國立交通大學

應 用 數 學 系

碩 士 論 文

應用逆推網絡在克拉茲猜想的探討

Study 3n+1 problem in a backward iteration net

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應用逆推網絡在克拉茲猜想的探討

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摘　要

本論文主要研究一個有名的數學問題：克拉茲猜想 (Collatz Conjecture)。雖然克拉茲猜想的形式與意義容易被理解，但它仍然尚未被證明。也因為如此，引發了我們對克拉茲猜想的興趣，也希望藉著研究對它有更多的了解。首先，我們會對克拉茲猜想做個簡單的介紹，並且整理一些其他人對克拉茲猜想做的研究。接下來，我們會建構一個特別的圖，也就是逆推網絡 (backward iteration net)，開始我們對克拉茲猜想的研究，同時我們也得到一些逆推網絡的特性。最後，我們嘗試建構一個模擬逆推網絡 (simulation backward iteration net)，去模擬實際的逆推網絡，根據兩者相互比較的數值結果做出分析結論。

關鍵詞：克拉茲猜想、逆推網絡、模擬逆推網絡。

Study $3n+1$ Problem in a Backward Iteration Net

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Abstract

3n+1 problem is one of the most famous conjectures in mathematics. It can be easily understood even younger children who just know how to divide by 2 and multiply by 3. However, the $3n+1$ problem still can not be proved yet. Therefore, we are so interested in this problem and hope to understand it completely. First, our study gives a brief introduction of the $3n+1$ problem, and we list some results in many research of the $3n+1$ problem. Then, we use a graph, a backward iteration net, to examine the $3n+1$ problem and obtain some properties in the backward iteration net. Finally, we simulate the backward iteration net and compare the numerical results with the net.

Keywords: $3n+1$ problem, backward iteration net, simulation backward iteration net.

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本篇論文的完成，最要感謝的就是我的指導教授張書銘老師，老師不僅是在研究上給我許多指導與方向，也教導我許多做人處事的道理與態度，在我遇到瓶頸的時候，老師總是不厭其煩的開導我鼓勵我，在我太過得意的時候，老師也會適時提醒該注意的小細節，在研究所生涯，受到老師的照顧與幫助實在是太多了老師，謝謝你帶給我的一切，真的非常謝謝老師。同時也要感謝我的口試委員楊一帆老師和何南國老師，謝謝你們給我許多指導，也提供了許多可以嘗試的方向與改進的方法，非常謝謝老師們。

在做研究與求學期間，我也要感謝所有修過課的老師們，謝謝你們教給我的許多概念與不同的知識領域，讓我有足夠的背景知識，讓這篇論文能夠有較爲完整的呈現，在此特別謝謝老師們的教導。

接下來，我要特別感謝媽媽、姑姑、叔叔、陳秋媛老師、吳慶堂老師、加鹽、耀宗、謝慶勳、貓達、小太、美女、GG 王、小 p、小吵、小泡、文昱、林志嘉、蜘蛛、劉崙欣、Blue，以及同師門的暉哥、林育賢、大鋼、定國、吳啓豪、黃柏綸、許尚。在論文難產的時候，多虧有你們的陪伴鼓勵開導，得到許多你們的溫暖與協助，最終這篇論文才得以順利誕生，在此獻上誠摯的感謝，也把這份榮耀與你們分享。

三年的研究所生涯，親身體悟了許多酸甜苦辣，人生中最好與最差的狀態在這期間也一併經歷。幸運的是，老師跟最親愛的家人朋友總是在我身邊，給我許多關懷與包容，讓我有足夠勇氣繼續往前走，希望我能好好記得這些深刻的種種在未來把它們都化爲正面力量，爲自己創造一個美好的人生，不讓自己與所有關心我的人們失望，並期許自己未來有能力當一個小太陽，帶給你們許多溫暖與歡樂。

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段俊旭

謹誌于交通大學

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1 Introduction

What's the $3n+1$ problem? The $3n+1$ problem can be presented by many ways. Attributed to L. Collatz, he defines the Collatz function as

$$C(n) = \begin{cases} 3n+1, & \texttt{if } n \texttt{ is odd.} \\ \frac{n}{2}, & \texttt{if } n \texttt{ is even.} \end{cases}$$

The $3n+1$ problem states that for each $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$, such that $C^{(k)}(n) = 1$, that is, any natural number will eventually iterate to 1 in finite iterations.

1.1 History of $3n+1$ Problem

The $3n+1$ problem is not only famous but historical. The $3n+1$ problem is presented about in the 1950s. It is often called "Syracuse conjecture". It is said that the $3n+1$ problem is first researched in American Syracuse University. [1]

For the same time, many mathematicians are interested in the $3n+1$ problem in the middle of the 20th century. American mathematics master, Martin Gardner, who called the $3n+1$ problem "Hail conjecture". Why did he call it? Because the $3n+1$ problem has a special phenomenon, sometimes the numbers will go larger but sometimes will go smaller. It's like that the hail influenced by air, it sometimes goes up and sometimes goes down between cloud layers in summer. Hence, it is called "Hail conjecture".

Afterward, Japanese scientist, Shizuo Katutani, brought the $3n+1$ problem from Europe to Japan, and he got some results: the $3n+1$ problem doesn't hold for negative integers. Thus the $3n+1$ problem is also called "Katutani conjecture". In 1996, Thwaites offered 1100 pounds for its solution, so the $3n+1$ problem had another name "Thwaites conjecture". [1]

Besides, the $3n+1$ problem has many kinds of names: Collatz conjecture, Hasse problem, Ulam problem. They are all about the mathematicians or places in researching the $3n+1$ problem. For the reason, the $3n+1$ problem has unique charisma in mathematics. [1]

Owing to the rapid develop of computer science in the 20th century' end, scientists use computers mighty computing speed to solve the $3n+1$ problem. Until now, about the natural numbers verification on $3n+1$ problem, from 1 to $20 \times 2^{58} \sim 5.764 \times 10^{18}$ [6], among these natural numbers, they all eventually will iterate to 1, and enter $1-4-2-1$ cycle. That is, they all hold for the $3n+1$ problem.

However, no matter computers can check how large natural numbers for the $3n+1$ problem, it still can't be proved by this way. How to solve the $3n+1$ problem is what we continue to strive.

1.2 Some Definitions and Results for The $3n+1$ Problem

For a long time, many mathematicians try kinds of ways to analysis $3n+1$ problem. Although $3n+1$ problem is still an open problem, they get some results about it. In this subsection, we will introduce some definitions, records, and results about their research for $3n+1$ problem.

Definition 1.1 ([16]). *Let $f(n) := n/2$, if n is even, and $f(n) := 3n+1$, if n is odd. Choosing a natural number x as staring number and applying f repeatedly produces a sequence of natural numbers, which is called $f-trajectory$ of x and denoted by*

$$\Gamma_f(x) := (x, f(x), f(f(x)), \ldots, f^k(x), \ldots).$$

For example, taking $x = 13$ gives the f-trajectory, then
$\Gamma_f(13) = (13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ...)$ which continues periodically with the cycle $(4, 2, 1)$. There are many starting numbers which have been tested. This leads to the $3n+1$ problem which asserts that any f-trajectory eventually runs into the limiting cycle (4,2,1).

In 2002, David Gluck and Brian D. Taylor had some results about the statistic of finite Collatz trajectory. The Theorem 1.2 gives more precise description.

Theorem 1.2 ([8, 11]). *If $a = (a_1, a_2, \ldots, a_n)$ is a finite Collatz trajectory starting from a_1, with $a_n = 1$ being the first time 1 is reached and the statistic*

$$C(a) = \frac{a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1}{(a_1)^2 + (a_2)^2 + \cdots + (a_n)^2}.$$

Then $\frac{9}{13} \le C(a) \le \frac{5}{7}$.

In the following, we will introduce the function T. In $3n+1$ problem research, several authors prefer to deal with T instead of f.

Definition 1.3 ([4, 11, 16]). *Let map $T : \mathbb{N} \to \mathbb{N}$ given by*

$$T(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2. \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

The $3x+1$ Conjecture states that every $m \ge 1$ has some iterate $T^{(k)}(m) = 1$.

This function T replaces the function f defined above without loss of information: if n is even, then $T(n) = f(n)$, and if n is odd, then $T(n) = f(f(n))$ (as $3n+1$ is even whenever n is odd). In this sense, T "shortens" the f-trajectories. Similar to f-trajectory, the T-trajectory of an integer n is the set

$$O^+(m) := (m, T(m), T(T(m)), \ldots, T^k(m), \ldots).$$

The structure of the positive integers forces any orbit of T to iterate to one of the following:

(1) the trivial cycle (1,2);

(2) a non-trivial cycle;

(3) infinity (the orbit is divergent).

The $3n+1$ problem claims that option (1) occurs in all cases. That is, the $3n+1$ conjecture asserts that for any starting number in $\mathbb{N}$, the T-trajectory eventually ends in the cycle (1,2). Moreover, there is a T-trajectory conjecture as follows:

No divergent trajectory conjecture.[16] There is no divergent $3n+1$ trajectory, i.e., there is no $y \in \mathbb{N}$ such that $\lim_{n\to\infty} T^{(n)}(y) = \infty$.

Definition 1.4 ([4, 11, 16]). *The stopping time $\sigma(n)$ of n is defined by*

$$\sigma(n) = \inf\{k : f^k(n) < n\}.$$

Why do we interested in $\sigma(n)$? If any $\sigma(n)$ is finite, then the $3n+1$ conjecture holds. Why? First, check that every positive integer up to $N-1$ iterates to 1, then consider the iterates of N. Once the iterates go below N, you are done. For this reason, one considers the so-called *stopping time* of n.

Definition 1.5 ([4, 11, 16]). *The total stopping time $\sigma_\infty(n)$ of n is defined by*

$$\sigma_\infty(n) = \inf\{k : f^k(n) = 1\}.$$

The $3n+1$ conjecture holds if any $\sigma_\infty(n)$ is finite.

Definition 1.6 ([4, 11, 16]). *The height h(n) of n is defined by*

$$h(n) = \sup\{f^k(n) : k \in \mathbb{Z}^+\}.$$

In [1], the author uses a interesting description to introduce $\sigma(n)$, $\sigma_\infty(n)$, $h(n)$. The author calls "the nth flight" as the starting value n, "keeping height distance" as $\sigma(n)$, "flight distance" as $\sigma_\infty(n)$, and "maximum flight height" as $h(n)$. For example, the 11st flight is

$$11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

By observing the 11st flight, we know the maximum flight height is 52, $h(11)$. Moreover, the flight distance is 14, $\sigma_\infty(11)$. Specially, the keeping height distance is 7, $\sigma(11)$. The author defines the flight distance of the 1st flight is 0. In the following, from $n = 1$ to $n = 30$, the author list $\sigma_(n)$, $\sigma_\infty(n)$, $h(n)$ as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sigma_\infty(n)$	0	1	7	2	5	8	16	3	19	6	14	9	9	17	17
$\sigma(n)$	0	0	5	0	2	0	10	0	2	0	7	0	2	0	10
$h(n)$	1	2	16	4	16	16	52	8	52	16	52	16	40	52	160

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\sigma_\infty(n)$	4	12	20	20	7	7	15	15	10	23	10	111	18	18	18
$\sigma(n)$	0	2	0	5	0	2	0	7	0	2	0	95	0	2	0
$h(n)$	16	52	52	88	20	64	52	160	24	88	40	9232	52	88	160

However, based on [16], L. E. Garner(1984) defined the *height* of a natural number n the least non-negative integer k such that $T^k = 1$ (if such k exists). That is,

$$h_T(n) := min\{k \in \mathbb{N} \cup \{0\} : T^k(n) = 1\}$$

L. E. Garner proved the theorem as follows:

Theorem 1.7 ([7]). *There are infinitely many pairs of consecutive positive integers $(n, n+1)$ whose T-trajectories coincide at a certain step, i.e. there is a non-negative integer k with the property $T^k(n) = T^k(n+1)$.*

About "consecutive integer pairs of the same height in the Collatz conjecture", Wu Jia Bang and Huang Guo Lin [17] have investigated pairs of successive integers of the same hight. They found that families of consecutive integer pairs of the same height occur infinitely often and in different patterns.

Remark 1.8 ([1]). *On Collatz function, the author calls $f(n) = n/2$ "even transformation", and calls $f(n) = 3n + 1$ "odd transformation". Moreover, for a starting number n, in the* *n iterating to 1 process, the author counts the total number of odd transformation, denoted $O(n)$, and counts the total number of even transformation, denoted $E(n)$. Thus, $\sigma_\infty(n) = O(n) + E(n)$.*

Based on [1], the relation between $O(n)$ and $E(n)$ was proved as the following theorem:

Theorem 1.9. *For any $n \in \mathbb{N}$, $O(n)/E(n) < \log 2/\log 3$.*

Definition 1.10 ([4, 16])**.** *For $a \in \mathbb{Z}^+$, the predecessor set of a is defined by*

$$P_T(a) := \{b \in \mathbb{Z}^+ : T^{(k)}(b) = a \text{ for some } k \in \mathbb{Z}^+\}.$$

In brief, $P_T(a)$ is set of numbers which approach a given number a on T-function. Thus, by the definition, we realize that the $3n+1$ conjecture holds if $P_T(1) \equiv \mathbb{N}$.

Definition 1.11 ([4, 16])**.** *For $a \in \mathbb{Z}^+$, the counting function of predecessor set of a not exceeding a given bound x is defined by*

$$Z_a(x) := |\{n \in P_T(a) : n \leq x\}|.$$

By the definition of $Z_a(x)$, we realize that the $3n+1$ conjecture holds if $Z_1(x) = x$ for all $x \geq 1$. The size of $Z_a(x)$ was first studied by Crandall, who proved the theorem below:

Theorem 1.12 ([5])**.** *There is a positive constant c such that $Z_1(x) > x^c$ for sufficiently large $x \in \mathbb{N}$.*

Based on [5, 16], Wirsching [15](1998) notes that this result extends to $Z_a(x)$ for all $a \neq 0$ (mod 3). Using the tree-search method of Crandall, Sander [13](1990) gave a specific lower bound, $c = 0.25$. This "tree-search method" has been improved by D. Applegate and J. C. Lagarias [2](1995), arriving at a computer-assisted proof for $c = 0.654$. Using functional difference inequalities, Krasikov [9](1989) succeeded in proving $Z_a(x) > x^c$ with $c = 3/7 \approx 0.42857$, for sufficiently large x and certain numbers $a \in \mathbb{N}$ (including $a = 1$). Wirsching [14](1993) used the same approach to obtain $c = 0.48$. Applegate and Lagarias [3](1995) superseded these results with $Z_1(x) > x^{0.81}$, for sufficiently large x, by enhancing Krasikov's approach with nonlinear programming. Recently, Krasikov and Lagarias [10](2002) streamlined this approach to obtain

$$Z_1(x) > x^{0.84}, \ \textit{x sufficiently large.}$$

On the other hand, there are some results about computational verification of $3n+1$ conjecture. Based on [6], in 1996, the team of Tomas Oliveira e Silva wrote a computer program to test the $3n+1$ conjecture. In 1999, they reported on computations verifying the $3x+1$ conjecture for $n < 3 \times 2^{53} \approx 2.702 \times 10^{16}$. In 2004, they devised an improved algorithm to test the $3n+1$ conjecture, about three times faster than their previous one. Thus, they restarted the verification efforts in 2004, however, they stopped the work in January 2009. Nevertheless, they have verified the $3n+1$ conjecture up to

$$20 \times 2^{58} = 5764607523034234880 > 5.764 \times 10^{18}.$$

Besides, the team of Eric Roodendaal [12] maintains an ongoing distributed search program for verifying the $3n+1$ conjecture to new records. As of February 2008 the $3n+1$ conjecture is verified up to $612 \times 2^{50} \approx 6.89 \times 10^{17}$. The current record happens in February 2011, they have verified the $3n+1$ conjecture up to

$$964 \times 2^{50} \approx 1.805 \times 10^{18}.$$

In the following, about the definitions mentioned above, we classify the equivalent propositions of $3n+1$ problem in Remark 1.13.

Remark 1.13. *If the $3n+1$ problem holds, then it is equivalent to the following propositions:*

(1) For any starting value $n \in \mathbb{N}$, the f-trajectory $\Gamma_f(n)$ *eventually runs into the limiting cycle* (4,2,1).

(2) For any starting value $n \in \mathbb{N}$, the T-trajectory $O^+(n)$ *eventually runs into the limiting cycle* (2,1).

(3) For any starting value $n \in \mathbb{N}$, the stopping time $\sigma(n)$ is finite.

(4) For any starting value $n \in \mathbb{N}$, the total stopping time $\sigma_\infty(n)$ is finite.

(5) For predecessor set of 1, $P_T(1) \equiv \mathbb{N}$.

(6) For all $x \geq 1$, $Z_1(x) = x$.

