國 立 交 通 大 學

應用數學系

碩　士　論　文

Lit-only σ-games 的代數結構

The Algebra Behind Lit-only σ-games

研　究　生：劉倫欣

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中華民國一百年六月

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摘　要

令 $S=\{s_1,s_2,\ldots,s_n\}$ 是一個有限的集合。如果給定一個函數 $m:S\times S\to \mathrm{N}\cup\{\infty\}$ 定義為 $m(s,s)=1$；而對不同的 s,s' 滿足 $m(s,s')=m(s',s)\in\{2,3\}$。那麼此集合 S 可以被聯想成一個圖(也把此圖用 S 表示)，圖的點集合為集合 S，邊集合為 $\{ss' \mid m(s,s')=3\}$。一個 *simply-laced Coxeter group* W_S 是一個跟 (S,m) 有關的群。在此篇論文中證明了當圖 S 是一個有 n 個頂點的路徑(path)時，W_S 是同構(isomorphic)於一個對稱群 S_{n+1} 的群。我們考慮一個很自然的同態函數(homomorphsim) $\sigma:W_S\to GL(R^n)$ 將 W_S 對映到線性群 $GL(R^n)$ 中，使得 $\sigma(\mathrm{W_S})$ 是一個可以作用在 R^n 空間上的線性群(矩陣所構成的群)。當我們把 $\sigma(\mathrm{W_S})$ 中的矩陣都轉置後，可得到這些轉置矩陣形成的群 $\sigma^*(\mathrm{W_S})$。若將群 $\sigma^*(\mathrm{W_S})$ 作用在 R^n 上，可證明群 $\sigma^*(\mathrm{W_S})$ 會同構(isomorphic)於一個對稱群 S_{n+1}。因為群 $\sigma^*(W_S)$ 中的矩陣都是整係數矩陣，若將這些整係數矩陣的係數同餘(modulo) 2，則可得到一些新的矩陣形成一個新的群。在此篇論文中，我們規定這個新的群只有左乘運算，且將這個群作用在一個二元體(binary field) F_2 所形成的 n 維空間 F_2^n，並佈於一個二元體 F_2 上。我們稱這個新的群作用在 F_2^n 上是一個作用在圖 S 的 *lit only σ-game*。我們討論當圖 S 是3個頂點的 cycle 時，W_S 中的子群 G 之生成集的樣子且 G 滿足 $\sigma^*(G)=\{I\}\ (\bmod 2)$。

The Algebra Behind Lit-only σ-games

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The Algebra Behind Lit-only σ-games

Abstract

Let $S=\{s_1,s_2,\ldots,s_n\}$ be a finite set and m be a function with $m:S\times S\rightarrow \mathrm{N}\bigcup\{\infty\}$ satisfying $m(s,s)=1$ and $m(s,s')=m(s',s)\in\{2,3\}$ for distinct $s,s'\in S$. The set S is associated with the graph, also denoted by S, with the vertex set S and the edge set $\{ss'\mid m(s,s')=3\}$. A *simply-laced Coxeter group* W_S associated with (S,m) is the group generated by S subject to the relations

$$(ss')^{m(s,s')}$$

for $s,s'\in S$. We consider a homomorphism $\sigma:W_S\rightarrow GL(R^n)$, which is referred as *canonical representation* of W_S, where $GL(R^n)$ is the group of invertible linear transformations of R^n into itself. We consider the canonical representation σ of W_S into R^n and use its dual representation σ^* to show that W_S is isomorphic to the symmetric group S_{n+1} if the graph S is an n-vertex path. The matrices $\sigma^*(W_S)$ have integral coefficients. The left multiplication of these matrices modulo 2 on the n-dimensional space F_2^n over a binary field is usually called the *lit only σ-game* on the graph S in literatures. In the special case when S is a 3-vertex cycle, we determine the subgroup G of W_S with $\sigma^*(G)=\{I\}\ (\mathrm{mod}\,2)$.

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目　　錄

1 Introduction

Assume that S is a finite set, and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for distinct $s, s' \in S$. Let $F(S)$ be the free group on the set S and N be the normal subgroup of $F(S)$ generated by all elements

$$(ss')^{m(s,s')},$$

where $s, s' \in S$. The group $W := F(S)/N$ is called the *Coxeter group associated with* (S, m), and the pair (W, S) is called a *Coxeter system*. A Coxeter group W can be represented by a *Coxeter graph* $\Gamma = (V, E)$ whose vertex set $V = S$ and edge set $E = \{ss' \mid m(s, s') \geq 3, s \neq s' \in S\}$. The edges with $m(s, s') > 3$ are labeled by the number but the label 3 be omitted. The Coxeter group is *simply-laced* if $m(s, s') \in \{1, 2, 3\}$ for $s, s' \in S$. The Coxeter graph of simply-laced Coxeter groups exactly coincide with simply-laced Dynkin diagrams[6]. For example a Coxeter group of type A_n has its Coxeter graph a path of order n. Figure 1 lists the simply-laced Dynkin diagrams.

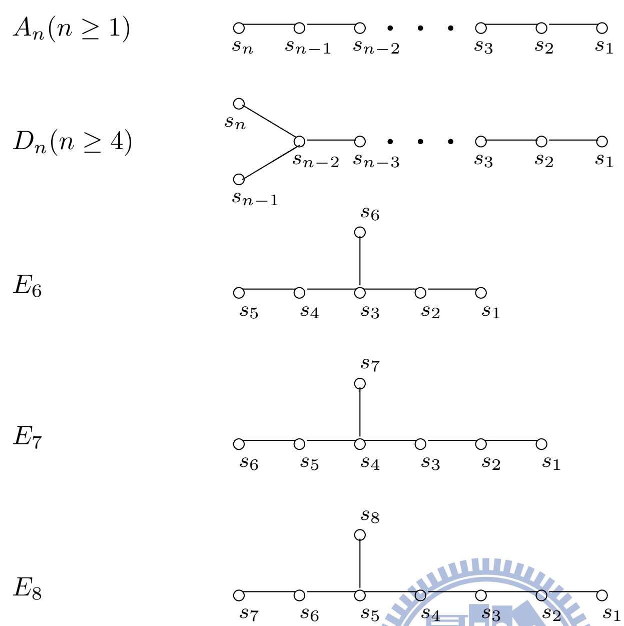

Figure 1: simply-laced Dynkin diagrams.

