\frac{1}{1-t^{q-1}}}\right)\frac{1}{\llbracket n \rrbracket},$$

and the error term $\xi(t,q)$ is a function of variables t, q with contribution 0 when q tends to 1. Note that the factor

$$\left(\frac{1}{t^{q-1}}\right)^{\frac{1}{1-t^{q-1}}}$$

is a $\tilde{q}$-analogue of nature exponential e since it tends to e as q tends to 1 and hence

$$\sum_{n=1}^{\infty} \frac{1}{\langle\langle n\rangle\rangle^2}$$

is a $\tilde{q}$-analogue of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The argument is quite similar to section 4.1 (but more complicated):

$$\tilde{\gamma}_{(t,q)}^{(n)} := \frac{1}{\langle\langle 1\rangle\rangle} + \frac{1}{\langle\langle 2\rangle\rangle} + ... + \frac{1}{\langle\langle n\rangle\rangle} - \log[\![n]\!]$$

$$\Longrightarrow [\![n]\!]^{-x} = e^{-x\log[\![n]\!]} = e^{-x(\frac{1}{\langle\langle 1\rangle\rangle}\frac{1}{\langle\langle 2\rangle\rangle}+...+\frac{1}{\langle\langle n\rangle\rangle}-\tilde{\gamma}_{(t,q)}^{(n)})} = e^{\frac{-x}{\langle\langle 1\rangle\rangle}}e^{\frac{-x}{\langle\langle 2\rangle\rangle}}...e^{\frac{-x}{\langle\langle n\rangle\rangle}}e^{x\tilde{\gamma}_{(t,q)}^{(n)}}$$

$$\begin{aligned}\Longrightarrow \frac{1}{\tilde{\Gamma}_{(t,q)}(x)} &= \lim_{n\to\infty}[\![x]\!](\frac{[\![x+1]\!]}{[\![1]\!]}\frac{[\![x+2]\!]}{[\![2]\!]}...\frac{[\![x+n]\!]}{[\![n]\!]})[\![n]\!]^{-x} \\ &= [\![x]\!]\lim_{n\to\infty} e^{\tilde{\gamma}_{(t,q)}^{(n)}x}(\frac{[\![x+1]\!]}{[\![1]\!]}e^{\frac{-x}{\langle\langle 1\rangle\rangle}}\frac{[\![x+2]\!]}{[\![2]\!]}e^{\frac{-x}{\langle\langle 2\rangle\rangle}}...\frac{[\![x+n]\!]}{[\![n]\!]}e^{\frac{-x}{\langle\langle n\rangle\rangle}}) \\ &= [\![x]\!]e^{\tilde{\gamma}_{(t,q)}x}\prod_{n=1}^{\infty}(\frac{[\![x+n]\!]}{[\![n]\!]}e^{\frac{-x}{\langle\langle n\rangle\rangle}})\end{aligned}$$

$$\Longrightarrow \log\frac{1}{\tilde{\Gamma}_{(t,q)}(x)} = \log[\![x]\!] + \tilde{\gamma}_{(t,q)}x + \sum_{n=1}^{\infty}(\log[\![x+n]\!] - \log[\![n]\!] - \frac{x}{\langle\langle n\rangle\rangle})$$

$$\because \frac{d}{dx}\log[\![x]\!] = (q^x\log(\frac{1}{q})^{\frac{1}{1-q}})((t^{q-1})^{[x]}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})[\![x]\!] = \frac{1}{\langle\langle x\rangle\rangle},$$

$$\begin{aligned}\frac{d}{dx}\log[\![x+n]\!] &= (q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}})((t^{q-1})^{[x+n]}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})\frac{1}{[\![x+n]\!]} \\ &= \frac{1}{\langle\langle x+n\rangle\rangle}\end{aligned}$$

$$\Longrightarrow \frac{d}{dx}\log\frac{1}{\tilde{\Gamma}_{(t,q)}(x)} = -\frac{\tilde{\Gamma}'_{(t,q)}(x)}{\tilde{\Gamma}_{(t,q)}(x)} = \tilde{\gamma}_{(t,q)} + \frac{1}{\langle\langle x\rangle\rangle} + \sum_{n=1}^{\infty}(\frac{1}{\langle\langle x+n\rangle\rangle} - \frac{1}{\langle\langle n\rangle\rangle})$$

