國 立 交 通 大 學

應用數學系

碩 士 論 文

二階投影平面架構能提供最好的群試設計

--------- 對七物件及最多兩感染物而論

A projective plane of order 2 offers the best group tests

---------for 7 items and at most 2 defectives

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指導教授：翁志文　教授

中 華 民 國 九 十 九 年 六 月

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中文摘要

我們證明一個由二階投影平面的關係矩陣, 刪去一列所得到大小爲6乘以7的二元矩陣其分離性爲二。我們同時證明不存在分離性爲二, 行數小於七, 列數小於六的二元矩陣。

A projective plane of order 2 offers the best group tests

- - - - - - - -for 7 items and at most 2 defectives

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Abstract

We prove that the matrix obtained by deleting a row of the incidence matrix of the projective plane of order 2 is $\overline{2}$-*separable*. We also prove that there is no $\overline{2}$-*separable* $s \times t$ matrix with $s < t < 7$.

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Contents

1 Introduction

In combinatorial group testing, a prototype problem called $(\overline{d}, n)$ problem is to assume that there are up to d defectives among n given items, and the problem is to separate the good items from the defective ones by group tests. A group test is administered on an arbitrary subset S of the items with two possible outcomes; a *negative* outcome means S contains no defectives and a *positive* outcome means S contains at least one defective, not knowing exactly how many or which ones. A group testing algorithm is *nonadaptive* if all tests must be specified at once. A nonadaptive algorithm can be represented by a 0-1 matrix where columns are items, rows are tests, and a 1-entry in cell (i, j) means item j is contained in test i. Note that a column can be viewed as a subset whose elements are indices of the rows incident to the column. Thus we can talk about the union of columns. S.H.Hung and F.K.Hwang [1] prove that what values of n, given d, individual testing is optimal on nonadaptive group testing.

Group testing has applications to biological experiments, DNA sequencing, electrical and chemical testing, coding, etc. The binary matrices have three types: *d-separable*, $\overline{\text{d}}$*-separable* and *d-disjunct* which have been found to be major tools in understanding and constructing a nonadaptive group testing. Hong-Bin Chen and Frank K. Hwang [3] proved that M is a *d-separable* matrix and $1 \le k \le d-1$, then M is $\overline{k+1}$*-separable*, if and only if M is *k-disjunct*. We use the property to prove that the matrix obtained by deleting a row of the incidence matrix of a projective plane of order n is $\overline{n}$*-separable*. In particular, $n = 2$, the matrix obtained by deleting a row of the incidence matrix of the projective plane of order 2 is $\overline{2}$*-separable*. In this paper, we want to show there is no $\overline{2}$-*separable* $s \times t$ matrix with $s < t < 7$. For example, there is a $\overline{2}$-*separable* matrix $M_{5\times7}$. Now, we get a matrix $M_{5\times6}$ by deleting a column from the matrix $M_{5\times7}$. Then, the matrix $M_{5\times6}$ must be not $\overline{2}$-*separable*.

2 The Matrix Representation

Consider a $s \times t$ 0-1 matrix M where R_i and C_j denote row i and column j, respectively. M is called d-*separable* if the boolean sums of d columns are all distinct. M is called $\overline{d}$-*separable* if the boolean sums of $\leq d$ columns are all distinct. M is called d-*disjunct* if the boolean sum of any d columns does not contain any other column. It is clear to know that $\overline{d}$-*separable* implies d-*separable* and d-*separable* implies k-*separable* for every $1 \leq k \leq d$.

Let $B(S)$ denote the boolean sum of a set S of columns.

Lemma 1. *[2] If the matrix M is d-disjunct then M is $\overline{d}$-separable.*

Proof. Suppose that M is not $\overline{d}$-*separable*, i.e., there exist a set K of k columns and another set K' of k' columns, $1 \leq k \leq k' \leq d$, such that $B(K) = B(K')$. Let C_j be a column in $K' \setminus K$. Then $C_j \subseteq B(K)$ and M is not k-*disjunct*, hence not d-*disjunct*. □

Lemma 2. *[2] Deleting any row R_i from a d-disjunct matrix M yields a d-separable matrix M_i.*