Throughout this paper, we assume W_S is a simply-laced Coxeter group associated with (S, m) and the set S is associated with the graph, also denoted by S, with the vertex set S and the edge set $\{ss' \mid m(s, s') = 3\}$. Note that, in section 3, the graph S of the Coxeter group W_S of type A_n is the Coxeter graph of W_S.

It is well known that W_S is finite if and only if S is simply-laced Dynkin diagrams [6]. The following is a well-known property, called the universal property of free groups. See [1, page 219] for details.

Theorem 1.1 (The universal property of free groups)**.** *If G is a group with identity 1, and ϕ is a map from S into G, then there is a unique homomorphism $\phi' : F(S) \to G$ such that $\phi'(s) = \phi(s)$ for $s \in S$.*

The next theorem is a direct result of Theorem 1.1

Theorem 1.2. *If G is a group with identity 1, and ϕ is a map from S into G such that $(\phi(s)\phi(s'))^{m(s,s')} = 1$ for $s,s' \in S$, then there is a unique homomorphism $\phi' : W_S \to G$ such that $\phi'(s) = \phi(s)$ for $s \in S$.*

Proof. By Theorem 1.1, there is a unique homomorphism $\phi' : F(S) \to G$ such that

$$\phi'((ss')^{m(s,s')}) = (\phi'(s)\phi'(s'))^{m(s,s')} = (\phi(s)\phi(s'))^{m(s,s')} = 1,$$

i.e. $N \subseteq \mathrm{Ker}(\phi')$. Hence ϕ' induces a unique homomorphism form $W_S = F(S)/N$ into G, which is still denoted by ϕ', $\phi' : W_S \to G$. □

We will use the same notation ϕ for ϕ' in the above theorem and say that *the domain S of the map ϕ lifts to the domain W_S.*

Let V_S denote the vector space over $\mathbb{R}$ with a given basis $\{\alpha_s \mid s \in S\}$ and $V_S^* := \{f \mid f : V_S \to \mathbb{R} \text{ is linear}\}$ be the *dual space* of V_S with the dual basis $\{\ \alpha_s^* \mid s \in S\}$, where $\alpha_s^* : V_S \to \mathbb{R}$ is the map satisfying

$$\alpha_s^* \alpha_{s'} = \begin{cases} 1, & \text{if } s' = s; \\ 0, & \text{else,} \end{cases}$$

for any $s, s' \in S$. The *linear representation* of W_S is a homomorphism $\sigma : W_S \to GL(V_S)$, where $GL(V_S)$ is the group of invertible linear transformations form V_S into V_S, with the composition. Since V_S, V_S^* are $|S|$-dimensional vector spaces, we may regard V_S, V_S^* as $\mathbb{R}^S$. In section 2, we introduce the linear representation σ of the Coxeter group W_S as described in [6, page 110] and replace $GL(V_S)$ with $GL(\mathbb{R}^n)$. In order to find the transpositions act on $\mathbb{R}^n$, we consider its dual representation σ^*. In Section 3, we use the dual representation σ^* of the Coxeter group W_S of type A_n into $\mathbb{R}^n$ to show that W_S is isomorphic to the symmetric group S_{n+1}. The matrices $\sigma^*(W_S)$ have integral coefficients as shown in Proposition 2.3. Let $\{e_s \mid s \in S\}$ denote the standard basis of $\mathbb{R}^S$. Then

$$\sigma^*(s)e_{s'} = \begin{cases} e_{s'}, & \text{if } s \neq s'; \\ -e_{s'} + \sum\limits_{m(s,s'')=3} e_{s''}, & \text{if } s = s'. \end{cases}$$

The left multiplication of the matrices $\sigma^*(s)$ modulo 2 on the set F_2^n is called the *lit only σ-game* on S, which was first studied in [2], and independently in [3, 4, 5, 7, 8].

Let G be a group with a generating set S such that $e \notin S$. The *Cayley graph* $\mathrm{Cay}(G, S)$ of G with respect to S has the vertex set G and the edge set $\{g(gs) \mid g \in G, s \in S\}$. The thesis focus on the special case when S is a 3-vertex cycle, and determines the subgroup G of W_S with $\sigma^*(G) = \{I\} \pmod 2$ in Theorem 4.9. The Cayley graph $\mathrm{Cay}(W_S/G, \{s_1, s_2, s_3\})$ is described in the end.

2 Representation of W_S

Let $S = \{s_1, s_2, \cdots, s_n\}$ be a finite set and let V_S denote the vector space over $\mathbb{R}$ with a given basis $\{\alpha_s \mid s \in S\}$. For $s, s' \in S$, we define a symmetric bilinear form B on V_S by

$$B(\alpha_s, \alpha_{s'}) := \ -\cos\tfrac{\pi}{m(s,s')},$$

and define a *reflection* $\sigma_s : V_S \to V_S$ by the rule:

$$\sigma_s\lambda := \lambda - 2B(\alpha_s, \lambda)\alpha_s,$$

where $\lambda \in V_S$. We have

$$\sigma_s(\alpha_{s'}) = \begin{cases} -\alpha_{s'}, & \text{if } m(s,s') = 1; \\ \alpha_{s'} + \alpha_s, & \text{if } m(s,s') = 3; \\ \alpha_{s'}, & \text{if } m(s,s') = 2, \end{cases} \tag{1}$$

where $\alpha_{s'}$ a basis vector of V_S. Note that $\sigma_s^2 = I$ and hence $\sigma_s \in GL(V_S)$ for $s \in S$.

