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碩士論文

薛丁格方程的 Strichartz 估計與 長波短波交互作用方程式的半古典極限

Strichartz Estimates for Schrödinger Equation and Semiclassical Limit of the Long Wave-Short Wave Interaction Equations

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摘　　要

此篇文章分為兩個部分。第一部分主要討論薛丁格方程上的Strichartz 估計，我們先從量綱分析的角度觀察不等式中指數對(p,q)所需滿足的關係式，再給予嚴格的証明。從而結論在推導中可允許的(p,q)符合量綱分析的結果。

第二部分討論長波短波交互作用方程式的半古典極限。首先利用Madelung 轉換，討論方程式的流體結構與守恆律。再透過修正的Madelung 轉換與能量估計，證明局部古典解的存在性與唯一性。最後證明半古典極限解的存在性。

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ABSTRACT

There are two parts in this paper. In part I, we discuss the Strichartz estimates on Schrödinger equation. First, we observe the restrictions on exponent pair (p,q) from the viewpoint of dimension. Then we also provide a rigid proof, and conclude that the so-called admissible pair coincides with the arguments of dimensional analysis.

In part II, we study the semiclassical limit of the three coupled long wave-short wave interaction equations. First, we employ the Madelung transformation to discuss the hydrodynamical structures and the conservation laws. Then, we apply the modified Madelung transformation and energy estimates to justify the existence and uniqueness of the local classical solution. Finally, we prove the existence of the semiclassical limit of the solution.

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此篇文章中所提及參考文獻的作者個個都是在該領域中偉大的人物，這些作者提供了富饒的研究成果，指引著我學習方向，除了敬佩，特此也表達感謝之意。

楊雅如小姐也在我寫作期間幫我檢查英文語法上的問題，沒有她的幫忙，此篇文章就不算完整。最後要感謝我的家人，從小家裡爸媽就很注重教育，不僅僅在學業上，更是在待人處事上對我都有所期許，是家人成就了現在的我。

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Part I
Strichartz Estimates for Schrödinger Equation

1 Introduction

In the part I of this paper, we consider the solution of the initial value problem for the nonhomogenerous Schrödinger equation in $\mathbb{R}^n$

$$\partial_t u(t,x) = i\Delta u(t,x) + f(t,x) \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0, \tag{1.2}$$

where $T > 0$, $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ and $f(t,x)$ is a real-valued function. By Duhamel principle, the solution u to (1.1),(1.2) can be described as the following integral equation

$$u(t,x) = e^{it\Delta} u_0(x) + \int_0^t e^{i(t-s)\Delta} f(s,x) ds \tag{1.3}$$

where the operator $e^{it\Delta}$ is defined as

$$e^{it\Delta} u_0(x) = \left(e^{-4\pi^2 it|\xi|^2} \widehat{u_0}(\xi)\right)^{\vee} = \frac{e^{-\frac{|x|^2}{4it}}}{(4\pi it)^{\frac{n}{2}}} * u_0(x). \tag{1.4}$$

The main subject here is to earn more inequalities, known as Strichartz estimates, from some existing decay estimates. We have the following results [3, 17] to answer the above question. Before that, we introduce the notion of admissible pair.

Definition 1.1. (1) *We say that the exponent pair (p,q) is admissible if*

$$\frac{n}{p} + \frac{2}{q} = \frac{n}{2} \tag{1.5}$$

and

$$\begin{cases} 2 \leqslant p \leqslant \infty & for\ n = 1, \\ 2 \leqslant p < \infty & for\ n = 2, \\ 2 \leqslant p \leqslant \frac{2n}{n-2} & for\ n \geqslant 3. \end{cases} \tag{1.6}$$

(2) *We say that the exponent pair (p,q) is an endpoint if*

$$\begin{cases} (p,q) = (\infty, 2) & for\ n = 2, \\ (p,q) = (\frac{2n}{n-2}, 2) & for\ n \geqslant 3. \end{cases} \tag{1.7}$$

Theorem 1.2 (Strichartz estimates)**.** *For admissible pair* (p, q)*, we have*

(1) $\left\|e^{it\Delta}u_0\right\|_{L_t^q L_x^p} \leqslant c_1\|u_0\|_{L_x^2}\,.$ (1.8)

(2) $\left\|\int_{-\infty}^{\infty} e^{it\Delta} f(t,x)dt\right\|_{L_x^2} \leqslant c_2\|f(t,x)\|_{L_t^{q'}L_x^{p'}}\,.$ (1.9)

(3) $\left\|\int_{-\infty}^{\infty} e^{i(t-s)\Delta} f(s,x)ds\right\|_{L_t^q L_x^p} \leqslant c_3\|f(t,x)\|_{L_t^{q'}L_x^{p'}}\,.$ (1.10)

This paper is organized as follows. In section 2, we collect some important preliminaries, including dimensional analysis which provides us an intuitional point of view to treat the equations and inequalities. Furthermore, it gives us a glance why we need the assumption, like the admissible pair. We also provide a rigid proof in section 3. In section 4, there are some remarks on Strichartz estimates.

Notations. $L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, represents the Lebesgue space with norm $\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p dx\right)^{\frac{1}{p}}$. $L^\infty(\mathbb{R}^n)$ is with norm $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{\mathbb{R}^n} |f|$. The mixed Lebesgue space $L_t^q L_x^p(I \times \mathbb{R}^n) = L^q(I; L^p(\mathbb{R}^n))$, $1 \leqslant q < \infty$, consists of $f : I \to L_x^p$ with $\|f\|_{L_t^q L_x^p} = \left(\int_I \|f(t)\|_{L_x^p}^q dt\right)^{\frac{1}{q}} < \infty$. $L_t^\infty L_x^p(I \times \mathbb{R}^n) = L^\infty(I; L^p(\mathbb{R}^n))$ consists of $f : I \to L_x^p$ with $\|f\|_{L_t^\infty L_x^p} = \operatorname{ess\,sup}_{t\in I} \|f\|_{L_x^p} < \infty$.

2 Preliminaries

2.1 Dimensional Analysis

Dimensional analysis is employed extensively in many fields in science especially physics and mathematics [7]. Here we establish some knowledge about applications on mathematical analysis.

Proposition 2.1 (Operation)**.** *We star from two basic operations, differentiation and integration. The notation* $[\cdot]$ *stands for the dimension of a function.*

(1) $\left[\dfrac{d^k f}{dx^k}\right] = \dfrac{\triangle f}{(\triangle x)^k}$ (1.11)

(2) $\left[\int_{\mathbb{R}^n} f dx\right] = (\triangle f)(\triangle x)^n$ (1.12)

Proposition 2.2 (Function space)**.** *We use the notation* $\approx$ *to describe the dimension of a function space.*

(1) (L^p). *If* $f \in L^p(\mathbb{R}^n)$, *then*

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p dx\right)^{\frac{1}{p}} < \infty.$$

Hence $[(\triangle f)^p(\triangle x)^n]^{\frac{1}{p}} = (\triangle f)(\triangle x)^{\frac{n}{p}}$, *and formally we say*

$$L^p \approx \frac{n}{p}. \tag{1.13}$$

(2) $\left(W^{k,p}\right)$. *If* $f \in W^{k,p}(\mathbb{R}^n)$, *then roughly we say that*

$$\left\|\frac{d^k f}{dx^k}\right\|_{L^p} = \left(\int_{\mathbb{R}^n} \left\|\frac{d^k f}{dx^k}\right\|^p dx\right)^{\frac{1}{p}} < \infty.$$

Hence $\left\{\left[(\triangle f)(\triangle x)^{-k}\right]^p (\triangle x)^n\right\}^{\frac{1}{p}} = (\triangle f)(\triangle x)^{\frac{n}{p}-k}$, *and formally we say*

$$W^{k,p} \approx \frac{n}{p} - k. \tag{1.14}$$

Proposition 2.3 (Differential equation). *A differential equation basically is an equality. If it makes sense, the dimension must be balanced. There, we can acquire some properties of this equation before applying any mathematical techniques. For example, the Schrödinger equation*

$$\partial_t u = i\Delta u.$$

Matching the dimension on both sides, we have

$$\frac{\triangle u}{\triangle t} = \frac{\triangle u}{(\triangle x)^2}$$

or

$$\triangle t = (\triangle x)^2 \tag{1.15}$$

This characterizes the relation between time variable and space variable in some sense.

Proposition 2.4 (Inequality). *In mathematical analysis, we usually need various inequalities to estimate our solutions of equations. These inequalities usually have annoying restrictions on its exponents. For example, the Hölder inequality: if* $1 \leqslant p, q \leqslant \infty, \frac{1}{p} + \frac{1}{q} = 1$ *and* $f \in L^p(\Omega), g \in L^q(\Omega)$ *then*

$$\int_\Omega |fg| dx \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

Checking the dimensions, we have

$$(\triangle f)(\triangle g)(\triangle x)^n = (\triangle f)(\triangle x)^{\frac{n}{p}}(\triangle g)(\triangle x)^{\frac{n}{q}}.$$

Hence $\frac{1}{p} + \frac{1}{q} = 1$ *is natural.*

2.2 Decay Estimates, Other Inequalities

In the following we present useful estimates in studying of Schrödinger equations as well as Strichartz estimates.

