國立交通大學

應用數學系

碩 士 論 文

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Decidability Problems of Triangle Edge-coloring

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學生：陳泓勳　　　　　　　　　　　　　　　　指導教授：林松山教授

國立交通大學應用數學學系（研究所）碩士班

摘　　　要

這個研究是關於用邊著色的正三角形與倒三角形拼湊整個平面。如果對每個正三角形與倒三角形相對應的邊都有相同的顏色，則這兩個三角形可以放在相鄰的位置。在這篇論文，我們考慮邊上著兩色與三色的三角形。我們研究的問題為：是否任意可佈滿整個平面的正三角形集合必存在週期性的拼法覆蓋整個平面。我們使用演算法來研究這個問題，然後藉由電腦計算得到結果。最後，這篇論文的主要結果為：在著兩色及三色的前提下，如果整個平面可以被邊著色的三角形拼滿，則整個平面就存在週期性的拼法覆蓋整個平面，反之亦然。

Decidability Problems of Triangle Edge-coloring

student：Hung-Shiun Chen Advisors：Dr. Song-Sun Lin

Department（Institute）of Applied Mathematics
National Chiao Tung University

ABSTRACT

This investigation is about tiling the whole plane with upper triangles and lower triangles which have colors on edges. Upper and lower triangles can be placed side by side if each of the intersections has the same color. In this paper, we consider upper and lower triangle with two and three colors on edges. The problem we studied is that: any set of triangle that can fill with the whole plane whether it can cover the whole plane periodically. We use an algorithm to do the problem and get the result by computers. Finally, the main result of this paper is that the whole plane can be tiling by triangle with two and three colors if and only if the whole plane is covered by the local pattern periodically.

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這篇論文的完成我必須要感謝許多人的教導與鼓勵。首先要感謝我的指導教授 — 林松山教授，感謝您兩年來的指導與勉勵。在研究與學業上的成長，待人處世的禮節與正直的人品更是老師特別著重的部分。老師常勉勵我們「做人要正直。」，學生銘記在心。

其次要感謝的是胡文貴學長。學長的指導與切實的建議，使我能夠解決許多研究上的問題。學長個性溫和有耐心，總是能在我有疑惑時細心的解惑。每當研究上遇到困難，學長都會給予新的方向，讓問題能夠迎刃而解。十分感謝學長的照顧。

其次要感謝的是我的夥伴賴德展。遇到問題能夠耐心的與我討論，提供不同的想法。認真積極的態度，是我可以學習的榜樣。接下來要感謝研究所的朋友們，一起打球、唸書的日子非常開心，生活中總是充滿歡笑。

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目　　錄

1 Introduction

The coloring of unit squares on $\mathbb{Z}^2$ has been studied for many years [6]. In 1961, in studying proving theorem by pattern recognition, Wang [12] started to study the square tiling of a plane. The unit squares with colored edges are arranged side by side so that the adjacent tiles have the same color; the tiles cannot be rotated or reflected. Today, such tiles are called Wang tiles or Wang dominos [4, 6].

The 2×2 unit squares is denoted by $\mathbb{Z}_{2\times 2}$. Let $\mathcal{S}_p$ be a set of p (≥ 1) colors. The total set of all Wang tiles is denoted by $\Sigma^w_{2\times 2}(p) \equiv \mathcal{S}_p^{\mathbb{Z}_{2\times 2}}$. A set $\mathcal{B}$ of Wang tiles, such that $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$, is called a basic set (of Wang tiles). Let $\Sigma(\mathcal{B})$ be the set of all global patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$. Clearly, $\mathcal{P}(\mathcal{B}) \subseteq \Sigma(\mathcal{B})$. The nonemptiness problem is to determine whether or not $\Sigma(\mathcal{B}) \neq \emptyset$. In [12], Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically, i.e.,

$$\text{if } \Sigma(\mathcal{B}) \neq \emptyset \quad \text{then } \mathcal{P}(\mathcal{B}) \neq \emptyset. \tag{1.1}$$

However, in 1966, Berger [4] proved that Wang's conjecture was wrong. He presented a set $\mathcal{B}$ of 20426 Wang tiles that could only tile the plane aperiodically:

$$\Sigma(\mathcal{B}) \neq \emptyset \quad \text{and} \quad \mathcal{P}(\mathcal{B}) = \emptyset. \tag{1.2}$$

Later, he reduced the number of tiles to 104. Now, the nonemptiness problem is called undicidable whenever (1.2) holds. Thereafter, smaller basic sets were found by Knuth, Läuchli, Robinson, Penrose, Ammann, Culik and Kari [5, 6, 7, 10, 11]. Currently, the smallest number of tiles that can tile the plane aperiodically is 13, with five colors: (1.2) holds and then (1.1) fails for $p = 5$ [5].

Recently, Hu and Lin [13] show that Wang's conjecture (1.1) holds provide $p = 2$: any set of Wang tiles with two colors that can tile a plane can tile the plane periodically.

In [13], statement (1.1) is understood by studying how periodic patterns can be generated from a given basic set. First, the minimal cycle generator is introduced. $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ whenever $\mathcal{B}' \subsetneqq \mathcal{B}$. $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$ is called a maximal non-cycle generator if $\mathcal{P}(\mathcal{B}) = \emptyset$ and $\mathcal{P}(\mathcal{B}'') \neq \emptyset$ for any $\mathcal{B}'' \supsetneqq \mathcal{B}$. Given $p \geq 2$, denote the set of all minimal cycle generators by $C(p)$ and the set of maximal non-cycle generators by $\mathcal{N}(p)$. Clearly,

$$C(p) \cap \mathcal{N}(p) = \emptyset. \tag{1.3}$$

Statement (1.1) follows for $p = 2$.