Theorem 2.1. *The domain of map $\sigma : S \to GL(V_S)$, defined by $s \to \sigma_s$ for $s \in S$, lifts to W_S.*

Proof. By Theorem 1.2, we need to check that $(\sigma(s)\sigma(s'))^{m(s,s')} = 1$. Assume $m(s,s') = 1$(i.e. $s' = s$). The result is hold, since $\sigma(s) = \sigma_s$ has order 2 in $GL(V_S)$. For any λ belongs to V_S, assume $m(s,s') = 2$. We have:

$$\begin{aligned}(\sigma_s\sigma_{s'})^{m(s,s')}\lambda &= (\sigma_s\sigma_{s'})^2\lambda \\ &= (\sigma_s\sigma_{s'})[\sigma_s\lambda - 2B(\alpha_{s'}, \lambda)\sigma_s\alpha_{s'}] \\ &= (\sigma_s\sigma_{s'})[\sigma_s\lambda - 2B(\alpha_{s'}, \lambda)\alpha_{s'}] \\ &= (\sigma_s\sigma_{s'})[\lambda - 2(B(\alpha_s, \lambda)\alpha_s + B(\alpha_{s'}, \lambda)\alpha_{s'})] \\ &= \lambda.\end{aligned}$$

Finally, assume $m(s, s') = 3$. We may compute that :

$$\begin{aligned}
(\sigma_s \sigma_{s'})^{m(s,s')}\lambda &= (\sigma_s \sigma_{s'})^3 \lambda \\
&= (\sigma_s \sigma_{s'})^2 [\sigma_s \lambda - 2B(\alpha_{s'}, \lambda)\sigma_s \alpha_{s'}] \\
&= (\sigma_s \sigma_{s'})[\lambda - 2B(\alpha_s, \lambda)(\alpha_s + \alpha_{s'}) - 2B(\alpha_{s'}, \lambda)\alpha_{s'}] \\
&= \lambda.
\end{aligned}$$

Thus the domain S of σ lifts to W_S. □

Indeed J.Humphreys proves the map σ is injective [6, Page 113].

Theorem 2.2. *$\sigma :\ W_S \to GL(V_S)$ is injective.*

We refer to the monomorphism σ as the *linear representation* of W_S.

As we fixed the Coxeter group W_S of type A_n (see in Section 3) and given an ordered basis $\{\alpha_{s_1}, \alpha_{s_2}, \ldots, \alpha_{s_n}\}$ of V_S. By the equation (1), σ_{s_i} has the matrix form:

$$\sigma_{s_i} \sim \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & 1 & -1 & 1 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}. \qquad (2)$$

Therefore, we consider the transpose of the above matrix. Back to the vector space V_S, if we let $L : V_S \to V_S$ be a linear transformation. Then the *dual map* $L^t : V_S^* \to V_S^*$ of L is defined by the following rule:

$$(L^t f)v = f(Lv)$$

for any $f \in V_S^*$ and $v \in V$. Note that if two maps $L_1, L_2 : V_S \to V_S$ are linear then it is easy to check $(L_1 L_2)^t = L_2^t L_1^t$.

Proposition 2.3. *For each $s \in S$, let $\sigma_s^t : V_S^* \to V_S^*$ be the dual map of the reflection $\sigma_s : V_S \to V_S$. Then for any $s, s' \in S$,*

$$\sigma_s^t(\alpha_{s'}^*) = \begin{cases} \alpha_{s'}^*, & \text{if } s \neq s'; \\ -\alpha_{s'}^* + \sum\limits_{k:m(s',k)=3} \alpha_k^*, & \text{if } s = s', \end{cases} \tag{3}$$

for $k \in S$. In particular, $(\sigma_s^t)^2 = I$ and hence $\sigma_s^t \in GL(V_S^*)$ for $s \in S$.

Proof. Let $\lambda = \sum\limits_{s'' \in S} c_{s''}\alpha_{s''} \in V_S$. First we may assume $s' = s$. Since σ_s^t is the dual map of σ_s and $\alpha_{s'}^*(\alpha_s) = 1$, then we have

$$\begin{aligned} [\sigma_s^t(\alpha_{s'}^*)](\lambda) &= \alpha_{s'}^*(\sigma_s\lambda) \\ &= \alpha_{s'}^*(\lambda) - 2B(\alpha_s, \lambda)\alpha_{s'}^*(\alpha_s) \\ &= \alpha_{s'}^*(\lambda) - 2B(\alpha_s, \lambda) \\ &= \alpha_{s'}^*(\lambda) - 2\sum_{s'' \in S} c_{s''} B(\alpha_s, \alpha_{s''}) \\ &= \alpha_{s'}^*(\lambda) - 2\sum_{\substack{s'' \in S \\ m(s,s'')=2}} c_s \cdot 0 - 2c_s \cdot 1 - 2\sum_{\substack{s'' \in S \\ m(s,s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}) \\ &= \alpha_{s'}^*(\lambda) - 2c_s - 2\sum_{\substack{s'' \in S \\ m(s,s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}). \end{aligned} \tag{4}$$

Since $\alpha_s^*(\lambda) = c_s$ and $B(\alpha_s, \alpha_{s''}) = -1/2$, for $m(s, s'') = 3$ and $s \in S$. Then

$$-2\sum_{\substack{s'' \in S \\ m(s,s'')=3}} c_{s''} B(\alpha_s, \alpha_{s''}) = \sum_{\substack{s'' \in S \\ m(s,s'')=3}} c_{s''} \cdot 1 = \sum_{\substack{s'' \in S \\ m(s,s'')=3}} \alpha_{s''}^*(\lambda)$$

and the equation (4) equal to $(-\alpha_{s'}^* + \sum\limits_{\substack{s'' \in S \\ m(s,s'')=3}} \alpha_{s''}^*)(\lambda)$

In the other case $s' \neq s$. Then $\alpha^*_{s'}(\alpha_s) = 0$. Thus,

$$[\sigma^t_s(\alpha^*_{s'})](\lambda) = \alpha^*_{s'}(\lambda) - 2B(\alpha_s, \lambda)\alpha^*_{s'}(\alpha_s) = \alpha^*_{s'}(\lambda).$$

□

We shall call $\sigma^t_s : V^*_S \to V^*_S$ a *dual reflection* of σ_s and refer the basis $\{\ \alpha^*_s \mid s \in S\}$ the *standard basis* of V^*_S.