Proof. Let S be a set of d columns and S' be an another set of d columns. We claim that $B(S)$ and $B(S')$ must differ in at least 2 rows. Suppose not, $B(S)$ and $B(S')$ differ in one row. Assume $B(S) \subseteq B(S')$, then there is a cloumn C_i in $S \backslash S'$ such that $C_i \subseteq B(S) \subseteq B(S')$. Since, M is *d-disjunct*. This is a contradiction. Hence, they are different even after the deletion of a row. □

Theorem 3. *[3] Let M be a d-separable matrix and $1 \leq k \leq d-1$. Then M is $\overline{k+1}$-separable, if and only if M is k-disjunct.*

Proof. *Sufficiency*:

Suppose to the contrary that there exist two distinct sets S and S' of columns in M, $\mid S \mid \leq k+1$, $\mid S' \mid \leq k+1$, such that $B(S) = B(S')$. By the *d-separable* property of M, we may assume $\mid S \mid < \mid S' \mid \leq k+1$. Then there exist a column $C \in S' \backslash S$. Since $C \subseteq B(S')$, we obtain $C \subseteq B(S)$, which violates the *k-disjunct* property of M.

Necessity:

Suppose M is not *k-disjunct*, i.e., there exist a column C and a set S of k other columns such that $C \subseteq B(S)$. Then $B(S) = B(S')$ where $S' = S \bigcup \{C\}$ and $\mid S \mid, \mid S' \mid \leq k+1$. Hence M is not $\overline{k+1}$-*separable*. □

3 Basic Definitions of BIBD

Definition 4. A *design* is a pair (X, B) such that the following properties are satisfied:

1. X is a set of elements called *points*, and
2. B is a collection of nonempty subsets of X called *blocks*.

Let v, k, and λ be positive integers such that $v > k \geq 2$. A (v, k, λ)-balanced incomplete block design (which we abbreviate to (v, k, λ)-BIBD) is a design (X, B) such that the following properties are satisfied:

1. $| X |= v$,

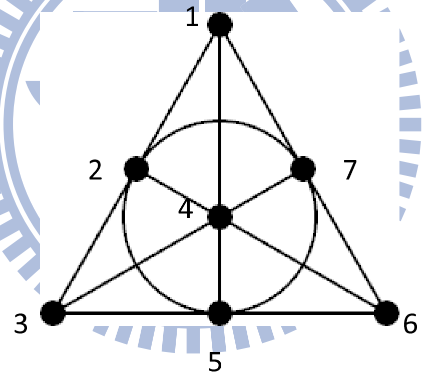
2. each block contains exactly k points, and

3. every pair of distinct points is contained in exactly λ blocks.

Example 5. *A (7, 3, 1)-BIBD*

$$X = \{1, 2, 3, 4, 5, 6, 7\},$$

$$B = \{123, 145, 167, 246, 257, 347, 356\}.$$



We will use the notation that $b =| B |$ and r_x is the number of blocks containing x, for all $x \in X$. In a (v, k, λ)-BIBD, every point has the same number of blocks which pass it. So, we called $r_x = r$.

Definition 6. The incidence matrix of (X, B) is the $v \times b$ 0-1 matrix $M = (m_{i,j})$ defined by the rule

$$m_{i,j} = \begin{cases} 1 & \text{if } x_i \in A_j, \\ 0 & \text{if } x_i \notin A_j, \end{cases}$$

where $A_1, \cdots, A_v$ are blocks. The incidence matrix, M of a (v, k, λ)-BIBD satisfies the following properties:

1. every column of M contains exactly k 1's ,

2. every row of M contains exactly $r = \frac{\lambda(v-1)}{k-1}$ 1's,

3. two distinct rows of M both contain 1 in exactly λ columns.

An $(n^2 + n + 1, n + 1, 1)$-BIBD with $n \geq 2$ is called a *projective plane* of order n and it is a symmetric BIBD $(b = v, r = k)$.

Corollary 7. *The incidence matrix of a projective plane of order n is n-disjunct.*

Proof. In the incidence matrix of a projective plane of order n, any two columns intersect in exactly 1 point and every column contains exactly $n+1$ 1's. Now, we take a set S of n columns. Suppose there exists another cloumn C_j with weight $n + 1$ such that $C_j \subseteq B(S)$. By *Pigeonhole Principle*, the column C_j and one of columns in S have two 1's in their intersection. This is a contradiction. $\square$

Corollary 8. *The matrix obtained by deleting a row of the incidence matrix of the projective plane of order n is $\overline{n}$-separable.*