In this section, we have preliminary concepts about the $3n+1$ problem. In the following, we will introduce the $3n+1$ backward iteration net, which is an important role in our study in the $3n+1$ problem.

2 The $3n+1$ Backward Iteration Net $\beta_k(n)$

What's the *3n+1 backward iteration net*? A $3n+1$ backward iteration net is constructed by a starting value $n \in \mathbb{N}$ and the formula of $3n+1$ backward iteration. Why are we interested in the $3n+1$ backward iteration net? In the beginning, our idea about the $3n+1$ problem is: if any natural number will iterate to 1 in finite transformations on Collatz function, then, in other words, if we start from 1 and apply the $3n+1$ backward iteration formula, can we strength all the natural numbers set? If we can, then we equally solve the $3n+1$ problem. Therefore, the $3n+1$ backward iteration net is the chief topic what we strive. In this section, we will introduce the $3n+1$ backward iteration formula I, the $3n+1$ backward iteration net $\beta_k(n)$, and relation between natural numbers in $\beta_k(n)$, and some properties of $\beta_k(n)$.

2.1 Formula

In this subsection, we will introduce the formula of *3n+1 backward iteration*. In fact, the $3n+1$ backward iteration is the inverse operator of Collatz function, denoted by I on the domain $\mathbb{Z}^+$, its formula is below:

$$I(n) = \begin{cases} \{2n\}, n \in \mathbb{N} \setminus \{6k+4 | k \in \mathbb{N} \cup \{0\}\}. \\ \{2n, (n-1)/3\}, n \in \{6k+4 | k \in \mathbb{N} \cup \{0\}\}. \end{cases}$$

We define $I^{(0)}(n) = n$, $I^{(k)}(n) = \{q | q \in I(p), p \in I^{(k-1)}(n)\}$, $k \in \mathbb{N}$. In the following, we will apply I to construct the $3n+1$ backward iteration net.

2.2 The $3n+1$ Backward Iteration net $\beta_k(n)$

In this subsection, we will introduce how we apply I to construct the $3n+1$ backward iteration net. We define $\beta_k(n)$ as a $3n+1$ backward iteration net constructed by a starting value $n \in \mathbb{N}$ and iterating k times on I, $k \in \mathbb{N}$. Moreover, $\beta_k(n)$ is a directed graph consisting of the vertex set $V(\beta_k(n))$ and the $E(\beta_k(n))$. In the following, we will introduce the definition of directed graph and define the corresponding notations for $\beta_k(n)$.

Definition 2.1. *A directed graph or digraph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge and ordered pair of vertices. In a digraph, we write (u,v) for an edge. If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v. We also write $u \to v$ for "there is an edge from u to v".*

By use of Definition 2.1, we give a more precise definition of the $3n+1$ backward iteration $\beta_k(n)$ as follows:

Definition 2.2. *In a $3n+1$ backward iteration net $\beta_k(n)$, the definitions of $V(\beta_k(n))$, $E(\beta_k(n))$, and the edge (p,g) for $\beta_k(n)$ as follows:*

1. $V(\beta_k(n))=\{p|p \in \{n\} \bigcup I^{(1)}(n) \bigcup I^{(2)}(n) \bigcup \cdots \bigcup I^{(k)}(n)\}$.

2. $E(\beta_k(n))=\{(p,q)|p,q \in V(\beta_k(n))$ *and* $q \in I(p)\}$.

3. *We denote an ordered pair (p,q) for an edge, if $q \in I(p)$, $p,q \in V(\beta_k(n))$. We also write $p \to q$ for "there is an edge from p to q".*

For convenience, we let $I^{(k)}(n)$ be L_k, called **level k**. In the following, we take $\beta_4(10)$ as an example to explain.

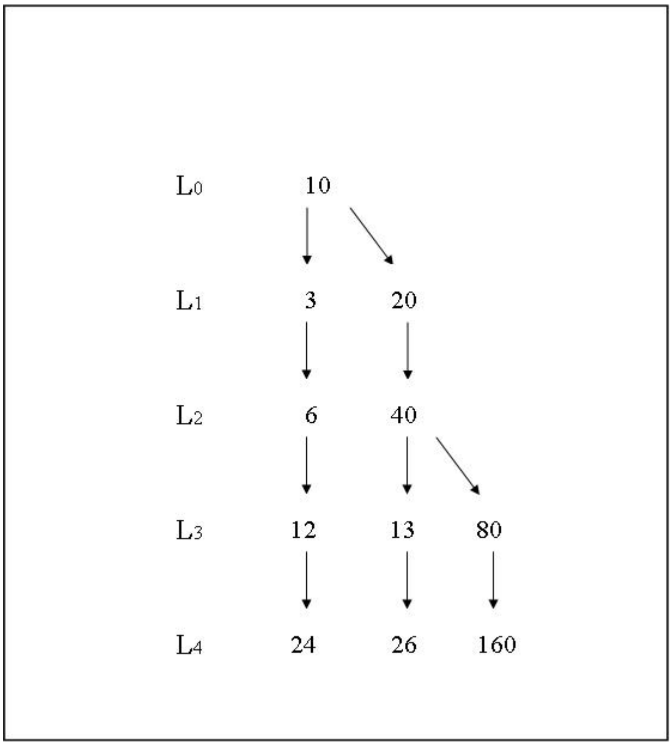

Figure 1: $\beta_4(10)$

By observing **Figure 1**, we can easily realize that $\beta_4(10)$ is constructed by the starting value 10 and iterating 4 times on I. Then, we will explain the vertex set $V(\beta_4(10))$ and the edge set $E(\beta_4(10))$ as follows:

1. Since $L_0 = I^{(0)}(10) = 10$, $L_1 = I^{(1)}(10) = \{3, 20\}$, $L_2 = I^{(2)}(10) = \{6, 40\}$, $L_3 = I^{(3)}(10) = \{12, 13, 80\}$, $L_4 = I^{(4)}(10) = \{24, 26, 160\}$. Hence, the vertex set $V(\beta_4(10)) = \{10, 3, 20, 6, 40, 12, 13, 80, 24, 26, 160\}$.

2. Since $V(\beta_4(10)) = \{10, 3, 20, 6, 40, 12, 13, 80, 24, 26, 160\}$, and $I(10) = \{3, 20\}$, $I(3) = \{6\}$, $I(20) = \{40\}$, $I(6) = \{12\}$, $I(40) = \{13, 80\}$, $I(12) = \{24\}$, $I(13) = \{26\}$, $I(80) = 160$. Hence, the edge set $E(\beta_4(10)) = \{$(10,3), (10,20), (3,6), (20,40), (6,12), (40,13), (40,80), (12,24), (13,26), (80,160)$\}$.

According to our initial idea about the $3n+1$ problem, we let 1 be the starting value and apply I to construct the $3n+1$ backward iteration net. That is, for a sufficient large natural number k, $\beta_k(1)$ is the main $3n+1$ backward iteration net which we study in $3n+1$ problem. Its partial framework is below:

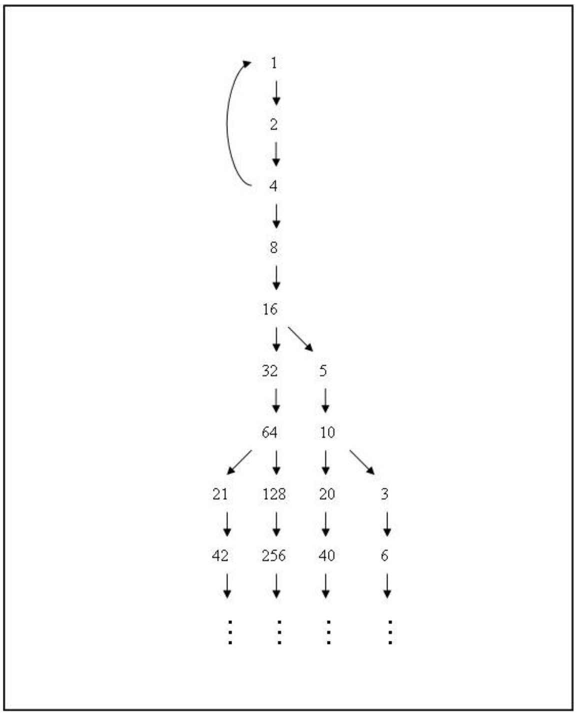

Figure 2: The partial framework of $\beta_k(1)$.

In order to know the $3n+1$ backward iteration net more, in the following, we will introduce the relation between natural numbers in the $3n+1$ backward iteration net.

2.3 The Relations Between Natural Numbers Under One Transformation On I

By observing the $3n+1$ backward iteration net, we realize that one natural number may backward iterate to another two different natural numbers. In this subsection, we partition the natural numbers set into six parts to realize the relations between natural numbers under **one** transformation on I in the $3n+1$ backward iteration net. In this way, we also can clearly realize that the "edges" are "from whom to whom" in the $3n+1$ backward iteration net. The six partitions of natural numbers set which we classify as follows:

$A := \{6k+1 | k \in \mathbb{N} \cup \{0\}\}$;
$B := \{6k+2 | k \in \mathbb{N} \cup \{0\}\}$;
$C := \{6k+3 | k \in \mathbb{N} \cup \{0\}\}$;
$D := \{6k+4 | k \in \mathbb{N} \cup \{0\}\}$;
$E := \{6k+5 | k \in \mathbb{N} \cup \{0\}\}$;
$F := \{6k+6 | k \in \mathbb{N} \cup \{0\}\}$.

For consistence, we let $a \in A$, $b \in B$, $c \in C$, $d \in D$, $e \in E$, $f \in F$. In the following, we will classify the relations between the natural numbers under **one** transformation on I in $3n+1$ backward iteration net.

(1) For $a \in A$, $\exists b \in B$ s.t. $I(a) = \{b\}$, that is, $a \to b$.

It shows that a must backward iterate to b under **one** transformation on I. Since $a = 6k+1$ for some $k \in \mathbb{N} \cup \{0\}$, then $I(a) = \{2(6k+1)\} = \{12k+2\} = \{6(2k)+2\}$. That is, there exists $b = 6(2k)+2 \in B$ s.t. $I(a) = \{b\}$. □

(2) For $b \in B$, $\exists d \in D$ s.t. $I(b) = \{d\}$, that is, $b \to d$.

It shows that b must backward iterate to d under **one** transformation on I. Since $b = 6k+2$ for some $k \in \mathbb{N} \cup \{0\}$, then $I(b) = \{2(6k+2)\} = \{12k+4\} = \{6(2k)+4\}$. That is, there exists $d = 6(2k)+4 \in D$ s.t. $I(b) = \{d\}$. □

(3) For $c \in C$, $\exists f \in F$ s.t. $I(c) = \{f\}$, that is, $c \to f$.

It shows that c must backward iterate to f under **one** transformation on I. Since $c = 6k + 3$ for some $k \in \mathbb{N} \cup \{0\}$, then $I(c) = \{2(6k + 3)\} = \{12k + 6\} = \{6(2k) + 6\}$. That is, there exists $f = 6(2k) + 6 \in F$ s.t. $I(c) = \{f\}$. □

(4) For $d \in D$: $d \to \{b, a\}$, or $d \to \{b, c\}$, or $d \to \{b, e\}$.

It shows that d may backward iterate to $\{b, a\}$, $\{b, c\}$, *or* $\{b, e\}$ under **one** transformation on I.

proof: Specially, d will backward iterate to another two different natural numbers. Since $d = 6k+4$ for some $k \in \mathbb{N} \cup \{0\}$, one is $I(d) = \{2(6k+4)\} = \{12k+8\} = \{6(2k+1)+2\}$, that is, there exists $b = 6(2k+1)+2 \in B$ s.t. $I(d) = \{b\}$. And another is $I(d) = \{((6k+4)-1)/3\} = \{2k + 1\}$. Now we consider k:

Case 1: If $k = 3m$, for $m \in \mathbb{N} \cup \{0\}$ s.t. $I(d) = \{((6k + 4) - 1)/3\} = \{2k + 1\} = \{2(3m) + 1\} = \{6m + 1\}$. That is, there exists $a = 6m + 1 \in A$ s.t. $I(d) = \{a\}$.

Case 2: If $k = 3m + 1$, for $m \in \mathbb{N} \cup \{0\}$ s.t. $I(d) = \{((6k + 4) - 1)/3\} = \{2k + 1\} = \{2(3m + 1) + 1\} = \{6m + 3\}$. That is, there exists $c = 6m + 3 \in C$ s.t. $I(d) = \{c\}$.

Case 3: If $k = 3m + 2$, for $m \in \mathbb{N} \cup \{0\}$ s.t. $I(d) = \{((6k + 4) - 1)/3\} = \{2k + 1\} = \{2(3m + 2) + 1\} = \{6m + 5\}$. That is, there exists $e = 6m + 5 \in E$ s.t. $I(d) = \{e\}$.

Therefore, It shows that d may backward iterate to $\{b, a\}$, $\{b, c\}$, or $\{b, e\}$ under **one** transformation on I. □

For completeness, we let the notations $d_a, d_c, d_e, D_a, D_c, D_e$ represent:

$d_a \to \{b, a\}$; $d_c \to \{b, c\}$; $d_e \to \{b, e\}$.

$D_a = \{d_a\} = \{18m + 4|m \in \mathbb{N} \cup \{0\}\}$;

$D_c = \{d_c\} = \{18m + 10|m \in \mathbb{N} \cup \{0\}\}$;

$D_e = \{d_e\} = \{18m + 16|m \in \mathbb{N} \cup \{0\}\}$;

That is, $D = D_a \cup D_c \cup D_e$.

(5) For $e \in E$, $\exists d \in D$ s.t. $I(e) = \{d\}$, that is, $e \to d$.

It shows that e must backward iterate to d under **one** transformation on I. Since $e = 6k + 5$ for some $k \in \mathbb{N} \cup \{0\}$, then $I(e) = \{2(6k + 5)\} = \{12k + 10\} = \{6(2k + 1) + 4\}$. That is, there exists $d = 6(2k + 1) + 4 \in D$ s.t. $I(e) = \{d\}$. □

(6) For $f \in F$, $\exists f_1 \in F$ s.t. $I(f) = \{f_1\}$, that is, $f \to f_1$.

It shows that f must backward iterate to f_1 under **one** transformation on I.

proof: Since $f = 6k + 6$ for some $k \in \mathbb{N} \bigcup \{0\}$, then $I(f) = \{2(6k + 6)\} = \{12k + 12\} = \{6(2k + 1) + 6\}$. That is, there exists another $f_1 = 6(2k + 1) + 6 \in F$ s.t. $I(f) = \{f_1\}$. □

In brief, we present their relations by two graphs:

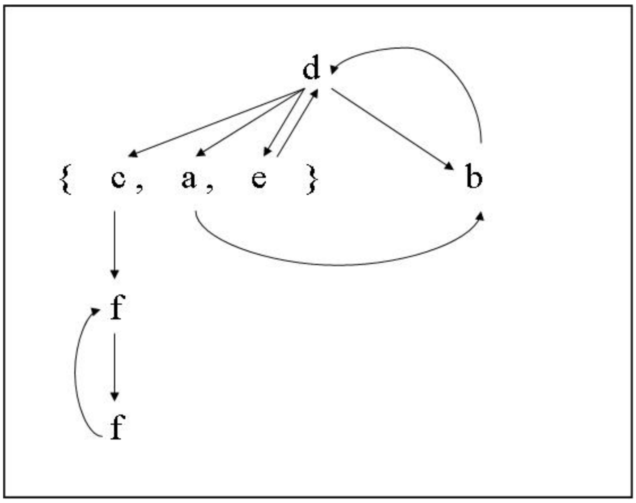

Figure 3: The number relation in the $3n + 1$ backward iteration net.

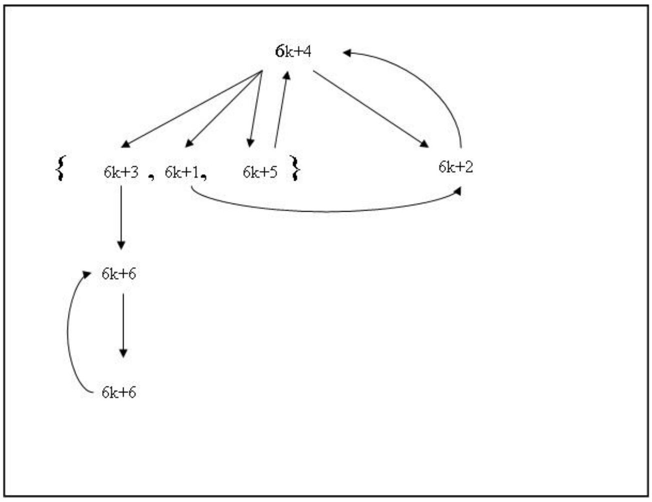

Figure 4: The number relation in the $3n + 1$ backward iteration net.