Proposition 2.5. *Let the operator* $e^{it\Delta}$ *be defined as* (1.4) *and* $t \neq 0$, *then*

(1) $(L^1 - L^\infty)$. $\left\|e^{it\Delta} f\right\|_{L^\infty} \leqslant c_4 |t|^{-\frac{n}{2}} \|f\|_{L^1}$. (1.16)

(2) $(L^2 - L^2)$. $\left\|e^{it\Delta} f\right\|_{L^2} = \|f\|_{L^2}$. (1.17)

(3) $(L^{p'} - L^p)$. $\left\|e^{it\Delta} f\right\|_{L^p} \leqslant c_5 |t|^{-\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)} \|f\|_{L^{p'}}$, (1.18)

$$\text{if } \frac{1}{p} + \frac{1}{p'} = 1 \text{ and } p' \in [1,2].$$

Proof. (1) By Young's inequality.
(2) By the nature of Fourier transform.
(3) Together with (1),(2) and Riesz-Thorin theorem. □

Proposition 2.6 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $0 < \alpha < n$, $1 < p < q < \infty$ *with* $\frac{n}{q} + \alpha = \frac{n}{p}$, *then*

$$\|I_\alpha f\|_{L^q} = \left\| c_\alpha \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right\|_{L^q} \leqslant c_{\alpha,n,p} \|f\|_{L^p}, \tag{1.19}$$

where $c_\alpha = \dfrac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}$.

We ignore the proof. However, from the viewpoint of dimension, we have $\left\{\left[(\triangle f)(\triangle x)^{-(n-\alpha)}(\triangle x)^n\right]^q (\triangle x)^n\right\}^{\frac{1}{q}} = \left[(\triangle f)^p (\triangle x)^n\right]^{\frac{1}{p}}$. Thus, the exponent (p,q) satisfies $\frac{n}{q} + \alpha = \frac{n}{p}$.

Proposition 2.7 (Minkowski integral inequality)**.** *For* $1 \leqslant p < \infty$,

$$\left\| \int_{\mathbb{R}^n} f(x,y) dx \right\|_{L^p_y} \leqslant \int_{\mathbb{R}^n} \|f(x,y)\|_{L^p_y} dx \tag{1.20}$$

Proposition 2.8 (Riesz Representation theorem)**.** *Let* $1 \leqslant p < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$(L^p(\Omega))^* = L^q(\Omega). \tag{1.21}$$

To be more precise, every $\mathcal{L} \in (L^p(\Omega))^*$ *is of the form*

$$\mathcal{L}(f) = \int_\Omega f g dx \quad \forall f \in L^p(\Omega) \tag{1.22}$$

for a unique $g \in L^q(\Omega)$. *Moreover, we have*

$$\|\mathcal{L}\| = \|g\|_{L^q}. \tag{1.23}$$

3 Proof of Theorem 1.2

Before setting to prove the theorem, we check the dimension of Theorem 1.2(a). We obtain that the exponent pair (p,q) satisfies $\frac{n}{p}+\frac{2}{q}=\frac{n}{2}$.

Proof of Theorem 1.2.
We only give the proof of (p,q) which is non-endpoint, i.e. $(p,q)\neq\left(\frac{2n}{n-2},2\right)$ for $n\geqslant 3$. As for endpoint estimates of admissible pair, we refer to [6].

(3) Employing Minkowski integral inequality, $L^{p'}-L^p$ estimate applying to space and Hardy-Littlewood-Sobolev inequality applying to time respectively, we have

$$\begin{aligned}\left\|\int_{\mathbb{R}} e^{i(t-s)\Delta} f(s,x)ds\right\|_{L_t^q L_x^p} &\leqslant \left\|\int_{\mathbb{R}} \left\|e^{i(t-s)\Delta} f(s,x)\right\|_{L_x^p} ds\right\|_{L_t^q}\\ &\leqslant c_{n,p'}\left\|\int_{\mathbb{R}} \frac{1}{|t-s|^{\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}}\|f(s,x)\|_{L_x^{p'}}ds\right\|_{L_t^q}\\ &\leqslant c_{n,p',q'}\|f(s,x)\|_{L_t^{q'}L_x^{p'}}.\end{aligned}$$

At $L^{p'}-L^p$ estimate, we need $\frac{1}{p}+\frac{1}{p'}=1$, $1\leqslant p'<2<p\leqslant\infty$ (for $p=p'=2$, we have (1.17)), and at Hardy-Littlewood-Sobolev inequality, we need $\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)>0$, $1<q'<q<\infty$ and $\frac{1}{q'}=\frac{1}{q}+\alpha=\frac{1}{q}+\left[1-\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)\right]$.

(2) By Hölder inequality and (3), we have

$$\begin{aligned}\left\|\int_{\mathbb{R}} e^{it\Delta} f(t,x)dt\right\|_{L_x^2}^2 &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}} e^{it\Delta} f(t,x)dt\right)\overline{\left(\int_{\mathbb{R}} e^{is\Delta} f(s,x)ds\right)}dx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} f(t,x)\left(\int_{\mathbb{R}} e^{i(t-s)\Delta}\overline{f(s,x)}ds\right)dtdx\\ &\leqslant \|f(t,x)\|_{L_t^{q'}L_x^{p'}}\left\|\int_{\mathbb{R}} e^{i(t-s)\Delta}\overline{f(s,x)}ds\right\|_{L_t^q L_x^p}\\ &\leqslant c_{n,p',q'}\|f(t,x)\|_{L_t^{q'}L_x^{p'}}^2.\end{aligned}$$

At Hölder inequality, we need $\frac{1}{q}+\frac{1}{q'}=1$, and hence (p,q) satisfies $\frac{n}{p}+\frac{2}{q}=\frac{n}{2}$.

(1) Applying Fubini theorem, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{\frac{i|x-y|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} u_0(y) dy \right) f(t,x) dx dt$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{\frac{i|x-y|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} f(t,x) dx \right) u_0(y) dy dt.$$

By Cauchy-Schwarz inequality and (2)

$$\begin{aligned} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(e^{it\Delta} u_0\right)(x) f(t,x) dx dt \right| &= \left| \int_{\mathbb{R}^n} u_0(x) \left(\int_{\mathbb{R}} e^{it\Delta} f(t,x) dt \right) dx \right| \\ &\leqslant \|u_0\|_{L^2_x} \left\| \int_{\mathbb{R}} e^{it\Delta} f(t,x) dt \right\|_{L^2_x} \\ &\leqslant c_{n,p',q'} \|u_0\|_{L^2_x} \|f(t,x)\|_{L^{q'}_t L^{p'}_x}. \end{aligned}$$

Using Riesz Representation theorem, we conclude that

$$\begin{aligned} \left\| e^{it\Delta} u_0 \right\|_{L^q_t L^p_x} &= \sup_{\|f\|_{L^{q'}_t L^{p'}_x} = 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(e^{it\Delta} u_0\right)(x) f(t,x) dx dt \right| \\ &\leqslant c_{n,p',q'} \|u_0\|_{L^2_x}. \end{aligned}$$

This completes the proof. □

From the process of the proof that we establish, we learn that the inequalities must be dimensional balanced as well as the results of the theorem. The admissible pair inherits from all the restriction on the exponents of these inequalities. On the other hand, if we conjecture on a phenomenon ahead, then apply dimensional analysis on it. Observing the relations between the dimensions of the units, it also help us to learn more knowledge about the nature of the phenomenon. It even points the way to the proof.

4 Remarks

Here are some observations. First, $\frac{1}{p}$ and $\frac{1}{q}$ are linear with slope $m_n = -\frac{n}{2}$, for fixed n. The increase of p costs the decrease of q. Second, they all pass through $(2, \infty)$ which also means that $(2, \infty)$ is always admissible for all n. We portray as in Figure 1.

Finally, we end Part I by going back to the Theorem 1.2. If the initial datum u_0 is given in L^2_x, the the solution u is in L^p_x with $p \geqslant 2$. We gain more integrability, that is the so-called smooth effect. This also reflects the dispersive nature of Schrödinger equation partially.

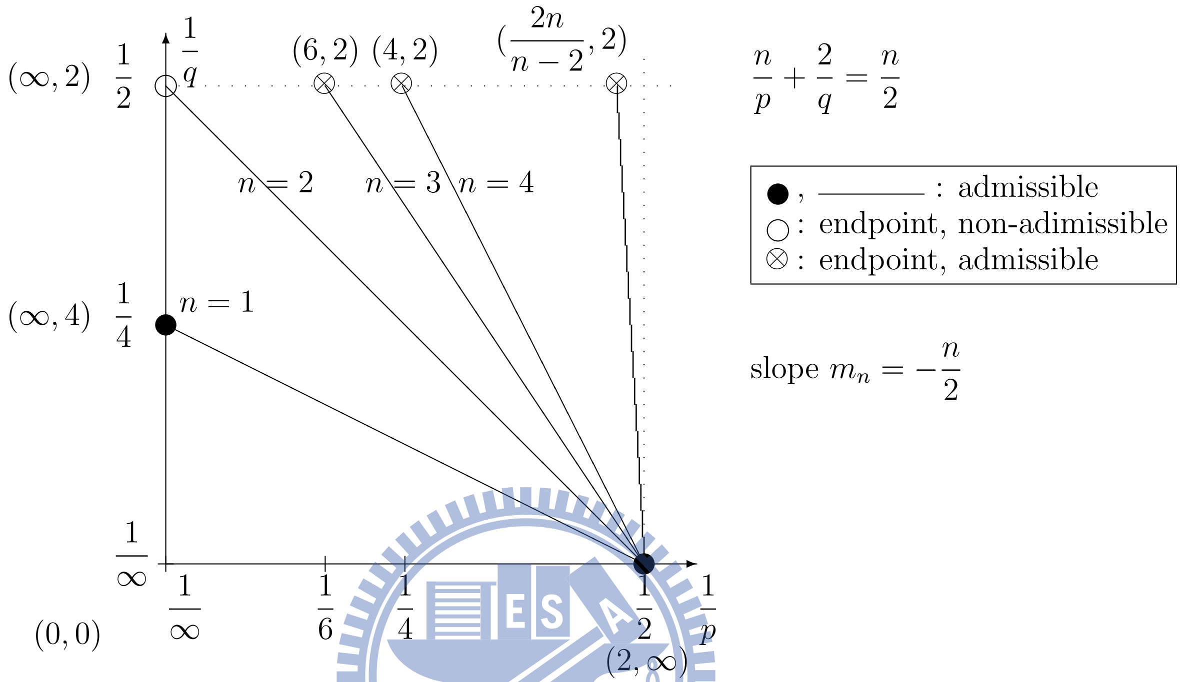

Figure 1: exponent pair

Dimension	$\frac{n}{p} + \frac{2}{q} = \frac{n}{2}$	Range of p	Range of q
$n = 1$	$\frac{1}{p} + \frac{2}{q} = \frac{1}{2}$	$2 \leqslant p \leqslant \infty$	$4 \leqslant q \leqslant \infty$
$n = 2$	$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$	$2 \leqslant p < \infty$	$2 < q \leqslant \infty$
$n \geqslant 3$	$\frac{n}{p} + \frac{2}{q} = \frac{n}{2}$	$2 \leqslant p \leqslant \frac{2n}{n-2}$	$2 \leqslant q \leqslant \infty$