Proof. Now, we delete a point from a projective plane, i.e.,deleting a row from the incidence matrix M. Let it be M'. By Lemma 2, M' is n-*separable*. Every column in M' contains $n+1$ or n 1's. Now, we claim that M' is $(n-1)$-*disjunct*. We take a set S of $n-1$ columns. Suppose there exists another cloumn C_j with weight n such that $C_j \subseteq B(S)$. By *Pigeonhole Principle*, the column C_j and one of columns in S have two 1's in their intersection. This is a contradiction. M' is $(n-1)$-*disjunct*. By Theorem 3, M' is $\overline{n}$-*separable*. □

In particular, when $n = 2$, this is a (7,3,1)-BIBD, i.e., this is a projective plane of order 2. The 6×7 matrix obtained by deleting a row of the incidence matrix of the projective plane of order 2 is $\overline{2}$-*separable*. Now, we prove that there is no $\overline{2}$-*separable* $s \times t$ matrix with $s < t < 7$.

4 The main result

Theorem 9. *There is no $\overline{2}$-separable $s \times t$ matrix with $s < t < 7$.*

Proof. If the $s \times t$ matrix is not $\overline{2}$-*separable*, the $k \times t$ matrix is not $\overline{2}$-*separable* for $k < s$, either. So, we just consider the condition $s = t - 1$. Suppose to the contrary that there exists a $\overline{2}$-*separable* matrix $M_{s\times t} = [m_{ij}]$. So, any two columns in $M_{s\times t}$ are different. Let (s,t) be such a pair of $M_{s\times t}$ that t is smallest.

First, we have two claims:

1. Each column in $M_{s\times t}$ has at least 2 1's

Suppose there is a zero column in $M_{s\times t}$. Any column union with the zero column is still itself. This is a contradiction.

Suppose there is a column with one 1 in $M_{s\times t}$. Then the other elements of the row corresponding to this 1 are all 0. Otherwise, $M_{s\times t}$ is not a $\overline{2}$-*separable* matrix. So, $M_{s\times t}$ has the following form.

$$\begin{pmatrix} & & & 0 & & & \\ & \ddots & & \vdots & & & \\ & & & 0 & & & \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ & & & 0 & & & \\ & & & \vdots & & \ddots & \\ & & & 0 & & & \end{pmatrix}$$

But we can get a $\overline{2}$-*separable* matrix $M_{s-1\times t-1}$ by deleting the row and the column corresponding to this 1. This is a contradiction to t be the smallest.

2. Each column in $M_{s\times t}$ has at most s-2 1's

Suppose there is a column with all 1's in $M_{s\times t}$. Any column union with the this column is this column. This is a contradiction.

Suppose there is a column with s-1 1's in $M_{s\times t}$. Say this is the last column as follows.

$$\begin{pmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \vdots \\ & & & & 1 \\ & & & & 1 \\ & & & & 0 \end{pmatrix}$$

Then $m_{sj} = 1$, where $1 \leq j \leq t-1$. Otherwise, if a $m_{sk} = 0$ for some $1 \leq k \leq t-1$, the union of column k and column t is identical with column t. So, $M_{s\times t}$=

$$\begin{pmatrix} & & & & & 1 \\ & & & & & 1 \\ & & & & & \vdots \\ & & & & & 1 \\ & & & & & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix}$$

But the union of any column from 1 to t-1 and the column t is a column with all 1's. This is a contradiction to $M_{s\times t}$ be $\overline{2}$-*separable*.

Since $M_{s\times t}$ is $\overline{2}$-*separable*. The columns which we choosed have

$$\binom{t}{1} + \binom{t}{2}$$

conditions. Since each column in $M_{5\times 6}$ has at least 2 1's, the boolean sum of the columns which we choosed have

$$\binom{s}{2} + \binom{s}{3} + \cdots + \binom{s}{s}$$

results. Since the number of results is more than the number of conditions, the $\overline{2}$-*separable* matrix $M_{s\times t}$ satisfies

$$\binom{t}{1}+\binom{t}{2}\leq\binom{s}{2}+\binom{s}{3}+\cdots+\binom{s}{s}.$$

Now we want to discuss the conditions for $t<7$.

1. When s=2, t=3; LHS= 6, RHS= 1.

   This is a contradiction.

2. When s=3, t=4; LHS= 10, RHS= 4.

   This is a contradiction.

3. When s=4, t=5; LHS= 15, RHS= 11.

   This is a contradiction.