$$\Longrightarrow -\frac{\tilde{\Gamma}'_{(t,q)}(1)}{\tilde{\Gamma}_{(t,q)}(1)} = \tilde{\gamma}_{(t,q)} + \frac{1}{\langle\langle 1\rangle\rangle} + \sum_{n=1}^{\infty}\left(\frac{1}{\langle\langle n+1\rangle\rangle} - \frac{1}{\langle\langle n\rangle\rangle}\right)$$

$$= \tilde{\gamma}_{(t,q)} + \lim_{n\to\infty}\frac{1}{\langle\langle n+1\rangle\rangle} = \tilde{\gamma}_{(t,q)}.$$

and

$$\frac{d^2}{dx^2}\log\frac{1}{\tilde{\Gamma}_{(t,q)}(x)}$$

$$= \frac{d}{dx}\{\tilde{\gamma}_{(t,q)} + \frac{1}{\langle\langle x\rangle\rangle} + \sum_{n=1}^{\infty}(\frac{1}{\langle\langle x+n\rangle\rangle} - \frac{1}{\langle\langle n\rangle\rangle})\}$$

$$= \frac{d}{dx}(\sum_{n=0}^{\infty}\frac{1}{\langle\langle x+n\rangle\rangle})$$

$$= \sum_{n=0}^{\infty}\frac{d}{dx}\{(q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}})(t^{(q^{x+n}-1)}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})\frac{1}{[\![x+n]\!]}\}$$

$$= -\sum_{n=0}^{\infty}\{(q^{n+x}\log(\frac{1}{q})^{\frac{1}{1-q}})(t^{(q^{x+n}-1)}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})\frac{1}{[\![x+n]\!]}\}^2 - \xi(t,q)$$

$$= -\sum_{n=0}^{\infty}\frac{1}{\langle\langle x+n\rangle\rangle^2} - \xi(t,q),$$

where

$$\xi(t,q) = \sum_{n=0}^{\infty}\frac{1-t^{(q^{x+n}-1)}}{[\![x+n]\!]^2}\cdot\{(q^{x+n}\log(\frac{1}{q})^{\frac{1}{1-q}}) (t^{(q^{x+n}-1)}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})(\log(\frac{1}{qt^{q-1}})^{\frac{1}{1-t^{q-1}}})\}$$

with contribution 0 as q tends to 1 since the factor

$$1 - t^{(q^{x+n}-1)}$$

tends to 0 and the factor

$$(q^{x+n}\log(\frac{1}{q})^{\frac{1}{1-q}})(t^{(q^{x+n}-1)}\log(\frac{1}{t^{q-1}})^{\frac{1}{1-t^{q-1}}})(\log(\frac{1}{qt^{q-1}})^{\frac{1}{1-t^{q-1}}})$$

tends to $(\log e)(\log e)(\log e)$ =1 as q tends to 1. Using the fact that

$$\frac{d^2}{dx^2}\log\frac{1}{\tilde{\Gamma}_{(t,q)}(x)} = -(\frac{\tilde{\Gamma}_{(t,q)}(x)\tilde{\Gamma}''_{(t,q)}(x) - \tilde{\Gamma}'_{(t,q)}(x)^2}{\tilde{\Gamma}^2_{(t,q)}(x)}),$$

we have

$$\begin{aligned}\frac{\tilde{\Gamma}''_{(t,q)}(x)}{\tilde{\Gamma}_{(t,q)}(x)} &= (\frac{\tilde{\Gamma}'_{(t,q)}(x)}{\tilde{\Gamma}_{(t,q)}(x)})^2 - \frac{d^2}{dx^2}\log\frac{1}{\tilde{\Gamma}_{(t,q)}(x)} \\ &= (\frac{\tilde{\Gamma}'_{(t,q)}(x)}{\tilde{\Gamma}_{(t,q)}(x)})^2 + \sum_{n=0}^{\infty}\frac{1}{\langle\langle x+n\rangle\rangle^2} + \xi(t,q)\end{aligned}$$

and then

$$\frac{\tilde{\Gamma}''_{(t,q)}(1)}{\tilde{\Gamma}_{(t,q)}(1)} = \tilde{\gamma}^2_{(t,q)} + \sum_{n=1}^{\infty}\frac{1}{\langle\langle n\rangle\rangle^2} + \xi(t,q).$$