In this work, the triangle edge-coloring of $p = 2$ and 3 are investigated. A square tile can be divided to a upper triangle and a lower triangle. Therefore, this problem is a special case of Wang tile for $p = 3$ which is still under investigation. We apply a similar method in [13] and this problem is deciable in $p = 2$ and 3, i.e, (1.1) holds.

In Section 2, for $p = 3$, the ordering matrix of all 54 local patterns on upper and lower triangle tiles is introduced. These local patterns are classified into two groups. The recurrence formula for patterns on $\mathbb{Z}_{m\times n}$ are derived which is important in proving that maximal non-cycle generator cannot generate global patterns.

In section 3, the procedure to determine the set of all minimum cycle generator $C(3)$ and maximum non-cycle generator $\mathcal{N}(3)$ are introduced. By the assistance of computer, the main result is proved.

2 Ordering Matrix of Triangle Patterns

In this section, the triangle tiles are classified into upper triangle tiles and lower triangle tiles. Denote by the upper triangle $\triangle$. The color of bottom, left and right edge on an upper triangle tile α are denoted $v_1(\alpha)$, $h_1(\alpha)$, $d_1(\alpha)$. Similarly, the lower triangle is denoted by ∇. The color of top, right and left edge on a lower triangle tile β are denoted $v_2(\beta)$, $h_2(\beta)$, $d_2(\beta)$, respectively.

When an upper triangle tile α and a lower triangle tile β satisfying $d_1(\alpha) = d_2(\beta)$, the parallelogram is formed which can be regarded as a square, denoted by $\mathbb{Z}_{2\times 2}$, see Fig 1.

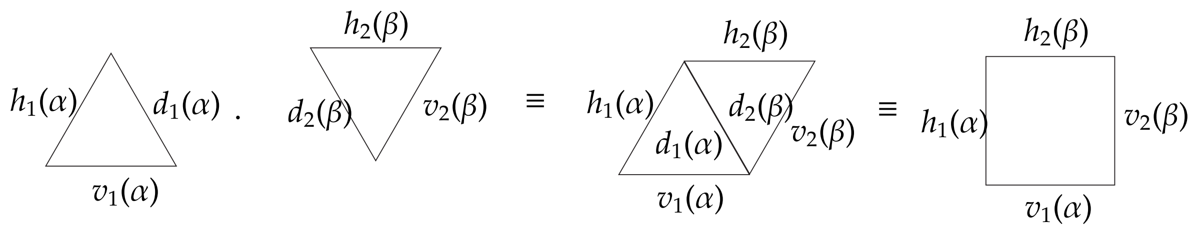


Figure 1.

Denote the set of p colors by $\mathcal{S}_p = \{0, 1, \cdots, p-1\}$. Then the set of all local patterns with colored edge on triangle tiles over $\mathcal{S}_p$ denoted by $\Sigma^T_{2\times 2}(p)$.

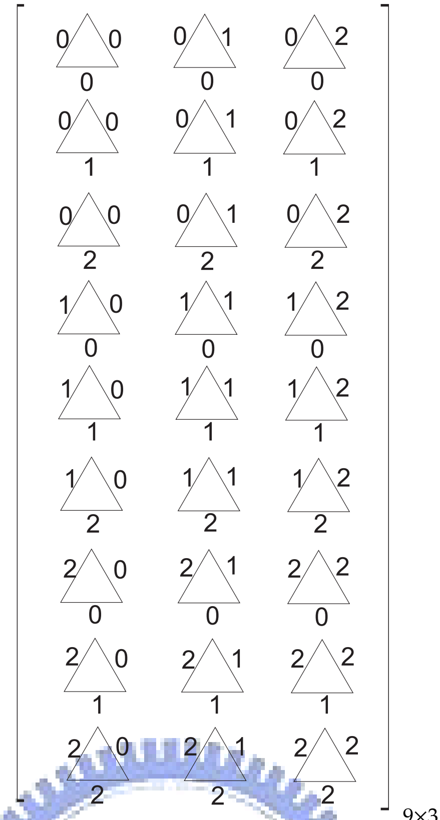
Given $\mathcal{B} \subset \Sigma^T_{2\times 2}(p)$, $\overline{\mathcal{B}}$ means the set of all square tiles can be formed by $\mathcal{B}$. Let $\Sigma_{m\times n}(\overline{\mathcal{B}})$ be the set of all local patterns on $\mathbb{Z}_{m\times n}$ generated by $\overline{\mathcal{B}}$; $\Sigma(\overline{\mathcal{B}})$ be the set of all global patterns generated by $\overline{\mathcal{B}}$, and $\mathcal{P}(\overline{\mathcal{B}})$ be the set of all periodic patterns generated by $\overline{\mathcal{B}}$.