Tabular 1: admissible pair

Part II

Semiclassical Limit of the Long Wave-Short Wave Interaction Equations

5 Introduction

In the Part II, we consider the existence and uniqueness of solutions of the initial value problem for the three coupled long wave-short wave interaction (LSI) equations

$$i\hbar\partial_t\psi^\hbar + \frac{\hbar^2}{2}\partial_{xx}\psi^\hbar = \beta(|\psi^\hbar|^2 + w^\hbar)\psi^\hbar \tag{5.1}$$

$$i\hbar\partial_t\phi^\hbar + \frac{\hbar^2}{2}\partial_{xx}\phi^\hbar = \beta(|\phi^\hbar|^2 + w^\hbar)\phi^\hbar \tag{5.2}$$

$$\partial_t w^\hbar = \beta\partial_x\left(|\psi^\hbar|^2 + |\phi^\hbar|^2\right) \tag{5.3}$$

with initial values

$$\psi^\hbar(0,x) = \psi_0^\hbar(x) \tag{5.4}$$

$$\phi^\hbar(0,x) = \phi_0^\hbar(x) \tag{5.5}$$

$$w^\hbar(0,x) = w_0^\hbar(x) \tag{5.6}$$

where $\beta > 0$, $w^\hbar$ is real-valued and $\psi^\hbar$, $\phi^\hbar$ are complex-valued. $w^\hbar$ characterizes the long wave and $\psi^\hbar$, $\phi^\hbar$ represent the short waves.This system describes the resonance when the group velocity of the short waves and the phase velocity of the long wave coincide.

In section 2, we employ the Madelung transformation to LSI equations (5.1)–(5.3) and rewrite them as a perturbation of the Euler equations. The conservation laws are also derived.

In section 3, we apply the modified Madelung transformation to LSI equations (5.1)–(5.3) and rewrite them as a perturbation of a quasilinear hyperbolic system. For suitable assumptions on initial data, there exists local classical solution to the quasilinear hyperbolic system as well as the LSI equations. Furthermore, the solution that we establish is uniformly bounded in $\hbar$. This allows us to pass to the limit $\hbar \to 0$.

Notations. $H^s = W^{s,2}$ represents the Sobolev space with norm $\|f\|_{H^s} = \|f\|_{W^{s,2}} = \left(\sum_{\alpha\leqslant s}\int |D^\alpha f|^2 dx\right)^{\frac{1}{2}}$ where $D^\alpha f$, the αth derivatives of f, exists in the weak sense. $C([0,T];X)$ consists of $f : [0,T] \to X$ with $\|f\|_{C([0,T];X)} = \max_{0\leqslant t\leqslant T}\|f\|_X < \infty$.

6 Hydrodynamical Structures and Conservation Laws

In this section, we will derive some conservation laws of the LSI equations (5.1)–(5.3) first. For further references (6.1)–(6.26),(6.46)–(6.51), we ignore the superscript $\hbar$.

By Madelung transformation, we introduce the complex-valued wave functions

$$\psi = A_1 \exp\left(i\frac{S_1}{\hbar}\right), \tag{6.1}$$

$$\phi = A_2 \exp\left(i\frac{S_2}{\hbar}\right), \tag{6.2}$$

where A_1, A_2, S_1 and S_2 are real-valued functions. A_1, A_2 are called the amplitudes, and S_1, S_2 the classical actions. Substituting (6.1) (resp.(6.2)) into (5.1) (resp.(5.2)), (A_1, S_1, A_2, S_2) obeys the following equations

$$\partial_t A_1 + \partial_x A_1 \partial_x S_1 + \frac{1}{2} A_1 \partial_{xx} S_1 = 0, \tag{6.3}$$

$$\partial_t S_1 + \frac{1}{2}(\partial_x S_1)^2 + \beta A_1^2 + \beta w = \frac{\hbar^2}{2}\frac{\partial_{xx} A_1}{A_1}, \tag{6.4}$$

$$\partial_t A_2 + \partial_x A_2 \partial_x S_2 + \frac{1}{2} A_2 \partial_{xx} S_2 = 0, \tag{6.5}$$

$$\partial_t S_2 + \frac{1}{2}(\partial_x S_2)^2 + \beta A_2^2 + \beta w = \frac{\hbar^2}{2}\frac{\partial_{xx} A_2}{A_2}. \tag{6.6}$$

Consider the new variables

$$\rho_1 \equiv A_1^2, \quad u_1 \equiv \partial_x S_1, \tag{6.7}$$

$$\rho_2 \equiv A_2^2, \quad u_2 \equiv \partial_x S_2, \tag{6.8}$$

we have the following two conservation laws

$$\partial_t \rho_1 + \partial_x(\rho_1 u_1) = 0, \tag{6.9}$$

$$\partial_t u_1 + \partial_x\left(\frac{1}{2}u_1^2 + \beta w\right) = \frac{\hbar^2}{2}\partial_x \frac{\partial_{xx}\sqrt{\rho_1}}{\sqrt{\rho_1}}, \tag{6.10}$$

$$\partial_t \rho_2 + \partial_x(\rho_2 u_2) = 0, \tag{6.11}$$

$$\partial_t u_2 + \partial_x\left(\frac{1}{2}u_2^2 + \beta w\right) = \frac{\hbar^2}{2}\partial_x \frac{\partial_{xx}\sqrt{\rho_2}}{\sqrt{\rho_2}}. \tag{6.12}$$

Equations (6.9)–(6.12) have the form of a perturbation of the Euler equations with w satisfying

$$\partial_t w = \beta\partial_x(\rho_1 + \rho_2), \tag{6.13}$$

which is equivalent to

$$w(t,x) = w_0(x) + \beta\int_0^t \partial_x(\rho_1 + \rho_2)d\tau. \tag{6.14}$$

Here (6.9) and (6.11) are conservation laws of mass. From (6.9), (6.10) (resp.(6.11), (6.12)), we can also derive the equation of the canonical momentum $\rho_1 u_1$ (resp. $\rho_2 u_2$)

$$\partial_t(\rho_1 u_1) + \partial_x\left(\rho_1 u_1^2 + \frac{\beta}{2}\rho_1^2\right) + \beta\rho_1\partial_x w = \frac{\hbar^2}{4}\partial_x(\rho_1\partial_{xx}\log\rho_1), \tag{6.15}$$

$$\partial_t(\rho_2 u_2) + \partial_x\left(\rho_2 u_2^2 + \frac{\beta}{2}\rho_2^2\right) + \beta\rho_2\partial_x w = \frac{\hbar^2}{4}\partial_x(\rho_2\partial_{xx}\log\rho_2), \tag{6.16}$$

which is not conservative. However, adding (6.15), (6.16) together and employing (6.13), we have the conservation law of momentum as follows

$$\begin{aligned}
&\partial_t\left(\rho_1 u_1 + \rho_2 u_2 - \frac{1}{2}w^2\right) \\
&+ \partial_x\left(\rho_1 u_1^2 + \frac{\beta}{2}\rho_1^2 + \beta\rho_1 w + \rho_2 u_2^2 + \frac{\beta}{2}\rho_2^2 + \beta\rho_2 w\right) \\
&= \frac{\hbar^2}{4}\partial_x(\rho_1\partial_{xx}\log\rho_1 + \rho_2\partial_{xx}\log\rho_2).
\end{aligned} \tag{6.17}$$

So far, we complete the conservation laws of mass and momentum. Next, we will seek for the conservation laws of energy. Multiply (6.9) by $-\frac{1}{2}u_1^2$ and βw respectively, and (6.15) by u_1, we have

$$-\frac{1}{2}u_1^2\,\partial_t\rho_1 - \frac{1}{2}u_1^2\,\partial_x(\rho_1 u_1) = 0, \tag{6.18}$$

$$\beta w\,\partial_t\rho_1 + \beta w\,\partial_x(\rho_1 u_1) = 0, \tag{6.19}$$

$$u_1\,\partial_t(\rho_1 u_1) + u_1\,\partial_x\left(\rho_1 u_1^2 + \frac{\beta}{2}\rho_1^2\right) + \beta\rho_1 u_1\partial_x w = \frac{\hbar^2}{4}u_1\partial_x(\rho_1\partial_{xx}\log\rho_1). \tag{6.20}$$