Chapter 5

Applications to Quantum Groups $SU_q(2)$

In mathematics and theoretical physics, the term "quantum groups"denotes various kinds of noncommutative algebra with additional structure and the corresponding representaion theory satisfies the Yang-Baxter equation. Mathematically, quantum groups is some special kind of Hopf algebra. However, the same term is sometimes considered as the "quantization"or "quantum deformation"of classical Lie groups or Lie algebras as well. In reality, there is no single, all-encompassing definition, but instead a family of broadly similar objects for Quantum Groups. Retrospect to the history, these objects first appeared in the theory of quantum integrable systems [18], and was then formalized by V. Drinfel'd [24] and M. Jimbo [25] as a "quantized"universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac-Moody algebra, in the category of Hopf algebras.

Nevertheless, our present knowlege to Quantum Groups is much less than that we just mentioned all above. We desire to skip all the terminologies and backgrounds of mathematics here and just consider Quantum Groups concretely as q-deformation of classical Lie Algebras and Lie Groups like the preceding chapters. Incomplete discussions and generalizations are considered in 5.1. Here we discuss different types of q-deformation of classical Lie Group SU(2) with the corresponding Casimir operators and generalize the deformation of q, q^{-1} to any q^{α}, q^{β}.

Additional works that we hope to achieve in later sections is to find the corresponding R-matrix together with Yang-Baxter equation and then to realize the structure of Quantum Groups.

5.1 Reviews and some extensions

The Lie Algebra of Lie Group $SU(2)$ is

$$\begin{aligned}
[j_x, j_y] &= j_x j_y - j_y j_x = i j_z \\
[j_y, j_z] &= j_y j_z - j_z j_y = i j_x \\
[j_z, j_x] &= j_z j_x - j_x j_z = i j_y,
\end{aligned}$$

or equivalently, for $j_x = \frac{j_+ + j_-}{\sqrt{2}}$, $j_y = -i\frac{(j_+ - j_-)}{\sqrt{2}}$, $j_z = j_0$,

$$\begin{aligned}
[j_0, j_+] &= j_0 j_+ - j_+ j_0 = j_+ \\
[j_+, j_-] &= j_+ j_- - j_- j_+ = j_0 \\
[j_-, j_0] &= j_- j_0 - j_0 j_- = j_-,
\end{aligned}$$

where [,] denotes the Lie commutator. The corresponding Casimir operator, which commutes with all generators, is

$$C = j_x^2 + j_y^2 + j_z^2$$
$$= j_+ j_- + j_- j_+ + j_0^2 \qquad = 2j_+ j_- + j_0(j_0 - 1) = 2j_- j_+ + j_0(j_0 + 1).$$

The first quantum deformation of SU(2), given by P. P. Kulish and N. Yu. Reshetikhin [18], is

$$\begin{aligned} [J_0, J_+] &= J_0 J_+ - J_+ J_0 = J_+ \\ [J_+, J_-] &= J_+ J_- - J_- J_+ = \frac{\sinh 2\hbar J_0}{2 \sinh \hbar} = \frac{1}{2} \cdot [2J_0]_q \\ [J_-, J_0] &= J_- J_0 - J_0 J_- = J_-, \end{aligned} \tag{5.1.1}$$

where

$$[X]_q := \frac{q^X - q^{-X}}{q - q^{-1}} \longrightarrow X$$

as $q \to 1$, or following the historical development $q = e^{\hbar} \to 1$ as $\hbar \to 0$. Other q-deformations of SU(2) are constructed by S. L. Woronowicz, E. Witten, and D. B. Fairlie: (We use the notations of T. L. Curtright and C. K. Zachos [19])

(i) Witten's 1st deformation [20]:

$$\begin{aligned} [E_0, E_+]_p &= pE_0 E_+ - \frac{1}{p} E_+ E_0 = E_+ \\ [E_+, E_-] &= E_+ E_- - E_- E_+ = E_0 - (p - \frac{1}{p}) E_0^2 \\ [E_-, E_0]_p &= pE_- E_0 - \frac{1}{p} E_0 E_- = E_- \end{aligned}$$

with Casimir operator

$$C_p = \frac{1}{p}E_+E_- + pE_-E_+ + E_0^2.$$

(ii) Witten's 2nd deformation [20]:

$$\begin{aligned}
[W_0, W_+]_r &= rW_0W_+ - \frac{1}{r}W_+W_0 = W_+ \\
[W_+, W_-]_{\frac{1}{r^2}} &= \frac{1}{r^2}W_+W_- - r^2W_-W_+ = W_0 \\
[W_-, W_0]_r &= rW_-W_0 - \frac{1}{r}W_0W_- = W_-.
\end{aligned}$$