Definition 2.4. The *dual representation* $\sigma^* : W_S \to GL(V^*_S)$ of σ is defined by

$$\sigma^*(w) := \sigma(w^{-1})^t \text{ , for } w \in W_S.$$

Proposition 2.5. *Then σ^* is a monomorphism.*

Proof. For any $w_1, w_2 \in W_S$,

$$\begin{aligned} \sigma^*(w_1 w_2) &= \sigma((w_1 w_2)^{-1})^t \\ &= (\sigma(w_2^{-1})\sigma(w_1^{-1}))^t \\ &= \sigma(w_1^{-1})^t \sigma(w_2^{-1})^t \\ &= \sigma^*(w_1)\sigma^*(w_2). \end{aligned}$$

Hence the map $\sigma^* : W_S \to GL(V^*_S)$ is a homomorphism. Next we need to prove σ^* is injective.

Let $\sigma^*(w) \in GL(V^*_S)$ be the identity linear transformation for some $w \in W_S$. Let $f \in V^*_S$ and for any $v \in V_S$,

$$\begin{aligned} (\sigma^*(w)f)v &= (\sigma(w^{-1})^t f)v \\ &= f(\sigma(w^{-1})v). \end{aligned}$$

Since $\sigma^*(w) \in GL(V_S^*)$ is a identity map, $(\sigma^*(w)f)v = fv$. This implies $\sigma(w^{-1})v = v$ for any $v \in V_S$. Then $\sigma(w^{-1}) = e \in GL(V_S)$.

And by Theorem 2.2, $\sigma : W_S \to GL(V_S)$ is a monomorphism, we must have $w = e \in W_S$. This shows that the map $\sigma^* : W_S \to GL(V_S^*)$ is injective.

□

The following lemma describes the mapping of σ^*.

Lemma 2.6. *For each $s \in S, \sigma^*(s) = \sigma_s^t$, where $\sigma^* : W_S \to GL(V_S^*)$.*

Proof. For each $s \in S, s$ has order 2 in W_S, then $\sigma^*(s) := \sigma(s^{-1})^t = \sigma(s)^t$. Since $\sigma(s) = \sigma_s$, we have $\sigma(s)^t = \sigma_s^t$. Thus, $\sigma^* : W_S \to GL(V_S^*)$ by sending s to σ_s^t. □

From Proposition 2.5 and Lemma 2.6, we had known that $\sigma^*(s_i s_j) = \sigma_{s_i}^t \sigma_{s_j}^t$, where $s_i, s_j \in S$. In the next section, we shall introduce the simply-laced Coxeter group W_S of type A_n and give a proof of W_S is the symmetric group on a set S^*.

3 Coxeter group W_S of type A_n

In this section, we consider the Coxeter group W_S of type A_n. We shall prove W_S is isomorphic to the symmetric group S_{n+1} on $n+1$ elements. Throughout this section, set $S = \{s_1, s_2, ..., s_n\}$, and $m : S \times S \to \{1, 2, 3\}$ is the function satisfying $m(s_i, s_j) = m(s_j, s_i)$ and

$$m(s_i, s_j) = \begin{cases} 1, & \text{if } j = i \text{ and } i \in \{1, 2, \dots, n\}; \\ 2, & \text{if } j \notin \{i-1, i+1\}, i, j \in \{1, 2, \dots, n\}; \\ 3, & \text{if } j \in \{i-1, i+1\}, i \in \{2, 3, \dots, n-1\}. \end{cases}$$

The (W_S, S) denotes the Coxeter system of type A_n and the set S is associated with the graph with the vertex set S and the edge set $\{ss' \mid m(s,s') = 3\}$, which exactly coincide with the Coxeter graph of type A_n as shown in Figure 1. Recall from Proposition 2.5, Lemma 2.6 and (3), there exists a monomorphism $\sigma^* : W_S \to GL(V_S^*)$ with $\sigma^*(s_i) = \sigma^t_{s_i}$ satisfying

$$\sigma^t_{s_i}(\alpha^*_{s_j}) = \begin{cases} \alpha^*_{s_j} & \text{if } j \neq i; \\ -\alpha^*_{s_j} + \sum\limits_{k:m(s_i,s_k)=3} \alpha^*_{s_k} & \text{if } j = i, \end{cases} \tag{5}$$

In the ordered basis $\{\alpha^*_{s_1}, \alpha^*_{s_2}, \ldots, \alpha^*_{s_n}\}$ of V_S^*, $\sigma^t_{s_i}$ has a matrix form:

$$\sigma^t_s \sim \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & 1 & & \\ & & & -1 & & \\ & & & 1 & 1 & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}. \tag{6}$$

If we define $\epsilon_1 = \alpha^*_{s_1}$, and for $2 \leq i \leq n+1$,

$$\epsilon_i = \sigma^t_{s_{i-1}} \ldots \sigma^t_{s_2} \sigma^t_{s_1} \alpha^*_{s_1}.$$

We call $S^* = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}\}$ a *simple set* of V_S^*. We will show later that any n elements of S^* form a basis of V_S^*.

Theorem 3.1. *If V_S^* is a dual space of vector space V_S, $\{\alpha^*_s \mid s \in S\}$ is a standard basis of V_S^*, and $S^* = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}\}$ is a subset of V_S^* satisfying* *the following relations:*

$$\begin{aligned}\epsilon_1 &= \alpha^*_{s_1},\\ \epsilon_i &= -\alpha^*_{s_{i-1}} + \alpha^*_{s_i} \text{ , for } 2 \leq i \leq n,\\ \epsilon_{n+1} &= -\alpha^*_{s_n},\end{aligned}$$

then any n elements of S^* is also a basis of V_S^*.