Clearly,

$$\text{if } \Sigma_{m\times n}(\overline{\mathcal{B}}) = \emptyset \text{ for some } m, n \geq 2 \quad \text{then } \Sigma(\overline{\mathcal{B}}) = \emptyset. \tag{2.1}$$

The ordering matrix $\mathbf{Y}^{\triangle} = [y_{\triangle}; i, j]$ of all upper triangle patterns in $\Sigma_{2\times 2}^{T}(3)$ is denoted by

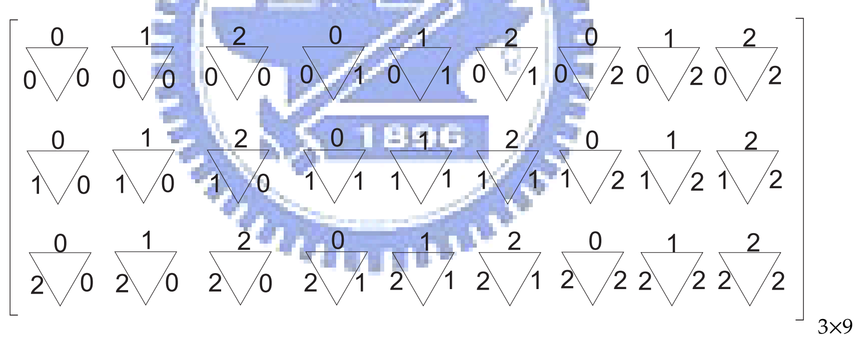
$\mathbf{Y}^{\triangle} =$



(2.2)

The ordering matrix $\mathbf{Y}^{\triangledown} = [y_{\triangledown}; i, j]$ of all lower triangle patterns in $\Sigma_{2\times 2}^{T}(3)$ is denoted by

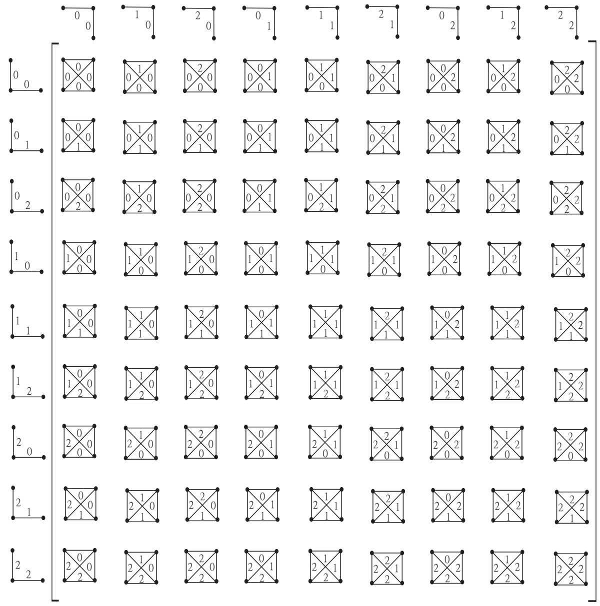
$\mathbf{Y}^{\triangledown} =$



(2.3)

Then, The vertical ordering matrix $\overline{\mathbf{Y}}_{2\times 2} = [y_{i,j}]$ of all local patterns on square in $\Sigma_{2\times 2}(3)$ is denoted by

$$\overline{\mathbf{Y}}_{2\times 2} = \quad (2.4)$$



$$\equiv \mathbf{Y}^{\triangle} \cdot \mathbf{Y}^{\triangledown} \quad (2.5)$$

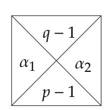
$$= \left[\begin{array}{ccc|ccc|ccc} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,5} & y_{1,6} & y_{1,7} & y_{1,8} & y_{1,9} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} & y_{2,5} & y_{2,6} & y_{2,7} & y_{2,8} & y_{2,9} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} & y_{3,5} & y_{3,6} & y_{3,7} & y_{3,8} & y_{3,9} \\ \hline y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} & y_{4,5} & y_{4,6} & y_{4,7} & y_{4,8} & y_{4,9} \\ y_{5,1} & y_{5,2} & y_{5,3} & y_{5,4} & y_{5,5} & y_{5,6} & y_{5,7} & y_{5,8} & y_{5,9} \\ y_{6,1} & y_{6,2} & y_{6,3} & y_{6,4} & y_{6,5} & y_{6,6} & y_{6,7} & y_{6,8} & y_{6,9} \\ \hline y_{7,1} & y_{7,2} & y_{7,3} & y_{7,4} & y_{7,5} & y_{7,6} & y_{7,7} & y_{7,8} & y_{7,9} \\ y_{8,1} & y_{8,2} & y_{8,3} & y_{8,4} & y_{8,5} & y_{8,6} & y_{8,7} & y_{8,8} & y_{8,9} \\ y_{9,1} & y_{9,2} & y_{9,3} & y_{9,4} & y_{9,5} & y_{9,6} & y_{9,7} & y_{9,8} & y_{9,9} \end{array}\right] \quad (2.6)$$

$$= \begin{bmatrix} \overline{\mathbf{Y}}_{2;1} & \overline{\mathbf{Y}}_{2;2} & \overline{\mathbf{Y}}_{2;3} \\ \overline{\mathbf{Y}}_{2;4} & \overline{\mathbf{Y}}_{2;5} & \overline{\mathbf{Y}}_{2;6} \\ \overline{\mathbf{Y}}_{2;7} & \overline{\mathbf{Y}}_{2;8} & \overline{\mathbf{Y}}_{2;9} \end{bmatrix}_{9\times 9} \quad (2.7)$$

where $\cdot$ means upper and lower triangle pattern is glued together as in Fig 1.

$$\overline{\mathbf{Y}}_2 = \sum_{i=1}^{9} \overline{\mathbf{Y}}_{2;i} \quad (2.8)$$

$$\overline{\mathbf{Y}}_{2;i} = \left[\, y_{2;i;p;q} \,\right]_{3\times 3} = \{ \quad \} \quad (2.9)$$

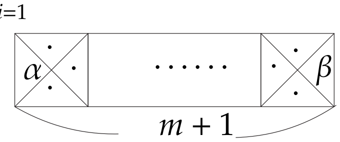


where $i = 1 + \alpha_1 \cdot 3^1 + \alpha_2 \cdot 3^0$, $\alpha_i \in \{0, 1, 2\}$.

