國立交通大學

應用數學系

碩 士 論 文

六角形邊著兩色的決定性問題

Decidability Problems of Hexagonal Edge-coloring with Two Colors

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摘　　　要

這個研究是關於用邊著色的正六角形拼湊整個平面。如果對每個相交處都有相同的顏色，則正六角形可以放在相鄰的位置。在這篇論文，我們考慮邊上只著兩色的正六角形。我們研究的問題為：任意可佈滿整個平面的六角形集合是否可以週期性地覆蓋整個平面。我們使用演算法來研究這個問題，然後藉由電腦計算得到結果。最後，這篇論文的主要結果為：如果整個平面可以被邊著兩色的正六角形拼滿，則整個平面就被週期性的區塊圖案覆蓋，反之亦然。

Decidability Problems of Hexagonal Edge-coloring with Two Colors

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ABSTRACT

This investigation is about tiling the whole plane with regular hexagons which have colors on edges. Regular hexagons can be placed side by side if each of the intersections has the same color. In this paper, we consider regular hexagons with only two colors on edges. The problem we studied is that: any set of hexagons that can fill with the whole plane whether it can cover the whole plane periodically. We use an algorithm to do the problem and get the result by computers. Finally, the main result of this paper is that the whole plane can be tiling by regular hexagons with two colors if and only if the whole plane is covered by the local pattern periodically.

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其次要感謝的是我的夥伴陳泓勳，研究室位置就在我的隔壁，遇到問題能快速回答我的疑惑，打球時也很可靠，總會帶東西回來給我吃。接下來要感謝研究所的朋友們，這兩年來的相處，製造了不少笑點，在一起非常快樂。其中要特別感謝林志嘉，瞎鬧中我交了第一個女朋友，改變了我的人生。

最後要感謝我的家人，能長到這麼大都是你們的功勞。謝謝所有幫助過我的貴人，謝謝。

目　　錄

1 Introduction

There are three possible cases such that the plane is covered by one kind of regular polygons: regular triangles or regular squares or regular hexagons. See the Figure 1.

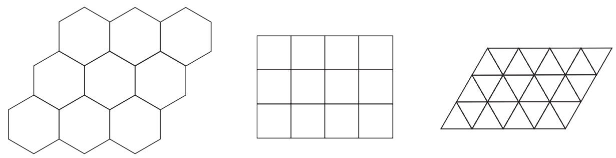


Figure 1.

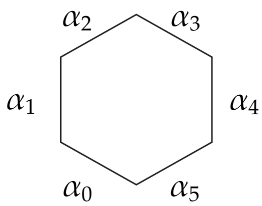
Squares with colors on edges for tiling the plane is well-known as Wang tile. In 1961, Wang [12] started to study the square tiling of a plane. Wang gave a conjecture: any set of tiles that can fill with the whole plane can cover the whole plane periodically. This conjecture was false proved by Berger [4]. Berger presented a set of 20426 Wang tiles that could only tile the plane aperiodically. Later, Berger reduced the number of tiles to 104. Thereafter, smaller sets of Wang tiles were found by Knuth, Läuchli, Robinson, Penrose, Ammann, Culik and Kari [5, 6, 7, 10, 11]. Currently, the smallest number of tiles that can tile the plane aperiodically is 13, with five colors[5]. Nevertheless, Wang's conjecture was true proved by Hu and Lin [13] with two colors on edges. The question we want to know is that Wang's conjecture is true or not if we use regular hexagons with two colors.

Now consider the plane covered by regular hexagons with two different colors, denoted by the numbers 0 or 1, on edges. Two hexagons can be arranged side by side if all intersections have the same color. There are $2^6 = 64$ such hexagons. For convenience, we encode these hexagons to numbers from 1 to 64 with the following rule, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \{0, 1\}$: see Figure 2.

Definition 1.1. The set $\{1, 2, 3, \ldots, 64\}$, denoted by $\mathcal{T}(2)$, is called the *total set* of hexagons with two colors. Each element of $\mathcal{T}(2)$ represents a specific hexagon.

Definition 1.2. The set contained by $\mathcal{T}(2)$ is called a *cycle generator* if it can cover the plane periodically. Let $C(2)$ be the set contains all cycle generators.

Definition 1.3. The set, B, contained by $\mathcal{T}(2)$ is called a *minimal cycle generator* if B satisfies the following two conditions:



$$\Longleftrightarrow W = 1 + \alpha_0 2^0 + \alpha_1 2^1 + \alpha_2 2^2 + \alpha_3 2^3 + \alpha_4 2^4 + \alpha_5 2^5$$

Figure 2.

1. $B \in C(2)$.
2. $\forall B' \subsetneqq B, B' \notin C(2)$.

Let $C_m(2)$ be the set contains all minimal cycle generators.

Definition 1.4. The set, B, contained by $\mathcal{T}(2)$ is called a *non-cycle generator* if $B \notin C(2)$. Let $N(2)$ be the set contains all non-cycle generator.

Definition 1.5. The set, B, contained by $\mathcal{T}(2)$ is called a *maximal non-cycle generator* if B satisfies the following two conditions:

1. $B \in N(2)$.
2. $\forall B' \supsetneqq B, B' \in C(2)$.

Let $N_M(2)$ be the set contains all maximal non-cycle generators.

Given $B \subseteq \mathcal{T}(2)$. Let $\Sigma(B)$ be the set of all global patterns on the plane that can be constructed by B and $\mathcal{P}(B)$ be the set of all periodic patterns on the plane that can be generated by B. Now our question can be written by

$$\text{if } \Sigma(B) \neq \emptyset \text{ whether } \mathcal{P}(B) \neq \emptyset \text{ or not.} \tag{1.1}$$

All we have to do is to investigate all subset of $\mathcal{T}(2)$ whether (1.1) is true or not. Though there are 2^{64} such cases. So in the following sections, some methods are introduced to reduce cases. In section 2, we introduce two useful matrices $\mathbb{T}$ and $\mathbb{Y}$ such that if given $B \subseteq \mathcal{T}(2)$, we can decide whether $B \in C(2)$ with $\mathbb{T}$ and whether $B \in N(2)$ with $\mathbb{X}$. In section 3, a method called classification can reduce possible cases. In section 4, we use an algorithm to deal with the problem. Finally, we derive a conclusion in section 5.

2 Ordering Matrix

Since the plane has two dimensions, we first can construct the pattern along horizontal direction and then second along vertical direction. In the following we show how to obtain the horizontal patterns. Let $\mathbb{X}_1^*$ be the matrix of the following form(see Figure 3) ,where $\beta \in \{0, 1\}$. More general form of $\mathbb{X}_1^*$ can be written as Figure 4 where $\alpha_1, \alpha_2, \alpha_1', \alpha_2', \beta \in \{0, 1\}$.