2.4 The Total Number Relations Between Consecutive Levels in The $3n+1$ Backward Iteration Net

In this subsection, by use of the relations between natural numbers under one transformation on I in the $3n+1$ backward iteration net, we classify the total number relations of A to F in two consecutive levels. We list some notations of the $3n+1$ backward iteration net.

L_i : The ith level, $i = 0, 1, 2, 3, \ldots$;
$L_i(N)$: The total natural numbers in L_i;
$L_i(A)$: The total numbers of A in L_i;
$L_i(B)$: The total numbers of B in L_i;
$L_i(C)$: The total numbers of C in L_i;
$L_i(D)$: The total numbers of D in L_i;
$L_i(E)$: The total numbers of E in L_i;
$L_i(F)$: The total numbers of F in L_i.

Then, the total number relations of A to F in two consecutive levels as follows:

1. $L_i(D) = L_{i+1}(A) + L_{i+1}(C) + L_{i+1}(E)$.

proof: By the relations of numbers in the backward iteration net, we have:

(1) $d_a \to \{b, a\}$; $d_c \to \{b, c\}$; $d_e \to \{b, e\}$.

(2) Moreover, the previous step of a must be d_a. The previous step is the original $3n+1$ rules. $C(a) = C(6k+1) = 3(6k+1)+1 = 18k+4 \in D_a$, that is, there exists a $d_a = 18k+4 \in D_a$ s.t. $C(a) = d_a$. Similarly, the previous step of c must be d_c, and the previous step of e must be d_e.

By (1) and (2) We have:
$L_{i+1}(A) = L_i(D_a)$; $L_{i+1}(C) = L_i(D_c)$; $L_{i+1}(E) = L_i(D_e)$.
$L_i(D) = L_i(D_a) + L_i(D_c) + L_i(D_e) = L_{i+1}(A) + L_{i+1}(C) + L_{i+1}(E)$.

□

2. $L_i(B) + L_i(E) = L_{i+1}(D)$.

proof: By the relations of numbers in the backward iteration net, we have:

(1) $b \to d$ and $e \to d$.

(2) Moreover, the previous step of d must be b or e. The previous step is the original $3n+1$ rules. Since $d = 6k+4 \in even$, for some $k \in \mathbb{N}$, $C(d) = (6k+4)/2 = 3k+2$.

Now we consider k:

Case 1: If $k = 2m$, $m \in \mathbb{N}$, then $C(d) = 3k+2 = 3 \cdot (2m) + 2 = 6m+2 \in B$.

Case 2: If $k = 2m+1$, $m \in \mathbb{N} \cup \{0\}$, then $C(d) = 3k+2 = 3 \cdot (2m+1) + 2 = 6m+5 \in E$.

By (1) and (2): We have $L_i(B) + L_i(E) = L_{i+1}(D)$.

□

3. $L_i(A) + L_i(D) = L_{i+1}(B)$.

proof: By the relations of numbers in the backward iteration net, we have:

(1) $a \to b$ and $d \to b$.

(2) Moreover, the previous step of b must be a or d. The previous step is the original $3n+1$ rules. Since $b = 6k+2 \in even$, for some $k \in \mathbb{N}$, $C(d) = (6k+2)/2 = 3k+1$.

Now we consider k:

Case 1: If $k = 2m$, $m \in \mathbb{N}$. Then $C(b) = 3k+1 = 3 \cdot (2m) + 1 = 6m+1 \in A$.

Case 2: If $k = 2m+1$, $m \in \mathbb{N}$. Then $C(b) = 3k+1 = 3 \cdot (2m+1) + 1 = 6m+4 \in D$.

By (1) and (2): We have $L_i(A) + L_i(D) = L_{i+1}(B)$.

□

4. $L_i(C) + L_i(F) = L_{i+1}(F)$.

proof: By the relations of numbers in the backward iteration net, we have:

(1) $c \to f$ and $f \to f$.

(2) Moreover, the previous step of f must be c or f. The previous step is the original $3n+1$ rules. Since $f = 6k+6 \in even$, for some $k \in \mathbb{N}$, $C(f) = (6k+6)/2 = 3k+3$.

Now we consider k:

Case 1: If $k = 2m$, $m \in \mathbb{N}$, then $C(f) = 3k+3 = 3 \cdot (2m) + 3 = 6m+3 \in C$.

Case 2: If $k = 2m+1$, $m \in \mathbb{N}$, then $C(f) = 3k+3 = 3 \cdot (2m+1) + 3 = 6m+6 \in F$.

By (1) and (2): We have $L_i(C) + L_i(F) = L_{i+1}(F)$.

□

5. $L_i(N) + L_i(D) = L_{i+1}(N)$.

proof: By the relations of numbers in $3n + 1$ backward iteration net, we have: under one transformation on I, a, b, c, e, f, will iterate to only one natural number respectively. However, for $d \in D$, d will iterate to two different natural numbers under one transformation on I. Hence, the extra numbers of L_{i+1} more than L_{i+1} equals $L_i(D)$. That is, $L_i(N) + L_i(D) = L_{i+1}(N)$.

□

2.5 Properties

In this subsection, we will show the relation between Collatz function $C(n)$ and its inverse operator, $I(n)$. This helps us realize the relations among more natural numbers in the $3n+1$ backward iteration net.

Remark 2.3. *For $p, q \in \mathbb{N}$, if $C(p) = q$, then $p \in I(q)$.*

proof: We will prove it by considering p.

Case 1: If $p = 6k$, for some $k \in \mathbb{N}$, since $C(p) = C(6k) = (6k)/2 = 3k = q$, then $I(q) = I(3k) = \{2(3k)\} = \{6k\}$. Thus, $p \in I(q)$.

Case 2: If $p = 6k+1$, for some $k \in \mathbb{N}\cup\{0\}$, since $C(p) = C(6k+1) = 3(6k+1)+1 = 18k+4 = q$, then $I(q) = I(18k+4) = \{\frac{(18k+4)-1}{3}, 2(18k+4)\} = \{6k+1, 36k+8\}$. Thus, $p \in I(q)$.

Case 3: If $p = 6k+2$, for some $k \in \mathbb{N} \cup \{0\}$, since $C(p) = C(6k+2) = (6k+2)/2 = 3k+1 = q$, then $I(q) = I(3k+1) = \{2(3k+1)\} = \{6k+2\}$. Thus, $p \in I(q)$.

Case 4: If $p = 6k+3$, for some $k \in \mathbb{N}\cup\{0\}$, since $C(p) = C(6k+3) = 3(6k+3)+1 = 18k+10 = q$, then $I(q) = I(18k+10) = \{\frac{(18k+10)-1}{3}, 2(18k+10)\} = \{6k+3, 36k+20\}$. Thus, $p \in I(q)$.

Case 5: If $p = 6k+4$, for some $k \in \mathbb{N} \cup \{0\}$, since $C(p) = C(6k+4) = (6k+4)/2 = 3k+2 = q$, then $I(q) = I(3k+2) = \{2(3k+2)\} = \{6k+4\}$. Thus, $p \in I(q)$.

Case 6: If $p = 6k+5$, for some $k \in \mathbb{N}\cup\{0\}$, since $C(p) = C(6k+5) = 3(6k+5)+1 = 18k+16 = q$, then $I(q) = I(18k+16) = \{\frac{(18k+16)-1}{3}, 2(18k+16)\} = \{6k+5, 36k+32\}$. Thus, $p \in I(q)$.

Because Case 1 to Case 6 all hold, we claim the remark.

□

Remark 2.4. *For $p, q \in \mathbb{N}$, if $p \in I(q)$, then $C(p) = q$.*

proof: We will prove it by considering q.

Case 1: If $q = 6k$, for some $k \in \mathbb{N}$, since $I(q) = I(6k) = \{2(6k)\} = \{12k\}$ and $p \in I(q)$, then $p = 12k$. Thus, $C(p) = C(12k) = (12k)/2 = 6k = q$.

Case 2: If $q = 6k+1$, for some $k \in \mathbb{N} \cup \{0\}$, since $I(q) = I(6k+1) = \{2(6k+1)\} = \{12k+2\}$ and $p \in I(q)$, then $p = 12k+2$. Thus, $C(p) = C(12k+2) = (12k+2)/2 = 6k+1 = q$.

Case 3: If $q = 6k+2$, for some $k \in \mathbb{N}$, since $I(q) = I(6k+2) = \{2(6k+2)\} = \{12k+4\}$ and $p \in I(q)$, then $p = 12k+4$. Thus, $C(p) = C(12k+4) = (12k+4)/2 = 6k+2 = q$.

Case 4: If $q = 6k+3$, for some $k \in \mathbb{N}$, since $I(q) = I(6k+3) = \{2(6k+3)\} = \{12k+6\}$ and $p \in I(q)$, then $p = 12k+6$. Thus, $C(p) = C(12k+6) = (12k+6)/2 = 6k+3 = q$.

Case 5: If $q = 6k+4$, for some $k \in \mathbb{N}$, since $I(q) = I(6k+4) = \{\frac{(6k+4)-1}{3}, 2(6k+4)\} = \{2k+1, 12k+8\}$ and $p \in I(q)$, then $p = 2k+1$ or $p = 12k+8$. Now we consider the two conditions:

Subcase 1: If $p = 2k+1$, then $C(p) = C(2k+1) = 3(2k+1)+1 = 6k+4 = q$.

Subcase 2: If $p = 12k+8$, then $C(p) = C(12k+8) = (12k+8)/4 = 6k+4 = q$.

By Subcase 1 and Subcase 2, the Case 5 holds.

Case 6: If $q = 6k+5$, for some $k \in \mathbb{N}$, since $I(q) = I(6k+5) = \{2(6k+5)\} = \{12k+10\}$ and $p \in I(q)$, then $p = 12k+10$. Thus, $C(p) = C(12k+10) = (12k+10)/2 = 6k+5 = q$.

Because Case 1 to Case 6 all hold, we claim the remark.

□

Proposition 2.5. *If $q \in \mathbb{N}$, then $I(q) = \{p | \ C(p) = q, p \in \mathbb{N}\}$.*

proof: By Remark 2.3 and Remark 2.4, we have: for $p, q \in \mathbb{N}$, $C(p) = q$ if and only if $p \in I(q)$. That is, $I(q) = \{p | \ C(p) = q\}$.

□

In Proposition 2.5, we know the relation between $C(n)$ and $I(n)$ under **one** transformation. In the following, by use of Remark 2.3 and Remark 2.4, we will show the relation between $C(n)$ and $I(n)$ under **k** transformations.

Remark 2.6. *For $p, q, k \in \mathbb{N}$, if $C^{(k)}(p) = q$, then $p \in I^{(k)}(q)$.*

proof: For $p, q, k \in \mathbb{N}$, if $C^{(k)}(p) = q$, then there exists a finite trajectory P on $C(n)$ with starting value p and length k, such that

$$P = (p, p_1, p_2, \ldots, p_{k-1}, q),\ \ p_i = C^{(i)}(p),\ \ i = 1, 2, 3, \ldots, k-1.$$

Since $C(p) = p_1$, by Remark 2.3, we have $p \in I(p_1)$. Similarly, we know $p_1 \in I(p_2)$, $p_2 \in I(p_3), \ldots, p_{k-2} \in I(p_{k-1})$, and $p_{k-1} \in I(q)$. By the relations, we have:

$$p \in I(p_1) \subseteq I(I(p_2)) \subseteq I(I(I(p_3))) \subseteq \cdots \subseteq I^{(k-1)}(p_{k-1}) \subseteq I^{(k)}(q).$$

Hence, $p \in I^{(k)}(q)$.

□

Remark 2.7. *For $p, q, k \in \mathbb{N}$, if $p \in I^{(k)}(q)$, then $C^{(k)}(p) = q$.*

proof: For $p, q, k \in \mathbb{N}$, if $p \in I^{(k)}(q)$, then there exists a path Q on I in $\beta_k(q)$ with length k, such that

$$Q = (q \to q_1 \to q_2 \to \cdots \to q_{k-1} \to p),\ \ q_i \in I^{(i)}(q),\ \ i = 0, 1, 2, 3, \ldots, k-1.$$

Moreover, by Remark 2.4, we have $q = C(q_1)$, $q_1 = C(q_2), \ldots, q_{k-2} = C(q_{k-1})$, and $q_{k-1} = C(p)$. Thus,

$$q = C(q_1) = C(C(q_2)) = \cdots = C^{(k-1)}(q_{k-1}) = C^{(k-1)}(C(p)) = C^{(k)}(p).$$

□

Proposition 2.8. *If $q \in \mathbb{N}$, then $I^{(k)}(q) = \{p |\ C^{(k)}(p) = q, p \in \mathbb{N}\}$.*

proof: By Remark 2.6 and Remark 2.7, we have: for $p, q \in \mathbb{N}$, $C^{(k)}(p) = q$ if and only if $p \in I^{(k)}(q)$. That is, $I^{(k)}(q) = \{p |\ C^{(k)}(p) = q\}$.

□

In Proposition 2.8, we realize the meaning of $I^k(n)$, that is, by use of $I^k(n)$, we can collect all the natural numbers which iterate to n under exactly k transformations on Collatz function. In section 2, we introduce the $3n+1$ backward iteration net, and the relation of the natural numbers in the net. However, for our initial idea, we wonder if 1 can strength all the natural numbers set. Thus, in section 3, we focus on a particular $3n+1$ backward iteration net.

3 A Particular $3n+1$ Backward Iteration Net $\beta_k(8)$

For our initial idea: if we let 1 be the starting value and make use of the $3n+1$ backward iteration formula I, can we strength all the natural numbers set $\mathbb{N}$? Therefore, $\beta_k(1)$ is the main net what we interested in. However, we find that there is a trivial cycle $1 \to 2 \to 4 \to 1$ in $\beta_k(1)$. For simplifying our work, the main net what we research transformed into $\beta_k(8)$. Why do we choose $\beta_k(8)$? Because observing $\beta_k(1)$, we find that our initial idea is equal to the following idea: if we let 8 be the starting value and make use of the $3n+1$ backward iteration formula I, can we strength $\mathbb{N} \setminus \{1, 2, 4\}$? Thus, in this section, $\beta_k(8)$ is the main $3n+1$ backward iteration net what we research. In this section, we will introduce $\beta_k(8)$ and some properties in $\beta_k(8)$.

3.1 Some Records of $\beta_k(8)$

By the definition of $3n+1$ backward iteration net in section 2, $\beta_k(8)$ is constructed by the starting value 8 and iterating k times on I, $k \in \mathbb{N}$. In this section, we suppose k is a sufficient large natural number, then its partial framework as follows:

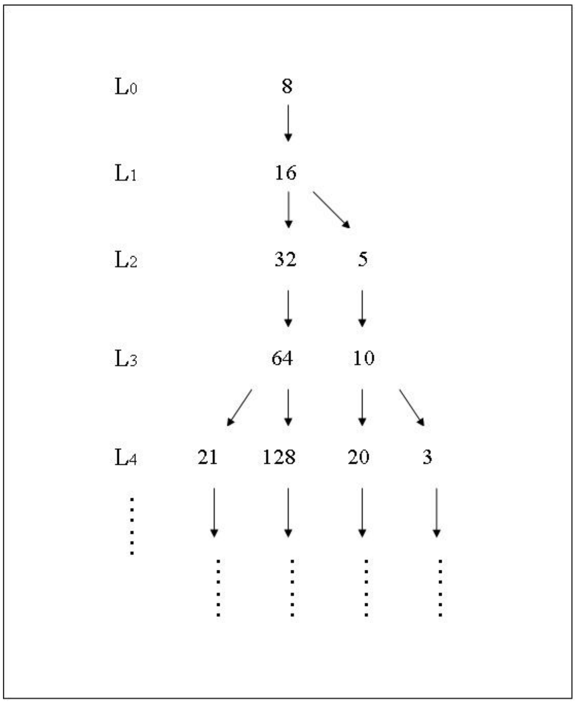

Figure 5: partial framework of $\beta_k(8)$.

For convenience, we let $I^{(i)}(8)$ be L_i, called level i, $i = 0, 1, 2, \ldots, k$. In the following, we list the first 12 levels of $\beta_k(8)$:

$L_0 : 8.$

$L_1 : 16.$

$L_2 : 5, 32.$

$L_3 : 10, 64.$

$L_4 : 3, 20, 21, 128.$

$L_5 : 6, 40, 42, 256.$

$L_6 : 12, 13, 80, 84, 85, 512.$

$L_7 : 24, 26, 160, 168, 170, 1024.$

$L_8 : 48, 52, 53, 320, 336, 340, 341, 2048.$

$L_9 : 96, 17, 104, 106, 640, 672, 113, 680, 682, 4096.$

$L_{10} : 192, 34, 208, 35, 212, 213, 1280, 1344, 226, 1360, 227, 1364, 1365, 8192.$

$L_{11} : 384, 11, 68, 69, 416, 70, 424, 426, 2560, 2688, 75, 452, 453, 2720, 454, 2728, 2730, 16384.$

In the following, for our best, we record L_0 to L_{59} in $\beta_k(8)$, by **Appendix** *A*. Being accompanied the increasing level, the total number $L_i(N)$ will be larger and larger. Therefore, we want to know the total number growing rate to realize $\beta_k(8)$ more. We analysis the total number growing rate in $\beta_k(8)$ as follows:

The total number growing rate $L_i(N)/L_{i-1}(N)$:

Owing to the proposition in 2.4: $L_{i+1}(N) - L_i(N) = L_i(D)$ and $L_i(D) \geq 0$. Therefore, we have $L_i(N)/L_{i-1}(N) \geq 1$. From L_0 to L_{10}, the growing rate varies much, falls in $[1, 2]$. However, accompanied the increasing level, the growing rate seems more and more stable. From L_{25} to L_{59}, the growing rate all falls in $(1.25, 1.27)$. Moreover, From L_{43} to L_{59}, the growing rate all falls in $(1.263, 1.265)$.