4. When s=5, t=6; LHS= 21, RHS= 26.

   This case satisfies the neccessary condition.

Hence, we just consider the matrix $M_{5\times 6}$.

By two claims, the weight of a column in $M_{5\times 6}$ is 2 or 3, so we have 5 conditions.

1. There are at least four columns with weight 3.

2. There are three columns with weight 3 and three columns with weight 2 in $M_{5\times 6}$.

3. There are two columns with weight 3 and four columns with weight 2 in $M_{5\times 6}$.

4. There are only one column with weight 3 in $M_{5\times 6}$.

5. The weight of every column in $M_{5\times 6}$ is 2.

Now we discuss the cases step-by-step.

First, we define $N = (n_1, n_2, n_3, n_4, n_5)$ where n_i is the number of zeros at the ith row in $M_{5\times 6}$.

Case 1: There are at least four columns with weight 3.

We just consider the matrix $M_{5\times 4}$ which consists of four columns with weight 3. In other word, every column in this $M_{5\times 4}$ has 2 0's. So, there are 8 0's in this $M_{5\times 4}$. Now, we want to discuss the conditions of N. If $N = (4, 4, 0, 0, 0)$, then $M_{5\times 4}$ as follows.

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

But there are two columns identical. This is a contradiction. Thus, we find N such that any two columns are different. So, when $N = (4, 4, 0, 0, 0), (4, 3, 1, 0, 0), (4, 2, 2, 0, 0), (4, 2, 1, 1, 0), (3, 3, 2, 0, 0)$ and $(3, 3, 1, 1, 0)$, they do not satisfy the condition. And we find five cases for N which satisfy the condition.

Case 1.1: $N = (4, 1, 1, 1, 1)$. W.L.O.G, we take $M_{5\times4}$ as follows.

A	B	C	D
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

But the unions of any two columns are identical. Hence, $M_{5\times4}$ is not $\overline{2}$-*separable*. Thus, $M_{5\times6}$ is not $\overline{2}$-*separable* in this case.

Case 1.2: $N = (3, 2, 2, 1, 0)$. W.L.O.G, we take $M_{5\times4}$ as follows.

A	B	C	D
0	0	0	1
1	1	0	0
1	0	1	0
0	1	1	1
1	1	1	1

But the union of column A and column B is identical with the union of column A and column C. Hence, $M_{5\times4}$ is not $\overline{2}$-*separable*. Thus, $M_{5\times6}$ is not $\overline{2}$-*separable* in this case.

Case 1.3: $N = (3, 2, 1, 1, 1)$. W.L.O.G, we take $M_{5\times4}$ as follows.

A	B	C	D
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	1
1	1	1	0

But the union of column A and column B is identical with the union of column A and column C. Hence, $M_{5\times4}$ is not $\overline{2}$-*separable*. Thus, $M_{5\times6}$ is not $\overline{2}$-*separable* in this case.

Case 1.4: $N = (2, 2, 2, 1, 1)$. W.L.O.G, we take $M_{5\times4}$ as follows.

A	B	C	D
0	0	1	1
1	0	0	1
1	1	0	0
0	1	1	1
1	1	1	0

But the union of column A and column C is identical with the union of column A and column D. Hence, $M_{5\times4}$ is not $\overline{2}$-*separable*. Thus, $M_{5\times6}$ is not $\overline{2}$-*separable* in this case.

Case 1.5: $N = (2, 2, 2, 2, 0)$. W.L.O.G, we take $M_{5\times 4}$ as follows.

A	B	C	D
0	0	1	1
1	0	0	1
1	1	0	0
0	1	1	0
1	1	1	1

But the union of column A and column C is identical with the union of column B and column D. Hence, $M_{5\times 4}$ is not $\overline{2}$-*separable*. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Note: A column with weight 2 has ten conditions.

A	B	C	D	E	F	G	H	I	J
1	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0
0	1	0	0	1	0	0	1	1	0
0	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	1	0	1	1

In the following cases, we will use it.

Case 2: There are the three columns with weight 3 and three columns with weight 2 in $M_{5\times 6}$.

First, we take three columns with weight 3. There are 6 0's in these cloumns. Now, we find N such that two columns are different. So, when $N = (3, 3, 0, 0, 0)$ and $(3, 2, 1, 0, 0)$, they do not satisfy the condition. And we find four cases for N which satisfy the condition.