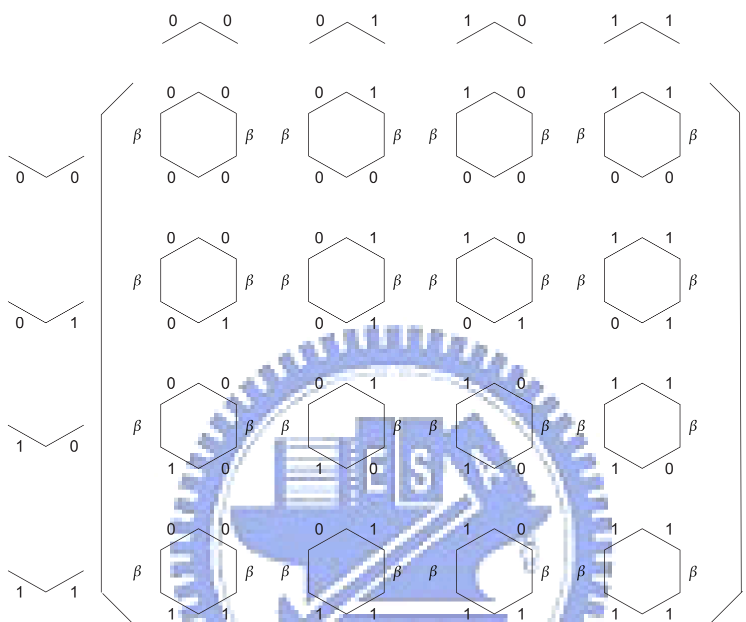


Figure 3. $\mathbb{X}_1^*$

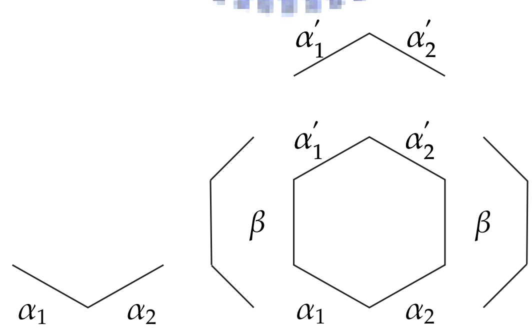
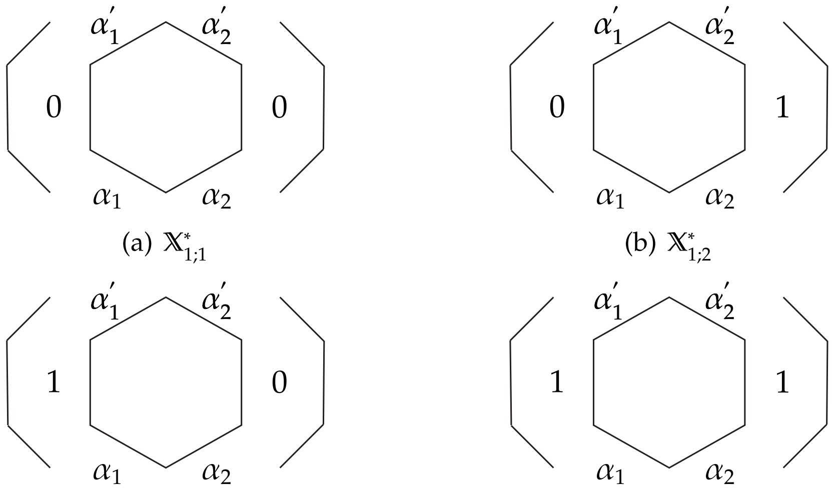


Figure 4. the general form of $\mathbb{X}_1^*$

Let $\mathbb{X}_{1;1}^*$ be the form of Figure 5(a). The complete matrix $\mathbb{X}_{1;1}^*$ can be easily obtained by replacing Figure 3′ $\beta\beta$ from left to right of the hexagon



(a) $\mathbb{X}^*_{1;1}$ (b) $\mathbb{X}^*_{1;2}$

(c) $\mathbb{X}^*_{1;3}$ (d) $\mathbb{X}^*_{1;4}$

Figure 5.

with 0 and 0. Each element of $\mathbb{X}^*_{1;1}$ is denoted by $x^*_{1;1;i,j}$. Similarly, $\mathbb{X}^*_{1;2}$, $\mathbb{X}^*_{1;3}$, $\mathbb{X}^*_{1;4}$ can be defined, and their matrix forms are shown in Figure 5(b), Figure 5(c), Figure 5(d). We can find

$$\mathbb{X}^*_1 = \sum_{i=1,4} \mathbb{X}^*_{1;i} \tag{2.1}$$

Now consider the general form $\mathbb{X}^*_2$ as Figure 6. We can define $\mathbb{X}^*_{2;1}$ by replacing Figure 6's $\beta\beta$ from left to right of the hexagon with 0 and 0. $\mathbb{X}^*_{2;1}$ can derive from $\mathbb{X}^*_1$ in the following way, see equations (2.2).

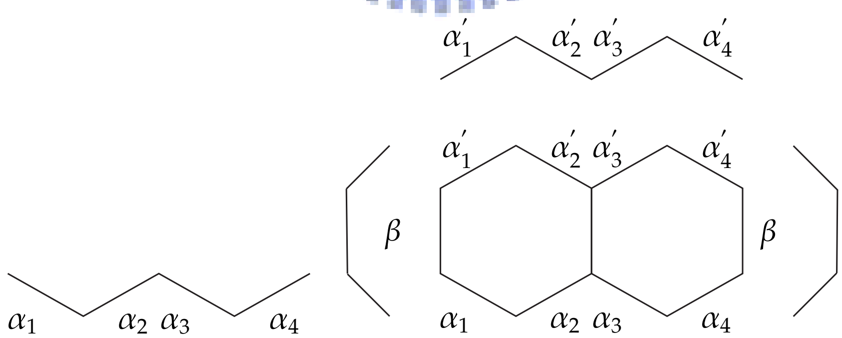


Figure 6. the general form of $\mathbb{X}^*_2$

Similarly, $\mathbb{X}^*_{2;2}$, $\mathbb{X}^*_{2;3}$, $\mathbb{X}^*_{2;4}$ can be derive from equations (2.3), (2.4), (2.5) alternatively.

$$\mathbb{X}^*_{2;1} = \left[\begin{array}{c|c|c|c}
x_{1;1;1,1}\mathbb{X}^*_{1;1}+ & x_{1;1;1,2}\mathbb{X}^*_{1;1}+ & x_{1;1;1,3}\mathbb{X}^*_{1;1}+ & x_{1;1;1,4}\mathbb{X}^*_{1;1}+ \\
x_{1;2;1,1}\mathbb{X}^*_{1;3} & x_{1;2;1,2}\mathbb{X}^*_{1;3} & x_{1;2;1,3}\mathbb{X}^*_{1;3} & x_{1;2;1,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;1;2,1}\mathbb{X}^*_{1;1}+ & x_{1;1;2,2}\mathbb{X}^*_{1;1}+ & x_{1;1;2,3}\mathbb{X}^*_{1;1}+ & x_{1;1;2,4}\mathbb{X}^*_{1;1}+ \\
x_{1;2;2,1}\mathbb{X}^*_{1;3} & x_{1;2;2,2}\mathbb{X}^*_{1;3} & x_{1;2;2,3}\mathbb{X}^*_{1;3} & x_{1;2;2,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;1;3,1}\mathbb{X}^*_{1;1}+ & x_{1;1;3,2}\mathbb{X}^*_{1;1}+ & x_{1;1;3,3}\mathbb{X}^*_{1;1}+ & x_{1;1;3,4}\mathbb{X}^*_{1;1}+ \\
x_{1;2;3,1}\mathbb{X}^*_{1;3} & x_{1;2;3,2}\mathbb{X}^*_{1;3} & x_{1;2;3,3}\mathbb{X}^*_{1;3} & x_{1;2;3,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;1;4,1}\mathbb{X}^*_{1;1}+ & x_{1;1;4,2}\mathbb{X}^*_{1;1}+ & x_{1;1;4,3}\mathbb{X}^*_{1;1}+ & x_{1;1;4,4}\mathbb{X}^*_{1;1}+ \\
x_{1;2;4,1}\mathbb{X}^*_{1;3} & x_{1;2;4,2}\mathbb{X}^*_{1;3} & x_{1;2;4,3}\mathbb{X}^*_{1;3} & x_{1;2;4,4}\mathbb{X}^*_{1;3}
\end{array}\right] \tag{2.2}$$

$$\mathbb{X}^*_{2;2} = \left[\begin{array}{c|c|c|c}
x_{1;1;1,1}\mathbb{X}^*_{1;2}+ & x_{1;1;1,2}\mathbb{X}^*_{1;2}+ & x_{1;1;1,3}\mathbb{X}^*_{1;2}+ & x_{1;1;1,4}\mathbb{X}^*_{1;2}+ \\
x_{1;2;1,1}\mathbb{X}^*_{1;4} & x_{1;2;1,2}\mathbb{X}^*_{1;4} & x_{1;2;1,3}\mathbb{X}^*_{1;4} & x_{1;2;1,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;1;2,1}\mathbb{X}^*_{1;2}+ & x_{1;1;2,2}\mathbb{X}^*_{1;2}+ & x_{1;1;2,3}\mathbb{X}^*_{1;2}+ & x_{1;1;2,4}\mathbb{X}^*_{1;2}+ \\
x_{1;2;2,1}\mathbb{X}^*_{1;4} & x_{1;2;2,2}\mathbb{X}^*_{1;4} & x_{1;2;2,3}\mathbb{X}^*_{1;4} & x_{1;2;2,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;1;3,1}\mathbb{X}^*_{1;2}+ & x_{1;1;3,2}\mathbb{X}^*_{1;2}+ & x_{1;1;3,3}\mathbb{X}^*_{1;2}+ & x_{1;1;3,4}\mathbb{X}^*_{1;2}+ \\
x_{1;2;3,1}\mathbb{X}^*_{1;4} & x_{1;2;3,2}\mathbb{X}^*_{1;4} & x_{1;2;3,3}\mathbb{X}^*_{1;4} & x_{1;2;3,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;1;4,1}\mathbb{X}^*_{1;2}+ & x_{1;1;4,2}\mathbb{X}^*_{1;2}+ & x_{1;1;4,3}\mathbb{X}^*_{1;2}+ & x_{1;1;4,4}\mathbb{X}^*_{1;2}+ \\
x_{1;2;4,1}\mathbb{X}^*_{1;4} & x_{1;2;4,2}\mathbb{X}^*_{1;4} & x_{1;2;4,3}\mathbb{X}^*_{1;4} & x_{1;2;4,4}\mathbb{X}^*_{1;4}
\end{array}\right] \tag{2.3}$$