Although we just record L_0 to L_{59} in $\beta_k(8)$, by the analysis of the total number growing rate, we have some sense of total number in $\beta_k(8)$. Moreover, according the growing rate trend, if $k \to \infty$, then the total number in $\beta_k(8)$ has the chance to meet the $\mathbb{N} \setminus \{1, 2, 4\}$.

3.2 Properties

In this subsection, by use of the remarks and propositions in 2.5, we will classify the properties in $\beta_k(8)$ more precisely.

Remark 3.1. *In* $\beta_k(8)$*,* $2^{n+3} \in L_n$*, for* $n = 0, 1, 2, \ldots, k$*.*

proof: We will prove it by mathematical induction.

(i) When $n = 0$, $2^3 = 8 \in L_0$. True!

(ii) Suppose $n = m$, $m < k$, $2^{m+3} \in L_m$ is true. Then by the formula $I(n)$, we have: $\exists m_1 \in L_{m+1}$ s.t. $m_1 = I(2^{m+3}) = 2 \cdot 2^{m+3} = 2^{m+4}$. Therefore, $2^{m+4} \in L_{m+1}$. True!

By mathematical induction, we prove it!

□

Remark 3.1 presents a trivial fact about 2-power natural numbers in $\beta_k(8)$.

Remark 3.2. *In* $\beta_k(8)$*, the elements in* L_i *are distinct,* $i \leq k$*.*

proof: If $p \in L_i$, then by Remark 2.7, we have: $C^{(i)}(p) = 8$, $i \leq k$. Suppose there exist two finite trajectories X, Y with the same length i on Collatz function such that

$$X = (p, p_1, p_2, \ldots, p_{i-1}, 8), C^{(n)}(p) = p_n, n = 0, 1, 2, \ldots, i-1.$$

$$Y = (p, q_1, q_2, \ldots, q_{i-1}, 8), C^{(n)}(p) = q_n, n = 0, 1, 2, \ldots, i-1.$$

Since Collatz function, $C^{(1)}(p) = p_1 = q_1$, $C^{(2)}(p) = p_2 = q_2$, similarly, $p_i = q_i$, for $i = 1, 2, \ldots, i-1$. That is, $X \equiv Y$. Thus, it shows that every element p in L_i exists an unique finite trajectory with length i from p to 8 on Collatz function. Therefore, the elements in L_i are distinct.

□

Remark 3.3. *In* $\beta_k(8)$*, if* $p \in L_s$ *and* $p \in L_t$*, then* $s = t$*.*

proof: Suppose $s \neq t$ and $p \in L_s$, $p \in L_t$, without loss of generality, we let $s < t \leq k$, then by Remark 2.7, we have: $C^{(s)}(p) = 8$ and $C^{(t)}(p) = 8$. Therefore, there exist a trajectory S with length s and a trajectory T with length t on Collatz function such that

$$S = (p, p_1, p_2, \ldots, p_{s-1}, 8), C^{(i)}(p) = p_i, i = 0, 1, 2, \ldots, s-1.$$

$$T = (p, q_1, q_2, \ldots, q_{t-1}, 8), C^{(i)}(p) = q_i, i = 0, 1, 2, \ldots, t-1.$$

Since Collatz function, $C^{(1)}(p) = p_1 = q_1$, $C^{(2)}(p) = p_2 = q_2$, similarly, $p_i = q_i$, for $i = 1, 2, \ldots, s-1$. Thus, T can be written as

$$T = (p, p_1, p_2, \ldots, p_{s-1}, 8, q_{s+1}, q_{s+2}, \ldots, q_{t-1}, 8).$$

Now we consider the finite trajectory $(8, q_{s+1}, q_{s+2}, \ldots, q_{t-1}, 8)$ in T. Since $C^{(1)}(8) = 4$, $C^{(2)}(8) = 2$, $C^{(3)}(8) = 1$, $C^{(4)}(8) = 4$, and so on. That is, $q_i \in \{1, 2, 4\}$ for $i > s$. Thus, the finite trajectory $(8, q_{s+1}, q_{s+2}, \ldots, q_{t-1}, 8)$ does not exist. It is contradiction to our assumption. Hence, we prove it!

□

Remark 3.4. *In $\beta_k(8)$, the elements in $V(\beta_k(8))$ are distinct.*

proof: By Remark 3.3, we know that every element belongs to an unique level. Moreover, by Remark 3.2, we know the elements in a level are distinct. Therefore, the elements in $V(\beta_k(8))$ are distinct.

□

Theorem 3.5. *For $k \in \mathbb{N}$, there is no cycle in $\beta_k(8)$.*

proof: By Remark 3.4, we know: the elements in $V(\beta_k(8))$ are distinct. Therefore, there is no cycle in $\beta_k(8)$.

□

The Theorem 3.5 reveals a fact: For any $n \in \mathbb{N} \setminus \{1, 2, 4\}$, if n can iterate to 1 on Collatz function, before getting to 1 at first time, then n don't enter any cycle. It matches the fact we have known.

About our initial idea: if we let 8 be the starting value and make use of the $3n + 1$ backward iteration formula, can we strength $\mathbb{N} \setminus \{1, 2, 4\}$? Although we still don't know whether the idea is right or not, we get some results in $\beta_k(8)$. In 3.1, we know that if $k \to \infty$, then the total number in $\beta_k(8)$ have the chance to meet $\mathbb{N} \setminus \{1, 2, 4\}$. In 3.2, we know all the natural numbers are different in $\beta_k(8)$. Moreover, in Theorem 3.5, we know there is no cycle in $\beta_k(8)$. Through the research of $\beta_k(8)$, we have another way to examine and interpret some facts about the $3n + 1$ problem.

4 Simulation Backward Iteration Net

In this section, in order to know more about $\beta_k(8)$, we construct a simulation expected matrix E to create a simulation backward net. We take the simulation backward net compared with $\beta_k(8)$ on three parts: total numbers, total number growing rate, and the ratio of A to F in every level. We will introduce E and how we use E to simulate $\beta_k(8)$. Moreover, we take another four different simulation expected value matrices to simulate $\beta_k(8)$. In the end, we analysis the simulation results and make conclusions.

4.1 Simulation Expected Value Matrix E

In the subsection, we introduce the *simulation expected value matrix* E. The idea of E is from the relations of numbers in $3n+1$ backward iteration net. In section 2, we realize the natural number relations are a to b, b to d, c to f, e to d, f to f under one transformation on I. Based on the relations, we construct the simulation expected value matrix E. Especially, d may be to $\{a,b\}$,$\{c,b\}$, or $\{e,b\}$ under one transformation on I. Therefore, in E, we suppose the expected value of d to a, c, or e is 1/3 respectively under one transformation. It is more precise description below:

$$E = \begin{bmatrix} 0 & 0 & 0 & 1/3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We will explain E by columns:

(1) The first column: $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

The first column is **a-column**. It shows that the *expected value* of "a iterating to b under one transformation" is 1. Moreover, the *expected value* of "a iterating to a, c, d, e, or f under one transformation on I" is 0 respectively.

(2) The second column: $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$.

The second column is **b-column.** It shows that the *expected value* of "b iterating to d under one transformation" is 1. Moreover, the *expected value* of "b iterating to a, b, c, e, or f under one transformation " is 0 respectively.

(3) The third column: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$.

The third column is **c-column.** It shows that the *expected value* of "c iterating to f under one transformation" is 1. Moreover, the *expected value* of "c iterating to a, b, c, d, or e under one transformation" is 0 respectively.

(4) The fourth column: $\begin{bmatrix} 1/3 & 1 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}^T$.

The fourth column is **d-column.** It shows that the *expected value* of "d iterating to b under one transformation" is 1. Moreover, the *expected value* of "d iterating to d, or f under one transformation" is 0. Especially, the *expected value* of "d iterating to a, c, or e under one transformation" is 1/3 respectively.

(5) The fifth column: $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$.

The fifth column is **e-column.** It shows that the *expected value* of "e iterating to d under one transformation" is 1. Moreover, the *expected value* of "e iterating to a, b, c, e, or f under one transformation" is 0 respectively.

(6) The sixth column: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$.

The sixth column is **f-column.** It shows that the *expected value* of "f iterating to f under one transformation" is 1. Moreover, the *expected value* of "f iterating to a, b, c, d, or e, under one transformation" is 0 respectively.

Besides E, we also define identity vectors $X_a, X_b, X_c, X_d, X_e, X_f$.

$X_a = (1, 0, 0, 0, 0, 0)^T$;
$X_b = (0, 1, 0, 0, 0, 0)^T$;
$X_c = (0, 0, 1, 0, 0, 0)^T$;
$X_d = (0, 0, 0, 1, 0, 0)^T$;
$X_e = (0, 0, 0, 0, 1, 0)^T$;
$X_f = (0, 0, 0, 0, 0, 1)^T$.

X_a shows that we choose a as starting value and make the $3n + 1$ backward iteration process. So do X_b, X_c, X_d, X_e, X_f. In the following, we take examples to explain how E and identity vectors work.

Example 4.1. $E^{(1)} * X_d = \left[\begin{array}{cccccc} 1/3 & 1 & 1/3 & 0 & 1/3 & 0 \end{array} \right]^T.$

It shows: If we choose d as starting value, under **one** transformation by E, the *expected values* of a is 1/3, of b is 1, of c is 1/3, of d is 0, of e is 1/3, of f is 0. That is, under one transformation by E, the next first level of starting value d consists $1/3 \cdot a$, $1 \cdot b$, $1/3 \cdot c$, $0 \cdot d$, $1/3 \cdot e$, and $0 \cdot f$.

Example 4.2. $E^{(2)} * X_d = \left[\begin{array}{cccccc} 0 & 1/3 & 0 & 4/3 & 0 & 1/3 \end{array} \right]^T.$

It shows: If we choose d as starting value, under **two** transformations by E, the *expected values* of a is 0, of b is 1/3, of c is 0, of d is 4/3, of e is 0, of f is 1/3. That is, under two transformations by E, the next second level of starting value d consists $0 \cdot a$, $1/3 \cdot b$, $0 \cdot c$, $4/3 \cdot d$, $0 \cdot e$, $1/3 \cdot f$.

Example 4.3. $E^{(3)} * X_d = \left[\begin{array}{cccccc} 4/9 & 4/3 & 4/9 & 1/3 & 4/9 & 1/3 \end{array} \right]^T.$

It shows: If we choose d as starting value, under **three** transformations by E, the *expected values* of a is 4/9, of b is 4/3, of c is 4/9, of d is 1/3, of e is 4/9, of f is 1/3. That is, under three transformations by E, the next third level of starting value d consists $4/9 \cdot a$, $4/3 \cdot b$, $4/9 \cdot c$, $1/3 \cdot d$, $4/9 \cdot e$, $1/3 \cdot f$.

In the following, we will introduce that how we use E to simulate $\beta_k(8)$.

4.2 The Simulation Backward Iteration Net

In this subsection, we will introduce the simulation backward iteration net constructed by E and X_b, then we use it to compare with $\beta_k(8)$. In our simulation work, we suppose the expected value of d to a, c, or e is 1/3 respectively under one transformation by E. But in $\beta_k(8)$, the behavior of d under one transformation on I is deterministic. Therefore, in the simulation backward iteration net by E, we take the **expected value** of A to F in every level as our simulation object. Since our simulation aim is $\beta_k(8)$ and $8 \in B$, we choose b as our simulation starting value. Thus, we use E and X_b to construct the simulation backward iteration net. The net compared with $\beta_k(8)$ is focused on three parts: total numbers, total number growing rate, and the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level. We try our best to simulate $\beta_k(8)$ with 60 levels ($L_0 - L_{59}$). In the following, we will introduce the simulation process and results.

The first way: Total numbers in every level.

Step 1: Owing to the simulation aim is $\beta_k(8)$, the starting value 8 belongs to B. Therefore, we choose E and X_b to construct the simulation backward iteration net.

Step 2: For $\beta_k(8)$, let L_0 be the starting value 8, L_1 be $\{16\}$, L_2 be $\{5, 32\}$, and so on. For our simulation backward iteration net, let L_0 be b, L_1 be $E^{(1)} * X_b$, L_2 be $E^{(2)} * X_b$, and so on.

Step 3: For our best, we can record the total numbers of L_0 to L_{59} in $\beta_k(8)$. On the other hand, for our simulation backward iteration net, we take the *sum* of $E^{(i)} * X_b$ as the **total expected value** in L_i, denoted $E_i(N)$, $i = 0, 1, 2, \ldots, 59$.

Step 4: We compare them for $(E_i(N) - L_i(N))$, $i = 0, 1, 2, \ldots, 59$.

For convenience, we denote $D_i(N) = E_i(N) - L_i(N)$. By **Appendix B**, the results as follows:

$D_i(N)$: From L_0 to L_3, $D_i(N)$ all are 0. But from L_4, $D_i(N)$ is not equal to 0 anymore. From L_4 to L_{15}, $D_i(N)$ may be positive or negative, and the absolute value of $D_i(N)$ less than 2. From L_{16} to L_{59}, surprisingly, we found that $D_i(N)$ is positive and increasing. $D_{16}(N)$ is about 0.5 , but $D_{59}(N)$ is about 65022.4.

In a word, accompanied by the increasing level, we can conclude that the $D_i(N)$ will be larger and larger. That is to say, on the total numbers, the simulation backward iteration net constructed by E and X_b will be further and further than $\beta_k(8)$.

The Second way: Total number growing rate in every level.

The simulation process is similar to total numbers simulation. For convenience, we denote $L_i(N)/L_{i-1}(N)$ as the total number growing rate of $\beta_k(8)$, denote $E_i(N)/E_{i-1}(N)$ as the total expected value growing rate of the simulation backward iteration net, and denote $K_i(N) = |(E_i(N)/E_{i-1}(N)) - (L_i(N)/L_{i-1}(N))|$ as the absolute value of the difference between the two growing rates. By **Appendix B**, the results as follows:

$K_i(N)$: $L_0 - L_9$: the minimum $K_i(N)$ is 0, the maximum $K_i(N)$ is 0.33333.

$L_{10} - L_{19}$: the minimum $K_i(N)$ is 0.01093, the maximum $K_i(N)$ is 0.0605.

$L_{20} - L_{29}$: the minimum $K_i(N)$ is 0.00009, the maximum $K_i(N)$ is 0.00893.

$L_{30} - L_{39}$: the minimum $K_i(N)$ is 0.00012, the maximum $K_i(N)$ is 0.00553.

$L_{40} - L_{49}$: the minimum $K_i(N)$ is 0.00011, the maximum $K_i(N)$ is 0.00135.

$L_{50} - L_{59}$: the minimum $K_i(N)$ is 0.00002, the maximum $K_i(N)$ is 0.00033.

In a word, accompanied by the increasing level, on average, the $K_i(N)$ trends smaller and smaller. That is, on the total number growing rate in every level, the simulation backward iteration net by E and X_b will be closer and closer than $\beta_k(8)$.

The third way: The ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level.

Step 1: Owing to the simulation aim is $\beta_k(8)$, 8 is the starting value and belongs to B. Therefore, we choose E and X_b to construct the simulation backward iteration net.

Step 2: For $\beta_k(8)$, let L_0 be the starting value 8, L_1 be $\{16\}$, L_3 be $\{5, 32\}$, and so on. For our simulation backward iteration net, let L_0 be b, L_1 be $E^{(1)} * X_b$, L_2 be $E^{(2)} * X_b$, and so on.