Now consider $\overline{\mathbf{Y}}_{m+1}$, for $m \geq 2$, the ordering matrix of all local patterns on $\mathbb{Z}_{(m+1)\times 2}$,

$$\overline{\mathbf{Y}}_{m+1} = \sum_{i=1}^{9} \overline{\mathbf{Y}}_{m+1;i} \quad (2.10)$$

$$\overline{\mathbf{Y}}_{m+1;i} = \{ \quad \} \quad (2.11)$$



where $i = 1 + \alpha \cdot 3^0 + \beta \cdot 3^1$, α, β and $\cdot \in \{0, 1, 2\}$.

The recurrence formula for $\overline{\mathbf{Y}}_{m+1;i}$ in terms of $\overline{\mathbf{Y}}_{m;j}$ are given as follows.

For i = 1, 4, 7

$$\overline{\mathbf{Y}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} y_{2;j+i-1;1;1}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;1;2}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;1;3}\overline{\mathbf{Y}}_{m;3j-2} \\ \sum_{j=1}^{3} y_{2;j+i-1;2;1}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;2;2}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;2;3}\overline{\mathbf{Y}}_{m;3j-2} \\ \sum_{j=1}^{3} y_{2;j+i-1;3;1}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;3;2}\overline{\mathbf{Y}}_{m;3j-2} & \sum_{j=1}^{3} y_{2;j+i-1;3;3}\overline{\mathbf{Y}}_{m;3j-2} \end{bmatrix}_{3^m \times 3^m}$$

For i = 2, 5, 8

$$\overline{\mathbf{Y}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} y_{2;j+i-2;1;1}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;1;2}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;1;3}\overline{\mathbf{Y}}_{m;3j-1} \\ \sum_{j=1}^{3} y_{2;j+i-2;2;1}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;2;2}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;2;3}\overline{\mathbf{Y}}_{m;3j-1} \\ \sum_{j=1}^{3} y_{2;j+i-2;3;1}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;3;2}\overline{\mathbf{Y}}_{m;3j-1} & \sum_{j=1}^{3} y_{2;j+i-2;3;3}\overline{\mathbf{Y}}_{m;3j-1} \end{bmatrix}_{3^m \times 3^m}$$

For i = 3, 6, 9

$$\overline{\mathbf{Y}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} y_{2;j+i-3;1;1}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;1;2}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;1;3}\overline{\mathbf{Y}}_{m;3j} \\ \sum_{j=1}^{3} y_{2;j+i-3;2;1}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;2;2}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;2;3}\overline{\mathbf{Y}}_{m;3j} \\ \sum_{j=1}^{3} y_{2;j+i-3;3;1}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;3;2}\overline{\mathbf{Y}}_{m;3j} & \sum_{j=1}^{3} y_{2;j+i-3;3;3}\overline{\mathbf{Y}}_{m;3j} \end{bmatrix}_{3^m \times 3^m}$$

Given $\mathcal{B} \subset \Sigma^T_{2\times 2}(3)$, the associated vertical transition matrix $\overline{\mathbf{V}}_m(\mathcal{B})$ is obtained from $\overline{\mathbf{Y}}_{m\times 2}$. Indeed, $\overline{\mathbf{V}}_2(\mathcal{B}) = [v_{i,j}]$, where $v_{i,j} = 1$ if and only if $y_{i,j} \in \mathcal{B}$.

The recurrence formula for higher order vertical triangle follow from (15).

$$\overline{\mathbf{V}}_2 = \sum_{i=1}^{9} \overline{\mathbf{V}}_{2;i};$$
$$\overline{\mathbf{V}}_{2;i} = \begin{bmatrix} v_{2;i;p;q} \end{bmatrix}_{3\times 3}$$

For $m \geq 2$,

$$\overline{\mathbf{V}}_{m+1} = \sum_{i=1}^{9} \overline{\mathbf{V}}_{m+1;i}$$

For i = 1, 4, 7

$$\overline{\mathbf{V}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} v_{2;j+i-1;1;1}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;1;2}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;1;3}\overline{\mathbf{V}}_{m;3j-2} \\ \sum_{j=1}^{3} v_{2;j+i-1;2;1}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;2;2}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;2;3}\overline{\mathbf{V}}_{m;3j-2} \\ \sum_{j=1}^{3} v_{2;j+i-1;3;1}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;3;2}\overline{\mathbf{V}}_{m;3j-2} & \sum_{j=1}^{3} v_{2;j+i-1;3;3}\overline{\mathbf{V}}_{m;3j-2} \end{bmatrix}_{3^m \times 3^m}$$