$$\mathbb{X}^*_{2;3} = \left[\begin{array}{c|c|c|c}
x_{1;3;1,1}\mathbb{X}^*_{1;1}+ & x_{1;3;1,2}\mathbb{X}^*_{1;1}+ & x_{1;3;1,3}\mathbb{X}^*_{1;1}+ & x_{1;3;1,4}\mathbb{X}^*_{1;1}+ \\
x_{1;4;1,1}\mathbb{X}^*_{1;3} & x_{1;4;1,2}\mathbb{X}^*_{1;3} & x_{1;4;1,3}\mathbb{X}^*_{1;3} & x_{1;4;1,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;3;2,1}\mathbb{X}^*_{1;1}+ & x_{1;3;2,2}\mathbb{X}^*_{1;1}+ & x_{1;3;2,3}\mathbb{X}^*_{1;1}+ & x_{1;3;2,4}\mathbb{X}^*_{1;1}+ \\
x_{1;4;2,1}\mathbb{X}^*_{1;3} & x_{1;4;2,2}\mathbb{X}^*_{1;3} & x_{1;4;2,3}\mathbb{X}^*_{1;3} & x_{1;4;2,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;3;3,1}\mathbb{X}^*_{1;1}+ & x_{1;3;3,2}\mathbb{X}^*_{1;1}+ & x_{1;3;3,3}\mathbb{X}^*_{1;1}+ & x_{1;3;3,4}\mathbb{X}^*_{1;1}+ \\
x_{1;4;3,1}\mathbb{X}^*_{1;3} & x_{1;4;3,2}\mathbb{X}^*_{1;3} & x_{1;4;3,3}\mathbb{X}^*_{1;3} & x_{1;4;3,4}\mathbb{X}^*_{1;3} \\ \hline
x_{1;3;4,1}\mathbb{X}^*_{1;1}+ & x_{1;3;4,2}\mathbb{X}^*_{1;1}+ & x_{1;3;4,3}\mathbb{X}^*_{1;1}+ & x_{1;3;4,4}\mathbb{X}^*_{1;1}+ \\
x_{1;4;4,1}\mathbb{X}^*_{1;3} & x_{1;4;4,2}\mathbb{X}^*_{1;3} & x_{1;4;4,3}\mathbb{X}^*_{1;3} & x_{1;4;4,4}\mathbb{X}^*_{1;3}
\end{array}\right] \tag{2.4}$$

$$
\mathbb{X}^*_{2;4} = \left[\begin{array}{c|c|c|c}
x_{1;3;1,1}\mathbb{X}^*_{1;2} + x_{1;4;1,1}\mathbb{X}^*_{1;4} & x_{1;3;1,2}\mathbb{X}^*_{1;2} + x_{1;4;1,2}\mathbb{X}^*_{1;4} & x_{1;3;1,3}\mathbb{X}^*_{1;2} + x_{1;4;1,3}\mathbb{X}^*_{1;4} & x_{1;3;1,4}\mathbb{X}^*_{1;2} + x_{1;4;1,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;3;2,1}\mathbb{X}^*_{1;2} + x_{1;4;2,1}\mathbb{X}^*_{1;4} & x_{1;3;2,2}\mathbb{X}^*_{1;2} + x_{1;4;2,2}\mathbb{X}^*_{1;4} & x_{1;3;2,3}\mathbb{X}^*_{1;2} + x_{1;4;2,3}\mathbb{X}^*_{1;4} & x_{1;3;2,4}\mathbb{X}^*_{1;2} + x_{1;4;2,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;3;3,1}\mathbb{X}^*_{1;2} + x_{1;4;3,1}\mathbb{X}^*_{1;4} & x_{1;3;3,2}\mathbb{X}^*_{1;2} + x_{1;4;3,2}\mathbb{X}^*_{1;4} & x_{1;3;3,3}\mathbb{X}^*_{1;2} + x_{1;4;3,3}\mathbb{X}^*_{1;4} & x_{1;3;3,4}\mathbb{X}^*_{1;2} + x_{1;4;3,4}\mathbb{X}^*_{1;4} \\ \hline
x_{1;3;4,1}\mathbb{X}^*_{1;2} + x_{1;4;4,1}\mathbb{X}^*_{1;4} & x_{1;3;4,2}\mathbb{X}^*_{1;2} + x_{1;4;4,2}\mathbb{X}^*_{1;4} & x_{1;3;4,3}\mathbb{X}^*_{1;2} + x_{1;4;4,3}\mathbb{X}^*_{1;4} & x_{1;3;4,4}\mathbb{X}^*_{1;2} + x_{1;4;4,4}\mathbb{X}^*_{1;4}
\end{array}\right] \tag{2.5}
$$

$$
\mathbb{X}^*_{n;1} = \left[\begin{array}{c|c|c|c}
x_{1;1;1,1}\mathbb{X}^*_{n-1;1} + x_{1;2;1,1}\mathbb{X}^*_{n-1;3} & x_{1;1;1,2}\mathbb{X}^*_{n-1;1} + x_{1;2;1,2}\mathbb{X}^*_{n-1;3} & x_{1;1;1,3}\mathbb{X}^*_{n-1;1} + x_{1;2;1,3}\mathbb{X}^*_{n-1;3} & x_{1;1;1,4}\mathbb{X}^*_{n-1;1} + x_{1;2;1,4}\mathbb{X}^*_{n-1;3} \\ \hline
x_{1;1;2,1}\mathbb{X}^*_{n-1;1} + x_{1;2;2,1}\mathbb{X}^*_{n-1;3} & x_{1;1;2,2}\mathbb{X}^*_{n-1;1} + x_{1;2;2,2}\mathbb{X}^*_{n-1;3} & x_{1;1;2,3}\mathbb{X}^*_{n-1;1} + x_{1;2;2,3}\mathbb{X}^*_{n-1;3} & x_{1;1;2,4}\mathbb{X}^*_{n-1;1} + x_{1;2;2,4}\mathbb{X}^*_{n-1;3} \\ \hline
x_{1;1;3,1}\mathbb{X}^*_{n-1;1} + x_{1;2;3,1}\mathbb{X}^*_{n-1;3} & x_{1;1;3,2}\mathbb{X}^*_{n-1;1} + x_{1;2;3,2}\mathbb{X}^*_{n-1;3} & x_{1;1;3,3}\mathbb{X}^*_{n-1;1} + x_{1;2;3,3}\mathbb{X}^*_{n-1;3} & x_{1;1;3,4}\mathbb{X}^*_{n-1;1} + x_{1;2;3,4}\mathbb{X}^*_{n-1;3} \\ \hline
x_{1;1;4,1}\mathbb{X}^*_{n-1;1} + x_{1;2;4,1}\mathbb{X}^*_{n-1;3} & x_{1;1;4,2}\mathbb{X}^*_{n-1;1} + x_{1;2;4,2}\mathbb{X}^*_{n-1;3} & x_{1;1;4,3}\mathbb{X}^*_{n-1;1} + x_{1;2;4,3}\mathbb{X}^*_{n-1;3} & x_{1;1;4,4}\mathbb{X}^*_{n-1;1} + x_{1;2;4,4}\mathbb{X}^*_{n-1;3}
\end{array}\right] \tag{2.8}
$$

So,

$$
\mathbb{X}^*_2 = \sum_{i=1,4} \mathbb{X}^*_{2;i} \tag{2.6}
$$

By observation, the recursive formula of $\mathbb{X}^*_n$ is

$$
\mathbb{X}^*_n = \sum_{i=1,4} \mathbb{X}^*_{n;i} \tag{2.7}
$$

where $\mathbb{X}^*_{n;1}$, $\mathbb{X}^*_{n;2}$, $\mathbb{X}^*_{n;3}$, $\mathbb{X}^*_{n;4}$ are derived from equations (2.8), (2.9), (2.10), (2.11) alternatively.

Remark 2.1. Now we get a recursive formula about $\mathbb{X}^*_{n;i}$ where $i = 1, 2, 3, 4$ which are useful to get matrices we want, i.e. $\mathbb{T}_n$ and $\mathbb{Y}_n$.