Step 3: For our best, we can record the ratio of A, B, C, D, E, F from L_0 to L_{59} respectively in $\beta_k(8)$. We denote $R_i(A)$ represents the ratio of $L_i(A)$ to $L_i(N)$, and so do $R_i(B)$, $R_i(C)$, $R_i(D)$, $R_i(E)$,and $R_i(F)$. For the convenience on comparison, we let:
$R_i(A) = (L_i(A)/L_i(N)) \times 100$;
$R_i(B) = (L_i(B)/L_i(N)) \times 100$;
$R_i(C) = (L_i(C)/L_i(N)) \times 100$;
$R_i(D) = (L_i(D)/L_i(N)) \times 100$;
$R_i(E) = (L_i(E)/L_i(N)) \times 100$;
$R_i(F) = (L_i(F)/L_i(N)) \times 100$.
On the other hand, in the simulation backward iteration net, in L_i, the first entry of $E^{(i)} * X_b$ represents the expected value of A, denoted $E_i(A)$. Similarly, the second entry represents the expected value of B in L_i, denoted $E_i(B)$, and so do $E_i(C)$, $E_i(D)$, $E_i(E)$, and $E_i(F)$. Moreover, we denote $W_i(A)$ represents the ratio of $E_i(A)$ to $E_i(N)$, and so do $W_i(B)$, $W_i(C)$, $W_i(D)$, $W_i(E)$,and $W_i(F)$. For the convenience on comparison, we let:
$W_i(A) = (E_i(A)/E_i(N)) \times 100$;
$W_i(B) = (E_i(B)/E_i(N)) \times 100$;
$W_i(C) = (E_i(C)/E_i(N)) \times 100$;
$W_i(D) = (E_i(D)/E_i(N)) \times 100$;
$W_i(E) = (E_i(E)/E_i(N)) \times 100$;
$W_i(F) = (E_i(F)/E_i(N)) \times 100$.

Step 4: For the ratio comparisons, we take absolute value of every ratio error, and we observe the maximum ratio error in every level. The notations as follows:

$Y_i(A) = |W_i(A) - R_i(A)|$;

$Y_i(B) = |W_i(B) - R_i(B)|$;

$Y_i(C) = |W_i(C) - R_i(C)|$;

$Y_i(D) = |W_i(D) - R_i(D)|$;

$Y_i(E) = |W_i(E) - R_i(E)|$;

$Y_i(F) = |W_i(F) - R_i(F)|$;

$Y_i(M) = max\,\{Y_i(A), Y_i(B), Y_i(C), Y_i(D), Y_i(E), Y_i(F)\}$.

Step 5: We compare them by use of $Y_i(M)$, $i = 0, 1, 2, \ldots, 59$.

By **Appendix C** and **Appendix D**, the results as follows:

$L_0 - L_9$: the maximum $Y_i(M)$ is 36.66667.

$L_{10} - L_{19}$: the maximum $Y_i(M)$ is 10.53939.

$L_{20} - L_{29}$: the maximum $Y_i(M)$ is 2.20515.

$L_{30} - L_{39}$: the maximum $Y_i(M)$ is 0.70776.

$L_{40} - L_{49}$: the maximum $Y_i(M)$ is 0.28697.

$L_{50} - L_{59}$: the maximum $Y_i(M)$ is 0.05015.

In a word, accompanied by the increasing level, on average, the $Y_i(M)$ trends smaller and smaller. That is to say, on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the simulation backward iteration net by E and X_b will be closer and closer than $\beta_k(8)$.

In this subsection, we use E and X_b to construct the simulation backward iteration net and compared with $\beta_k(8)$. Accompanied by the increasing level, the total number simulation is further and further, however, on average, total number growing rate and the ratio simulation are closer and closer. It reveals that the framework of A to F in every level seems similar. Although we take the expected value to simulate, the assumption of d to a, c, e is 1/3 respectively under one transformation by E which may be meaningful.

4.3 Other Simulation Backward Iteration Nets

In this subsection, we take another four different expected value matrices and use them to construct four different simulation backward iteration nets to simulate $\beta_k(8)$. In E, we suppose that the expected value of d to a, c, or e is 1/3 respectively under one transformation.

Because we have the fact that the *sum of expected value* of d to a, c, and e is 1 under one transformation on I. Therefore, we modify the expected value of d to a, c, and e rather than $1/3, 1/3, 1/3$. The modify expected value matrix P is below:

$$P = \begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

x: The expected value of d to a under one transformation, $0 < x < 1$.

y: The expected value of d to c under one transformation, $0 < y < 1$.

z: The expected value of d to e under one transformation, $0 < z = 1 - x - y < 1$.

Key: $x + y + z = 1$ means that the sum of expected value of d to a, c, and e is 1 under one transformation on I.

In the simulation backward iteration net constructed by E and X_b, we find the total expected value growing rate will nearly equal the maximum absolute value of eigenvalues of E (about 1.2637626) if the level i is sufficient large (about larger than 100). On the other hand, we observe the total number growing rate of L_{50} to L_{59} in $\beta_k(8)$. Therefore, we write a program to get some sets of (x, y, z) of P whose maximum eigenvalues falls in $(1.2632, 1.2638)$. The four different simulation expected value matrices which we choose as follows:

(1) P_1: $(x, y, z) = (\frac{1}{3} + 0.0001, \frac{1}{3} - 0.0001, \frac{1}{3})$.

(2) P_2: $(x, y, z) = (\frac{1}{3} - 0.0001, \frac{1}{3} + 0.0001, \frac{1}{3})$.

(3) P_3: $(x, y, z) = (\frac{1}{3} + 0.0002, \frac{1}{3} - 0.0002, \frac{1}{3})$.

(4) P_4: $(x, y, z) = (\frac{1}{3} - 0.0002, \frac{1}{3} + 0.0002, \frac{1}{3})$.

The simulation process by P is similar to E, we will compare them by three ways: total numbers, total number growing rate, and the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level. In the following, we will introduce the four simulation results respectively.

4.3.1 The simulation results of P_1

Case 1. P_1: $(x, y, z) = (\frac{1}{3} + 0.0001, \frac{1}{3} - 0.0001, \frac{1}{3})$. The results as follows:

(A)Total numbers in every level:

For consistence, we use the same notations which are used on the E simulation. $D_i(N) = P_i(N) - L_i(N)$. By **Appendix E**, the results as follows:

$D_i(N)$: From L_0 to L_3, $D_i(N)$ all are 0. But from L_4, $D_i(N)$ is not equal to 0 anymore. From L_4 to L_{15}, $D_i(N)$ may be positive or negative, and the absolute value of $D_i(N)$ less than 2. However, from L_{16} to L_{59}, surprisingly, we found that $D_i(N)$ is positive and increasing. $D_{16}(N)$ is about 0.5, but $D_{59}(N)$ is about 66679.3.

In a word, accompanied by the increasing level, we can conclude that the $D_i(N)$ will be larger and larger. That is to say, on the total numbers, the simulation backward iteration net constructed by P_1 and X_b will be further and further than $\beta_k(8)$.

(B)Total number growing rate in every level:

For consistence, we use the same notations which are used on the E simulation. $K_i(N) = |(E_i(N)/E_{i-1}(N)) - (L_i(N)/L_{i-1}(N))|$. By **Appendix E**, the results as follows:

$K_i(N)$: $L_0 - L_9$: the minimum $K_i(N)$ is 0, the maximum $K_i(N)$ is 0.33333.

$L_{10} - L_{19}$: the minimum $K_i(N)$ is 0.01095, the maximum $K_i(N)$ is 0.06045.

$L_{20} - L_{29}$: the minimum $K_i(N)$ is 0.00006, the maximum $K_i(N)$ is 0.00895.

$L_{30} - L_{39}$: the minimum $K_i(N)$ is 0.00015, the maximum $K_i(N)$ is 0.00556.

$L_{40} - L_{49}$: the minimum $K_i(N)$ is 0.00008, the maximum $K_i(N)$ is 0.00138.

$L_{50} - L_{59}$: the minimum $K_i(N)$ is 0.00005, the maximum $K_i(N)$ is 0.00030.

In a word, accompanied by the increasing level, on average, the $K_i(N)$ trends smaller and smaller. That is, on the total number growing rate in every level, the simulation backward iteration net constructed by P_1 and X_b will be closer and closer than $\beta_k(8)$.

(C)The ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level:

For consistence, we use the same notations which are used on the E simulation. By **Appendix C** and **Appendix F**, the results as follows:

$L_0 - L_9$: the maximum $Y_i(M)$ is 36.67067.

$L_{10} - L_{19}$: the maximum $Y_i(M)$ is 10.54013.

$L_{20} - L_{29}$: the maximum $Y_i(M)$ is 2.20345.

$L_{30} - L_{39}$: the maximum $Y_i(M)$ is 0.70626.

$L_{40} - L_{49}$: the maximum $Y_i(M)$ is 0.28757.

$L_{50} - L_{59}$: the maximum $Y_i(M)$ is 0.05163.

In a word, accompanied by the increasing level, on average, the $Y_i(M)$ trends smaller and smaller. That is to say, on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the simulation backward iteration net constructed by P_1 and X_b will be closer and closer than $\beta_k(8)$.

4.3.2 The simulation results of P_2

Case 2. P_2: $(x, y, z) = (\frac{1}{3} - 0.0001, \frac{1}{3} + 0.0001, \frac{1}{3})$. The results as follows:

(A)Total numbers in every level:

For consistence, we use the same notations which are used on the E simulation. $D_i(N) = P_i(N) - L_i(N)$. By **Appendix G**, the results as follows:

$D_i(N)$: From L_0 to L_3, $D_i(N)$ all are 0. But from L_4, $D_i(N)$ is not equal to 0 anymore. From L_4 to L_{15}, $D_i(N)$ may be positive or negative, and the absolute value of $D_i(N)$ less than 2. However, from L_{16} to L_{59}, surprisingly, we found that $D_i(N)$ is positive and increasing. $D_{16}(N)$ is about 0.4 , but $D_{59}(N)$ is about 63367.4.

In a word, accompanied by the increasing level, the $D_i(N)$ will be larger and larger. That is to say, on the total numbers, the simulation backward iteration net constructed by P_2 and X_b will be further and further than $\beta_k(8)$.

(B)Total number growing rate in every level:

For consistence, we use the same notations which are used on the E simulation. $K_i(N) = |(E_i(N)/E_{i-1}(N)) - (L_i(N)/L_{i-1}(N))|$. By **Appendix G**, the results as follows:

$K_i(N)$: $L_0 - L_9$: the minimum $K_i(N)$ is 0, the maximum $K_i(N)$ is 0.33333.

$L_{10} - L_{19}$: the minimum $K_i(N)$ is 0.01092, the maximum $K_i(N)$ is 0.06055.

$L_{20} - L_{29}$: the minimum $K_i(N)$ is 0.00011, the maximum $K_i(N)$ is 0.00890.

$L_{30} - L_{39}$: the minimum $K_i(N)$ is 0.00009, the maximum $K_i(N)$ is 0.00550.

$L_{40} - L_{49}$: the minimum $K_i(N)$ is 0.00013, the maximum $K_i(N)$ is 0.00132.

$L_{50} - L_{59}$: the minimum $K_i(N)$ is 0.00005, the maximum $K_i(N)$ is 0.00036.

In a word, accompanied by the increasing level, on average, the $K_i(N)$ trends smaller and smaller. That is, on the total number growing rate in every level, the simulation backward iteration net constructed by P_2 and X_b will be closer and closer than $\beta_k(8)$.

(C) The ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level:

For consistence, we use the same notations which are used on the E simulation. By **Appendix C** and **Appendix H**, the results as follows:

$L_0 - L_9$: the maximum $Y_i(M)$ is 36.66267.

$L_{10} - L_{19}$: the maximum $Y_i(M)$ is 10.53865.

$L_{20} - L_{29}$: the maximum $Y_i(M)$ is 2.2068.

$L_{30} - L_{39}$: the maximum $Y_i(M)$ is 0.70926.

$L_{40} - L_{49}$: the maximum $Y_i(M)$ is 0.28637.

$L_{50} - L_{59}$: the maximum $Y_i(M)$ is 0.04866.

In a word, accompanied by the increasing level, on average, the $Y_i(M)$ trends smaller and smaller. That is to say, on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the simulation backward iteration net constructed by P_2 and X_b will be closer and closer than $\beta_k(8)$.

4.3.3 The simulation results of P_3

Case 3. P_3: $(x, y, z) = (\frac{1}{3} + 0.0002, \frac{1}{3} - 0.0002, \frac{1}{3})$. The results as follows:

(A) Total numbers in every level:

For consistence, we use the same notations which are used on the E simulation. $D_i(N) = P_i(N) - L_i(N)$. By **Appendix I**, the results as follows:

$D_i(N)$: From L_0 to L_3, $D_i(N)$ all are 0. But from L_4, $D_i(N)$ is not equal to 0 anymore. From L_4 to L_{15}, $D_i(N)$ may be positive or negative, and the absolute value of $D_i(N)$ less than 2. However, from L_{16} to L_{59}, surprisingly, we found that $D_i(N)$ is positive and increasing. $D_{16}(N)$ is about 0.5 , but $D_{59}(N)$ is about 68338.1.

In a word, accompanied by the increasing level, we can conclude that the $D_i(N)$ will be larger and larger. That is to say, on the total numbers, the simulation backward iteration net constructed by P_3 and X_b will be further and further than $\beta_k(8)$.

(B) Total number growing rate in every level:

For consistence, we use the same notations which are used on the E simulation. $K_i(N) = |(E_i(N)/E_{i-1}(N)) - (L_i(N)/L_{i-1}(N))|$. By **Appendix I**, the results as follows:

$K_i(N)$: $L_0 - L_9$: the minimum $K_i(N)$ is 0, the maximum $K_i(N)$ is 0.33333.

$L_{10} - L_{19}$: the minimum $K_i(N)$ is 0.01097, the maximum $K_i(N)$ is 0.06040.

$L_{20} - L_{29}$: the minimum $K_i(N)$ is 0.00004, the maximum $K_i(N)$ is 0.00898.

$L_{30} - L_{39}$: the minimum $K_i(N)$ is 0.00018, the maximum $K_i(N)$ is 0.00558.

$L_{40} - L_{49}$: the minimum $K_i(N)$ is 0.00005, the maximum $K_i(N)$ is 0.00141.

$L_{50} - L_{59}$: the minimum $K_i(N)$ is 0.00004, the maximum $K_i(N)$ is 0.00028.

In a word, accompanied by the increasing level, on average, the $K_i(N)$ trends smaller and smaller. That is, on the total number growing rate in every level, the simulation backward iteration net constructed by P_3 and X_b will be closer and closer than $\beta_k(8)$.

(C)The ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level:

For consistence, we use the same notations which are used on the E simulation. By **Appendix C** and **Appendix J**, the results as follows:

$L_0 - L_9$: the maximum $Y_i(M)$ is 36.67467.

$L_{10} - L_{19}$: the maximum $Y_i(M)$ is 10.54087.

$L_{20} - L_{29}$: the maximum $Y_i(M)$ is 2.20175.

$L_{30} - L_{39}$: the maximum $Y_i(M)$ is 0.70475.

$L_{40} - L_{49}$: the maximum $Y_i(M)$ is 0.28817.

$L_{50} - L_{59}$: the maximum $Y_i(M)$ is 0.05311.

In a word, accompanied by the increasing level, on average, the $Y_i(M)$ trends smaller and smaller. That is to say, on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the simulation backward iteration net constructed by P_3 and X_b will be closer and closer than $\beta_k(8)$.

4.3.4 The simulation results of P_4

Case 4. P_4: $(x, y, z) = (\frac{1}{3} - 0.0002, \frac{1}{3} + 0.0002, \frac{1}{3})$. The results as follows:

(A)Total numbers in every level:

For consistence, we use the same notations which are used on the E simulation. $D_i(N) = P_i(N) - L_i(N)$. By **Appendix K**, the results as follows:

$D_i(N)$: From L_0 to L_3, $D_i(N)$ all are 0. But from L_4, $D_i(N)$ is not equal to 0 anymore. From L_4 to L_{15}, $D_i(N)$ may be positive or negative, and the absolute value of $D_i(N)$ less than 2. However, from L_{16} to L_{59}, surprisingly, we found that $D_i(N)$ is positive and increasing. $D_{16}(N)$ is about 0.4 , but $D_{59}(N)$ is about 61714.2.

In a word, accompanied by the increasing level, we can conclude that the $D_i(N)$ will be larger and larger. That is to say, on the total numbers, the simulation backward iteration net constructed by P_4 and X_b will be further and further than $\beta_k(8)$.

(B)Total number growing rate in every level:

For consistence, we use the same notations which are used on the E simulation. $K_i(N) = |(E_i(N)/E_{i-1}(N)) - (L_i(N)/L_{i-1}(N))|$. By **Appendix K**, the results as follows:

$K_i(N)$: F$L_0 - L_9$: the minimum $K_i(N)$ is 0, the maximum $K_i(N)$ is 0.33333.

$L_{10} - L_{19}$: the minimum $K_i(N)$ is 0.01090, the maximum $K_i(N)$ is 0.06060.

$L_{20} - L_{29}$: the minimum $K_i(N)$ is 0.00013, the maximum $K_i(N)$ is 0.00887.

$L_{30} - L_{39}$: the minimum $K_i(N)$ is 0.00006, the maximum $K_i(N)$ is 0.00547.

$L_{40} - L_{49}$: the minimum $K_i(N)$ is 0.00010, the maximum $K_i(N)$ is 0.00129.