Proof. Let the subset $S_1^* = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ of S. First we prove that S_1^* is a linear independent set. Assume

$$\begin{aligned}0 &= \sum_{i=1}^{n} a_i \epsilon_i\\ &= a_1\alpha^*_{s_1} + a_2(-\alpha^*_{s_1} + \alpha^*_{s_2}) + \cdots + a_n(-\alpha^*_{s_{n-1}} + \alpha^*_{s_n})\\ &= (a_1 - a_2)\alpha^*_{s_1} + (a_2 - a_3)\alpha^*_{s_2} + \cdots + a_n\alpha^*_{s_n}.\end{aligned}$$

Since $\{\alpha^*_s \mid s \in S\}$ is a basis of V_S^*, we have

$$\begin{cases} 0 &= a_1 - a_2,\\ \vdots & \\ 0 &= a_{n-1} - a_{n-2},\\ 0 &= a_n.\end{cases}$$

Thus, $a_1 = a_2 = \ldots = a_n = 0$.

By definition, ϵ_i can be written as a linear combinations of $\{\alpha^*_{s_i} \mid s_i \in S\}$ for $1 \leq i \leq n$, and S_1^* is a linear independent set. Thus, S_1^* is also a basis of V_S^*.

Next, by the definition of ϵ_i for $1 \leq i \leq n+1$, we had known that $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n+1} = 0$. Hence any n elements of $S^* = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_{n+1}\}$ form a basis of V_S^*. □

Theorem 3.2. *Let (W_S, S) denote the Coxeter system of type A_n. Then W_S is the symmetric group on S^*.*

Proof. By proposition 2.5, we had known that $\sigma^*(W_S)$ is isomorphic to W_S. Thus, we just prove that $\sigma^*(W_S)$ is the symmetric group on S^*. For each $\sigma^*(s_i) = \sigma^t_{s_i} \in GL(V_S^*), 1 \leq i \leq n$.

$$\sigma^t_{s_i}\epsilon_j = \begin{cases} \epsilon_{i+1}, & \text{if } j = i; \\ \epsilon_i, & \text{if } j = i+1; \\ \epsilon_j, & \text{others.} \end{cases}$$

for $1 \leq j \leq n+1$. Thus, $\sigma^t_{s_i}$ is a transposition $(\epsilon_i, \epsilon_{i+1})$ for $1 \leq i \leq n$. Since $\{\ \sigma^t_{s_i} \mid 1 \leq i \leq n\}$ is a generating set of $\sigma^*(W_S)$. Hence $\sigma^*(W_S)$ is the symmetric group on S^*. □

Then W_S is isomorphic to the symmetric group S_{n+1}, since $|S^*| = n+1$.

4 The Coxeter group associated with K_3

In this section, we consider the Coxeter group W with its associated graph K_3 of three vertices and three edges. That is $W = W_S$, where $S = \{s_1, s_2, s_3\}$ and $m(s_i, s_i) = 1, m(s_i, s_j) = m(s_j, s_i) = 3$ for distinct $i, j \in \{1, 2, 3\}$. Recall from Proposition 2.5 and (3), there exists a monomorphism $\sigma^* : W_S \to GL(V_S^*)$ with the matrices of $\sigma^*(s_1)$, $\sigma^*(s_2)$, $\sigma^*(s_3)$ as

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \tag{7}$$

with respect to the standard ordered basis $\{\alpha^*_{s_1}, \alpha^*_{s_2}, \alpha^*_{s_3}\}$ of V^*_S. Note that the above three matrices generate an infinite group over $\mathbb{R}$. Let F_2 be the field of two elements $0, 1$. We define an action of W on F_2^3 by $w \cdot u = \sigma^*(w)u$ (mod 2) for $w \in W$ and $u \in F_2^3$. Let $\{e_1, e_2, e_3\}$ be the standard ordered basis of F_2^3. We shall determine the stabilizer W_{e_i} of e_i under the above action, and then determine $W_{e_1} \cap W_{e_2} \cap W_{e_3}$. Note that

$$s_i \cdot e_j := \begin{cases} e_1 + e_2 + e_3, & \text{if } i = j; \\ e_j, & \text{otherwise,} \end{cases} \tag{8}$$

for $1 \leq i, j \leq 3$. Hence with respect to the ordered basis $\{e_1, e_2, e_3\}$, the action of s_1, s_2, s_3 has the following matrix form

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

respectively. The action of a Coxeter group W on F_2^S is also called *lit-only σ-game on Γ*, where Γ is the Coxeter graph associated with (S, m).

From (8), we have the following two Lemmas.

Lemma 4.1. *Let $S = \{s_1, s_2, s_3\}$. Then for distinct two elements $s_i, s_j \in S$,*

$$s_i s_j s_i(e_t) = \begin{cases} e_j, & \text{if } t = i; \\ e_i, & \text{if } t = j; \\ e_t, & \text{otherwise.} \end{cases}$$

In particular, $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W. Thus, $s_i s_j s_i$ and $s_j s_i s_j$ act on the subset $\{e_1, e_2, e_3\}$ of F_2^3 as the same transpositions (e_i, e_j) of the symmetric group on $\{e_1, e_2, e_3\}$ for distinct $i, j \in \{1, 2, 3\}$.

Lemma 4.2. *Let $S = \{s_1, s_2, s_3\}$. Then for distinct three elements $s_i, s_j, s_t \in S$,*

$$s_i s_t s_j s_i(e_k) = \begin{cases} e_j, & \text{if } k = i; \\ e_t, & \text{if } k = j; \\ e_i, & \text{if } k = t. \end{cases}$$

Hence, $s_i s_t s_j s_i$ and $s_i s_j s_t s_i$ are permutations (e_i, e_j, e_t) and (e_i, e_t, e_j) respectively for distinct three numbers $i, j, t \in \{1, 2, 3\}$. □

Definition 4.3. We use the following notations.

$$\begin{aligned} H_{s_1} &= \{s_2, s_3, s_1 s_2 s_3 s_2 s_1\}, \\ H_{s_2} &= \{s_3, s_1, s_2 s_3 s_1 s_3 s_2\}, \\ H_{s_3} &= \{s_1, s_2, s_3 s_1 s_2 s_1 s_3\}. \end{aligned}$$

From the above, we may discover that H_{s_2} can be obtained from H_{s_1} by replacing 1 with 2, 2 with 3 and 3 with 1. In the same way, H_{s_3} can be obtained from H_{s_1} by replacing 1 with 3, 2 with 1 and 3 with 2.