Definition 2.2. A function φ is called a *counting function* if

$$
\varphi(\alpha_1\alpha_2\cdots\alpha_n) = \alpha_1 2^{n-1} + \alpha_2 2^{n-2} + \cdots + \alpha_n 2^0 + 1
$$

, where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \{0, 1\}$.

$$\mathbb{X}^*_{n;2} = \left[\begin{array}{c|c|c|c}
\begin{array}{l} x_{1;1;1,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;1,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;1,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;1,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;1,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;1,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;1,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;1,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;1;2,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;2,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;2,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;2,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;2,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;2,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;2,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;2,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;1;3,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;3,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;3,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;3,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;3,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;3,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;3,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;3,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;1;4,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;4,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;4,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;4,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;4,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;4,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;1;4,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;2;4,4}\mathbb{X}^*_{n-1;4} \end{array}
\end{array}\right] \quad (2.9)$$

$$\mathbb{X}^*_{n;3} = \left[\begin{array}{c|c|c|c}
\begin{array}{l} x_{1;3;1,1}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;1,1}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;1,2}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;1,2}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;1,3}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;1,3}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;1,4}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;1,4}\mathbb{X}^*_{n-1;3} \end{array} \\ \hline
\begin{array}{l} x_{1;3;2,1}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;2,1}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;2,2}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;2,2}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;2,3}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;2,3}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;2,4}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;2,4}\mathbb{X}^*_{n-1;3} \end{array} \\ \hline
\begin{array}{l} x_{1;3;3,1}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;3,1}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;3,2}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;3,2}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;3,3}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;3,3}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;3,4}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;3,4}\mathbb{X}^*_{n-1;3} \end{array} \\ \hline
\begin{array}{l} x_{1;3;4,1}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;4,1}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;4,2}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;4,2}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;4,3}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;4,3}\mathbb{X}^*_{n-1;3} \end{array} & \begin{array}{l} x_{1;3;4,4}\mathbb{X}^*_{n-1;1}+ \\ x_{1;4;4,4}\mathbb{X}^*_{n-1;3} \end{array}
\end{array}\right] \quad (2.10)$$

$$\mathbb{X}^*_{n;4} = \left[\begin{array}{c|c|c|c}
\begin{array}{l} x_{1;3;1,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;1,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;1,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;1,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;1,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;1,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;1,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;1,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;3;2,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;2,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;2,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;2,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;2,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;2,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;2,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;2,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;3;3,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;3,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;3,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;3,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;3,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;3,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;3,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;3,4}\mathbb{X}^*_{n-1;4} \end{array} \\ \hline
\begin{array}{l} x_{1;3;4,1}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;4,1}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;4,2}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;4,2}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;4,3}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;4,3}\mathbb{X}^*_{n-1;4} \end{array} & \begin{array}{l} x_{1;3;4,4}\mathbb{X}^*_{n-1;2}+ \\ x_{1;4;4,4}\mathbb{X}^*_{n-1;4} \end{array}
\end{array}\right] \quad (2.11)$$

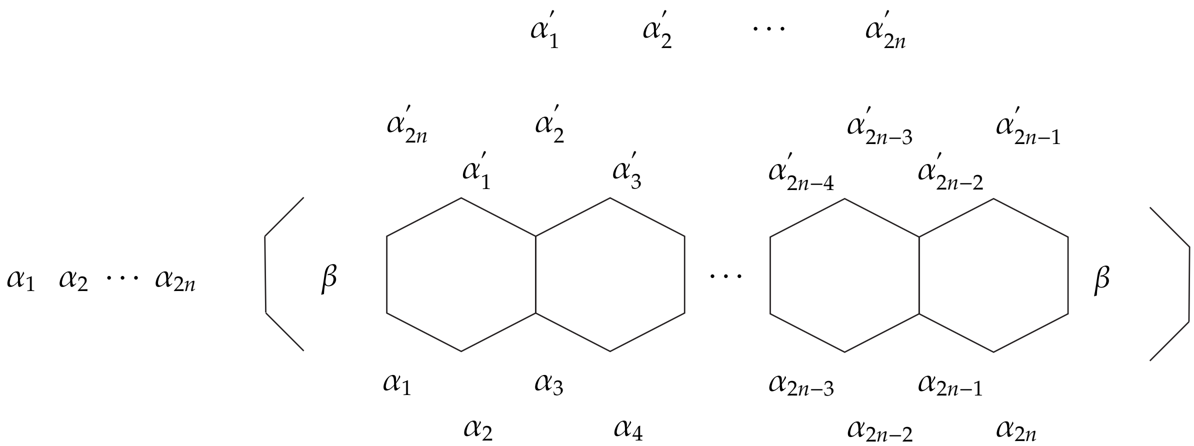


Figure 7. the general form of $\mathbb{T}_n$

2.1 Construct $\mathbb{T}_n$

The general form of $\mathbb{T}_n$ is shown in Figure 7. Notice that the upper colors are circularly shifting one position such that the patterns can tile above the original patterns if we multiply $\mathbb{T}_n$ and $\mathbb{T}_n$.

Define a permutation matrix $\mathbb{P}_n$.

$$\mathbb{P}_n = \left[\; \mathbb{P}_{n;i,j} \; \right]_{2^{2n} \times 2^{2n}} \tag{2.12}$$

Given $1 \leq j \leq 2^{2n}$, $\exists \alpha_1, \cdots, \alpha_{2n}$ such that $j = \varphi(\alpha_1 \cdots \alpha_{2n})$, then

$$\begin{cases} \mathbb{P}_{n;i,j} = 1 & \text{, if } i = \varphi(\alpha_{2n}\alpha_1 \cdots \alpha_{2n-1}) \\ \mathbb{P}_{n;i,j} = 0 & \text{, otherwise} \end{cases} \tag{2.13}$$

By changing variables, we get more simple form of $\mathbb{P}_n$ where

$$\begin{cases} \mathbb{P}_{n;i,2i-1} = 1 \text{ and } \mathbb{P}_{n;i+2^{2n-1},2i} = 1 & \text{, if } 1 \leq i \leq 2^{2n-1} \\ \mathbb{P}_{n;i,j} = 0 & \text{, otherwise} \end{cases} \tag{2.14}$$

Thus $\mathbb{T}_n = \mathbb{X}_n^* \mathbb{P}_n$.

2.2 Construct $\mathbb{Y}_n$

The general form of $\mathbb{Y}_n = \left[\; \mathbb{Y}_{n;i,j} \; \right]_{2^{2n-1} \times 2^{2n-1}}$ is shown in Figure 8. Notice the difference between $\mathbb{Y}_n$ and $\mathbb{X}_n$.