(iii) Woronowicz's deformation [21]:

$$\begin{aligned}
[V_0, V_+]_{s^2} &= s^2V_0V_+ - \frac{1}{s^2}V_+V_0 = V_+ \\
[V_+, V_-]_{\frac{1}{s}} &= \frac{1}{s}V_+V_- - sV_-V_+ = V_0 \\
[V_-, V_0]_{s^2} &= s^2V_-V_0 - \frac{1}{s^2}V_0V_- = V_-.
\end{aligned}$$

In reality, deformation (ii) and (iii) are special limits of a two-parameter generalization of Fairlie.

(iv) Fairlie's generalization [22]:

$$\begin{aligned}
[I_0, I_+]_r &= rI_0I_+ - \frac{1}{r}I_+I_0 = I_+ \\
[I_+, I_-]_{\frac{1}{s}} &= \frac{1}{s}I_+I_- - sI_-I_+ = I_0 \\
[I_-, I_0]_r &= rI_-I_0 - \frac{1}{r}I_0I_- = I_-.
\end{aligned}$$

(v) The cyclically symmetric deformation [22]: (Fairlie)

$$\begin{aligned}
[X,Y]_q &= qXY - \frac{1}{q}YX = Z \\
[Y,Z]_q &= qYZ - \frac{1}{q}ZY = X \\
[Z,X]_q &= qZX - \frac{1}{q}XZ = Y
\end{aligned}$$

with Casimir operator

$$C_q = (q^3 + \frac{2}{q})(XYZ + YZX + ZXY) - (\frac{1}{q^3} + 2q)(XZY + ZYX + YXZ).$$

Our simple generalization is to extend the factors q, q^{-1} of the preceding algebras to arbitrary q^α, q^β. For Witten's 1st form, we have

$$\begin{aligned}
[E_0, E_+]_{(q^\alpha,q^\beta)} &= q^\alpha E_0 E_+ - q^\beta E_+ E_0 = E_+ \\
[E_+, E_-] &= E_+ E_- - E_- E_+ = E_0 - (q^\alpha - q^\beta) E_0^2 \\
[E_-, E_0]_{(q^\alpha,q^\beta)} &= q^\alpha E_- E_0 - q^\beta E_0 E_- = E_-
\end{aligned}$$

with Casimir operator

$$C_{(q^\alpha,q^\beta)} = q^\beta E_+ E_- + q^\alpha E_- E_+ + q^{\alpha+\beta} E_0^2.$$

For Fairlie's form, we have

$$[X,Y]_{(q^\alpha,q^\beta)} = q^\alpha XY - q^\beta YX = Z$$

$$[Y,Z]_{(q^\alpha,q^\beta)} = q^\alpha YZ - q^\beta ZY = X$$

$$[Z,X]_{(q^\alpha,q^\beta)} = q^\alpha ZX - q^\beta XZ = Y$$

with Casimir operator

$$\begin{aligned}
C_{(q^\alpha,q^\beta)} &= (q^{\alpha-\beta}+\frac{1}{q^{\alpha-\beta}})(X^2+Y^2+Z^2) \\
&\quad +\{X(q^\beta YZ-q^\alpha ZY)+Y(q^\beta ZX-q^\alpha XZ)+Z(q^\beta XY-q^\alpha YX)\} \\
&= (q^{2\alpha-\beta}+q^\beta)(XYZ+YZX+ZXY) \\
&\quad -(q^\alpha+q^{-\alpha+2\beta})(XZY+ZYX+YXZ).
\end{aligned}$$

Note that the geometric interpretation of $C_{(q^\alpha,q^\beta)}$ is interesting. The tangent plane of the first equality is of three spheres: Two concentric inversion S^2 with factor $q^{\alpha-\beta}$ of degree 1 and -1, respectively. One distorted sphere S_q^2 with degree 0. In fact, the distorted one would return to normal sphere when q tends to 1 since $YZ-ZY=X$, $ZX-XZ=Y$, and $XY-YX=Z$ in classical SU(2). The specific expression with irregular notations is that

$$C_{(q^\alpha,q^\beta)}=(q^{\alpha-\beta})^1S^2+(q^{\alpha-\beta})^0S_q^2+(q^{\alpha-\beta})^{-1}S^2.$$