Case 2.1: $N = (3, 1, 1, 1, 0)$. W.L.O.G, we take the three columns as follows.

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

But the union of any two columns are identical. Thus, $M_{5\times6}$ is not $\overline{2}$-*separable* in this case.

Case 2.2: $N = (2, 2, 1, 1, 0)$. W.L.O.G, we take the three columns as follows.

$$\begin{array}{ccc} X & Y & Z \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

Since $M_{5\times6}$ is $\overline{2}$-*separable* matrix, we can't take columns B, D, F, G, H, I, J. Hence, we have $M_{5\times6}$ as follows.

$$\begin{array}{cccccc} X & Y & Z & A & C & E \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

But the union of column X and column A is identical with the union of column Y and column Z. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 2.3: $N = (2, 2, 2, 0, 0)$. W.L.O.G, we take the three columns as follows.

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

Since $M_{5\times 6}$ is $\overline{2}$-*separable*, we can't take columns C, D, F, G, H, I, J. Hence, we have $M_{5\times 6}$ as follows.

$$\begin{array}{cccccc} & & & A & B & E \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

But the union of column A and column B is identical with the union of column A and column E. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 2.4: $N = (2, 1, 1, 1, 1)$. W.L.O.G, we take the three columns as follows.

X	Y	Z
0	0	1
1	1	0
1	0	1
0	1	1
1	1	0

But the union of column X and column Z is identical with the union of column Y and column Z. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 3: There are two columns with weight 3 and four columns with weight 2 in $M_{5\times 6}$.

First, we take two columns with weight 3. There are 4 0's in these cloumns. Now, we find N such that two columns are different. So, when $N = (2, 2, 0, 0, 0)$, it do not satisfy the condition. And we find two cases for N which satisfy the condition.

Case 3.1: $N = (1, 1, 1, 1, 0)$. W.L.O.G, we take the two columns as follows.

0	1
0	1
1	0
1	0
1	1

Since $M_{5\times 6}$ is $\overline{2}$-*separable*, we can't take columns A, D, G, H, I, J. Hence, we have $M_{5\times 6}$ as follows.

$$\begin{array}{cccccc} & & B & C & E & F \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

But the union of column B and column F is identical with the union of column C and column E. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 3.2: $N = (2, 1, 1, 0, 0)$. W.L.O.G, we take the two columns as follows.

$$\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{array}$$

Since $M_{5\times 6}$ is $\overline{2}$-*separable*, we can't take columns F, G, H, I, J and we just can take one of columns B, C, D. But we only have five columns. This is a contradiction. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 4: There are only one column with weight 3 in $M_{5\times 6}$.

First, we take the column with weight 3. W.L.O.G, we take one column as follow.

$$\begin{matrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{matrix}$$

Since $M_{5\times 6}$ is $\overline{2}$-*separable*, we can't take columns A, B, E. And we just can take one of columns C, F, H, we just can take one of columns D, G and we just can take two of columns H, I, J. But we only have five columns. This is a contradiction. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Case 5: The weight of every column in $M_{5\times 6}$ is 2.

There are $2\times 6 = 12$ 1's in $M_{5\times 6}$. But there are five rows. By *Pigeonhole Principle*, there must be a row that contains 3 1's in $M_{5\times 6}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

There are $3\times 3 = 9$ 0's in these columns. It remains four rows. By *Pigeonhole*

Principle, there must be a row that contains 3 0's.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

W.L.O.G, the three columns are

$$\begin{array}{c} \begin{array}{ccc} B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$M_{5\times 6}$ is $\overline{2}$-*separable*. But the union of column C and column H is identical with the union of column B and column H, the union of column B and column I is identical with the union of column D and column I and the union of column C and column J is identical with the union of column D and column J. So, we can not take cloumns H, I and J. Since the union of column A and column D is identical with the union of column G and column D, we just can take one of columns A, G. Since the union of column C and column E is identical with the union of column B and column F, we just can take one of columns E, F. But we only have five columns. This is a contradiction. Thus, $M_{5\times 6}$ is not $\overline{2}$-*separable* in this case.

Through above discussion, $M_{5\times 6}$ is not $\overline{2}$-*separable*. Thus, there is no $\overline{2}$-*separable* $s \times t$ matrix with $s < t < 7$. $\square$

Conjecture 10. There is no $\overline{d}$-*separable* matrix of size $s \times t$ with $s < t < d^2 + d + 1$.

5 References

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