Let

$$\mathbb{Y}_n^* = \sum_{i=1}^{4} \mathbb{X}_{n;i}^*. \tag{2.15}$$

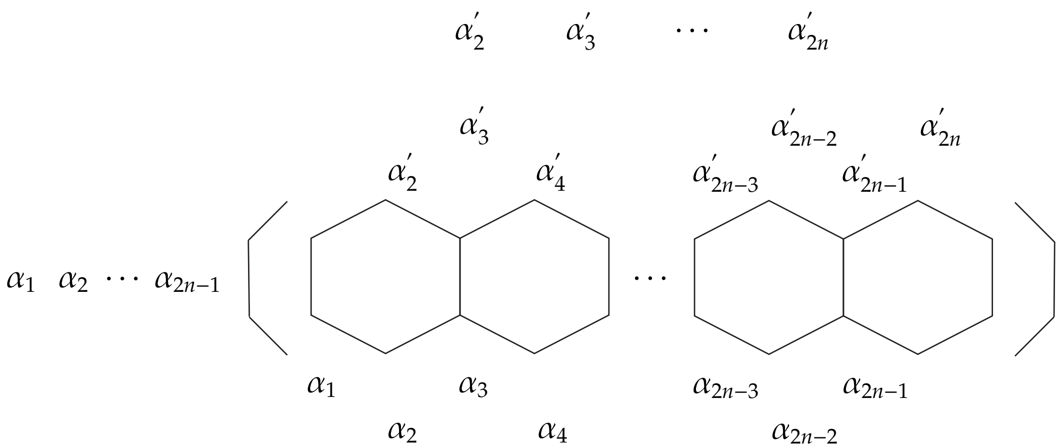


Figure 8. the general form of Υ_n

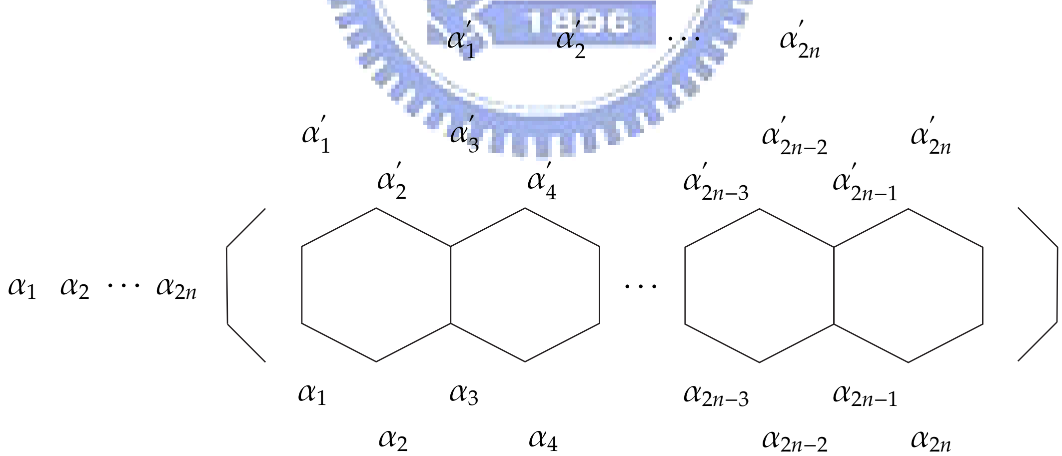


Figure 9. the general form of Υ_n^*

Now $\mathbb{Y}_n^*$ has the matrix form in Figure 9.

We construct $\mathbb{Y}_n$ from $\mathbb{Y}_n^*$. Given $1 \leq i \leq 2^{2n-1}$, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$ such that $i = \varphi(\alpha_1\alpha_2\cdots\alpha_{2n-1})$. Define the set , $C_{n;1}(i)$, such that

$$\begin{aligned} C_{n;1}(i) &= \{\varphi(\alpha_1\alpha_2\cdots\alpha_{2n-1}\alpha_{2n}) : \alpha_{2n} = 0, 1\} \\ &= \{2i - 1 + 0, 2i - 1 + 1\} \\ &= \{2i - 1, 2i\} \end{aligned}$$

Given $1 \leq j \leq 2^{2n-1}$, then there exist $\alpha_2', \alpha_3', \cdots, \alpha_{2n}'$ such that $j = \varphi(\alpha_2'\alpha_3'\cdots\alpha_{2n}')$. Define the set, $C_{n;2}(j)$, such that

$$\begin{aligned} C_{n;2}(j) &= \{\varphi(\alpha_1'\alpha_2'\cdots\alpha_{2n-1}'\alpha_{2n}') : \alpha_1' = 0, 1\} \\ &= \{j, j + 2^{2n-1}\} \end{aligned}$$

Then

$$\mathbb{Y}_n = \left[\ \mathbb{Y}_{n;i,j}\ \right]_{2^{2n-1}\times 2^{2n-1}} \text{ where } \mathbb{Y}_{n;i,j} = \sum_{\substack{p\in C_{n;1}(i)\\ q\in C_{n;2}(j)}} \mathbb{Y}_{n;p,q}^*$$

3 Symmetry of Hexagons

The symmetry group of the hexagon is D_6, the dihedral group where $\mid D_6 \mid= 12$. The group $D_6 = \{I, \rho, \rho^2, \rho^3, \rho^4, \rho^5, m, m\rho, m\rho^2, m\rho^3, m\rho^4, m\rho^5\}$ where ρ is rotating $\frac{\pi}{3}$ counterclockwise, m is reflecting about the y-axis. See Figure 10.

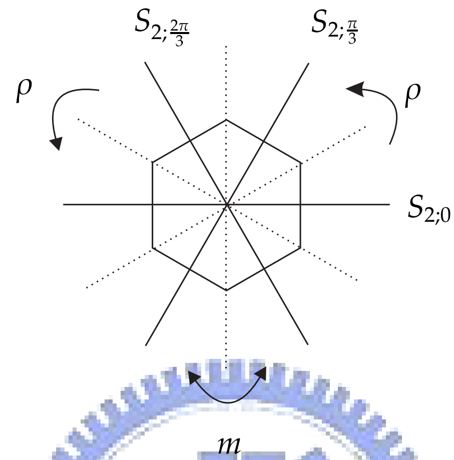


Figure 10.

For example, see Figure 11.

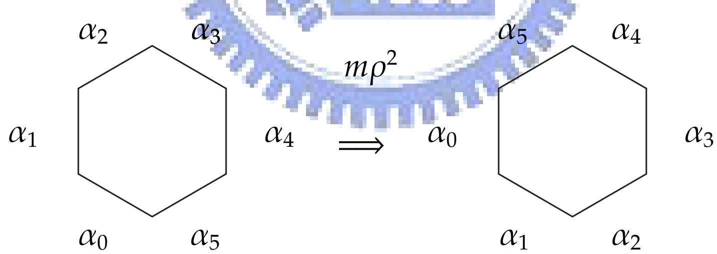


Figure 11.

There are three permutation groups S_2 in three directions, denoted by $S_{2;0}, S_{2;\frac{\pi}{3}}, S_{2;\frac{2\pi}{3}}$, on edge coloring of hexagons as Figure 10. According to the symmetry group and the permutation group, we define an equivalent class of a set $B \subseteq \mathcal{T}(2)$, denoted by $[B]$, as follows:

Let $A = \{\tau\eta_o\eta_{\frac{\pi}{3}}\eta_{\frac{2\pi}{3}} \mid \tau \in D_6, \eta_0 \in S_{2;0}, \eta_{\frac{\pi}{3}} \in S_{2;\frac{\pi}{3}}, \eta_{\frac{2\pi}{3}} \in S_{2;\frac{2\pi}{3}}\}$, then obviously $|A| = (2 \times 6) \times 2 \times 2 \times 2 = 96$. Given $B \subseteq \mathcal{T}(2)$, then

$$[B] = \{\xi(B) \mid \xi \in A\}.$$

Some observations about an equivalent class:

1. if $P(B) = \emptyset$ ($\neq \emptyset$), then $P(B') = \emptyset$ ($\neq \emptyset$) $\forall B' \in [B]$
2. if $\Sigma(B) = \emptyset$ ($\neq \emptyset$), then $\Sigma(B') = \emptyset$ ($\neq \emptyset$) $\forall B' \in [B]$

In addition, $\mathcal{T}(2)$ can be decomposed into four types, denoted by $\mathcal{K}_0$, $\mathcal{K}_I$, $\mathcal{K}_{II}$, $\mathcal{K}_{III}$, according to equivalent classes as Figure 12.

We know

1. if $t \in \mathcal{K}_I$, then $\xi(t) \in \mathcal{K}_I$ $\forall \xi \in A$
2. if $t \in \mathcal{K}_{II}$, then $\xi(t) \in \mathcal{K}_{II}$ $\forall \xi \in A$
3. if $t \in \mathcal{K}_{III}$, then $\xi(t) \in \mathcal{K}_{III}$ $\forall \xi \in A$

So we construct two groups $\mathcal{G}_I$ and $\mathcal{G}_{II}$. Let $\mathcal{G}_I = \mathcal{K}_{II}$ and $\mathcal{G}_{II} = \mathcal{K}_I \cup \mathcal{K}_{III}$, then

1. if $t \in \mathcal{G}_I$, then $\xi(t) \in \mathcal{G}_I$ $\forall \xi \in A$
2. if $t \in \mathcal{G}_{II}$, then $\xi(t) \in \mathcal{G}_{II}$ $\forall \xi \in A$

Lemma 3.1. *Given $B = B_I \cup B_{II}$, where $B_I \subseteq \mathcal{G}_I$, $B_{II} \subseteq \mathcal{G}_{II}$. For any $\overline{B_{II}} \in [B_{II}]$, there exists $B'_I \subseteq \mathcal{G}_I$ such that $B'_I \cup \overline{B_{II}} \in [B]$*

Proof. Since $\overline{B_{II}} \in [B_{II}]$, there exists $\xi \in A$ such that $\overline{B_{II}} = \xi(B_{II})$. We know

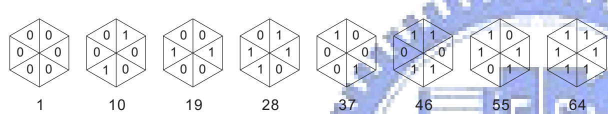
$$\xi(B) = \xi(B_I) \cup \xi(B_{II}) \in [B]$$

where $\xi(B_I) \subseteq \mathcal{G}_I$ and $\xi(B_{II}) \subseteq \mathcal{G}_{II}$. Therefore,