Summing (6.18), (6.19) and (6.20), we obtain

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\rho_1 u_1^2 + \frac{\hbar^2}{8}\frac{(\partial_x\rho_1)^2}{\rho_1}\right) + \partial_x\left(\frac{1}{2}\rho_1 u_1^3 + \frac{\hbar^2}{8}\frac{u_1(\partial_x\rho_1)^2}{\rho_1} + \beta\rho_1 u_1 w\right) \\
&+ \beta w\partial_t\rho_1 + u_1\partial_x\left(\frac{\beta}{2}\rho_1^2\right) = \frac{\hbar^2}{4}\partial_x\left(\frac{\rho_1 u_1\partial_{xx}\rho_1 - \partial_x(\rho_1 u_1)\partial_x\rho_1}{\rho_1}\right).
\end{aligned} \tag{6.21}$$

Also, from the symmetry point of view, we have

$$
\partial_t\left(\frac{1}{2}\rho_2 u_2^2+\frac{\hbar^2}{8}\frac{(\partial_x\rho_2)^2}{\rho_2}\right)+\partial_x\left(\frac{1}{2}\rho_2 u_2^3+\frac{\hbar^2}{8}\frac{u_2(\partial_x\rho_2)^2}{\rho_2}+\beta\rho_2 u_2 w\right)
+\beta w\partial_t\rho_2+u_2\partial_x\left(\frac{\beta}{2}\rho_2^2\right)=\frac{\hbar^2}{4}\partial_x\left(\frac{\rho_2 u_2\partial_{xx}\rho_2-\partial_x(\rho_2 u_2)\partial_x\rho_2}{\rho_2}\right). \tag{6.22}
$$

Equations (6.21) and (6.22) are not in the conservative forms yet. Adding (6.21), (6.22) together and employing (6.13), we then have the conservation law of energy

$$
\begin{aligned}
&\partial_t\left(\frac{1}{2}\rho_1 u_1^2+\frac{\hbar^2}{8}\frac{(\partial_x\rho_1)^2}{\rho_1}+\frac{\beta}{2}\rho_1^2+\beta\rho_1 w\right.\\
&\qquad\left.+\frac{1}{2}\rho_2 u_2^2+\frac{\hbar^2}{8}\frac{(\partial_x\rho_2)^2}{\rho_2}+\frac{\beta}{2}\rho_2^2+\beta\rho_2 w\right)\\
&+\partial_x\left(\frac{1}{2}\rho_1 u_1^3+\frac{\hbar^2}{8}\frac{u_1(\partial_x\rho_1)^2}{\rho_1}+\beta\rho_1^2 u_1+\beta\rho_1 u_1 w\right.\\
&\qquad\left.+\frac{1}{2}\rho_2 u_2^3+\frac{\hbar^2}{8}\frac{u_2(\partial_x\rho_2)^2}{\rho_2}+\beta\rho_2^2 u_2+\beta\rho_2 u_2 w-\frac{\beta^2}{2}(\rho_1+\rho_2)^2\right)\\
&=\frac{\hbar^2}{4}\partial_x\left(\frac{\rho_1 u_1\partial_{xx}\rho_1-\partial_x(\rho_1 u_1)\partial_x\rho_1}{\rho_1}-\frac{\rho_2 u_2\partial_{xx}\rho_2-\partial_x(\rho_2 u_2)\partial_x\rho_2}{\rho_2}\right).
\end{aligned}\tag{6.23}
$$

Define energy densities E_ψ, E_ϕ by

$$
\begin{aligned}
E_\psi &= E_{\psi,1}+E_{\psi,2}+E_{\psi,3}+E_{\psi,4}\\
&\equiv \frac{1}{2}\rho_1 u_1^2+\frac{\hbar^2}{8}\frac{(\partial_x\rho_1)^2}{\rho_1}+\frac{\beta}{2}\rho_1^2+\beta\rho_1 w, \qquad (6.24)\\
E_\phi &= E_{\phi,1}+E_{\phi,2}+E_{\phi,3}+E_{\phi,4}\\
&\equiv \frac{1}{2}\rho_2 u_2^2+\frac{\hbar^2}{8}\frac{(\partial_x\rho_2)^2}{\rho_2}+\frac{\beta}{2}\rho_2^2+\beta\rho_2 w, \qquad (6.25)
\end{aligned}
$$

then we can rewrite (6.23) as

$$
\begin{aligned}
&\partial_t\left(E_\psi+E_\phi\right)\\
&+\partial_x\left((E_\psi+E_{\psi,3})u_1+(E_\phi+E_{\phi,3})u_2-\frac{\beta^2}{2}(\rho_1+\rho_2)^2\right)\\
&=\frac{\hbar^2}{4}\partial_x\left(\frac{\rho_1 u_1\partial_{xx}\rho_1-\partial_x(\rho_1 u_1)\partial_x\rho_1}{\rho_1}-\frac{\rho_2 u_2\partial_{xx}\rho_2-\partial_x(\rho_2 u_2)\partial_x\rho_2}{\rho_2}\right).
\end{aligned}\tag{6.26}
$$

The total energy of the LSI equations (5.1)–(5.3) is constituted by the classical part, $E_{\psi,1}+E_{\phi,1}$ the kinetic energy, $E_{\psi,3}+E_{\psi,4}+E_{\phi,3}+E_{\phi,4}$ the potential energy, and the quantum part $E_{\psi,2}+E_{\phi,2}$ which is of order $O(\hbar^2)$.

The general problem of the semiclassical limit is to determine the limiting behavior of any function of the field $\psi^\hbar$, $\phi^\hbar$ and $w^\hbar$ as $\hbar \to 0$. It is natural to conjecture that the dispersive term $O(\hbar^2)$ which appears in (6.15) and (6.16) is negligible as $\hbar \to 0$ and the limiting density $(\rho_1, u_1, \rho_2, u_2)$ satisfies the limiting Euler system with initial values

$$\partial_t \rho_1 + \partial_x(\rho_1 u_1) = 0, \tag{6.27}$$

$$\partial_t(\rho_1 u_1) + \partial_x \left(\rho_1 u_1^2 + \frac{\beta}{2}\rho_1^2 \right) + \beta \rho_1 \partial_x w = 0, \tag{6.28}$$

$$\partial_t \rho_2 + \partial_x(\rho_2 u_2) = 0, \tag{6.29}$$

$$\partial_t(\rho_2 u_2) + \partial_x \left(\rho_2 u_2^2 + \frac{\beta}{2}\rho_2^2 \right) + \beta \rho_2 \partial_x w = 0, \tag{6.30}$$

with initial values

$$\rho_{1,0}(x) = \rho_1(0,x) = A_{1,0}^2(x), \tag{6.31}$$

$$u_{1,0}(x) = u_1(0,x) = \partial_x S_{1,0}(x), \tag{6.32}$$

$$\rho_{2,0}(x) = \rho_2(0,x) = A_{2,0}^2(x), \tag{6.33}$$

$$u_{2,0}(x) = u_2(0,x) = \partial_x S_{2,0}(x), \tag{6.34}$$

which w satisfies

$$\partial_t w = \beta \partial_x(\rho_1 + \rho_2), \tag{6.35}$$

$$w(0,x) = w_0(x). \tag{6.36}$$

This argument is self-consistent only if the limiting Euler system (6.27)–(6.36) remains classical. Furthermore, the limiting energy densities will be given by

$$\begin{aligned} E_\psi &= E_{\psi,1} + E_{\psi,3} + E_{\psi,4} \\ &= \frac{1}{2}\rho_1 u_1^2 + \frac{\beta}{2}\rho_1^2 + \beta\rho_1 w, \end{aligned} \tag{6.37}$$

$$\begin{aligned} E_\phi &= E_{\phi,1} + E_{\phi,3} + E_{\phi,4} \\ &= \frac{1}{2}\rho_2 u_2^2 + \frac{\beta}{2}\rho_2^2 + \beta\rho_2 w, \end{aligned} \tag{6.38}$$

and will satisfy

$$\begin{aligned} &\partial_t \left(E_\psi + E_\phi \right) \\ &+ \partial_x \left((E_\psi + E_{\psi,3}) u_1 + (E_\phi + E_{\phi,3}) u_2 - \frac{\beta^2}{2}(\rho_1 + \rho_2)^2 \right) \\ &= 0. \end{aligned} \tag{6.39}$$

Moreover we introduce the modified Madelung transformation as follows

$$\psi = A_1 \exp\left(i\frac{S_1}{\hbar}\right), \tag{6.40}$$

$$A_1 = \sqrt{\rho_1}\exp(i\theta_1),\ u_1 = \partial_x S_1, \tag{6.41}$$

$$\phi = A_2 \exp\left(i\frac{S_2}{\hbar}\right), \tag{6.42}$$

$$A_2 = \sqrt{\rho_2}\exp(i\theta_2),\ u_2 = \partial_x S_2, \tag{6.43}$$

which A_1 and A_2 are complex-valued. Plugging (6.40)–(6.43) into (5.1),(5.2), $(\rho_1, \theta_1, u_1, \rho_2, \theta_2, u_2)$ satisfies

$$\partial_t \rho_1 + \partial_x(\rho_1 u_1 + \hbar \rho_1 \partial_x \theta_1) = 0, \tag{6.44}$$

$$\partial_t \theta_1 + u_1 \partial_x \theta_1 + \frac{\hbar}{2}(\partial_x \theta_1)^2 = \frac{\hbar}{2}\frac{\partial_{xx}\sqrt{\rho_1}}{\sqrt{\rho_1}}, \tag{6.45}$$

$$\partial_t u_1 + u_1 \partial_x u_1 + \beta \partial_x(\rho_1 + w) = 0, \tag{6.46}$$

$$\partial_t \rho_2 + \partial_x(\rho_2 u_2 + \hbar \rho_2 \partial_x \theta_2) = 0, \tag{6.47}$$

$$\partial_t \theta_2 + u_2 \partial_x \theta_2 + \frac{\hbar}{2}(\partial_x \theta_2)^2 = \frac{\hbar}{2}\frac{\partial_{xx}\sqrt{\rho_2}}{\sqrt{\rho_2}}, \tag{6.48}$$

$$\partial_t u_2 + u_2 \partial_x u_2 + \beta \partial_x(\rho_2 + w) = 0, \tag{6.49}$$

which w is given by

$$\partial_t w = \beta \partial_x(\rho_1 + \rho_2), \tag{6.50}$$

or is equivalent to

$$w(t, x) = w_0(x) + \beta \int_0^t \partial_x(\rho_1 + \rho_2) d\tau. \tag{6.51}$$

It is remarkable that the quantum effect in this system is of order $O(\hbar)$ different from the perturbation of the Euler equations (6.9)–(6.14) of order $O(\hbar^2)$.