For i = 2, 5, 8

$$\overline{\mathbf{V}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} v_{2;j+i-2;1;1}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;1;2}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;1;3}\overline{\mathbf{V}}_{m;3j-1} \\ \sum_{j=1}^{3} v_{2;j+i-2;2;1}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;2;2}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;2;3}\overline{\mathbf{V}}_{m;3j-1} \\ \sum_{j=1}^{3} v_{2;j+i-2;3;1}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;3;2}\overline{\mathbf{V}}_{m;3j-1} & \sum_{j=1}^{3} v_{2;j+i-2;3;3}\overline{\mathbf{V}}_{m;3j-1} \end{bmatrix}_{3^m \times 3^m}$$

For i = 3, 6, 9

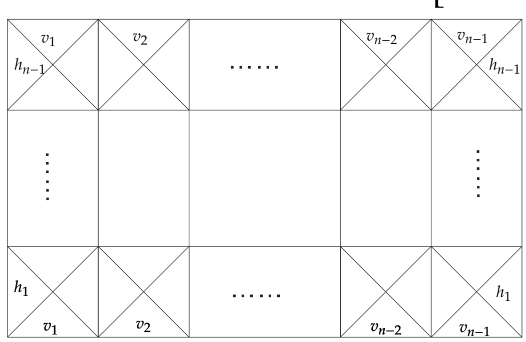
$$\overline{\mathbf{V}}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^{3} v_{2;j+i-3;1;1}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;1;2}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;1;3}\overline{\mathbf{V}}_{m;3j} \\ \sum_{j=1}^{3} v_{2;j+i-3;2;1}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;2;2}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;2;3}\overline{\mathbf{V}}_{m;3j} \\ \sum_{j=1}^{3} v_{2;j+i-3;3;1}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;3;2}\overline{\mathbf{V}}_{m;3j} & \sum_{j=1}^{3} v_{2;j+i-3;3;3}\overline{\mathbf{V}}_{m;3j} \end{bmatrix}_{3^m \times 3^m}$$

Then

$$|\ \Sigma_{(m+1)\times n}(\mathcal{B})\ | = |\ \overline{\mathbf{V}}_{m+1}^{n-1}\ |; \tag{2.12}$$

Now, two set of periodic patterns are studied. Given a periodic sequence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1})^\infty$. Define shift function σ by $\sigma(\alpha_i) = (\alpha_{i+1})$.

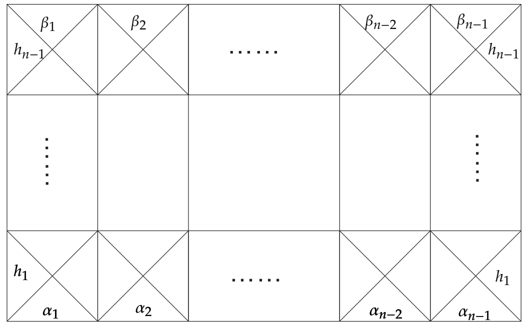
Denote the periodic set of $\mathbb{Z}_{n\times k} = \mathcal{P}_{\mathcal{B}}(\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix})$

= {  }.

which $v_i, h_i \in S_p$.

$$\text{The set of } \mathbb{Z}_{n\times k} \text{ with } l \text{ shift } = \mathcal{P}_{\mathcal{B}}(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix})$$

$= \{$



$\}$

which $\sigma^l(\alpha_1, \cdots, \alpha_{n-1}) = (\beta_1, \cdots, \beta_{n-1})$, $v_i, h_i \in S_p$.

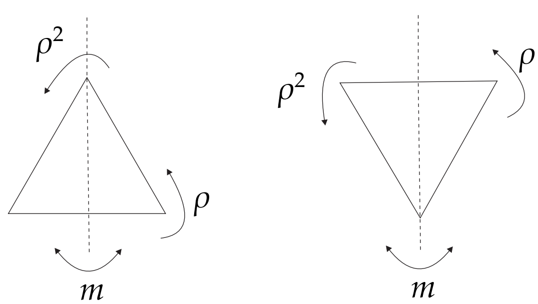
Denoted by $\overline{\mathbf{T}}_m$, periodic of patterns in $\overline{\mathbf{Y}}_{m+1;i}$.

$$\begin{aligned} \overline{\mathbf{T}}_1 &= \sum_{i=1,5,9} \overline{\mathbf{V}}_{2;i}; \\ \overline{\mathbf{T}}_m &= \sum_{i=1,5,9} \overline{\mathbf{V}}_{m+1;i} \end{aligned}$$

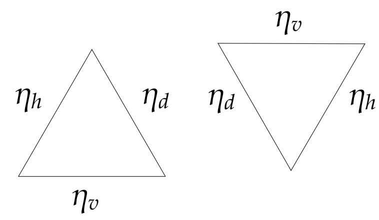
and the # of $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$ periodic is $\Gamma_{\mathcal{B}}(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}) = tr(\mathbf{T}_m^k \mathbf{R}_m^l)$, $0 \le l \le m-1$,
where $\mathbf{R}_m = \begin{bmatrix} r_{m;i,j} \end{bmatrix}$ is the rotational matrix for $p = 3$.
More precisely,

$$r_{ij} = \begin{cases} r_{m;i,3i-2} = 1, \ r_{m;3^{m-1}+i,3i-1} = 1 r_{m;2\cdot 3^{m-1}+i,3i} = 1 & 1 \le i \le 3^{m-1} \\ r_{m;i,j} = 0 & \text{otherwise.} \end{cases}$$

Now, the symmetry of the upper and lower triangle is introduced. The symmetry group of the triangle is D_3, the dihedral group of order six. The group D_3 is generated by the rotation ρ, through $\frac{2\pi}{3}$, and the reflection m about the y-axis. Denote by $D_3 = \{I, \rho, \rho^2, m, m\rho, m\rho^2\}$.