$L_{50} - L_{59}$: the minimum $K_i(N)$ is 0.00003, the maximum $K_i(N)$ is 0.00039.

In a word, accompanied by the increasing level, on average, the $K_i(N)$ trends smaller and smaller. That is, on the total number growing rate in every level, the simulation backward iteration net constructed by P_4 and X_b will be closer and closer than $\beta_k(8)$.

(C)The ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level:

For consistence, we use the same notations which are used on the E simulation. By **Appendix C** and **Appendix L**, the results as follows:

$L_0 - L_9$: the maximum $Y_i(M)$ is 36.65867.

$L_{10} - L_{19}$: the maximum $Y_i(M)$ is 10.53792.

$L_{20} - L_{29}$: the maximum $Y_i(M)$ is 2.20854.

$L_{30} - L_{39}$: the maximum $Y_i(M)$ is 0.71077.

$L_{40} - L_{49}$: the maximum $Y_i(M)$ is 0.28577.

$L_{50} - L_{59}$: the maximum $Y_i(M)$ is 0.04980.

In a word, accompanied by the increasing level, on average, the $Y_i(M)$ trends smaller and smaller. That is to say, on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the simulation backward iteration net constructed by P_4 and X_b will be closer and closer than $\beta_k(8)$.

Accompanied by the increasing level, On the total numbers in every level, the four different simulation backward iteration nets all are further and further than $\beta_k(8)$; however, on average, on the total number growing rate and on the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and

$L_i(F)$ to $L_i(N)$ respectively in every level, the four different simulations backward iteration nets all are closer and closer than $\beta_k(8)$.

4.4 Simulation Results

In this section, we construct five different simulation backward iteration nets to simulate $\beta_k(8)$. We take the expected values of A to F in the net to simulate the "real" natural numbers of A to F in $\beta_k(8)$. Among the five simulation results, accompanied by the increasing level, on the total numbers in every level, the five simulation nets all are further and further than $\beta_k(8)$, however, on average, on the total number growing rate and the ratios of $L_i(A)$, $L_i(B)$, $L_i(C)$, $L_i(D)$, $L_i(E)$, and $L_i(F)$ to $L_i(N)$ respectively in every level, the five simulation nets all are closer and closer than $\beta_k(8)$. It reveals that the framework of A to F in every level seems similar. Through the five simulation backward iteration nets, we have more idea about the number framework in every level of $\beta_k(8)$. The expected value assumption of d to a, c, e is $1/3$ respectively under one transformation by E may be meaningful. Because the expected value is a long-term concept, although d iterating to $\{b, a\}$, $\{b, c\}$, or $\{b, e\}$ under one transformation on I is deterministic in $\beta_k(8)$, we believe that the set of ratio $((L_{i+1}(A)/L_i(D)), (L_{i+1}(C)/L_i(D)), (L_{i+1}(E)/L_i(D)))$ will trend toward $(1/3, 1/3, 1/3)$ accompanied by the increasing level in $\beta_k(8)$.

5 Conclusion and Future Work

In this thesis, we study $3n+1$ problem in a backward iteration net. We classify the relations between natural numbers and some properties in the net. In section 3, in order to solve the $3n+1$ problem, we research a particular $3n+1$ backward iteration net $\beta_k(8)$. Although we still don't know whether the natural number set $\mathbb{N} \setminus \{1, 2, 4\}$ is in $\beta_k(8)$ or not, we prove some properties of $\beta_k(8)$ and it offers another way for us to examine and interpret the facts about the $3n+1$ problem. In section 4, we construct a simulation backward iteration net compared with $\beta_k(8)$. Although the expected value of A to F in the net is different from the real natural numbers of A of F in $\beta_k(8)$, through the simulation results, the simulation backward iteration nets still help us know more about the number framework in every level of $\beta_k(8)$.

Through the research in $3n+1$ backward iteration net $\beta_k(n)$ and $\beta_k(8)$, we get some results and know more about the $3n+1$ problem. However, we still can't solve it. In the future, we hope to dominate more rules and properties in $\beta_k(n)$, $\beta_k(8)$, and even $\beta_k(1)$. Most importantly, we hope the idea of $3n+1$ backward iteration net will help mathematicians solve the $3n+1$ conjecture some day.

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Appendix A

Level	$L_i(A)$	$L_i(B)$	$L_i(C)$	$L_i(D)$	$L_i(E)$	$L_i(F)$	$L_i(N)$	$L_i(N)/L_{i-1}(N)$
0	0	1	0	0	0	0	1	
1	0	0	0	1	0	0	1	1
2	0	1	0	0	1	0	2	2
3	0	0	0	2	0	0	2	1
4	0	2	2	0	0	0	4	2
5	0	0	0	2	0	2	4	1
6	2	2	0	0	0	2	6	1.5
7	0	2	0	2	0	2	6	1
8	0	2	0	2	2	2	8	1.33333
9	0	2	0	4	2	2	10	1.25
10	0	4	2	4	2	2	14	1.4
11	0	4	3	6	1	4	18	1.28571
12	3	6	2	5	1	7	24	1.33333
13	4	8	1	7	0	9	29	1.20833
14	2	11	2	8	3	10	36	1.24138
15	1	10	3	14	4	12	44	1.22222
16	5	15	5	14	4	15	58	1.31818
17	6	19	5	19	3	20	72	1.24138
18	7	25	7	22	5	25	91	1.26389
19	8	29	7	30	7	32	113	1.24176
20	13	38	7	36	10	39	143	1.26549
21	14	49	11	48	11	46	179	1.25175
22	12	62	19	60	17	57	227	1.26816
23	19	72	19	79	22	76	287	1.26432
24	29	98	30	94	20	95	366	1.27526
25	34	123	29	118	31	125	460	1.25683
26	40	152	36	154	42	154	578	1.25652
27	50	194	50	194	54	190	732	1.26644
28	62	244	61	248	71	240	926	1.26503

Level	$L_i(A)$	$L_i(B)$	$L_i(C)$	$L_i(D)$	$L_i(E)$	$L_i(F)$	$L_i(N)$	$L_i(N)/L_{i-1}(N)$
29	76	310	92	315	80	301	1174	1.26782
30	110	391	110	390	95	393	1489	1.26831
31	135	500	132	486	123	503	1879	1.26192
32	167	621	148	623	171	635	2365	1.25865
33	202	790	203	792	218	783	2988	1.26342
34	245	994	280	1008	267	986	3780	1.26506
35	328	1253	354	1261	326	1266	4788	1.26667
36	446	1589	397	1579	418	1620	6049	1.26337
37	531	2025	518	2007	530	2017	7628	1.26103
38	649	2538	677	2555	681	2535	9635	1.26311
39	862	3204	854	3219	839	3212	12190	1.26518
40	1101	4081	1090	4043	1028	4066	15409	1.26407
41	1376	5144	1347	5109	1320	5156	19452	1.26238
42	1696	6485	1694	6464	1719	6503	24561	1.26265
43	2165	8160	2108	8204	2191	8197	31025	1.26318
44	2730	10369	2743	10351	2731	10305	39229	1.26443
45	3371	13081	3486	13100	3494	13048	49580	1.26386
46	4360	16471	4322	16575	4418	16534	62680	1.26422
47	5537	20935	5575	20889	5463	20856	79255	1.26444
48	6944	26426	6983	26398	6962	26431	100144	1.26357
49	8856	33342	8729	33388	8813	33414	126542	1.26360
50	11103	42244	11159	42155	11126	42143	159930	1.26385
51	13990	53258	14009	53370	14156	53302	202085	1.26358
52	17773	67360	17824	67414	17773	67311	255455	1.26410
53	22437	85187	22624	85133	22353	85135	322869	1.26390
54	28499	107570	28339	107540	28295	107759	408002	1.26368
55	35884	136039	35864	135865	35792	136098	515542	1.26358
56	45288	171749	45074	171831	45503	171962	651407	1.26354
57	57339	217119	57065	217252	57427	217036	823238	1.26378
58	72095	274591	72795	274546	72362	274101	1040490	1.26390
59	91542	346641	91408	346953	91596	346896	1315036	1.26386

Appendix B

Total number comparison of $E^{(i)} * X_b$ and $\beta_k(8)$.

Level	$\beta_k(8)$	$E^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$E_i(N)/E_{i-1}(N)$	$K_i(N)$
0	1	1	0			
1	1	1	0	1	1	0
2	2	2	0	2	2	0
3	2	2	0	1	1	0
4	4	3.3	-0.7	2	1.66667	0.33333
5	4	3.67	-0.3	1	1.1	0.1
6	6	5.4	-0.6	1.5	1.48485	0.01515
7	6	6.3	0.3	1	1.16327	0.16327
8	8	8.8	0.8	1.33333	1.39181	0.05848
9	10	10.6	0.6	1.25	1.20168	0.04832
10	14	14.2	0.2	1.4	1.34033	0.05967
11	18	17.4	-0.6	1.28571	1.22522	0.06050
12	24	22.8	-1.2	1.33333	1.31039	0.02295
13	29	28.3	-0.7	1.20833	1.23975	0.03142
14	36	36.5	0.5	1.24138	1.29246	0.05108
15	44	45.6	1.6	1.22222	1.24878	0.02656
16	58	58.5	0.5	1.31818	1.28154	0.03664
17	72	73.3	1.3	1.24318	1.25440	0.01302
18	91	93.5	2.5	1.26389	1.27482	0.01093
19	113	117.6	4.6	1.24176	1.25791	0.01615
20	143	149.4	6.4	1.26549	1.27066	0.00517
21	179	188.3	9.3	1.25175	1.26010	0.00835
22	227	238.7	11.7	1.26816	1.26807	0.00009
23	287	301.2	14.2	1.26432	1.26147	0.00285
24	366	381.4	15.4	1.27526	1.26646	0.00881
25	460	481.4	21.4	1.25683	1.26233	0.00550
26	578	609.2	31.2	1.25652	1.26545	0.00893
27	732	769.4	37.4	1.26644	1.26286	0.00357
28	926	973.1	47.1	1.26503	1.26482	0.00021

Level	$\beta_k(8)$	$E^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$E_i(N)/E_{i-1}(N)$	$K_i(N)$
29	1174	1229.3	55.3	1.26782	1.26320	0.00462
30	1489	1554.3	65.3	1.26831	1.26442	0.00389
31	1879	1963.8	84.8	1.26192	1.26341	0.00149
32	2365	2482.5	117.5	1.25865	1.26418	0.00553
33	2988	3136.8	148.8	1.26342	1.26354	0.00012
34	3780	3965.0	185.0	1.26506	1.26402	0.00104
35	4788	5010.2	222.2	1.26667	1.26362	0.00304
36	6049	6332.5	283.5	1.26337	1.26392	0.00056
37	7628	8002.3	374.3	1.26103	1.26368	0.00264
38	9635	10113.8	478.8	1.26311	1.26386	0.00075
39	12190	12780.9	590.9	1.26518	1.26371	0.00147
40	15409	16152.8	743.8	1.26407	1.26383	0.00024
41	19452	20412.8	960.8	1.26238	1.26373	0.00135
42	24561	25797.7	1236.7	1.26265	1.26380	0.00116
43	31025	32601.6	1576.6	1.26318	1.26374	0.00056
44	39229	41201.5	1972.5	1.26443	1.26379	0.00064
45	49580	52068.4	2488.4	1.26386	1.26375	0.00011
46	62680	65802.9	3122.9	1.26422	1.26378	0.00044
47	79255	83158.7	3903.7	1.26444	1.26375	0.00068
48	100144	105093.6	4949.6	1.26357	1.26377	0.00021
49	126542	132812.8	6270.8	1.26360	1.26376	0.00016
50	159930	167844.7	7914.7	1.26385	1.26377	0.00008
51	202085	212115.3	10030.3	1.26358	1.26376	0.00018
52	255455	268064.2	12609.2	1.26410	1.26377	0.00033
53	322869	338769	15900.0	1.26390	1.26376	0.00014
54	408002	428124.4	20122.4	1.26368	1.26377	0.00009
55	515542	541047.1	25505.1	1.26358	1.26376	0.00018
56	651407	683755.9	32348.9	1.26354	1.26376	0.00023
57	823238	864104.6	40866.6	1.26378	1.26376	0.00002
58	1040490	1092024.9	51534.9	1.26390	1.26376	0.00014
59	1315036	1380058.4	65022.4	1.26386	1.26376	0.00010

Appendix C

The distribution of $\beta_k(8)$.

Level	$R_i(A)$	$R_i(B)$	$R_i(C)$	$R_i(D)$	$R_i(E)$	$R_i(F)$	$R_i(N)$
0	0	100	0	0	0	0	100
1	0	0	0	100	0	0	100
2	0	50	0	0	50	0	100
3	0	0	0	100	100	0	100
4	0	50	50	0	0	0	100
5	0	0	0	50	0	50	100
6	33.33333	33.33333	0	0	0	33.33333	100
7	0	33.33333	0	33.33333	0	33.33333	100
8	0	25	0	25	25	25	100
9	0	20	0	40	20	20	100
10	0	28.57143	14.28571	28.57143	14.28571	14.28571	100
11	0	22.22222	16.66667	33.33333	5.55556	22.22222	100
12	12.5	25	8.33333	20.83333	4.16667	29.16667	100
13	13.79310	27.58621	3.44828	24.13793	0	31.03448	100
14	5.55556	30.55556	5.55556	22.22222	8.33333	27.77778	100
15	2.27273	22.72727	6.81818	31.81818	9.09091	27.27273	100
16	8.62069	25.86207	8.62069	24.13793	6.89655	25.86207	100
17	8.33333	26.38889	6.94444	26.38889	4.16667	27.77778	100
18	7.69231	27.47253	7.69231	24.17582	5.49451	27.47253	100
19	7.07965	25.66372	6.19469	26.54867	6.19469	28.31858	100
20	9.09091	26.57343	4.89511	25.17483	6.99301	27.27273	100
21	7.82123	27.37430	6.14525	26.81564	6.14525	25.69832	100
22	5.28634	27.31278	8.37004	26.43172	7.48899	25.11013	100
23	6.62021	25.08711	6.62021	27.52613	7.66551	26.48084	100
24	7.92350	26.77596	8.19672	25.68306	5.46448	25.95628	100
25	7.39130	26.73913	6.30435	25.65217	6.73913	27.17391	100
26	6.92042	26.29758	6.22837	26.64360	7.26644	26.6436	100
27	6.83060	26.50273	6.83060	26.50273	7.37705	25.95628	100
28	6.69546	26.34989	6.58747	26.78186	7.66739	25.91793	100

Level	$R_i(A)$	$R_i(B)$	$R_i(C)$	$R_i(D)$	$R_i(E)$	$R_i(F)$	$R_i(N)$
29	6.47360	26.40545	7.83646	26.83135	6.81431	25.63884	100
30	7.38751	26.25923	7.38751	26.19208	6.38012	26.39355	100
31	7.18467	26.60990	7.02501	25.86482	6.54604	26.76956	100
32	7.06131	26.25793	6.25793	26.34249	7.23044	26.84989	100
33	6.76038	26.43909	6.79384	26.50602	7.29585	26.20482	100
34	6.48148	26.29630	7.40741	26.66667	7.06349	26.08466	100
35	6.85046	26.16959	7.39348	26.33668	6.80869	26.44110	100
36	7.37312	26.26880	6.56307	26.10349	6.91023	26.78129	100
37	6.96120	26.54693	6.79077	26.31096	6.94809	26.44206	100
38	6.73586	26.34146	7.02647	26.5179	7.06798	26.31033	100
39	7.07137	26.28384	7.00574	26.40689	6.88269	26.34947	100
40	7.14518	26.48452	7.07379	26.23791	6.67143	26.38718	100
41	7.07382	26.44458	6.92474	26.26465	6.78594	26.50627	100
42	6.90526	26.40365	6.89711	26.31815	6.99890	26.47693	100
43	6.97824	26.30137	6.79452	26.44319	7.06205	26.42063	100
44	6.95914	26.43198	6.99228	26.38609	6.96169	26.26883	100
45	6.79911	26.38362	7.03106	26.42194	7.04720	26.31706	100
46	6.95597	26.27792	6.89534	26.44384	7.04850	26.37843	100
47	6.98631	26.41474	7.03426	26.35670	6.89294	26.31506	100
48	6.93402	26.38800	6.97296	26.36004	6.95199	26.39299	100
49	6.99847	26.34856	6.89811	26.38492	6.96449	26.40546	100
50	6.94241	26.41406	6.97743	26.35841	6.95680	26.35090	100
51	6.92283	26.35426	6.93223	26.40968	7.00497	26.37603	100
52	6.95739	26.36864	6.97735	26.38978	6.95739	26.34945	100
53	6.94926	26.38438	7.00718	26.36766	6.92324	26.36828	100
54	6.98502	26.36507	6.94580	26.35771	6.93502	26.41139	100
55	6.96044	26.38757	6.95656	26.35382	6.94260	26.39901	100
56	6.95234	26.36585	6.91948	26.37844	6.98534	26.39855	100
57	6.96506	26.37378	6.93177	26.38994	6.97576	26.36370	100
58	6.92895	26.39055	6.99622	26.38622	6.95461	26.34345	100
59	6.96118	26.35981	6.95099	26.38354	6.96529	26.37920	100

Appendix D

The distribution of $E^{(i)} * X_b$.