Proposition 4.4. *For $1 \le i \le 3$, $H_{s_i} \subseteq W_{e_i}$*

Proof. Without loss of generality, we may assume $i = 1$, then by the action of W on F_2^3. We may check that s_2, s_3 fix the vector e_1. Next we check that $s_1 s_2 s_3 s_2 s_1 \in W_{e_1}$. By the equation (8), $s_2 s_1 \cdot e_1 = s_2(e_1 + e_2 + e_3) = e_2$. Then, we have that $s_1 s_2 s_3 \cdot e_2 = s_1 s_2 \cdot e_2 = e_1$. Thus $H_{s_1} \subseteq W_{e_1}$. □

We shall prove that H_{s_i} is a generating set of W_{e_i}, for $1 \le i \le 3$. Before this, we introduce the *length function* and the *reduced form* of an element in W.

Definition 4.5. Let $S = \{s_1, s_2, s_3\}$ and W be the Coxeter group associated with (S, m). For each $w \in W$, let r be the smallest integer such that

$$w = s'_1 s'_2 \cdots s'_r$$

for some $s'_i \in S$. r is called the *length* of w, denoted by $\ell(w)$, and call any expression of w as a product of r elements of S a *reduced form*. By convention, $\ell(1) = 0$, and $\ell(s) = 1$ for $s \neq 1$ and $s \in S$. Note that for any reduced form $s'_1 s'_2 \cdots s'_r$, $s'_i \neq s'_{i+1}$ for $i \in \{1, 2, \ldots, r-1\}$.

Proposition 4.6. *For* $1 \leq i \leq 3$, H_{s_i} *generates* W_{e_i}

Proof. We provide the case $i = 1$, and the remaining can be done by symmetry. By Proposition 4.4, $H_{s_1} \subseteq W_{e_1}$.

To prove $W_{e_1} \subseteq \langle H_{s_1} \rangle$, we pick $u \in W_{e_1}$. We show $u \in \langle H_{s_1} \rangle$. Proved by the length $\ell(u)$ of u. This is clear when $\ell(u) = 0$ since $u = 1$ in this case.

By induction, assume that $u \in \langle H_{s_1} \rangle$ if $\ell(u) \leq k - 1$. Suppose $\ell(u) = k$ and $u = s_{i_1} s_{i_2} \cdots s_{i_k} \in W_{e_1}$ in a reduced form, for some $s_{i_j} \in S$. We divide the argument into two cases: $i_k = 2$ or 3 and the other case $i_k = 1$.

Case1 Suppose $s_{i_k} = s_3$(or s_2,). We choose $\alpha = s_3 \in H_{s_1}$(resp. $s_2 \in H_{s_1}$.) Thus, $\ell(u\alpha) < \ell(u)$, and we have $u\alpha \in \langle H_{s_1} \rangle$. Hence $u\alpha\alpha^{-1} \in \langle H_{s_1} \rangle$ i.e. $u \in \langle H_{s_1} \rangle$.

Case2 Suppose $s_{i_k} = s_1$. Clearly $i_k \neq 1$ since $s_1 \notin W_{e_1}$. Now we discuss two cases $s_{i_{k-1}} = s_2$ and $s_{i_{k-1}} = s_3$.

1. Suppose $s_{i_{k-1}} = s_2$. Note that $s_2 s_1 \notin W_{e_1}$, since the first column of the matrix $\sigma^*(s_2 s_1)$ is not e_1. Hence $k \geq 3$. Then $s_{i_{k-2}} = s_1$ or s_3.

(a) Suppose $s_{i_{k-2}} = s_1$. We choose $\alpha = s_2 \in H_{s_1}$. Then

$$u\alpha = (s_{i_1} \cdots s_{i_{k-3}} s_1 s_2 s_1) s_1 s_2 s_1 s_2 s_1$$
$$= s_{i_1} \cdots s_{i_{k-3}} s_2 s_1.$$

Since $s_2 = s_1 s_2 s_1 s_2 s_1$ in the Coxeter group W. Thus $\ell(u\alpha) < \ell(u), u\alpha \in \langle H_{s_1} \rangle$. Hence $u \in \langle H_{s_1} \rangle$, since $\alpha^{-1} \in \langle H_{s_1} \rangle$.

(b) Suppose $s_{i_{k-2}} = s_3$. We choose $\alpha = s_1 s_2 s_3 s_2 s_1, \in H_{s_1}$. Then

$$u\alpha = (s_{i_1} \cdots s_{i_{k-3}} s_3 s_2 s_1) s_1 s_2 s_3 s_2 s_1$$
$$= s_{i_1} \cdots s_{i_{k-3}} s_2 s_1.$$

Thus $\ell(u\alpha) < \ell(u), u\alpha \in \langle H_{s_1} \rangle \Rightarrow u \in \langle H_{s_1} \rangle$.

2. Suppose $s_{i_{k-1}} = s_3$. This can be done similarly by replacing s_2 by s_3 in the above proof, and notice that $s_1 s_2 s_3 s_2 s_1 = s_1 s_3 s_2 s_3 s_1$ in the Coxeter group W_S.

□

Definition 4.7. For a subgroup $G \subseteq W$, let G^i be the set of elements of length i in G. In particular, we list the elements of $W^i_{e_j}$ for $0 \le i \le 4, 1 \le j \le 3$ as following table.