The factors $(q^{\alpha-\beta})^1$, $(q^{\alpha-\beta})^0$, $(q^{\alpha-\beta})^{-1}$ are meaningful in some sense. For Woronowicz's and Witten's 2nd form, we desire to find the Casimir operator $C_{(q^\alpha,q^\beta,q^\gamma,q^\delta)}$ of the algebra

$$\begin{aligned}
[W_0,W_+]_{(q^\alpha,q^\beta)} &= q^\alpha W_0W_+-q^\beta W_+W_0=W_+ \\
[W_+,W_-]_{(q^\gamma,q^\delta)} &= q^\gamma W_+W_--q^\delta W_-W_+=W_0 \\
[W_-,W_0]_{(q^\alpha,q^\beta)} &= q^\alpha W_-W_0-q^\beta W_0W_-=W_-.
\end{aligned}$$

Chapter 6

Conclusion and future works

Nearly three decades, characterized by thousands of investigators from different approaches of physics and mathematics during their careers, the feature of Quantum Groups was visualized gradually nowadays. Two different approachs to our superficial study are the followings. One, which relates the representations of Quantum Groups with q-special functions, is in [23]. This approach is the primary motivation for our studying in Chapter 1 to Chapter 4. However, the aims were not completely achieved eventually. Another approach, which starts with the construction of quantum linear problem, is considered by Kulish in [18]. As Algebra (5.1.1) corresponding to Sine-Gordon equation in Kulish's works. A direct question is that what happens when we consider a quantum linear problem for differential equations other than Sine-Gordon equation?

For this reason, we desire to construct a quantum linear problem for the

"Liouville equations "

$$u_{tt} \pm u_{xx} = \pm e^{\pm u}$$

and hope to find the representation together with the Casimir operator of the corresponding "expected"quantum algebra $SU_q(2)$ or whatever. Further works are to find the corresponding R-matrix together with Yang-Baxter equation and finally to visualize the structures of Quantum Groups under quantum linear problem of this kind. We hope to generalize the Liouville equation to q- and $\tilde{q}$-analogue and to visualize the evolutions between different structures of Quantum Groups.

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Appendix A

$$\frac{\Gamma_q^2(\frac{1}{2})}{\Gamma_q(x)\Gamma_q(1-x)} = \frac{(q^x;q)_\infty(q^{1-x};q)_\infty}{(q^{\frac{1}{2}};q)_\infty(q^{\frac{1}{2}};q)_\infty}$$

$$= \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n^2+2n(x-\frac{1}{2}))}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2}} \quad (A.0.1)$$

$$= q^{-\frac{1}{2}(x-\frac{1}{2})^2}\frac{\sum_{n=-\infty}^{\infty}(-1)^n r^{(n+\frac{1}{2})^2}(-i)e^{i(2n+1)\pi x}}{\sum_{n=-\infty}^{\infty} r^{(n+\frac{1}{2})^2}} \quad (A.0.2)$$

$$= q^{-\frac{1}{2}(x-\frac{1}{2})^2}\frac{\sum_{n=0}^{\infty}(-1)^n r^{(n+\frac{1}{2})^2}\sin(2n+1)\pi x}{\sum_{n=0}^{\infty} r^{(n+\frac{1}{2})^2}} \quad (A.0.3)$$

$$= q^{-\frac{1}{2}(x-\frac{1}{2})^2}\sin\pi x\prod_{n=1}^{\infty}(\frac{1-2r^{2n}\cos 2\pi x+r^{4n}}{1+2r^{2n}+r^{4n}}). \quad (A.0.4)$$

(A.0.1) is direct from the results of Jacobi Triple Product

$$(1-t)(qt;q)_\infty(q;q)_\infty(\frac{q}{t};q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}}t^n$$

for $t = q^x$ and $t = q^{\frac{1}{2}}$, that is,

$$(q^x;q)_\infty(q;q)_\infty(q^{1-x};q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n^2+2n(x-\frac{1}{2}))}$$

and

$$(q^{\frac{1}{2}};q)_\infty(q;q)_\infty(q^{\frac{1}{2}};q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2}.$$

(A.0.2) is from the Poisson's Theta Transformation Formula [15]