7 Semiclassical Limit

In this section, we will derive the existence and uniqueness of local classical solutions for LSI equations (5.1)–(5.3) with initial values (5.4)–(5.6). Then we will study their semiclassical limit by utilizing the hydrodynamical structures presented in the previous section.

First, we employ the modified Madelung transformation [4] to rewrite (5.1)–(5.3) into a perturbation of a quasilinear hyperbolic system [5, 14]. Let

$$\psi^\hbar = A_1^\hbar \exp\left(i\frac{S_1^\hbar}{\hbar}\right), \tag{7.1}$$

$$A_1^\hbar = a_1^\hbar + ib_1^\hbar, \quad u_1^\hbar = \partial_x S_1^\hbar, \tag{7.2}$$

$$\phi^\hbar = A_2^\hbar \exp\left(i\frac{S_2^\hbar}{\hbar}\right), \tag{7.3}$$

$$A_2^\hbar = a_2^\hbar + ib_2^\hbar, \quad u_2^\hbar = \partial_x S_2^\hbar, \tag{7.4}$$

then substituting (7.1) (resp.(7.3)) into (5.1) (resp.(5.2)), we have

$$\partial_t A_1^\hbar + \partial_x S_1^\hbar \partial_x A_1^\hbar + \frac{1}{2} A_1^\hbar \partial_{xx} S_1^\hbar = i\frac{\hbar}{2}\partial_{xx} A_1^\hbar, \tag{7.5}$$

$$\partial_t S_1^\hbar + \frac{1}{2}(\partial_x S_1^\hbar)^2 + \beta |A_1^\hbar|^2 + \beta w^\hbar = 0, \tag{7.6}$$

$$\partial_t A_2^\hbar + \partial_x S_2^\hbar \partial_x A_2^\hbar + \frac{1}{2} A_2^\hbar \partial_{xx} S_2^\hbar = i\frac{\hbar}{2}\partial_{xx} A_2^\hbar, \tag{7.7}$$

$$\partial_t S_2^\hbar + \frac{1}{2}(\partial_x S_2^\hbar)^2 + \beta |A_2^\hbar|^2 + \beta w^\hbar = 0. \tag{7.8}$$

Differentiating (7.6) (resp.(7.8)) w.r.t. x and replacing $(A_1^\hbar, S_1^\hbar)$ (resp.$(A_2^\hbar, S_2^\hbar)$) by (7.2) (resp.(7.4)), we have

$$\partial_t a_1^\hbar + u_1^\hbar \partial_x a_1^\hbar + \frac{1}{2} a_1^\hbar \partial_x u_1^\hbar = -\frac{\hbar}{2}\partial_{xx} b_1^\hbar, \tag{7.9}$$

$$\partial_t b_1^\hbar + u_1^\hbar \partial_x b_1^\hbar + \frac{1}{2} b_1^\hbar \partial_x u_1^\hbar = \frac{\hbar}{2}\partial_{xx} a_1^\hbar, \tag{7.10}$$

$$\partial_t u_1^\hbar + u_1^\hbar \partial_x u_1^\hbar + 2\beta a_1^\hbar \partial_x a_1^\hbar + 2\beta b_1^\hbar \partial_x b_1^\hbar + \beta \partial_x w^\hbar = 0, \tag{7.11}$$

$$\partial_t a_2^\hbar + u_2^\hbar \partial_x a_2^\hbar + \frac{1}{2} a_2^\hbar \partial_x u_2^\hbar = -\frac{\hbar}{2}\partial_{xx} b_2^\hbar, \tag{7.12}$$

$$\partial_t b_2^\hbar + u_2^\hbar \partial_x b_2^\hbar + \frac{1}{2} b_2^\hbar \partial_x u_2^\hbar = \frac{\hbar}{2}\partial_{xx} a_2^\hbar, \tag{7.13}$$

$$\partial_t u_2^\hbar + u_2^\hbar \partial_x u_2^\hbar + 2\beta a_2^\hbar \partial_x a_2^\hbar + 2\beta b_2^\hbar \partial_x b_2^\hbar + \beta \partial_x w^\hbar = 0, \tag{7.14}$$

with initial values

$$a_1^\hbar(0,x) = a_{1,0}^\hbar(x),\ b_1^\hbar(0,x) = b_{1,0}^\hbar(x),\ u_1^\hbar(0,x) = u_{1,0}^\hbar x = \partial_x S_1^\hbar(0,x), \tag{7.15}$$

$$a_2^\hbar(0,x) = a_{2,0}^\hbar(x),\ b_2^\hbar(0,x) = b_{2,0}^\hbar(x),\ u_2^\hbar(0,x) = u_{2,0}^\hbar x = \partial_x S_2^\hbar(0,x). \tag{7.16}$$

According to (5.3), $w^\hbar$ is given explicitly by

$$w^\hbar(x,t) = w_0^\hbar(x) + \beta \int_0^t \partial_x \left[(a_1^\hbar)^2 + (b_1^\hbar)^2 + (a_2^\hbar)^2 + (b_2^\hbar)^2\right] d\tau. \tag{7.17}$$

Hence, (7.9)–(7.17) form a quasilinear hyperbolic system which is equivalent to the LSI equations (5.1)–(5.3) with initial values (5.4)–(5.6). The system can be rewritten in the vector form

$$\partial_t U^\hbar + A(U^\hbar)\partial_x U^\hbar + G(w^\hbar) = \frac{\hbar}{2}\mathcal{L}U^\hbar, \tag{7.18}$$

$$U^\hbar(0,x) = U_0^\hbar(x) = (a_{1,0}^\hbar(x), b_{1,0}^\hbar(x), u_{1,0}^\hbar(x), a_{2,0}^\hbar(x), b_{2,0}^\hbar(x), u_{2,0}^\hbar(x))^t, \tag{7.19}$$

$$w^\hbar(0,x) = w_0(x), \tag{7.20}$$

where $U^\hbar = (a_1^\hbar, b_1^\hbar, u_1^\hbar, a_2^\hbar, b_2^\hbar, u_2^\hbar)^t$, $G(w^\hbar) = (0, 0, \beta\partial_x w^\hbar, 0, 0, \beta\partial_x w^\hbar)^t$,

$$A(U^\hbar) = \begin{bmatrix} u_1^\hbar & 0 & \frac{a_1^\hbar}{2} & 0 & 0 & 0 \\ 0 & u_1^\hbar & \frac{b_1^\hbar}{2} & 0 & 0 & 0 \\ 2\beta a_1^\hbar & 2\beta b_1^\hbar & u_1^\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^\hbar & 0 & \frac{a_2^\hbar}{2} \\ 0 & 0 & 0 & 0 & u_2^\hbar & \frac{b_2^\hbar}{2} \\ 0 & 0 & 0 & 2\beta a_2^\hbar & 2\beta b_2^\hbar & u_2^\hbar \end{bmatrix},$$

and

$$\mathcal{L} = \begin{bmatrix} 0 & -\partial_{xx} & 0 & 0 & 0 & 0 \\ \partial_{xx} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_{xx} & 0 \\ 0 & 0 & 0 & \partial_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we introduce S,

$$S = \begin{bmatrix} 4\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{7.21}$$

which is symmetry and positive define for $\beta > 0$. Multiplying (7.18) by S, we have the quasilinear symmetry hyperbolic system

$$S\partial_t U^\hbar + \widetilde{A}(U^\hbar)\partial_x U^\hbar + \widetilde{G}(w^\hbar) = \frac{\hbar}{2}\widetilde{\mathcal{L}}U^\hbar, \tag{7.22}$$

where $\widetilde{G}(w^\hbar) = SG(w^\hbar)$, $\widetilde{\mathcal{L}} = S\mathcal{L}$ and $\widetilde{A}^\hbar = SA^\hbar$ is symmetry. The local existence in time for the initial values (7.19) of the quasilinear symmetry hyperbolic system (7.22) follows the iteration scheme as below. For convenience, we ignore the superscript $\hbar$ in (7.23)–(7.30) and some calculating process. Define $U^0(t,x) = U_0(x)$, $w^0(t,x) = w_0(x)$ where $U_0(x)$, $w_0(x)$ are the given initial values and define $U^{k+1}(t,x)$, $w^{k+1}(t,x)$ inductively as the solution of the linear initial value problem

$$S\partial_t U^{k+1} + \widetilde{A}(U^k)\partial_x U^{k+1} + \widetilde{G}(w^{k+1}) = \frac{\hbar}{2}\widetilde{\mathcal{L}}U^{k+1}, \tag{7.23}$$

$$w^{k+1}(t,x) = w_0(x) + \beta\int_0^t \partial_x\left[(a_1^k)^2 + (b_1^k)^2 + (a_2^k)^2 + (b_2^k)^2\right]d\tau, \tag{7.24}$$