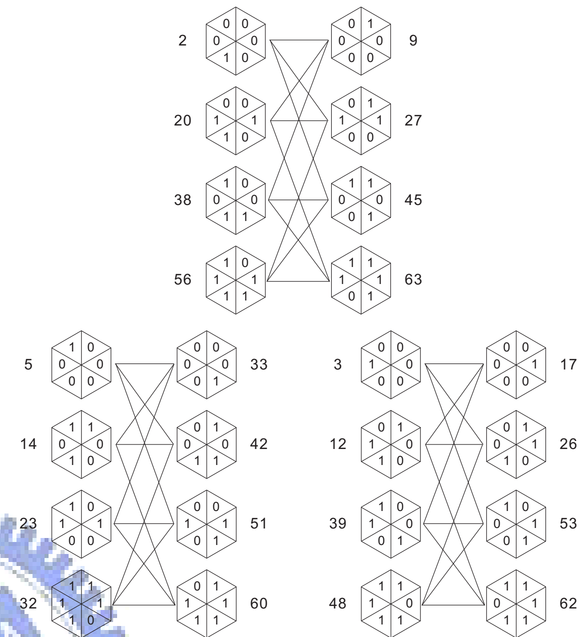
$$\xi(B_I) \cup \xi(B_{II}) = \xi(B_I) \cup \overline{B_{II}} \in [B]$$

where $\xi(B_I) \subseteq \mathcal{G}_I$ □

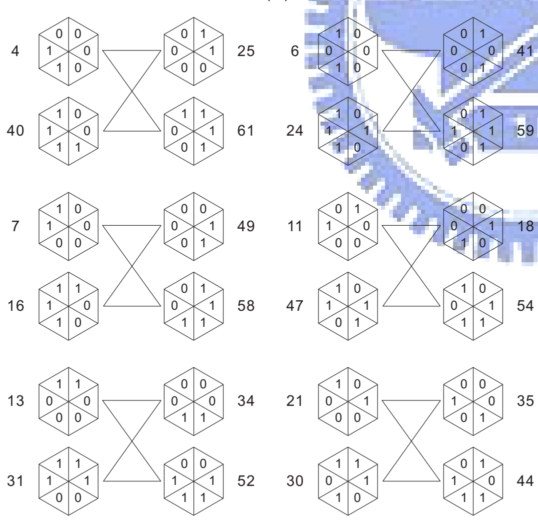
Remark 3.2. This lemma tells us that we can classify $\mathcal{G}_{II}$ with A to reduce computations.



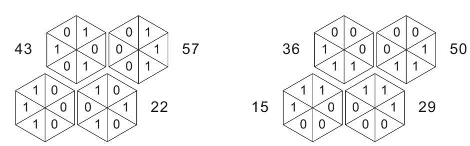
(a) $\mathcal{K}_0$



(b) $\mathcal{K}_I$



(c) $\mathcal{K}_{II}$



(d) $\mathcal{K}_{III}$

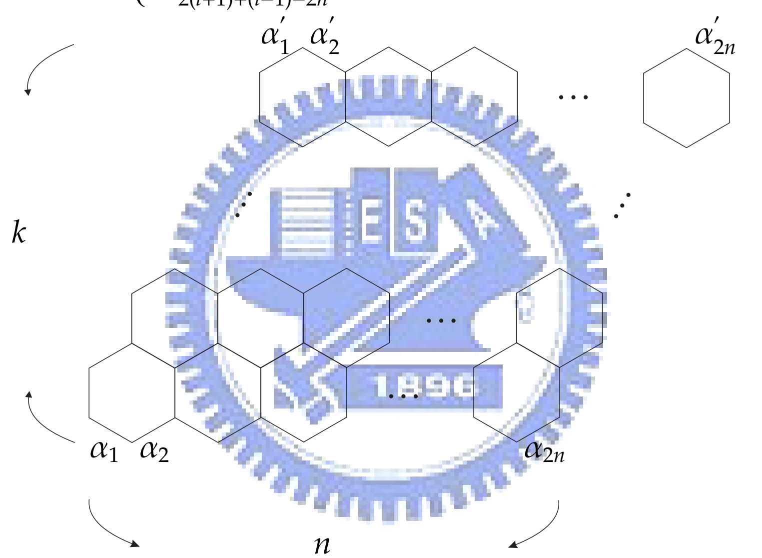
Figure 12.

4 Algorithm

According to the above discussions, we now have $\mathbb{T}$ and $\mathbb{Y}$ to check whether $B \in C(2)$ or $B \in N(2)$ or not for $B \subseteq \mathcal{T}(2)$ and a method called classification to reduce computations. Using the following algorithm to achieve our goal.

Definition 4.1. $P(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix})$ is the set of sets, i.e. each element is a set. Each element contains tiles that can tile the local pattern such that the width of the local pattern is n hexagons, the height of the local pattern is k hexagons and the shift of the local pattern is l. See Figure **??**, where i is from 1 to $2n$ and

$$\alpha_i = \begin{cases} \alpha'_{2(l+1)+(i-1)} & \text{,if } 2(l+1)+(i-1) \le 2 \\ \alpha'_{2(l+1)+(i-1)-2n} & \text{,otherwise} \end{cases}$$



Define two sets C^* and $\mathcal{N}^*$ which mean cycle and non-cycle alternatively such that

$$C^*(m) = \{B : B \in P(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}) \text{ where } m = nk, 0 \le l \le n-1, \\ n, k, l \in \mathbb{N}\}$$

and a recursive definition of $\mathcal{N}^*$

$$\begin{cases} \mathcal{N}^*(0) = \{B_1 \cup B_2 : B_1 \subseteq \mathcal{G}_I, B_2 \subseteq \mathcal{G}_{II}\} \\ \mathcal{N}^*(m) = \{B : B \in \mathcal{N}^*(m-1) \text{ and } c \not\subseteq B \text{ for all } c \in C^*(m)\} \end{cases}$$

Since the condition $\Sigma(B) = \emptyset$ for all $B \in \mathcal{N}^*(m)$ is hard to check and the algorithm perhaps does not stop, we modify the above algorithm.

Algorithm 1 Algorithm

```
m ← 0
repeat
  m ← m + 1
  C*(m)
  𝒩*(m)
until Σ(B) = ∅ for all B ∈ 𝒩*(m)
```

Algorithm 2 Modified Algorithm

```
C(2) ← ∅                      // cycle generator
N(2) ← ∅                      // Non-cycle generator
U⋆ ← ∅                        // Unknown
m ← 0                         // Loop index
repeat
  m ← m + 1
  C*(m)
  C(2) ← C(2) ∪ C*(m)
  𝒩*(m)
until m == 12

repeat
  pick B ∈ 𝒩*(12)
  𝒩*(12) ← 𝒩*(12) ∖ {B}
  if time is acceptable and P(B) ≠ ∅ then
    C(2) ← C(2) ∪ {B}
  else if time is acceptable and Σ(B) = ∅ then
    N(2) ← N(2) ∪ {B}
  else
    U⋆ ← U⋆ ∪ {B}
  end if
until | 𝒩*(12) |== 0
```

5 Conclusion

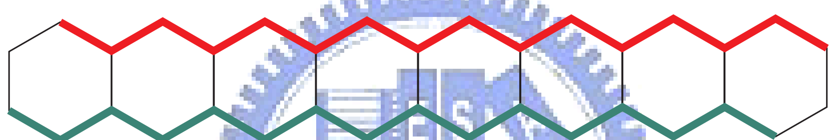
According to the modified algorithm, if $| U^{\star} |= 0$ and $\Sigma(B) = \emptyset \ \forall B \in N(2)$, then Wang's conjecture is true for hexagons with two colors, i.e.

$$\text{if } \Sigma(B) \neq \emptyset \text{ then } \mathcal{P}(B) \neq \emptyset.$$

Unfortunately, there are three cases we can't decide.

1. 2 4 15 20 21 56 59
2. 2 15 20 25 35 38
3. 2 15 20 25 35 56

Since $\mathbb{T}_n$ and Υ_n is too large to compute for n is sufficient large (6), another method for these three cases is introduced in the following.
Method:



1. Find all possible situations that can tile like the graph above, and then record the informations of colored edges.
2. Use the informations to tile the plane with the width is 8 tile.
3. If all possible cases is checked and can't tile the plane with some height, then the set $\Sigma(B) = \emptyset$.

The last three cases are shown.