Next, consider the permutation S_p on triangle tiles. The three edge of trinagle tile are mutually independent. If two directions of trinagle are periodic, the remaing one is also periodic [14]. Since, in edge coloring, the permutation of colors in the horizontal, vertical and diagonal directions are mutually independent. Denote the permutations of colors in the horizontal, vertical and diagonal edges by $\eta_h \in S_p$, $\eta_v \in S_p$ and $\eta_d \in S_p$, respectively.



Finally, the upper triangle and lower triangle can be exchanged to each other simultaneously. Denote this act by ξ.

Then for any $\mathcal{B} \subset \Sigma_{2\times2}^{T}(p)$, define the equivalent class $[\mathcal{B}]$ of $\mathcal{B}$ by

$$[\mathcal{B}] = \left\{ \mathcal{B}' \subset \Sigma_{2\times2}^{T}(p) : \mathcal{B}' = \Big(((((\mathcal{B})_{\tau})_{\eta_h})_{\eta_v} \Big)_{\eta_d})_{\xi}, \tau \in D_3 \,, \eta_h, \eta_v, \eta_d \in S_p \text{ and } \xi \right\}.$$

3 Main Result

In this section, we only consider $p = 3$, the result in $p = 2$ is given in Appendix Table 1 and Table 2. Now, we need some definitions.

Definition 3.1. For $\mathcal{B} \subset \Sigma^T_{2\times 2}(p)$,

(i) $\mathcal{B}$ is called a cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$.

(ii) $\mathcal{B}$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ for all $\mathcal{B}' \subsetneqq \mathcal{B}$.

(iii) $\mathcal{B}$ is called a non-cycle generator if $\mathcal{P}(\mathcal{B}) = \varnothing$.

(iv) $\mathcal{B}$ is called a maximal non-cycle generator if $\mathcal{P}(\mathcal{B}) = \emptyset$ and $\mathcal{P}(\mathcal{B}'') \neq \emptyset$ for all $\mathcal{B}'' \supsetneqq \mathcal{B}$.

(v) $C(p)$ is the set of all minimal cycle generators that are subsets of $\Sigma^T_{2\times 2}(p)$.

(vi) $\mathcal{N}(p)$ is the set of all maximal non-cycle generators that are subsets of $\Sigma^T_{2\times 2}(p)$.

Notably, if $\mathcal{B}$ is a cycle generator, then it has a subset of minimal cycle generator. In contrast, if $\mathcal{B}'$ is a non-cycle generator, then $\mathcal{B}'$ is a subset of a maximal non-cycle generator.

The total 27 local patterns on upper triangle tile $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ with three colors $S_3 = \{0, 1, 2\}$ can be ordered as follow:

$$\phi_1((\alpha_0, \alpha_1, \alpha_2)) = 1 + \alpha_0 \cdot 3^0 + \alpha_1 \cdot 3^1 + \alpha_3 \cdot 3^2$$

Hence, the upper triangle tiles are given by $1 \leq \phi_1(\alpha) \leq 27$.

Similarly, the total 27 local patterns on lower triangle $\beta = (\beta_0, \beta_1, \beta_2)$ with S_3 can be ordered by

$$\phi_2((\beta_0, \beta_1, \beta_2)) = 28 + \beta_0 \cdot 3^0 + \beta_1 \cdot 3^1 + \beta_3 \cdot 3^2$$

Hence, the lower triangle tiles are given by $28 \leq \phi_1(\alpha) \leq 54$.

Clearly, ϕ_1 and ϕ_2 are one to one and onto on upper and lower triangle tiles, respectably. Hence, the order of local patterns of triangle tiles from 1 to 54.

Since a local pattern (α) in $\mathbb{Z}_{2\times 2}$ with $h_1(\alpha) = h_2(\alpha)$, $v_1(\alpha) = v_2(\alpha)$ is the periodic pattern which is formed by an upper triagnle tile and a lower triangle tile. We use this idea to divided all 54 local patterns on triangle into two such sets G_1 and G_2 as follows.

Definition 3.2. All 54 local patterns on triangle tile into two sets G_1 and G_2.

$$\begin{aligned} G_1 &= \{\, 1, 2, 3, 10, 11, 12, 19, 20, 21, 28, 29, 30, 37, 38, 39, 46, 47, 48 \,\}. \\ G_2 &= \{\, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, \\ &\quad\; 31, 32, 33, 34, 35, 36, 40, 41, 42, 43, 44, 45, 49, 50, 51, 52, 53, 54 \,\} \end{aligned}$$

There are many ways to choose G_1 and G_2. We want every upper triangle tiles can joint every lower triangle tiles in G_1, then these tiles are easily to form periodic patterns on horizontal direction. The iterative method to obtain $C(3)$ and $\mathcal{N}(3)$ are introduced as follows.