$$U^{k+1}(0,x) = U_0^{k+1}(x) = U_0(x), \tag{7.25}$$

for $k = 0,1,2,\ldots$. Assume $U_0 \in H^s$ and $w_0 \in H^{s+1}$ where s is to be determined. Let U be a solution of (7.18) and belongs to $C^1([0,T];C^2(\Omega))$ which is of compact support for each t. The canonical energy associated with the quasilinear symmetry hyperbolic system (7.18) is defined by

$$(SU,U) = \int U^t SU\,dx. \tag{7.26}$$

The classical energy estimate follows immediately by the symmetry of S, $\widetilde{A}$ and antisymmetry of $\widetilde{\mathcal{L}}$. Indeed,

$$\begin{aligned}
(\widetilde{\mathcal{L}}U,U) &= \int U^t\widetilde{\mathcal{L}}U\,dx = \int (U^t\widetilde{\mathcal{L}}U)^t dx \\
&= \int U^t\left(\widetilde{\mathcal{L}}\right)^t U\,dx = -\int U^t\widetilde{\mathcal{L}}U\,dx \\
&= -(\widetilde{\mathcal{L}}U,U)
\end{aligned}$$

and this implies $(\widetilde{\mathcal{L}}U,U) = 0$. So, if $\widetilde{A}$ together with its derivatives of any desire order are continuous and bounded uniformly in $[0,T]\times\Omega$, by integration by parts, then

$$\begin{aligned}
\frac{d}{dt}(SU,U) &= (S\partial_t U,U) + (SU,\partial_t U) \\
&= 2(S\partial_t U,U) \\
&= \hbar(\widetilde{\mathcal{L}}U,U) - 2(\widetilde{A}\partial_x U,U) - 2(\widetilde{G},U) \\
&= 0 + ((\partial_x\widetilde{A})U,U) - 2(\widetilde{G},U) \\
&\leqslant c_1(t)(SU,U).
\end{aligned}$$

By applying Gronwall inequality, we deduce the energy inequality

$$(SU, U) \leq (SU_0, U_0)e^{\int_0^t c_1(\tau)d\tau}, \tag{7.27}$$

and hence

$$\max_{0\leqslant t\leqslant T} \|U^\hbar(t)\|_{L^2} \leqslant c_2\|U_0^\hbar\|_{L^2}. \tag{7.28}$$

The higher energy estimate can be obtained in the similar way. We differentiate (7.18) w.r.t. x, then multiply on both sides by S, we have

$$S\partial_x\partial_t U + \widetilde{A}\partial_x^2 U + \partial_x\widetilde{A}\partial_x U + \partial_x\widetilde{G} = \frac{\hbar}{2}\widetilde{\mathcal{L}}\partial_x U, \tag{7.29}$$

$$\partial_x U(0, x) = \partial_x U_0(x). \tag{7.30}$$

With similar calculation,

$$\begin{aligned}
\frac{d}{dt}(S\partial_x U, \partial_x U) &= (S\partial_t\partial_x U, \partial_x U) + (S\partial_x U, \partial_t\partial_x U) \\
&= 2(S\partial_t\partial_x U, \partial_x U) \\
&= \hbar(\widetilde{\mathcal{L}}\partial_x U, \partial_x U) - 2(\partial_x\widetilde{A}\partial_x U, \partial_x U) - 2(\widetilde{A}\partial_x\partial_x U, \partial_x U) - 2(\partial_x\widetilde{G}, \partial_x U) \\
&= 0 - 2(\partial_x\widetilde{A}\partial_x U, \partial_x U) + (\partial_x\widetilde{A}\partial_x U, \partial_x U) - 2(\partial_x\widetilde{G}, \partial_x U) \\
&= -(\partial_x\widetilde{A}\partial_x U, \partial_x U) - 2(\partial_x\widetilde{G}, \partial_x U) \\
&\leqslant c_3(t)(S\partial_x U, \partial_x U).
\end{aligned}$$

By Gronwall inequality again, we have

$$\max_{0\leqslant t\leqslant T} \|\partial_x U^\hbar(t)\|_{L^2} \leqslant c_4\|\partial_x U_0^\hbar\|_{L^2}. \tag{7.31}$$

Moreover, the estimate of the time derivative $\partial_t U$ is directly derived from the equation (7.18) itself.

$$\begin{aligned}
\max_{0\leqslant t\leqslant T} \|\partial_t U^\hbar\|_{H^{s-2}} &= \max_{0\leqslant t\leqslant T} \left\| \frac{\hbar}{2}\mathcal{L}U^\hbar - A\partial_x U^\hbar - G(w^\hbar) \right\|_{H^{s-2}} \\
&\leqslant c_5 \max_{0\leqslant t\leqslant T} \|U^\hbar\|_{H^s} + c_6 \max_{0\leqslant t\leqslant T} \|G(w^\hbar)\|_{H^s}.
\end{aligned} \tag{7.32}$$

$\partial_t U^\hbar$ only belongs to H^{s-2} because of the twice derivative appearing in $\mathcal{L}$.

So far, we have shown that for fixed $\hbar$,

$$U^{\hbar,k} \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}) \tag{7.33}$$

for all k. Hence $\left\{U^{\hbar,k}\right\}_{k\in\mathbb{N}}$ is uniformly bounded in k. Moreover, by mean value theorem,

$$\begin{aligned}
&\max_{0\leqslant t\leqslant T}\|U^{\hbar,k}(t+h)-U^{\hbar,k}(t)\|_{H^{s-2}}\\
&=\max_{0\leqslant t\leqslant T}\|\partial_t U^{\hbar,k}(\xi)\cdot h\|_{H^{s-2}},\quad \xi\in(t,t+h)\subset[0,T]\\
&=h\cdot\max_{0\leqslant t\leqslant T}\|\partial_t U^{\hbar,k}(t)\|_{H^{s-2}}
\end{aligned}$$

tends to 0 as h goes to 0, for all k. Thus the sequence $\left\{U^{\hbar,k}\right\}_{k\in\mathbb{N}}$ is equicontinuous. Following the Arzela-Ascoli theorem, there exists

$$U^{\hbar}\in L^{\infty}([0,T];H^s)\cap \mathrm{Lip}([0,T];H^{s-2}),$$

such that as $k\to\infty$

$$U^{\hbar,k}\to U^{\hbar}\quad \text{in}\quad C([0,T];H^{s-2}).$$

Thus, by interpolation inequality,

$$\begin{aligned}
\max_{0\leqslant t\leqslant T}\|U^{\hbar,k_1}-U^{\hbar,k_2}\|_{H^{s-\theta}}&\leqslant c_7\max_{0\leqslant t\leqslant T}\|U^{\hbar,k_1}-U^{\hbar,k_2}\|_{H^{s-2}}\max_{0\leqslant t\leqslant T}\|U^{\hbar,k_1}-U^{\hbar,k_2}\|_{H^{s}}\\
&\leqslant c_8\max_{0\leqslant t\leqslant T}\|U^{\hbar,k_1}-U^{\hbar,k_2}\|_{H^{s-2}}
\end{aligned}$$

for $0<\theta<2$, we have the convergence

$$U^{\hbar,k}\to U^{\hbar}\quad \text{in}\quad C([0,T];H^{s-\theta}).$$

In addition, we discuss the convergence $A(U^k)\partial_x U^{k+1}$ to $A(U)\partial_x U$. Indeed, it can be done with the fact that

$$\partial_x U^{\hbar,k}\to\partial_x U^{\hbar},$$

as $k\to\infty$, since

$$\begin{aligned}
&\|A(U^k)\partial_x U^{k+1}-A(U)\partial_x U\|_{H^{s-1}}\\
&=\|A(U^k)\partial_x U^{k+1}-A(U^k)\partial_x U+A(U^k)\partial_x U-A(U)\partial_x U\|_{H^{s-1}}\\
&\leqslant\|A(U^k)\|_{H^{s-1}}\|\partial_x U^{k+1}-\partial_x U\|_{H^{s-1}}+\|A(U^k)-A(U)\|_{H^{s-1}}\|\partial_x U\|_{H^{s-1}}
\end{aligned}$$

Consequently, we have

$$U^{\hbar}\in C([0,T];H^s).$$

Then the original equation (7.18) implies $U^\hbar \in C^1([0,T]; H^{s-2})$; hence we have the solution

$$U^\hbar \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}). \tag{7.34}$$

Also, from the relation between $U^\hbar$ and $w^\hbar$ in (7.17), we have

$$w^\hbar \in C([0,T]; H^{s-1}) \cap C^1([0,T]; H^{s-3}). \tag{7.35}$$

Furthermore, by Sobolev type inequality, if $s > \frac{1}{2} + 4$ then

$$H^{s-2} \hookrightarrow C^2.$$

This can be easily checked by the dimensions of two function spaces H^{s-2} and C^2, $\frac{1}{2} - (s-2) < \frac{1}{\infty} - 2$. Then we have

$$U^\hbar \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}) \hookrightarrow C^1([0,T]; C^2), \tag{7.36}$$

$$w^\hbar \in C([0,T]; H^{s-1}) \cap C^1([0,T]; H^{s-3}) \hookrightarrow C^1([0,T]; C^1), \tag{7.37}$$

and hence the solution $(U^\hbar, w^\hbar)$ of the quasilinear hyperbolic system (7.18)–(7.20) is classical.