Table:

i	$W^i_{e_1}$	$W^i_{e_2}$	$W^i_{e_3}$
0	$\{e\}$	$\{e\}$	$\{e\}$
1	$\{s_2, s_3\}$	$\{s_3, s_1\}$	$\{s_1, s_2\}$
2	$\{s_2s_3, s_3s_2\}$	$\{s_3s_1, s_1s_3\}$	$\{s_1s_2, s_2s_1\}$
3	$\{s_2s_3s_2\}$	$\{s_3s_1s_3\}$	$\{s_1s_2s_1\}$
4	$\emptyset$	$\emptyset$	$\emptyset$

From the above, $W^i_{e_2}$ can be obtained from $W^i_{e_1}$ by replacing 1 with 2, 2 with 3 and 3 with 1. In the same way, $W^i_{e_3}$ can be obtained from $W^i_{e_1}$ by replacing 1 with 3, 2 with 1 and 3 with 2. We may use $W^i_{e_1} \cap W^i_{e_2} \cap W^i_{e_3}$ to find the set of element of length i in W to fix $\{e_1, e_2, e_3\}$. In addition, we may prove that $W^4_{e_j} = \emptyset$, for $j = 1, 2, 3$. Assume that $e_j = e_1$ and $W^4_{e_1} \neq \emptyset$. Then there is a $s_i s_j s_k s_t$ satisfying $(s_i s_j s_k s_t)e_1 = e_1$ and any adjacent s_i, s_i are distinct. Then $(s_j s_k s_t)e_1 = e_1 + e_2 + e_3$ and $s_i = s_1$. That is, $(s_k s_t)e_1 = e_k, e_k \neq e_1$ for $s_t = s_1$; otherwise $(s_j s_k s_t)e_1 \neq e_1 + e_2 + e_3$. Thus $(s_j s_k s_t)e_1 = e_1 + e_2 + e_3$, for $s_j = s_k$, contradiction.

Definition 4.8.

$$H = \{(s_i s_j s_i s_t)^2 \mid \text{for distinct } i, j, t \in \{1, 2, 3\}\}.$$

In particular, $|H| = 3$ since $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W for distinct $i, j \in \{1, 2, 3\}$.

Theorem 4.9. *$G = W_{e_1} \cap W_{e_2} \cap W_{e_3}$ is the normal subgroup of W generated by the set H.*

Proof. First we prove that G is generated by the set H. By the group action of W on F_2^3, then we can easy to check that H is contained in $W_{e_1} \cap W_{e_2} \cap W_{e_3}$. Thus, $H \subseteq G$.

To prove $G \subseteq \langle H \rangle$, we pick $w \in G$. We show $w \in \langle H \rangle$. Proved by the length $\ell(w)$ of w. This is clear when $\ell(w) = 0$ since $w = 1$ in this case. For any element w of length, $\ell(w) \in \{1, 2, 3, 4\}$ one can check that $w \notin G$ by above Table. So we have known the assertion holds for $\ell(w) \leq 4$.

By induction, assume that $w \in \langle H \rangle$ if $\ell(w) \leq k-1$. Suppose $\ell(w) = k \geq 5$ and $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in G$ in a reduced form, for some $s_{i_j} \in S$. For any $i \in \{1,2,3\}$, we let $s_{i_k} = s_i$ then $s_{i_{k-1}} = s_j$ or s_t, where $j \neq t \in \{1,2,3\} - \{i\}$. By symmetry, we may assume $s_{i_{k-1}} = s_j$. Then we divide $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k}$ into the following 8 cases: (1)$s_is_js_is_js_i$, (2)$s_ts_js_is_js_i$, (3)$s_is_ts_is_js_i$, (4)$s_js_ts_is_js_i$, (5)$s_js_is_ts_js_i$, (6)$s_ts_is_ts_js_i$, (7)$s_is_js_ts_js_i$, (8)$s_ts_js_ts_js_i$, for distinct $i,j,t \in \{1,2,3\}$.

(1) Suppose $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k} = s_is_js_is_js_i$. We choose $\alpha = e \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= we \\ &= s_{i_1}\cdots s_{i_{k-5}}s_is_js_is_js_i(s_is_j)^3 \\ &= s_{i_1}\cdots s_{i_{k-5}}s_js_is_j. \end{aligned}$$

Since $(s_is_j)^3 = e$ in the Coxeter group W. Then, $\ell(w) = \ell(s_{i_1}\cdots s_{i_{k-5}}s_js_is_j) < \ell(w)$, contradiction. Thus, $s_{i_{k-4}}\cdots s_{i_k} \neq s_is_js_is_js_i$.

For the case (2), we may use the same way to prove that $s_{i_{k-4}}\cdots s_{i_k} \neq s_ts_js_is_js_i$.

(3) Suppose $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k} = s_is_ts_is_js_i$. We replace $\alpha = (s_is_js_is_t)^2 \in \langle H \rangle$. Then

$$\begin{aligned} w\alpha &= s_{i_1}\cdots s_{i_{k-5}}s_is_ts_is_js_i(s_is_js_is_t)^2 \\ &= s_{i_1}\cdots s_{i_{k-5}}s_js_is_t. \end{aligned}$$

Thus, $\ell(w\alpha) = \ell(s_{i_1}\cdots s_{i_{k-5}}s_js_is_t) < \ell(w)$. Then $w\alpha \in \langle H \rangle$. Hence $w \in \langle H \rangle$, since $\alpha^{-1} \in \langle H \rangle$.

(4) Suppose $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k} = s_js_ts_is_js_i$. We replace $\alpha = (s_js_is_js_t)^2 \in$

$\langle H \rangle$. Then

$$w\alpha = s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i s_j s_i (s_j s_i s_j s_t)^2$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_j s_t s_i s_j s_i (s_i s_j)^3 (s_j s_i s_j s_t)^2$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_t.$$

Since $(s_j s_i s_j s_t)^2 = (s_i s_j)^3 (s_j s_i s_j s_t)^2$ in the Coxeter group W. Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_i s_j s_t) < \ell(w)$. Hence $w \in \langle H \rangle$.

(5) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_j s_i s_t s_j s_i$. We replace $\alpha = (s_i s_j s_t s_j)^2 \in \langle H \rangle$. Then

$$w\alpha = s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_j)^2$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_j s_i s_j s_t s_j)$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t s_j s_i (s_i s_j s_t s_i s_j s_i s_t s_j)$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_i s_t s_j.$$

Since $(s_i s_j s_t s_j)^2 = ((s_j s_t s_j s_i)^2)^{-1} \in \langle H \rangle$ and $s_i s_j s_i = s_j s_i s_j$ in the Coxeter group W. Thus, $\ell(w\alpha) = \ell(s_{i_1} \cdots s_{i_{k-5}} s_i s_t s_j) < \ell(w)$. Hence $w \in \langle H \rangle$.