Algorithm 1

$m = 0$
repeat
  $m = m + 1$
  $C(m)$
  $\mathcal{N}(m)$
until ($\Sigma(\mathcal{B}) = \emptyset$ for all $\mathcal{B} \in \mathcal{N}(m)$)

Define

$$\begin{aligned} C(0) &= \{\emptyset\} \\ C(m) &= \{\mathcal{B} : \mathcal{B} \in \mathcal{P}_B(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}), \text{which } m = n \times k, 0 \leq l \leq n-1\} \\ \mathcal{N}(0) &= \{\mathcal{B}_1 \cup \mathcal{B}_2 : \mathcal{B}_1 \subsetneqq G_1, \mathcal{B}_2 \subsetneqq G_2\} \\ \mathcal{N}(m) &= \{\mathcal{B} : \mathcal{B} \in \mathcal{N}(m-1), c \nsubseteq \mathcal{B}, \forall c \in C(m)\} \end{aligned}$$

If this algorithm stops, then $C(3) = C(m)$, $\mathcal{N}(3) = \mathcal{N}(m)$ and this problem is decidable.

Lemma 3.3. *Given $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, which $\mathcal{B}_1 \in G_1$, $\mathcal{B}_2 \in G_2$. For any $\overline{\mathcal{B}_2} \in [\mathcal{B}_2]$, $\exists \mathcal{B}_1^{'} \subseteq \Sigma(G_1)$ such that $\mathcal{B}_1^{'} \cup \overline{\mathcal{B}_2} \in [\mathcal{B}]$.*

Proof. Since $\overline{\mathcal{B}_2} \in [\mathcal{B}_2]$, there exists $\xi \in \mathcal{A}$ s.t $\overline{\mathcal{B}_2} = \xi(\mathcal{B}_2)$.
We know $\xi(\mathcal{B}) = \xi(\mathcal{B}_1) \cup \xi(\mathcal{B}_2) \in [\mathcal{B}]$, where $\xi(\mathcal{B}_1) \subseteq \Sigma(G_1)$ and $\xi(\mathcal{B}_2) \subseteq \Sigma(G_2)$.
Therefore, $\xi(\mathcal{B}_1) \cup \xi(\mathcal{B}_2) = \xi(\mathcal{B}_1) \cup \overline{\mathcal{B}_2} \in [\mathcal{B}]$. □

We can use the above lemma to reduce the algorithm's computation. From lemma, G_2 can be replace by $[G_2]$.

Now, the following theorem gives the classes of minimal cycle generators in $C(3)$ and the classes of maximal non-cycle generators in $\mathcal{N}(3)$. Table 1 and Table 2 present the details of equivalent classes of minimal cycle generators in $C(2)$ and maximal non-cycle generators in $\mathcal{N}(2)$.

Theorem 3.4. *(i) The classes of minimal cycle generators in $C(3)$ are given in Table 3.*

(ii) The classes of maximal non-cycle generators in $\mathcal{N}(3)$ are given in Table 4.

(iii) If $\mathcal{B} \in \mathcal{N}(3)$, then $\Sigma(\mathcal{B}) = \emptyset$.
Furthermore, (1.1) holds for $p = 3$.

Proof. The basic sets in Table 3 are easily seen to be minimal cycle generators. The basic sets in Table 4 are obtained from the minimal cycle generators in Table 3 by finding all maximal basic sets $\mathcal{B} \subset \Sigma_{2\times 2}^{T}(3)$ that do not contain any minimal cycle generator in Table 3.

Then, to prove (i), (ii) and (iii), only $\Sigma(\mathcal{B}) = \emptyset$ for all $\mathcal{B} \in \mathcal{N}(3)$ need to be proven. From the transition matrix $\overline{\mathbf{V}}_m$, all the case in Table 4 has be straightforwardly proven by $\Gamma_{7\times 10}(\mathcal{B}) = 0$ for all $\mathcal{B} \in \mathcal{N}(3)$; then, $\Sigma(\mathcal{B}) = \emptyset$ for all $\mathcal{B} \in \mathcal{N}(3)$. Therefore, the results (i), (ii) and (iii) hold.

Finally, from (iii), $\Sigma(\mathcal{B}) = \emptyset$ is easily seen for any $\mathcal{B} \subset \Sigma_{2\times 2}^{T}(3)$ with $\mathcal{P}(\mathcal{B}) = \emptyset$. Therefore, (1.1) holds for $p = 3$ in edge coloring of triangle. The proof is complete. □

A Table1

Table 1: Minimal Data with P = 2(Classify)

Tile	$\mathcal{B}$	$\mathbb{Z}_{m\times n}$
2tile	{ 1, 9 }	$\mathbb{Z}_{2\times 2}$
4tile	{ 1, 7, 11, 13 }	$\mathbb{Z}_{3\times 2}$
	{ 1, 8, 11, 14 }	$\mathbb{Z}_{3\times 2}$
5tile	{ 1, 8, 10, 11, 13 }	$\mathbb{Z}_{4\times 2}$

B Table2

Table 2: Maximal Data with P = 2(Classify)

Tile	$\mathcal{B}$
8tile	{ 1, 2, 3, 4, 5, 6, 7, 8 }
	{ 1, 2, 3, 4, 5, 6, 7, 16 }
	{ 1, 2, 3, 5, 6, 7, 12, 16 }
	{ 1, 2, 3, 5, 7, 12, 14, 16 }
	{ 1, 2, 3, 5, 12, 14, 15, 16 }
	{ 1, 3, 5, 7, 10, 12, 14, 16 }

C Table3

Table 3: Minimal Data (Classify)

Tile	$\mathcal{B}$
2tile	{ 1, 28 }
4tile	{ 1, 5, 29, 31 }
	{ 1, 14, 29, 40 }
5tile	{ 1, 5, 11, 29, 40 }
6tile	{ 1, 2, 15, 30, 31, 38 }
	{ 1, 5, 9, 29, 33, 34 }
	{ 1, 5, 12, 29, 33, 37 }
	{ 1, 5, 18, 29, 33, 43 }
	{ 1, 5, 18, 30, 35, 40 }
	{ 1,14, 27, 29, 42, 52 }
	{ 1,14, 27, 33, 43, 47 }
	{ 1, 5, 12, 29, 33, 40 }
	{ 1, 5, 18, 36, 38, 40 }
7tile	{ 1, 5, 9, 11, 29, 33, 43 }
	{ 1, 2, 6, 18, 30, 34, 41 }
	{ 1, 2, 13, 18, 33, 35, 37 }

Continued...