The uniqueness of the classical solution of (7.18) follows from the energy estimate for the difference of two given solutions. Make U and V two solutions with the same initial data. Define $U^* = U - V$, and we have the equation

$$S\partial_t U^* + \widetilde{A}(U)\partial_x U^* + [\widetilde{A}(U) - \widetilde{A}(V)]\partial_x V = \frac{\hbar}{2}\widetilde{\mathcal{L}}U^*. \tag{7.38}$$

With previously similar arguments and U, V are of compact support, we have

$$\begin{aligned}
\frac{d}{dt}(SU^*, U^*) &= (S\partial_t U^*, U^*) + (SU^*, \partial_t U^*) \\
&= 2(S\partial_t U^*, U^*) \\
&= \hbar(\widetilde{\mathcal{L}}U^*, U^*) - 2(\widetilde{A}(U)\partial_x U^*, U^*) - 2([\widetilde{A}(U) - \widetilde{A}(V)]\partial_x V, U^*) \\
&= 0 + ((\partial_x \widetilde{A}(U))U^*, U^*) - 2([\widetilde{A}(U) - \widetilde{A}(V)]\partial_x V, U^*) \\
&\leqslant c_9(t)(SU^*, U^*).
\end{aligned}$$

By Gronwall inequality, we have

$$(SU^*, U^*) \leqslant (SU_0^*, U_0^*)e^{\int_0^t c_9(\tau)d\tau} = 0. \tag{7.39}$$

This implies $U^* = 0$ and hence $U = V$. Therefore the classical solution $(U^\hbar, w^\hbar)$ is unique.

To summarize all this, we have the following result:

Theorem 7.1. *Let $s > \frac{1}{2} + 4$. Assume the initial values*

$$U_0^\hbar = (a_{1,0}^\hbar, b_{1,0}^\hbar, u_{1,0}^\hbar, a_{2,0}^\hbar, b_{2,0}^\hbar, u_{2,0}^\hbar) \in H^s \times H^s \times H^s \times H^s \times H^s \times H^s, \tag{7.40}$$

$$w_0^\hbar \in H^{s+1}, \tag{7.41}$$

then there exists $T > 0$ such that the quasilinear hyperbolic system (7.18) *with initial values* (7.19),(7.20) *has a unique classical solution*

$$U^\hbar \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}) \hookrightarrow C^1([0,T]; C^2), \tag{7.42}$$

$$w^\hbar \in C([0,T]; H^{s-1}) \cap C^1([0,T]; H^{s-3}) \hookrightarrow C^1([0,T]; C^1), \tag{7.43}$$

for all $t \in [0,T]$.

As an immediate consequence, we have the similar result for the LSI equations (5.1)–(5.6).

Theorem 7.2. *Let $s > \frac{1}{2} + 4$. Assume the initial values*

$$(A_{1,0}^\hbar, S_{1,0}^\hbar, A_{2,0}^\hbar, S_{2,0}^\hbar, w_0^\hbar) \in H^s \times H^{s+1} \times H^s \times H^{s+1} \times H^{s+1}, \tag{7.44}$$

then there exists $T > 0$ such that the LSI equations (5.1)–(5.3) *with initial values* (5.4)–(5.6) *have a unique classical solution $(\psi^\hbar, \phi^\hbar, w^\hbar)$ of the form*

$$\psi^\hbar = A_1^\hbar \exp\left(i\frac{S_1^\hbar}{\hbar}\right),$$

$$\phi^\hbar = A_2^\hbar \exp\left(i\frac{S_2^\hbar}{\hbar}\right),$$

$$w^\hbar(t,x) = w_0^\hbar(x) + \beta \int_0^t \partial_x \left[(A_1^\hbar)^2 + (A_2^\hbar)^2\right] d\tau,$$

which $A_1^\hbar$, $S_1^\hbar$, $A_2^\hbar$, $S_2^\hbar$ (resp. $w^\hbar$) are bounded in $L^\infty([0,T]; H^s)$ (resp. $L^\infty([0,T]; H^{s-1})$) uniformly in $\hbar$.

Proof. Since

$$\psi^\hbar = A_1^\hbar \exp\left(i\frac{S_1^\hbar}{\hbar}\right) \quad \textit{and} \quad \phi^\hbar = A_2^\hbar \exp\left(i\frac{S_2^\hbar}{\hbar}\right)$$

where $A_1^\hbar = a_1^\hbar + ib_1^\hbar$, $u_1^\hbar = \partial_x S_1^\hbar$, $A_2^\hbar = a_2^\hbar + ib_2^\hbar$ and $u_2^\hbar = \partial_x S_2^\hbar$, by Theorem 7.1, we have

$$A_1^\hbar \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}),$$

$$\partial_x S_1^\hbar \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-2}),$$

and hence $$S_1^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}).$$

Similarly,

$$A_2^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}),$$
$$\partial_x S_2^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}),$$

and hence $$S_2^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}).$$

By Moser type calculus inequality, we conclude that

$$\psi^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}) \hookrightarrow C^1([0,T];C^2),$$
$$\phi^\hbar \in C([0,T];H^s)\cap C^1([0,T];H^{s-2}) \hookrightarrow C^1([0,T];C^2).$$

Moreover,

$$\begin{aligned} w^\hbar(t,x) &= w_0^\hbar(x) + \beta\int_0^t \partial_x\left[(A_1^\hbar)^2 + (A_2^\hbar)^2\right] d\tau \\ &\in C^1([0,T];C^1), \end{aligned}$$

and thus the theorem follows. □

Because of the nature of the antisymmetry of $\widetilde{\mathcal{L}}$, the term $\hbar(\mathcal{L}U,U)$ vanishs in our estimates. The time interval $[0,T]$ and the boundary for $U^\hbar$ in H^s are independent of $\hbar$. These will allow us to pass to the limit $\hbar\to 0$ in (7.18).

Proposition 7.3. *Let $(\rho_1^\hbar,\theta_1^\hbar,u_1^\hbar,\rho_2^\hbar,\theta_2^\hbar,u_2^\hbar,w^\hbar)$ be in $C^1([0,T];C^2)$ and be the solution of equations (6.44)–(6.51). For $i=1,2$, if $\rho_{i,0}^\hbar(x)>0$ then $\rho_i^\hbar(t,x)>0$, $\forall t\geqslant 0$. Furthermore, when the $\hbar$ varies, $\rho_i^\hbar$ will not be too small; that is, too closed to zero.*

Proof. Since $u_i^\hbar,\ \theta_i^\hbar \in C^1([0,T];C^2)$, $u_i^\hbar+\hbar\partial_x\theta_i^\hbar \in C^1([0,T]\times\mathbb{R})$. From (6.44), we have

$$\partial_t\rho_i^\hbar + \partial_x\left[\rho_i^\hbar(u_i^\hbar+\hbar\partial_x\theta_i^\hbar)\right] = 0, \tag{7.45}$$

or

$$\partial_t\rho_i^\hbar + (u_i^\hbar+\hbar\partial_x\theta_i^\hbar)\partial_x\rho_i^\hbar = -\rho_i^\hbar\partial_x(u_i^\hbar+\hbar\partial_x\theta_i^\hbar). \tag{6.46}$$

In addition, the ordinary differential equations

$$\frac{dx}{dt} = u_i^\hbar + \hbar\partial_x\theta_i^\hbar, \tag{7.47}$$
$$x(\tau) = \xi, \tag{7.48}$$

has a unique solution $x = \Gamma(t)$ which belongs to $C^1([0,T] \times \mathbb{R})$. Equation (7.46) implies

$$\frac{d}{dt}\rho_i^\hbar(t,\Gamma(t)) = -\rho_i^\hbar(t,\Gamma(t))\partial_x(u_i^\hbar + \hbar\partial_x\theta_i^\hbar). \tag{7.49}$$

Integrating over $[0,\tau]$, we have

$$\rho_i^\hbar(\tau,\xi) = \rho_i^\hbar(0,\Gamma(0)) \exp\left[-\int_0^\tau \partial_x(u_i^\hbar + \hbar\partial_x\theta_i^\hbar)dt\right]. \tag{7.50}$$

Hence $\rho_i^\hbar(t,x) > 0$ if $\rho_{i,0}^\hbar(x) > 0$. Moreover, the integration in the r.h.s. of (7.50) will not tend to the infinity when the $\hbar$ varies, hence $\rho_i^\hbar$ will not be too closed to zero. □

The limiting system of the quasilinear hyperbolic system (7.18) with initial value (7.19) is also a quasilinear hyperbolic system as the following shows: (formally letting $\hbar \to 0$)

$$U_t + A(U)U_x + G(w) = 0 \tag{7.51}$$
$$U(0,x) = U_0(x) \tag{7.52}$$
$$w(0,x) = w_0(x) \tag{7.53}$$

where w is given by

$$\partial_t w = \beta\partial_x(a_1^2 + b_1^2 + a_2^2 + b_2^2), \tag{7.54}$$

or is equivalent to

$$w(t,x) = w_0(x) + \beta\int_0^t \partial_x(a_1^2 + b_1^2 + a_2^2 + b_2^2)d\tau. \tag{7.55}$$

This is equivalent to the limiting Euler system (6.27)–(6.36) as long as the solutions are smooth. Next, we will show the existence and uniqueness of the local smooth solution to the system (6.27)–(6.36).

Theorem 7.4. *Let $s > \frac{1}{2} + 4$ and $[0,T]$ be the fixed time interval determined in* Theorem 3.1. *Given initial values $U_0^\hbar$, $U_0 \in H^s$, and $U_0^\hbar$ converges to U_0 in H^s as $\hbar \to 0$. Then, there exists*

$$U \in C([0,T];H^s) \cap C^1([0,T];H^{s-2}) \hookrightarrow C^1([0,T];C^2),$$
$$w \in C([0,T];H^{s-1}) \cap C^1([0,T];H^{s-3}) \hookrightarrow C^1([0,T];C^1),$$

which is a classical solution to the IVP for the limiting quasilinear hyperbolic system (7.51)–(7.55), *and so is to the IVP for the limiting Euler system* (6.27)–(6.36).