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+z)^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}+2\pi inz}$$

with the substitutions $q = e^{-2\pi t}$, $r = e^{\frac{-\pi}{t}}$, $z = (x - \frac{1}{2}) + \frac{i}{2t}$. The remainder details are the following:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{-\pi t(n+z)^2} &= \sum_{n=-\infty}^{\infty} e^{-\pi t(n+(x-\frac{1}{2})+\frac{i}{2t})^2} \\
&= \sum_{n=-\infty}^{\infty} e^{-\pi t(n^2+2n(x-\frac{1}{2})+\frac{ni}{t}+(x-\frac{1}{2})^2+\frac{i}{t}(x-\frac{1}{2})-\frac{1}{4t^2})} \\
&= \sum_{n=-\infty}^{\infty} e^{-\pi t(n^2+2n(x-\frac{1}{2}))} \cdot (e^{-\pi i})^n \cdot e^{-\pi t(x-\frac{1}{2})^2} \cdot e^{-\pi i(x-\frac{1}{2})} \cdot e^{\frac{\pi}{4t}} \\
&= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n^2+2n(x-\frac{1}{2}))} \cdot (-1)^n \cdot q^{\frac{1}{2}(x-\frac{1}{2})^2} \cdot e^{-\pi i(x-\frac{1}{2})} \cdot r^{\frac{-1}{4}},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{\frac{-\pi}{t}n^2+2\pi inz} &= \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} r^{n^2} \cdot e^{2\pi in(x-\frac{1}{2}+\frac{i}{2t})} \\
&= \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} r^{n^2+n} \cdot e^{2\pi in(x-\frac{1}{2})},
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n^2+2n(x-\frac{1}{2}))} &= q^{-\frac{1}{2}(x-\frac{1}{2})^2} \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} r^{(n+\frac{1}{2})^2} \cdot e^{\pi i(2n+1)(x-\frac{1}{2})} \\
&= q^{-\frac{1}{2}(x-\frac{1}{2})^2} \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} (-1)^n r^{(n+\frac{1}{2})^2} \cdot (-ie^{\pi i(2n+1)x}) \\
&= q^{-\frac{1}{2}(x-\frac{1}{2})^2} \frac{1}{\sqrt{t}} \cdot (2 \sum_{n=0}^{\infty} (-1)^n r^{(n+\frac{1}{2})^2} \sin(2n+1)\pi x),
\end{aligned}$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2} = \frac{1}{\sqrt{t}} \cdot (2\sum_{n=0}^{\infty}(-1)^n r^{(n+\frac{1}{2})^2})$$

for the substitution $x = \frac{1}{2}$ in last identity. Then (A.0.3) satisfies. Finally, (A.0.4) is from the fact [16] that

$$\begin{aligned}\theta_1(z;r) &= 2\sum_{n=0}^{\infty}(-1)^n r^{(n+\frac{1}{2})^2}\sin(2n+1)z \\ &= 2r^{\frac{1}{4}}(r^2;r^2)_\infty \sin z \prod_{n=1}^{\infty}(1-2r^{2n}\cos 2z + r^{4n})\end{aligned}$$

with $z = \pi x$ and

$$\begin{aligned}\theta_2(z;r) &= 2\sum_{n=0}^{\infty} r^{(n+\frac{1}{2})^2}\cos(2n+1)z \\ &= 2r^{\frac{1}{4}}(r^2;r^2)_\infty \cos z \prod_{n=1}^{\infty}(1+2r^{2n}\cos 2z + r^{4n})\end{aligned}$$

with $z = 0$. Then the proof is accomplished.

The slight differences from Askey's proof [11] are the substitutions $t = q^{\frac{1}{2}+x}$, $t = q^{\frac{1}{2}}$ in the Jacobi Triple Product, the substitutions $q = e^{-2\pi t}$, $r = e^{\frac{-\pi}{t}}$, $z = x + \frac{i}{2t}$ in the Poisson's Theta Transformation Formula and the use of

$$\begin{aligned}\theta_2(z;r) &= 2\sum_{n=0}^{\infty} r^{(n+\frac{1}{2})^2}\cos(2n+1)z \\ &= 2r^{\frac{1}{4}}(r^2;r^2)_\infty \cos z \prod_{n=1}^{\infty}(1+2r^{2n}\cos 2z + r^{4n})\end{aligned}$$

with $z = \pi x$ and $z = 0$ respectively.