1. 2 4 15 20 21 56 59 (8, 17)
2. 2 15 20 25 35 38 (8, 15)
3. 2 15 20 25 35 56 (8, 13)

Theorem 5.1. *Given* $B \subseteq \mathcal{T}(2)$.

$$\textit{if } \Sigma(B) \neq \emptyset \textit{ then } \mathcal{P}(B) \neq \emptyset.$$

The other cases are decided and show in Table 1. Only show $[C_m(2)]$ in Table 1 since $[N_M(2)]$ are too many to append.

A Table1

TileNum	$[C_m(2)]$
3tile	{ 2 5 41 }
	{ 2 3 25 }
	{ 2 5 59 }
	{ 2 29 35 }
	{ 2 11 53 }
	{ 4 13 49 }
	{ 2 11 25 }
	{ 4 29 57 }
	{ 2 29 43 }
4tile	{ 2 3 5 57 }
	{ 2 4 13 57 }
	{ 2 5 11 49 }
	{ 2 5 11 58 }
	{ 4 11 21 49 }
	{ 2 11 22 59 }
	{ 2 11 20 61 }
	{ 2 4 29 59 }
	{ 2 3 21 59 }
	{ 2 4 29 41 }
	{ 4 11 22 57 }
	{ 4 11 21 58 }
	{ 2 5 60 63 }
	{ 2 22 43 63 }
	{ 2 11 56 61 }
	{ 4 15 54 57 }

Continued. . .

TileNum	$[C_m(2)]$
	{ 2 20 47 61 }
	{ 8 15 50 57 }
	{ 2 5 11 57 }
	{ 2 11 30 41 }
	{ 2 3 21 57 }
	{ 2 7 25 36 }
	{ 2 5 43 58 }
	{ 2 11 21 59 }
	{ 2 21 36 63 }
	{ 2 5 43 57 }
	{ 2 7 21 41 }
	{ 2 11 29 41 }
	{ 2 11 22 57 }
	{ 2 7 25 44 }
	{ 2 15 29 60 }
	{ 2 7 21 59 }
	{ 2 5 17 43 }
	{ 2 11 38 61 }
	{ 8 29 43 50 }
	{ 2 11 24 57 }
	{ 2 11 31 41 }
	{ 2 11 22 41 }
	{ 2 8 25 43 }
	{ 2 7 25 51 }
	{ 2 7 41 62 }
	{ 2 7 25 60 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 7 41 53 }
	{ 2 5 25 43 }
	{ 4 15 21 58 }
	{ 2 7 41 54 }
	{ 2 7 25 52 }
	{ 2 8 21 41 }
	{ 4 6 31 57 }
	{ 2 22 31 43 }
	{ 2 7 43 53 }
	{ 2 15 21 60 }
	{ 2 5 44 57 }
	{ 2 11 29 60 }
	{ 2 7 29 60 }
	{ 2 5 25 44 }
	{ 2 3 21 41 }
	{ 4 6 11 57 }
	{ 4 13 50 57 }
	{ 2 11 21 62 }
	{ 2 11 22 61 }
	{ 4 11 21 50 }
	{ 2 7 44 61 }
	{ 4 15 21 57 }
	{ 2 11 21 57 }
	{ 4 21 47 58 }
5tile	{ 2 3 29 39 58 }
	{ 2 7 25 38 59 }
	{ 2 7 25 35 62 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 4 29 60 63 }
	{ 2 5 15 52 57 }
	{ 2 7 29 36 57 }
	{ 2 21 31 36 59 }
	{ 4 6 15 49 57 }
	{ 2 15 22 35 57 }
	{ 4 13 22 43 58 }
	{ 2 11 22 43 62 }
	{ 2 4 21 59 63 }
	{ 2 11 13 20 57 }
	{ 2 7 25 41 50 }
	{ 2 5 12 15 49 }
	{ 2 7 21 43 57 }
	{ 2 7 29 44 57 }
	{ 2 3 23 29 42 }
	{ 2 4 21 60 63 }
	{ 2 8 11 41 62 }
	{ 2 3 21 43 61 }
	{ 2 7 25 43 62 }
	{ 2 15 22 43 57 }
	{ 4 7 18 21 41 }
	{ 2 7 25 50 59 }
	{ 4 6 11 31 49 }
	{ 2 11 31 38 57 }
	{ 2 7 41 50 61 }
	{ 2 7 52 57 61 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 11 29 38 59 }
	{ 2 3 21 43 60 }
	{ 2 22 31 40 59 }
	{ 2 8 20 43 61 }
	{ 2 5 22 43 62 }
	{ 2 22 31 35 61 }
	{ 2 3 8 60 61 }
	{ 2 21 24 43 62 }
	{ 2 11 20 29 57 }
	{ 2 4 15 17 42 }
	{ 2 4 11 31 57 }
	{ 2 3 11 29 49 }
	{ 2 8 15 25 51 }
	{ 2 4 15 41 62 }
	{ 2 5 15 49 57 }
	{ 2 7 22 41 57 }
	{ 2 7 25 43 50 }
	{ 2 3 11 21 61 }
	{ 2 3 7 41 61 }
	{ 2 8 20 41 61 }
	{ 2 4 5 15 58 }
	{ 2 3 5 15 58 }
	{ 2 4 21 42 43 }
	{ 2 7 13 51 57 }
	{ 2 4 13 47 58 }
	{ 2 8 11 57 61 }
	{ 2 3 7 22 41 }

Continued. . .

TileNum	$[C_m(2)]$
	{ 2 7 30 57 60 }
	{ 2 3 29 60 63 }
	{ 2 4 11 31 49 }
	{ 2 7 41 50 63 }
	{ 2 7 36 57 61 }
	{ 2 4 11 31 58 }
	{ 2 7 25 50 63 }
	{ 2 7 22 57 59 }
	{ 2 3 29 47 58 }
	{ 2 7 41 56 57 }
	{ 2 7 25 56 57 }
	{ 2 7 29 52 57 }
	{ 2 3 24 57 61 }
	{ 2 7 44 57 62 }
	{ 2 5 29 48 60 }
	{ 2 3 8 13 57 }
	{ 2 11 29 36 57 }
	{ 2 4 5 16 57 }
	{ 2 15 20 43 61 }
	{ 2 3 8 58 61 }
	{ 2 3 8 59 61 }
	{ 2 7 41 52 61 }
	{ 2 7 20 25 57 }
	{ 2 4 16 23 57 }
	{ 2 3 16 38 57 }
	{ 2 11 29 40 52 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 11 29 36 52 }
	{ 2 7 21 43 61 }
	{ 2 4 15 53 57 }
	{ 2 7 21 25 43 }
	{ 2 7 21 47 57 }
	{ 2 8 21 43 57 }
	{ 2 15 21 36 57 }
	{ 2 7 29 52 59 }
	{ 2 4 17 32 43 }
	{ 2 15 21 43 62 }
	{ 2 4 15 53 60 }
	{ 2 3 16 29 57 }
	{ 2 3 7 20 57 }
	{ 2 4 29 44 49 }
	{ 2 7 35 52 61 }
	{ 2 5 15 57 58 }
	{ 2 3 29 32 59 }
	{ 2 8 11 20 57 }
	{ 2 8 29 54 59 }
	{ 2 15 22 54 59 }
	{ 2 7 24 35 61 }
	{ 2 3 6 44 61 }
	{ 2 4 31 44 53 }
	{ 2 4 17 31 44 }
	{ 2 3 24 29 57 }
	{ 2 4 21 30 41 }
	{ 2 3 29 59 60 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 4 16 53 59 }
	{ 2 8 24 43 61 }
	{ 2 8 25 31 51 }
	{ 2 4 11 21 41 }
	{ 2 3 8 13 58 }
	{ 2 3 13 38 57 }
	{ 2 3 13 39 57 }
	{ 2 7 13 35 57 }
	{ 4 8 11 29 58 }
	{ 2 8 11 29 57 }
	{ 2 4 22 41 63 }
	{ 2 11 21 36 41 }
	{ 2 3 8 14 57 }
	{ 2 3 8 23 57 }
	{ 2 11 29 38 52 }
	{ 2 4 13 50 59 }
	{ 4 8 29 47 49 }
	{ 2 11 30 36 59 }
	{ 2 7 47 57 62 }
	{ 2 3 30 60 61 }
	{ 2 11 24 43 62 }
	{ 2 4 31 43 62 }
	{ 2 3 22 42 63 }
	{ 2 8 11 52 61 }
	{ 2 5 12 50 63 }
	{ 2 3 16 57 61 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 3 7 52 61 }
	{ 2 15 22 43 59 }
	{ 2 4 31 38 57 }
	{ 2 22 43 47 59 }
	{ 2 4 22 32 41 }
	{ 2 11 29 38 57 }
	{ 4 15 22 43 57 }
	{ 2 11 29 36 61 }
	{ 2 4 13 22 59 }
	{ 4 8 29 43 49 }
	{ 2 4 5 58 63 }
	{ 2 4 21 42 63 }
	{ 2 4 13 35 58 }
	{ 2 4 7 30 41 }
	{ 2 7 11 57 62 }
	{ 2 4 5 15 57 }
	{ 2 7 36 38 61 }
	{ 4 6 11 15 49 }
	{ 2 7 20 22 41 }
	{ 4 7 18 22 41 }
	{ 2 11 32 36 57 }
	{ 2 15 22 47 57 }
	{ 2 15 21 43 57 }
	{ 2 5 15 44 58 }
	{ 2 4 21 31 59 }
	{ 2 4 21 31 41 }
	{ 2 21 31 43 59 }