Proof. Since $\{U^\hbar\}_\hbar$ is bounded uniformly in $\hbar$, by Arzela-Ascoli theorem and interpolation inequality, we have a function U such that, as $\hbar \to 0$

$$U^\hbar \to U \quad \text{in} \ \ C([0,T];H^{s-\theta}),$$

for $0<\theta<2$. Also, from the equation (7.18) itself, we have

$$U^\hbar \to U \quad \text{in} \ \ C^1([0,T];H^{s-2-\theta}),$$

for $0<\theta<2$. $\mathcal{L}U^\hbar$ is uniformly bounded in H^{s-2}, so the perturbation term $\frac{\hbar}{2}\mathcal{L}U^\hbar$ tends to 0 as $\hbar \to 0$. Hence the sequence converges to a solution of the limiting quasilinear hyperbolic system (7.51)–(7.55). The solution w is then given by (7.55) and belongs to $C^1([0,T];C^1)$. □

Theorem 7.5. *Let* $(\rho_1,u_1,\rho_2,u_2,w)$ *be a solution of the limiting Euler system* (6.27)–(6.36) *on* $[0,T]$*, which initial value* $(\rho_{1,0},u_{1,0},\rho_{2,0},u_{2,0},w_0)$ *belongs to* $H^s\times H^s\times H^s\times H^s\times H^{s+1}$*. Assume* $A^\hbar_{1,0}$ *(resp.* $A^\hbar_{2,0}$*,* $w^\hbar_0$*) converges strongly to* $A_{1,0}$ *(resp.* $A_{2,0}$*,* w_0*) in* H^s *(resp.* H^s*,* H^{s+1}*) as* $\hbar\to 0$*. Then, for* $\hbar$ *small enough, there exists a unique classical solution* $(\psi^\hbar,\phi^\hbar,w^\hbar)$ *to the IVP for the LSI equations* (5.1)–(5.6).

Proof. Consider the difference of (7.18) and (7.51). Define $\widetilde{U}^\hbar = U^\hbar - U$, then we have

$$\partial_t\widetilde{U}^\hbar + A(\widetilde{U}^\hbar+U)\partial_x\widetilde{U}^\hbar + [A(\widetilde{U}^\hbar+U)-A(U)]\partial_x U + \left[G(w^\hbar)-G(w)\right] = \frac{\hbar}{2}\mathcal{L}(\widetilde{U}^\hbar+U). \tag{7.56}$$

We introduce $S=S(\widetilde{U}^\hbar+U)$ which is symmetry, positive define and can symmetrize $A(\widetilde{U}^\hbar+U)$. Multiplying (7.56) by S, we have

$$S\partial_t\widetilde{U}^\hbar + SA(\widetilde{U}^\hbar+U)\partial_x\widetilde{U}^\hbar + S[A(\widetilde{U}^\hbar+U)-A(U)]\partial_x U + S\left[G(w^\hbar)-G(w)\right] = \frac{\hbar}{2}S\mathcal{L}(\widetilde{U}^\hbar+U). \tag{7.57}$$

The energy associated with (7.56) is defined by

$$(S\widetilde{U}^\hbar,\widetilde{U}^\hbar) = \int (\widetilde{U}^\hbar)^t S\widetilde{U}^\hbar dx. \tag{7.58}$$

We apply the energy estimate again.

$$\begin{aligned}\frac{d}{dt}(S\widetilde{U}^\hbar,\widetilde{U}^\hbar) &= (S\partial_t\widetilde{U}^\hbar,\widetilde{U}^\hbar)+(S\widetilde{U}^\hbar,\partial_t\widetilde{U}^\hbar)\\ &= 2(S\partial_t\widetilde{U}^\hbar,\widetilde{U}^\hbar)\\ &= \hbar(S\mathcal{L}(\widetilde{U}^\hbar+U),\widetilde{U}^\hbar)-2(SA(\widetilde{U}^\hbar+U)\partial_x\widetilde{U}^\hbar,\widetilde{U}^\hbar)\\ &\quad -2(S[A(\widetilde{U}^\hbar+U)-A(U)]\partial_xU,\widetilde{U}^\hbar)-2(S[G(w^\hbar)-G(w)],\widetilde{U}^\hbar).\end{aligned}$$

By the antisymmetry of $\mathcal{L}$, we have

$$\hbar(S\mathcal{L}\widetilde{U}^\hbar,\widetilde{U}^\hbar)=0.$$

The Cauchy-Schwarz inequality implies

$$\hbar(S\mathcal{L}U,\widetilde{U}^\hbar)\leqslant \hbar c_{10}\|\mathcal{L}U\|_{L^2}\|\widetilde{U}^\hbar\|_{L^2}\leqslant \hbar c_{11}\|U\|_{H^2}\|\widetilde{U}^\hbar\|_{L^2}\leqslant c_{12}\|\widetilde{U}^\hbar\|^2_{L^2}\ ;$$

$$-2(SA(\widetilde{U}^\hbar+U)\partial_x\widetilde{U}^\hbar,\widetilde{U}^\hbar)=(S(\partial_xA(\widetilde{U}^\hbar+U))\widetilde{U}^\hbar,\widetilde{U}^\hbar)\leqslant c_{13}\|\widetilde{U}^\hbar\|^2_{L^2}\ ;$$

$$\begin{aligned}(S[A(\widetilde{U}^\hbar+U)-A(U)]\partial_xU,\widetilde{U}^\hbar) &\leqslant c_{14}\|[A(\widetilde{U}^\hbar+U)-A(U)]\partial_xU\|_{L^2}\|\widetilde{U}^\hbar\|_{L^2}\\ &\leqslant c_{15}\|\partial_xU\|_{L^2}\|\widetilde{U}^\hbar\|_{L^2}\leqslant c_{16}\|U\|_{H^1}\|\widetilde{U}^\hbar\|_{L^2}\\ &\leqslant c_{17}\|\widetilde{U}^\hbar\|^2_{L^2}\ ;\end{aligned}$$

$$(S[G(w^\hbar)-G(w)],\widetilde{U}^\hbar)\leqslant c_{18}\|\widetilde{U}^\hbar\|^2_{L^2}\ .$$

Hence we have the inequality

$$\frac{d}{dt}(S\widetilde{U}^\hbar,\widetilde{U}^\hbar)\leqslant c_{19}(t)(S\widetilde{U}^\hbar,\widetilde{U}^\hbar).$$

By Gronwall inequality,

$$(S\widetilde{U}^\hbar,\widetilde{U}^\hbar)\leqslant (S\widetilde{U}^\hbar_0,\widetilde{U}^\hbar_0)e^{\int_0^t c_{19}(\tau)d\tau}, \tag{7.59}$$

which the r.h.s. tends to 0 as $\hbar\to 0$ because of $\widetilde{U}^\hbar_0=U^\hbar_0-U_0$ tends to 0. Then the theorem follows. □

We conclude that the behavior of the quasilinear hyperbolic system (7.18) resembles the limiting system (7.51). That is to say, the $\hbar$ appearing in the Euler equations (6.9)–(6.13) is negligible. Hence the quantum equations can be depicted by the classical hydrodynamics equations.

References

[1] B. Desjardins, C.-K. Lin, and T.-C. Tso. Semiclassical limit of the derivative nonlinear schrödinger equation. *Math. Models Methods Appl. Sci.*, 10:261–285, 2000.

[2] J. Dias, M. Figueira, and F. Oliveira. Existence of local strong solutions for a quasilinear benney system. *C. R. Math. Acad. Sci. Paris*, 344:493–496, 2007.

[3] J. Ginibre and G. Velo. Smoothing properties and retarded estimates for some dispersive evolution equations. *Comm. Math. Phys.*, 144:163–188, 1992.

[4] E. Grenier. Semiclassical limit of the nonlinear schrödinger equation in small time. *Proc. Amer. Math. Soc.*, 126:523–530, 1998.

[5] T. Kato. The cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58:181–205, 1975.

[6] M. Keel and T. Tao. Endpoint strichartz estimates. *Amer. J. Math.*, 120:955–980, 1998.

[7] 林琦焜. *Riesz* 位勢與 *Sobolev* 不等式. 交大出版社, 2008.

[8] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.* CBMS-NSF Regional Conference series in applied mathematics, 1973.

[9] J.-H. Lee and C.-K. Lin. The behaviour of solutions of nls equation of derivative type in the semiclassical limit. *Chaos Solitons Fractals*, 13:1475–1492, 2002.

[10] C.-K. Lin. On the fluid-dynamical analogue of the general nonlinear schrödinger equation. *Southeast Asian Bull. Math.*, 22:45–56, 1998.

[11] C.-K. Lin. Singular limit of the modified nonlinear schr0̈dinger equation. *CRM Proc. Lecture Notes, 27*, pages 97–109, 2001.

[12] C.-K. Lin and Y.-S. Wong. Zero-dispersion limit of the short-wave–long-wave interaction equations. *J. Differential Equations*, 228:87–110, 2006.

[13] F. Linares and G. Ponce. *Introduction to nonlinear dispersive equations.* New York : Springer-Verlag, 2009.

[14] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* New York : Springer-Verlag, 1984.

[15] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970.

[16] Walter A. Strauss. *Nonlinear wave equations.* Providence, Rhode Island : American Mathematical Society, 1989.

[17] K. Yajima. Existence of solutions for schrÖdinger evolution equations. *Comm. Math. Phys.*, 110:415–426, 1987.