Tile	$\mathcal{B}$
	{ 1, 5, 12, 26, 29, 43, 51 }
	{ 1, 2, 15, 27, 30, 43, 50 }
	{ 1, 5, 9, 11, 29, 42, 43 }
	{ 1, 5, 9, 11, 30, 35, 40 }
8tile	{ 1, 2, 13, 26, 32, 34, 38, 46 }
	{ 1, 2, 13, 27, 32, 34, 39, 46 }
	{ 1, 2, 13, 27, 30, 31, 44, 46 }
	{ 1, 2, 13, 27, 31, 35, 39, 46 }
	{ 1, 2, 13, 27, 31, 34, 39, 47 }
	{ 1, 2, 13, 27, 30, 32, 43, 46 }
	{ 1, 5, 11, 27, 30, 35, 40, 47 }
	{ 1, 5, 12, 25, 29, 34, 40, 48 }
	{ 1, 5, 11, 27, 29, 33, 44, 46 }
	{ 1, 2, 6, 12, 27, 30, 43, 50 }
	{ 1, 5, 12, 26, 29, 33, 43, 46 }
	{ 1, 5, 12, 26, 29, 37, 45, 51 }
	{ 1, 5, 12, 26, 29, 33, 44, 49 }
	{ 1, 2, 15, 27, 30, 31, 43, 47 }
	{ 1, 5, 11, 27, 29, 33, 43, 50 }
	{ 1, 2, 13, 27, 31, 39, 43, 47 }
	{ 1, 2, 15, 27, 30, 31, 44, 46 }
	{ 1, 2, 13, 18, 26, 35, 37, 51 }
	{ 1, 5, 11, 18, 24, 30, 41, 52 }
	{ 1, 5, 12, 26, 29, 33, 43, 49 }
	{ 1, 5, 11, 27, 29, 33, 43, 49 }
	{ 1, 5, 12, 26, 29, 36, 37, 51 }
	{ 1, 5, 12, 25, 29, 36, 43, 49 }
	{ 1, 5, 12, 26, 30, 38, 43, 49 }
	{ 1, 5, 11, 27, 29, 42, 43, 49 }
	{ 1, 5, 11, 18, 29, 36, 42, 43 }
9tile	{ 1, 2, 6, 13, 26, 32, 34, 39, 46 }
	{ 1, 2, 13, 17, 24, 30, 34, 41, 46 }
	{ 1, 2, 13, 14, 27, 31, 39, 44, 47 }
	{ 1, 5, 11, 18, 24, 30, 40, 44, 50 }
	{ 1, 2, 13, 17, 24, 34, 39, 41, 47 }
	{ 1, 2, 13, 27, 33, 34, 41, 45, 46 }
	{ 1, 2, 13, 17, 24, 32, 39, 43, 47 }
	{ 1, 2, 13, 18, 24, 32, 36, 39, 49 }
	{ 1, 2, 13, 18, 26, 33, 35, 38, 52 }
	{ 1, 2, 4, 15, 27, 30, 35, 45, 49 }
	{ 1, 5, 11, 18, 24, 30, 34, 41, 46 }
	{ 1, 5, 11, 27, 30, 34, 42, 44, 47 }
	{ 1, 2, 13, 18, 24, 26, 32, 39, 52 }
	{ 1, 5, 11, 18, 24, 25, 30, 40, 53 }
	{ 1, 5, 12, 17, 24, 25, 29, 40, 54 }

Continued. . .

Tile	$\mathcal{B}$
	{ 1, 5, 11, 18, 24, 30, 34, 41, 54 }
	{ 1, 2, 13, 27, 33, 34, 39, 41, 46 }
	{ 1, 5, 11, 18, 24, 33, 39, 40, 53 }
	{ 1, 5, 11, 15, 25, 34, 40, 45, 47 }
	{ 1, 5, 11, 18, 24, 29, 36, 42, 49 }
	{ 1, 2, 6, 13, 27, 31, 45, 48, 53 }
	{ 1, 2, 13, 18, 24, 30, 34, 42, 47 }
	{ 1, 5, 11, 18, 24, 30, 34, 41, 53 }
10tile	{ 1, 5, 11, 18, 24, 33, 35, 39, 40, 47 }
	{ 1, 2, 13, 17, 24, 32, 36, 38, 39, 49 }
	{ 1, 2, 6, 13, 18, 25, 26, 34, 41, 48 }
	{ 1, 2, 13, 17, 24, 32, 36, 39, 47, 49 }
	{ 1, 5, 11, 18, 24, 25, 29, 42, 43, 49 }
11tile	{ 1, 2, 4, 10, 15, 17, 23, 27, 33, 43, 47 }

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