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
0	0	100	0	0	0	0	100	0
1	0	0	0	100	0	0	100	0
2	16.66667	50	16.66667	0	16.66667	0	100	33.33333
3	0	16.66667	0	66.66667	0	16.66667	100	33.33333
4	13.33333	40	13.33333	10	13.33333	10	100	36.66667
5	3.03030	21.21212	3.03030	48.48485	3.03030	21.21212	100	28.78788
6	10.88435	34.69388	10.88435	16.32653	10.88435	16.32653	100	22.44898
7	4.67836	23.39181	4.67836	39.18129	4.67836	23.39181	100	9.94152
8	9.38375	31.51261	9.38375	9.38375	9.38375	9.38375	100	15.61625
9	5.59441	24.59207	5.59441	34.03263	5.59441	24.59207	100	14.40559
10	8.46377	29.56522	8.46377	22.52174	8.46377	22.52174	100	8.463768
11	6.12728	25.28980	6.12728	31.03856	6.12728	25.28980	100	10.53939
12	7.89553	28.36252	7.89553	23.97545	7.89553	23.97545	100	5.19122
13	6.44629	25.70749	6.44629	29.24615	6.44629	25.70749	100	7.34681
14	7.54275	27.61586	7.54275	24.87794	7.54275	24.87794	100	2.93969
15	6.64060	25.96190	6.64060	28.15439	6.64060	25.96190	100	4.36787
16	7.32304	27.15084	7.32304	25.44002	7.32304	25.44002	100	1.30209
17	6.76021	26.11851	6.76021	27.48236	6.76021	26.11851	100	2.59354
18	7.18590	26.86063	7.18590	25.79080	7.18590	25.79080	100	1.69142
19	6.83431	26.21553	6.83431	27.06601	6.83431	26.21553	100	2.10306
20	7.10025	26.67930	7.10025	26.00997	7.10025	26.00997	100	2.20515
21	6.88040	26.27588	6.88040	26.80705	6.88040	26.27588	100	1.09843
22	7.04668	26.56591	7.04668	26.14703	7.04668	26.14703	100	1.76033
23	6.90914	26.31351	6.90914	26.64556	6.90914	26.31351	100	1.22640
24	7.01316	26.49497	7.01316	26.23278	7.01316	26.23278	100	1.54868
25	6.92709	26.33701	6.92709	26.54471	6.92709	26.33701	100	0.89254
26	6.99218	26.45058	6.99218	26.28644	6.99218	26.28644	100	0.76381
27	6.93831	26.35170	6.93831	26.48160	6.93831	26.35170	100	0.43874
28	6.97905	26.42279	6.97905	26.32003	6.97905	26.32003	100	0.68833

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
29	6.94533	26.36089	6.94533	26.44224	6.94533	26.36089	100	0.89113
30	6.97083	26.40539	6.97083	26.34105	6.97083	26.34105	100	0.59071
31	6.94972	26.36664	6.94972	26.41756	6.94972	26.36664	100	0.55274
32	6.96569	26.39450	6.96569	26.35422	6.96569	26.35422	100	0.70776
33	6.95247	26.37024	6.95247	26.40212	6.95247	26.37024	100	0.34338
34	6.96247	26.38768	6.96247	26.36246	6.96247	26.36246	100	0.48099
35	6.95419	26.37249	6.95419	26.39245	6.95419	26.37249	100	0.43929
36	6.96045	26.38341	6.96045	26.36762	6.96045	26.36762	100	0.41367
37	6.95527	26.37390	6.95527	26.38640	6.95527	26.37390	100	0.17303
38	6.95919	26.38074	6.95910	26.37085	6.95910	26.37085	100	0.22333
39	6.95594	26.37478	6.95594	26.38261	6.95594	26.37478	100	0.11543
40	6.95840	26.37906	6.95840	26.37287	6.95840	26.37287	100	0.28697
41	6.95637	26.37534	6.95637	26.38023	6.95637	26.37534	100	0.17043
42	6.95790	26.37802	6.95790	26.37414	6.95790	26.37414	100	0.10280
43	6.95663	26.37568	6.95663	26.37875	6.95663	26.37568	100	0.16211
44	6.95759	26.37736	6.95759	26.37493	6.95759	26.37493	100	0.10610
45	6.95680	26.37590	6.95680	26.37782	6.95680	26.37590	100	0.15768
46	6.95740	26.37695	6.95740	26.37543	6.95740	26.37543	100	0.09903
47	6.95690	26.37603	6.95690	26.37724	6.95690	26.37603	100	0.07736
48	6.95728	26.37669	6.95728	26.37574	6.95728	26.37574	100	0.02326
49	6.95696	26.37612	6.95696	26.37687	6.95696	26.37612	100	0.05886
50	6.95720	26.37653	6.95720	26.37594	6.95720	26.37594	100	0.03753
51	6.95700	26.37617	6.95700	26.37664	6.95700	26.37617	100	0.04797
52	6.95715	26.37643	6.95715	26.37606	6.95715	26.37606	100	0.02660
53	6.95703	26.37621	6.95703	26.37650	6.95703	26.37621	100	0.05015
54	6.95712	26.37637	6.95712	26.37613	6.95712	26.37613	100	0.03526
55	6.95705	26.37623	6.95705	26.37641	6.95705	26.37623	100	0.02279
56	6.95710	26.37633	6.95710	26.37618	6.95710	26.37618	100	0.03762
57	6.95706	26.37624	6.95706	26.37636	6.95706	26.37624	100	0.02528
58	6.95709	26.37630	6.95709	26.37621	6.95709	26.37621	100	0.03913
59	6.95700	26.37625	6.95700	26.37632	6.95700	26.37625	100	0.01644

Appendix E

$(x, y, z) = (1/3 + 0.0001, 1/3 - 0,0001, 1/3)$. Total number comparison of $P_1^{(i)} * X_b$ and $\beta_k(8)$.

Level	$\beta_k(8)$	$P_1^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
0	1	1	0			
1	1	1	0	1	1	0
2	2	2	0	2	2	0
3	2	2	0	1	1	0
4	4	3.3	-0.7	2	1.66667	0.33333
5	4	3.7	-0.3	1	1.10003	0.10003
6	6	5.4	-0.6	1.5	1.48484	0.01516
7	6	6.3	0.3	1	1.16331	0.16331
8	8	8.8	0.8	1.33333	1.39180	0.05847
9	10	10.6	0.6	1.25	1.20173	0.04827
10	14	14.2	0.2	1.4	1.34032	0.05968
11	18	17.4	-0.6	1.28571	1.22527	0.06045
12	24	22.8	-1.2	1.33333	1.31039	0.02295
13	29	28.3	-0.7	1.20833	1.23980	0.03147
14	36	36.5	0.5	1.24138	1.29247	0.05109
15	44	45.6	1.6	1.22222	1.24882	0.0266
16	58	58.5	0.5	1.31818	1.28156	0.03662
17	72	73.3	1.3	1.24138	1.25444	0.01306
18	91	93.5	2.5	1.26389	1.27484	0.01095
19	113	117.6	4.6	1.24176	1.25794	0.01619
20	143	149.4	6.4	1.26549	1.27068	0.00519
21	179	188.3	9.3	1.25175	1.26013	0.00839
22	227	238.8	11.8	1.26816	1.26809	0.00006
23	287	301.3	14.3	1.26432	1.26150	0.00281
24	366	381.5	15.5	1.27526	1.26648	0.00878
25	460	481.7	21.7	1.25683	1.26236	0.00553
26	578	609.5	31.5	1.25652	1.26547	0.00895
27	732	769.8	37.8	1.26644	1.26290	0.00354
28	926	973.6	47.6	1.26503	1.26484	0.00018

Level	$\beta_k(8)$	$P_1^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
29	1174	1229.9	55.9	1.26782	1.26323	0.00459
30	1489	1555.2	66.2	1.26831	1.26445	0.00386
31	1879	1964.9	85.9	1.26192	1.26344	0.00152
32	2365	2484.0	119.0	1.25865	1.26420	0.00556
33	2988	3138.7	150.7	1.26342	1.26357	0.00015
34	3780	3967.4	187.4	1.26506	1.26405	0.00101
35	4788	5013.5	225.5	1.26667	1.26365	0.00301
36	6049	6336.8	287.8	1.26337	1.26395	0.00059
37	7628	8007.8	379.8	1.26103	1.26371	0.00267
38	9635	10121.1	486.1	1.26311	1.26389	0.00078
39	12190	12790.4	600.4	1.26518	1.26374	0.00144
40	15409	16165.2	756.2	1.26407	1.26385	0.00021
41	19452	20428.8	976.8	1.26238	1.27376	0.00138
42	24561	25818.6	1257.6	1.26265	1.26383	0.00118
43	31025	32628.8	1603.8	1.26318	1.26377	0.00059
44	39229	41236.8	2007.8	1.26443	1.26382	0.00062
45	49580	52114.2	2534.2	1.26386	1.26378	0.00008
46	62680	65862.3	3182.3	1.26422	1.26381	0.00041
47	79255	83235.6	3980.6	1.26444	1.26378	0.00066
48	100144	105193.3	5049.3	1.26357	1.26380	0.00023
49	126542	132941.9	6399.9	1.26360	1.26379	0.00019
50	159930	168011.6	8081.6	1.26385	1.26380	0.00005
51	202085	212331.1	10246.1	1.26358	1.26379	0.00020
52	255455	268343.1	12888.1	1.26410	1.26380	0.00030
53	322869	339129.2	16260.2	1.26390	1.26379	0.00011
54	408002	428589.4	20587.4	1.26368	1.26379	0.00012
55	515542	541647.1	26105.1	1.26358	1.26379	0.00021
56	651407	684529.8	33122.8	1.26354	1.26379	0.00025
57	823238	865102.5	41864.5	1.26378	1.26379	0.00006
58	1040490	1093310.0	52820.0	1.26390	1.26379	0.00011
59	1315036	1381715.3	66679.3	1.26386	1.26379	0.00007

Appendix F

$(x, y, z) = (1/3 + 0.0001, 1/3 - 0.0001, 1/3)$. The distribution of $P^{(i)} * X_b$.

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
0	0	100	0	0	0	0	100	0
1	0	0	0	100	0	0	100	0
2	16.6717	50	16.66167	0	16.6667	0	100	33.33333
3	0	16.67167	0	66.66667	0	16.66167	100	33.33333
4	13.3373	40	13.32933	10.003	13.3333	9.997	100	36.67067
5	3.03204	21.21791	3.03022	48.48353	3.03113	21.20518	100	28.79482
6	10.8874	34.69446	10.88089	16.33113	10.8842	16.32194	100	22.44591
7	4.68090	23.39748	4.67809	39.18007	4.67950	23.38397	100	9.94936
8	9.38636	31.51383	9.38073	20.17313	9.38354	20.16241	100	15.61646
9	5.59725	24.59742	5.59390	34.03205	5.59557	24.58382	100	14.40443
10	8.46620	29.56703	8.46112	22.52670	8.46366	22.51530	100	8.46620
11	6.13022	25.29481	6.12654	31.03870	6.12838	25.28136	100	10.54013
12	7.89792	28.36484	7.89319	23.98008	7.89556	23.96842	100	5.19826
13	6.44923	25.7122	6.44536	29.24695	6.44729	25.69897	100	7.34387
14	7.54517	27.61858	7.54065	24.88221	7.54291	24.87047	100	2.93698
15	6.64351	25.96637	6.63952	28.15573	6.64151	25.95335	100	4.37078
16	7.32551	27.15387	7.32111	25.44396	7.32331	25.43224	100	1.30603
17	6.76307	26.12279	6.75901	27.48413	6.76104	26.10995	100	2.59438
18	7.18844	26.86389	7.18413	25.79445	7.18629	25.78279	100	1.69178
19	6.83713	26.21968	6.83303	27.06811	6.83508	26.20698	100	2.11160
20	7.10281	26.68273	7.09855	26.01342	7.10068	26.00181	100	2.20345
21	6.88319	26.27992	6.87906	26.80938	6.88112	26.26733	100	1.09438
22	7.04927	26.56946	7.04505	26.15031	7.04716	26.13875	100	1.76293
23	6.91190	26.31748	6.90776	26.64807	6.90983	26.30497	100	1.23037
24	7.01578	26.4986	7.01158	26.23594	7.01368	26.22442	100	1.54920
25	6.92983	26.34093	6.92567	26.54734	6.92775	26.32847	100	0.89517
26	6.99483	26.45427	6.99063	26.28952	6.99273	26.27803	100	0.76226
27	6.94104	26.35559	6.93687	26.48438	6.93895	26.34317	100	0.43809
28	6.98171	26.42652	6.97752	26.32305	6.97962	26.31158	100	0.68777

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
29	6.94804	26.36475	6.94388	26.44500	6.94596	26.35236	100	0.89258
30	6.97350	26.40915	6.96932	26.34403	6.97141	26.33259	100	0.59129
31	6.95243	26.37049	6.94826	26.42037	6.95034	26.35812	100	0.55555
32	6.96836	26.39828	6.96418	26.35717	6.96627	26.34573	100	0.70626
33	6.95517	26.37407	6.95100	26.40495	6.95309	26.36172	100	0.34276
34	6.96515	26.39147	6.96097	26.36539	6.96306	26.35396	100	0.48367
35	6.95689	26.37632	6.95272	26.39530	6.95480	26.36397	100	0.44077
36	6.96313	26.38721	6.95896	26.37054	6.96105	26.35912	100	0.42217
37	6.95796	26.37772	6.95379	26.38926	6.95588	26.36538	100	0.16921
38	6.96187	26.38454	6.95770	26.37376	6.95979	26.36234	100	0.22601
39	6.95864	26.37860	6.95446	26.38548	6.95655	26.36627	100	0.11273
40	6.96108	26.38287	6.95691	26.37578	6.95900	26.36436	100	0.28757
41	6.95906	26.37915	6.95488	26.38312	6.95697	26.36682	100	0.17104
42	6.96059	26.38183	6.95642	26.37704	6.95850	26.36563	100	0.11130
43	6.95932	26.37950	6.95515	26.38163	6.95723	26.36717	100	0.16063
44	6.96028	26.38117	6.95611	26.37783	6.95819	26.36642	100	0.09759
45	6.95949	26.37971	6.95531	26.38071	6.95740	26.36738	100	0.16037
46	6.96009	26.38076	6.95591	26.37833	6.95800	26.36692	100	0.10284
47	6.95959	26.37985	6.95542	26.38013	6.95750	26.36752	100	0.07884
48	6.95997	26.3805	6.95579	26.37863	6.95788	26.36723	100	0.02595
49	6.95965	26.37993	6.95548	26.37976	6.95757	26.36760	100	0.05738
50	6.95989	26.38034	6.95572	26.37883	6.95780	26.36742	100	0.03372
51	6.95969	26.37999	6.95552	26.37953	6.95761	26.36766	100	0.04737
52	6.95984	26.38024	6.95567	26.37895	6.95775	26.36754	100	0.02169
53	6.95972	26.38002	6.95555	26.37939	6.95763	26.36769	100	0.05163
54	6.95981	26.38018	6.95564	26.37903	6.95773	26.36762	100	0.04377
55	6.95974	26.38004	6.95556	26.3793	6.95765	26.36771	100	0.03130
56	6.95979	26.38014	6.95562	26.37907	6.95771	26.36767	100	0.03614
57	6.95975	26.38005	6.95557	26.37925	6.95766	26.36772	100	0.02380
58	6.95978	26.38012	6.95561	26.37910	6.95769	26.36770	100	0.04062
59	6.95975	26.38006	6.95558	26.37921	6.95766	26.36773	100	0.02025

Appendix G

$(x, y, z) = (1/3 - 0.0001, 1/3 + 0,0001, 1/3)$. Total number comparison of $P_2^{(i)} * X_b$ and $\beta_k(8)$.