Continued…

TileNum	$[C_m(2)]$
	{ 2 11 21 43 61 }
	{ 2 11 21 43 60 }
	{ 2 4 29 39 42 }
	{ 2 3 23 42 61 }
	{ 2 5 44 47 58 }
	{ 2 3 23 38 57 }
	{ 2 5 44 58 63 }
	{ 2 11 29 56 57 }
	{ 2 11 21 41 63 }
	{ 15 25 35 38 56 }
	{ 2 15 20 44 61 }
	{ 2 20 29 47 58 }
	{ 2 23 35 44 61 }
6tile	{ 2 7 22 25 35 41 }
	{ 2 3 8 29 57 60 }
	{ 2 3 8 29 58 59 }
	{ 2 5 12 15 50 57 }
	{ 2 8 11 30 57 60 }
	{ 2 8 11 13 50 57 }
	{ 2 3 13 22 43 49 }
	{ 2 5 15 36 58 61 }
	{ 2 5 15 36 57 62 }
	{ 2 5 29 44 47 50 }
	{ 2 7 30 38 41 59 }
	{ 4 7 18 21 43 57 }
	{ 2 7 25 35 38 61 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 8 11 31 50 57 }
	{ 2 8 29 47 50 59 }
	{ 2 7 31 36 57 62 }
	{ 2 15 22 36 59 61 }
	{ 2 3 29 38 39 57 }
	{ 4 6 31 47 49 58 }
	{ 2 7 13 35 50 61 }
	{ 2 4 5 23 57 63 }
	{ 2 4 5 29 44 58 }
	{ 2 3 5 29 44 58 }
	{ 2 4 13 43 58 62 }
	{ 2 3 11 20 30 49 }
	{ 2 4 15 32 57 62 }
	{ 2 4 32 47 57 62 }
	{ 2 4 31 47 49 62 }
	{ 2 4 21 47 59 62 }
	{ 2 7 20 29 41 44 }
	{ 2 8 11 50 57 63 }
	{ 2 7 24 25 35 57 }
	{ 2 4 15 31 49 58 }
	{ 2 8 15 25 56 57 }
	{ 2 7 29 36 41 61 }
	{ 2 4 15 30 41 49 }
	{ 2 7 22 25 41 43 }
	{ 2 4 13 31 51 58 }
	{ 2 3 6 29 47 49 }
	{ 2 7 22 25 43 59 }

Continued...

TileNum	$[C_m(2)]$
	{ 2 3 8 29 57 59 }
	{ 2 8 11 30 57 59 }
	{ 2 3 13 40 59 62 }
	{ 2 3 8 31 41 58 }
	{ 2 3 22 47 60 61 }
	{ 4 8 13 43 54 57 }
	{ 2 4 21 41 48 63 }
	{ 2 3 7 54 60 61 }
	{ 2 3 16 54 59 61 }
	{ 2 3 24 47 58 61 }
	{ 2 4 29 31 36 49 }
	{ 2 4 8 21 57 59 }
	{ 2 4 21 31 36 57 }
	{ 2 4 29 40 49 63 }
	{ 2 3 15 56 57 62 }
	{ 2 3 8 29 58 60 }
	{ 2 3 15 24 57 58 }
	{ 2 3 6 47 58 61 }
	{ 2 4 15 22 57 63 }
	{ 2 4 29 47 49 56 }
	{ 2 4 13 21 51 59 }
	{ 2 4 17 21 41 48 }
	{ 2 3 6 13 43 49 }
	{ 2 3 8 31 49 59 }
	{ 2 4 5 17 48 57 }
	{ 2 3 14 21 43 62 }

Continued...

TileNum	[$C_m(2)$]
	{ 2 4 30 31 41 51 }
	{ 2 3 6 11 49 61 }
	{ 2 3 29 40 59 62 }
	{ 2 3 8 31 57 58 }
	{ 2 3 7 29 50 59 }
	{ 2 3 7 60 61 62 }
	{ 4 8 13 54 57 59 }
	{ 2 3 5 11 34 61 }
	{ 2 3 13 43 54 58 }
	{ 2 8 25 30 41 51 }
	{ 4 7 18 22 43 57 }
	{ 2 7 29 36 51 61 }
	{ 2 3 13 22 47 58 }
	{ 2 7 22 25 35 43 }
	{ 2 3 5 44 61 62 }
	{ 2 3 6 39 49 61 }
	{ 2 4 29 31 44 50 }
	{ 2 3 7 18 29 59 }
	{ 2 4 13 31 58 59 }
	{ 2 7 25 38 41 56 }
	{ 2 4 21 31 43 57 }
	{ 2 4 15 18 21 59 }
	{ 2 7 29 38 41 59 }
	{ 2 7 25 38 43 61 }
	{ 2 22 30 40 47 59 }
	{ 2 4 29 39 44 58 }
	{ 2 4 15 29 39 58 }

Continued...

TileNum	[$C_m(2)$]
	{ 2 3 29 39 60 61 }
	{ 2 4 5 23 48 57 }
	{ 4 8 13 31 57 58 }
	{ 2 3 5 12 58 63 }
	{ 2 4 11 13 56 58 }
	{ 2 3 8 29 38 57 }
	{ 2 5 12 23 48 57 }
	{ 4 7 22 30 41 57 }
	{ 2 7 29 36 41 51 }
	{ 2 5 15 44 52 61 }
	{ 2 7 21 47 60 61 }
	{ 2 11 29 31 44 57 }
	{ 2 3 13 24 54 57 }
	{ 2 7 29 40 41 51 }
	{ 2 8 15 21 52 57 }
	{ 2 11 30 39 52 57 }
	{ 2 7 11 20 30 57 }
	{ 2 7 29 41 44 59 }
	{ 2 7 22 25 41 56 }
	{ 2 7 22 41 56 61 }
	{ 2 7 20 25 56 59 }
	{ 2 7 20 36 41 61 }
	{ 2 7 30 38 41 51 }
	{ 2 4 13 43 56 58 }
	{ 2 4 13 40 53 59 }
	{ 2 15 21 40 57 59 }

Continued. . .

TileNum	$[C_m(2)]$
	{ 2 4 15 30 41 56 }
	{ 2 4 15 21 38 59 }
	{ 2 3 21 47 60 62 }
	{ 2 11 29 44 47 57 }
	{ 2 7 22 29 41 56 }
	{ 2 5 15 17 36 57 }
	{ 2 7 29 41 44 51 }
	{ 2 22 29 40 47 59 }
	{ 2 24 31 43 54 61 }
	{ 2 7 20 36 43 61 }
	{ 2 5 20 23 44 61 }
	{ 2 11 24 29 44 61 }
	{ 2 5 20 23 47 58 }
	{ 3 8 22 34 44 61 }
	{ 2 3 16 29 38 59 }
	{ 15 22 25 36 49 59 }
	{ 2 15 24 35 54 61 }
	{ 2 8 31 44 54 61 }
7tile	{ 4 7 8 30 41 54 57 }
	{ 2 7 22 30 40 41 59 }
	{ 2 5 15 32 36 56 57 }
	{ 2 7 22 32 40 41 59 }
	{ 2 7 22 32 40 43 57 }
	{ 2 3 12 22 31 48 49 }
	{ 2 7 20 29 41 56 61 }
	{ 4 6 24 29 40 43 50 }
	{ 2 3 12 22 31 39 49 }

Continued. . .

TileNum	[$C_m(2)$]
	{ 2 3 22 31 39 48 49 }
	{ 2 11 29 30 39 47 52 }
	{ 2 5 24 29 47 52 58 }
	{ 2 7 16 22 29 40 59 }

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