(6) Suppose $s_{i_{k-4}} \cdots s_{i_{k-1}} s_{i_k} = s_t s_i s_t s_j s_i$. We replace $\alpha = (s_i s_j s_i s_t)^2 \in \langle H \rangle$. Then

$$w\alpha = s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_i s_t)^2$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_i s_t s_i s_j s_i s_t)$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_t s_i s_t s_j s_i (s_i s_j s_t s_i s_t s_j s_i s_t)$$
$$= s_{i_1} \cdots s_{i_{k-5}} s_j s_i s_t.$$

Since $s_is_js_i = s_js_is_j$ in the Coxeter group W. Thus, $\ell(w\alpha) = \ell(s_{i_1}\cdots s_{i_{k-5}}s_js_is_t) < \ell(w)$. Hence $w \in \langle H\rangle$.

(7) Suppose $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k} = s_is_js_ts_js_i$. We replace $\alpha = (s_is_js_ts_j)^2 \in \langle H\rangle$. Then

$$\begin{aligned} w\alpha &= s_{i_1}\cdots s_{i_{k-5}}s_is_js_ts_js_i(s_is_js_ts_j)^2 \\ &= s_{i_1}\cdots s_{i_{k-5}}s_js_ts_j. \end{aligned}$$

Since $(s_is_js_ts_j)^2 = ((s_js_ts_js_i)^2)^{-1} \in \langle H\rangle$ in the Coxeter group W. Thus, $\ell(w\alpha) = \ell(s_{i_1}\cdots s_{i_{k-5}}s_js_ts_j) < \ell(w)$. Hence $w \in \langle H\rangle$.

(8) Suppose $s_{i_{k-4}}\cdots s_{i_{k-1}}s_{i_k} = s_ts_js_ts_js_i$. We replace $\alpha = e \in \langle H\rangle$. Then

$$\begin{aligned} w\alpha &= we \\ &= s_{i_1}\cdots s_{i_{k-5}}s_ts_js_ts_js_i(s_i(s_js_t)^3s_i) \\ &= s_{i_1}\cdots s_{i_{k-5}}s_js_ts_i. \end{aligned}$$

Since $e = s_i(s_js_t)^3s_i$ in the Coxeter group W. Then, $\ell(w) = \ell(s_{i_1}\cdots s_{i_{k-5}}s_js_ts_i) < \ell(w)$, contradiction. Thus, $s_{i_{k-4}}\cdots s_{i_k} \neq s_ts_js_ts_js_i$.

Next we need to prove that G is normal in W. Recall $\sigma^* : W \to GL(V_S^*)$ is a monomorphism and W acts on F_2^3 by $w \cdot u = \sigma^*(w)u \pmod 2$ for $w \in W$ and $u \in F_2^3$. Let $x \in G, w \in W$. Then for any $i \in \{1,2,3\}$

$$\begin{aligned} w^{-1}xw \cdot e_i &= \sigma^*(w^{-1}xw)e_i \pmod 2 \\ &= \sigma^*(w^{-1})\sigma^*(x)\sigma^*(w)e_i \pmod 2 \\ &= \sigma^*(w^{-1})\sigma^*(w)e_i \pmod 2 \\ &= e_i. \end{aligned}$$

Since $x \in G = W_{e_1} \cap W_{e_2} \cap W_{e_3} \Rightarrow \sigma^*(x)$ fixes any vectors in F_2^3. Thus, $w^{-1}xw \in G$. □

Remark 4.10. *By Theorem 4.9, we may discover some relations in the group W/G as follows. For distinct $i, j, t \in \{1, 2, 3\}$.*

(i) $s_i s_j s_i s_t = s_j s_i s_j s_t = s_t s_i s_j s_i$.

(ii) $s_i s_j s_t s_i = s_j s_t s_i s_j = s_t s_i s_j s_t$.

Then we have the following Cayley graph Cay$(W/G, \{s_1, s_2, s_3\})$ of group W/G with respect to S has vertex set W/G and edge set $\{g(gs) \mid g \in G, s \in S\}$.

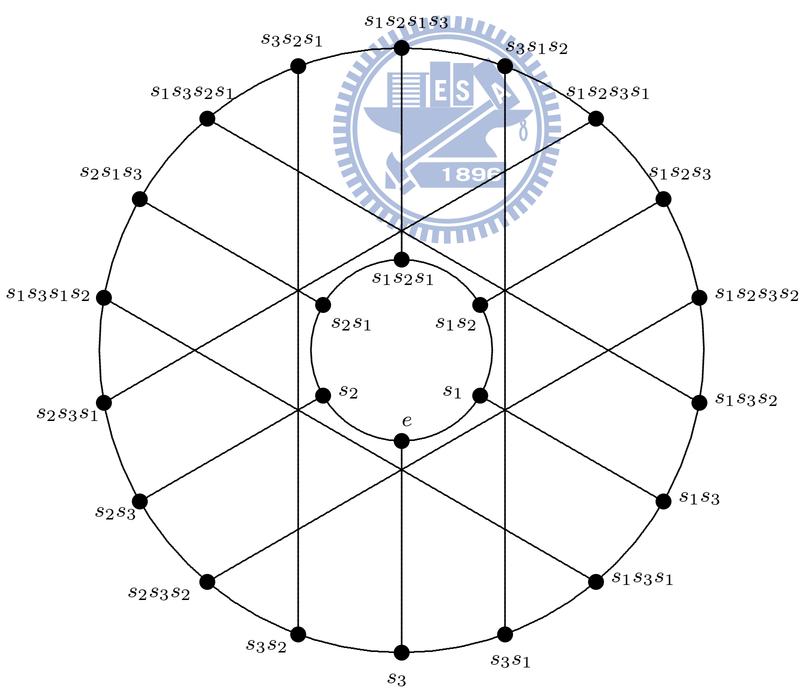

Figure 3. The Cayley graph Cay$(W/G, \{s_1, s_2, s_3\})$.

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