Level	$\beta_k(8)$	$P_2^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
0	1	1	0			
1	1	1	0	1	1	0
2	2	2	0	2	2	0
3	2	2	0	1	1	0
4	4	3.3	-0.7	2	1.66667	0.33333
5	4	3.7	-0.3	1	1.09997	0.09997
6	6	5.4	-0.6	1.5	1.48486	0.01514
7	6	6.3	0.3	1	1.16322	0.16322
8	8	8.8	0.8	1.33333	1.39183	0.05849
9	10	10.6	0.6	1.25	1.20163	0.04837
10	14	14.2	4.2	1.4	1.34033	0.05967
11	18	17.4	-0.6	1.28571	1.22517	0.06055
12	24	22.8	-1.2	1.33333	1.31038	0.02295
13	29	28.3	-0.7	1.20833	1.23971	0.03137
14	36	36.5	0.5	1.24138	1.29245	0.05107
15	44	45.6	1.6	1.22222	1.24874	0.02651
16	58	58.4	0.4	1.31818	1.28153	0.03665
17	72	73.3	1.3	1.24138	1.25436	0.01298
18	91	93.4	2.4	1.26389	1.27481	0.01092
19	113	117.5	4.5	1.24176	1.25787	0.01611
20	143	149.4	6.4	1.26549	1.27064	0.00515
21	179	188.2	9.2	1.25175	1.26007	0.00832
22	227	238.6	11.6	1.26816	1.26805	0.00011
23	287	301.0	14.0	1.26432	1.26144	0.00288
24	366	381.2	15.2	1.27526	1.26643	0.00883
25	460	481.2	21.2	1.25683	1.26230	0.00547
26	578	609.0	31.0	1.25652	1.26542	0.00890
27	732	769.0	37.0	1.26644	1.26283	0.00360
28	926	972.7	46.7	1.26503	1.26479	0.00024

Level	$\beta_k(8)$	$P_2^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
29	1174	1228.6	54.6	1.26782	1.26317	0.00465
30	1489	1553.5	64.5	1.26831	1.26439	0.00392
31	1879	1962.7	83.7	1.26192	1.26338	0.00146
32	2365	2481.1	116.1	1.25865	1.25415	0.00550
33	2988	3134.9	146.9	1.26342	1.26351	0.00009
34	3780	3962.5	182.5	1.26506	1.26399	0.00107
35	4788	5007.0	219.0	1.26667	1.26360	0.00307
36	6049	6328.3	279.3	1.26337	1.26390	0.00053
37	7628	7996.7	368.7	1.26103	1.26365	0.00261
38	9635	10106.5	471.5	1.26311	1.26384	0.00073
39	12190	12771.4	581.4	1.26518	1.26368	0.00150
40	15409	16140.4	731.4	1.26407	1.26380	0.00027
41	19452	20396.7	944.7	1.26238	1.26370	0.00132
42	24561	25776.8	1215.8	1.26265	1.26377	0.00113
43	31025	32574.4	1549.4	1.26318	1.26371	0.00053
44	39229	41166.2	1937.2	1.26443	1.26376	0.00067
45	49580	52022.6	2442.6	1.26386	1.26372	0.00014
46	62680	65743.5	3063.5	1.26422	1.26375	0.00047
47	79255	83081.7	3826.7	1.26444	1.26373	0.00071
48	100144	104994.0	4850.0	1.26357	1.26374	0.00018
49	126542	132683.9	6141.9	1.26360	1.26373	0.00013
50	159930	167678.0	7748.0	1.26385	1.26374	0.00011
51	202085	211900.0	9815.0	1.26358	1.26373	0.00015
52	255455	267785.6	12330.6	1.26410	1.26374	0.00036
53	322869	338409.2	15540.2	1.26390	1.26373	0.00017
54	408002	427660.0	19658.0	1.26368	1.26374	0.00006
55	515542	540447.7	24905.7	1.26358	1.26373	0.00016
56	651407	682982.8	39869.8	1.26354	1.26373	0.00020
57	823238	863107.8	39869.8	1.26378	1.26373	0.00005
58	1040490	1090739.2	50249.2	1.26390	1.26373	0.00016
59	1315036	1378403.4	63367.4	1.26386	1.26373	0.00013

Appendix H

$(x, y, z) = (1/3 - 0.0001, 1/3 + 0.0001, 1/3)$. The distribution of $P_2^{(i)} * X_b$.

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
0	0	100	0	0	0	0	100	0
1	0	0	0	100	0	0	100	0
2	16.6717	50	16.6717	0	16.6717	0	100	33.32833
3	0	16.6617	0	66.66667	0	16.67167	100	33.33333
4	13.3293	40	13.3373	9.997	13.3333	10.003	100	36.66267
5	3.02857	21.20634	3.03039	48.48617	3.02948	21.21906	100	28.78094
6	10.8813	34.69329	10.8878	16.32193	10.8846	16.33112	100	22.45205
7	4.67583	23.38615	4.67863	39.18250	4.67723	23.39966	100	9.94718
8	9.38115	31.51138	9.38678	20.16301	9.38396	20.17372	100	15.61604
9	5.59156	24.58673	5.59492	34.03322	5.59324	24.60033	100	14.40676
10	8.46134	29.56340	8.46642	22.51678	8.46388	22.52818	100	8.46134
11	6.12434	25.28480	6.12801	31.03843	6.12618	25.29825	100	10.53865
12	7.89313	28.36020	7.89787	23.97081	7.89550	23.98248	100	5.18419
13	6.44335	25.70278	6.44722	29.24536	6.44528	25.71601	100	7.34975
14	7.54033	27.61314	7.54486	24.87367	7.54259	24.88540	100	2.94241
15	6.6377	25.95743	6.64168	28.15304	6.63969	25.97046	100	4.36497
16	7.32057	27.14781	7.32496	25.43609	7.32277	25.44780	100	1.30012
17	6.75735	26.11422	6.76140	27.48059	6.75938	26.12706	100	2.59271
18	7.18341	26.85737	7.18772	25.78714	7.18556	25.79881	100	1.69106
19	6.83149	26.21138	6.83559	27.06392	6.83354	26.22408	100	2.09450
20	7.09769	26.67587	7.10195	26.00653	7.09982	26.01814	100	2.2068
21	6.87760	26.27180	6.88170	26.8047	6.87970	26.28440	100	1.10250
22	7.04408	26.56236	7.04831	26.14375	7.04619	26.15531	100	1.75774
23	6.90638	26.30953	6.91053	26.64306	6.90845	26.32205	100	1.22242
24	7.01053	26.49134	7.01474	26.22961	7.01264	26.24113	100	1.54816
25	6.92435	26.33308	6.92851	26.54209	6.92643	26.34554	100	0.88992
26	6.98954	26.44688	6.99373	26.28336	6.99164	26.29485	100	0.76536
27	6.93559	26.34781	6.93975	26.47896	6.93767	26.36023	100	0.43938
28	6.97640	26.41906	6.98058	26.31701	6.97849	26.32847	100	0.68890

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
29	6.94262	26.35702	6.94678	26.43947	6.9447	26.36941	100	0.88967
30	6.96817	26.40163	6.97235	26.33807	6.97026	26.34952	100	0.59014
31	6.94701	26.36279	6.95118	26.41475	6.94910	26.37516	100	0.54993
32	6.96301	26.39072	6.96719	26.35127	6.96510	26.36270	100	0.70926
33	6.94977	26.36640	6.95394	26.39928	6.95185	26.37876	100	0.34400
34	6.95979	26.38389	6.96396	26.35953	6.96188	26.37095	100	0.47831
35	6.95149	26.36866	6.95566	26.38959	6.95358	26.38101	100	0.43782
36	6.95777	26.37961	6.96194	26.36470	6.95985	26.37612	100	0.41535
37	6.95257	26.37008	6.95675	26.38353	6.95466	26.38242	100	0.17686
38	6.95650	26.37694	6.96068	26.36794	6.95859	26.37936	100	0.22064
39	6.95325	26.37096	6.95742	26.37973	6.95534	26.38330	100	0.11812
40	6.95571	26.37526	6.95988	26.36997	6.95780	26.38138	100	0.28637
41	6.95367	26.37152	6.95785	26.37735	6.95576	26.38385	100	0.16982
42	6.95521	26.37421	6.95939	26.37124	6.95730	26.38265	100	0.09428
43	6.95394	26.37187	6.95811	26.37586	6.95602	26.38420	100	0.16359
44	6.95490	26.37355	6.95908	26.37204	6.95699	26.38345	100	0.11462
45	6.95410	26.37208	6.95828	26.37493	6.95619	26.38442	100	0.15499
46	6.95471	26.37314	6.95888	26.37253	6.95679	26.38394	100	0.09522
47	6.95421	26.37222	6.95838	26.37435	6.95629	26.38455	100	0.07588
48	6.95459	26.37288	6.95876	26.37285	6.95667	26.38426	100	0.02057
49	6.95427	26.37230	6.95845	26.37398	6.95636	26.38464	100	0.06034
50	6.95451	26.37272	6.95868	26.37304	6.95660	26.38445	100	0.04134
51	6.95431	26.37236	6.95849	26.37375	6.95640	26.38469	100	0.04857
52	6.95446	26.37262	6.95864	26.37316	6.95655	26.38457	100	0.03512
53	6.95434	26.37239	6.95851	26.37361	6.95643	26.38472	100	0.04866
54	6.95443	26.37255	6.95861	26.37324	6.95652	26.38465	100	0.03058
55	6.95435	26.37241	6.95853	26.37352	6.95644	26.38474	100	0.01970
56	6.95441	26.37251	6.95859	26.37329	6.95650	26.38470	100	0.03910
57	6.95436	26.37243	6.95854	26.37346	6.95645	26.38476	100	0.02676
58	6.95440	26.37249	6.95857	26.37332	6.95649	26.38473	100	0.04127
59	6.95437	26.37243	6.95854	26.37343	6.95646	26.38476	100	0.01262

Appendix I

$(x, y, z) = (1/3 + 0.0002, 1/3 - 0, 0002, 1/3)$. Total number comparison of $P_3^{(i)} * X_b$ and $\beta_k(8)$.

Level	$\beta_k(8)$	$P_3^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
0	1	1	0			
1	1	1	0	1	1	0
2	2	2	0	2	2	0
3	2	2	0	1	1	0
4	4	3.3	-0.7	2	1.66667	0.33333
5	4	3.7	-0.3	1	1.10006	0.10006
6	6	5.4	-0.6	1.5	1.48482	0.01518
7	6	6.3	0.3	1	1.16336	0.16336
8	8	8.8	0.8	1.33333	1.39179	0.05846
9	10	10.6	0.6	1.25	1.20178	0.04822
10	14	14.2	0.2	1.4	1.34031	0.05969
11	18	17.4	-0.6	1.28571	1.22532	0.06040
12	24	22.8	-1.2	1.33333	1.31039	0.02294
13	29	28.3	-0.73	1.20833	1.23985	0.02151
14	36	36.5	0.54	1.24138	1.29248	0.05110
15	44	45.6	1.6	1.22222	1.24886	0.02664
16	58	58.5	0.5	1.31818	1.28157	0.03661
17	72	73.4	1.4	1.24138	1.25448	0.01310
18	91	93.5	2.5	1.26389	1.27486	0.01097
19	113	117.6	4.6	1.24176	1.25798	0.01622
20	143	149.5	6.5	1.26549	1.27070	0.00522
21	179	188.4	9.4	1.25175	1.26017	0.00842
22	227	238.9	11.9	1.26816	1.26812	0.00004
23	287	301.4	14.4	1.26432	1.26154	0.00278
24	366	381.7	15.7	1.27526	1.26651	0.00876
25	460	481.9	21.9	1.25683	1.26239	0.00556
26	578	609.8	31.8	1.25652	1.26550	0.00898
27	732	770.1	38.1	1.26644	1.26293	0.00351
28	926	974.1	48.1	1.26503	1.26487	0.00016

Level	$\beta_k(8)$	$P_3^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
29	1174	1230.5	56.5	1.26782	1.26326	0.00456
30	1489	1556.0	67.0	1.26831	1.26448	0.00384
31	1879	1966.0	87.0	1.26192	1.26347	0.00155
32	2365	2485.4	120.4	1.25865	1.26423	0.00558
33	2988	3140.6	152.6	1.26342	1.26360	0.00018
34	3780	3970.0	190.0	1.26506	1.26408	0.00098
35	4788	5016.7	228.7	1.26667	1.26368	0.00298
36	6049	6341.1	292.1	1.26337	1.26398	0.00061
37	7628	8013.4	385.4	1.26103	1.26373	0.00270
38	9635	10128.3	493.3	1.26311	1.26392	0.00081
39	12190	12800.0	610.0	1.26518	1.26377	0.00141
40	15409	16177.5	768.5	1.26407	1.26388	0.00019
41	19452	20445.0	993.0	1.26238	1.26379	0.00141
42	24561	25839.6	1278.6	1.26265	1.26386	0.00121
43	31025	32656.0	1631.0	1.26318	1.26380	0.00062
44	39229	41272.2	2043.2	1.26443	1.26385	0.00059
45	49580	52160.0	2580.0	1.26386	1.26381	0.00005
46	62680	65921.7	3241.7	1.26422	1.26384	0.00038
47	79255	83312.7	4057.7	1.26444	1.26381	0.00063
48	100144	105293.1	5149.1	1.26357	1.26383	0.00026
49	126542	133071.0	6529.0	1.26360	1.26382	0.00021
50	159930	168178.7	8248.7	1.26385	1.26383	0.00002
51	202085	212547.1	10462.1	1.26358	1.26382	0.00023
52	255455	268622.2	13167.2	1.26410	1.26382	0.00027
53	322869	339489.7	16620.7	1.26390	1.26382	0.00008
54	408002	429054.8	21052.8	1.26368	1.26382	0.00015
55	515542	542247.7	26705.7	1.26358	1.26382	0.00024
56	651407	685304.6	33897.6	1.26354	1.26382	0.00028
57	823238	866101.4	42863.4	1.26378	1.26382	0.00004
58	1040490	1094597.4	54107.4	1.26390	1.26382	0.00008
59	1315036	1383374.1	68338.1	1.26386	1.26382	0.00004

Appendix J

$(x, y, z) = (1/3 + 0.0002, 1/3 - 0.0002, 1/3)$. The distribution of $P_3^{(i)} * X_b$.

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
0	0	100	0	0	0	0	100	0
1	0	0	0	100	0	0	100	0
2	16.6767	50	16.6567	0	16.6667	0	100	33.33333
3	0	16.67667	0	66.66667	0	16.65667	100	33.33333
4	13.3413	40	13.3253	10.006	13.3333	9.994	100	36.67467
5	3.03378	21.22369	3.03014	48.4822	3.03196	21.19824	100	28.80176
6	10.8905	34.69505	10.8774	16.33573	10.884	16.31736	100	22.44285
7	4.68344	23.40314	4.67782	39.17886	4.68063	23.37612	100	9.95721
8	9.38897	31.51505	9.37771	20.17818	9.38334	20.15676	100	15.61666
9	5.6001	24.60276	5.59338	34.03146	5.59674	24.57556	100	14.40326
10	8.46863	29.56885	8.45847	22.53165	8.46355	22.50885	100	8.46863
11	6.13316	25.29981	6.12580	31.03884	6.12948	25.27292	100	10.54087
12	7.90032	28.36716	7.89084	23.98471	7.89558	23.96138	100	5.20529
13	6.45217	25.71691	6.44443	29.24775	6.44830	25.69045	100	7.34093
14	7.5476	27.62131	7.53855	24.88647	7.54307	24.86301	100	2.93425
15	6.64641	25.97084	6.63844	28.15707	6.64243	25.94481	100	4.37368
16	7.32798	27.15690	7.31919	25.44789	7.32358	25.42446	100	1.30996
17	6.76593	26.12708	6.75782	27.4859	6.76187	26.10140	100	2.59521
18	7.19096	26.86715	7.18234	25.79811	7.18665	25.77478	100	1.69775
19	6.83995	26.22382	6.83175	27.07020	6.83585	26.19842	100	2.12015
20	7.10538	26.68616	7.09685	26.01686	7.10112	25.99364	100	2.20175
21	6.88597	26.28397	6.87772	26.81171	6.88185	26.25879	100	1.09033
22	7.05187	26.57301	7.04341	26.15359	7.04764	26.13047	100	1.76553
23	6.91466	26.32146	6.90637	26.65057	6.91052	26.29643	100	1.23435
24	7.01841	26.50223	7.00999	26.23910	7.01420	26.21607	100	1.54972
25	6.93257	26.34486	6.92426	26.54996	6.92841	26.31994	100	0.89779
26	6.99747	26.45796	6.98908	26.29260	6.99328	26.26962	100	0.76071
27	6.94376	26.35948	6.93543	26.48709	6.93960	26.33464	100	0.43745
28	6.98437	26.43025	6.97599	26.32607	6.98018	26.30314	100	0.68721

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
29	6.95076	26.36862	6.94242	26.44777	6.94659	26.34384	100	0.89403
30	6.97617	26.41291	6.96781	26.34701	6.97199	26.32412	100	0.59187
31	6.95514	26.37433	6.94679	26.42318	6.95097	26.34959	100	0.55836
32	6.97104	26.40205	6.96268	26.36012	6.96686	26.33725	100	0.70475
33	6.95787	26.37791	6.94953	26.40779	6.95370	26.35320	100	0.34215
34	6.96783	26.39526	6.95947	26.36832	6.96365	26.34547	100	0.48635
35	6.95959	26.38015	6.95124	26.39816	6.95541	26.35545	100	0.44224
36	6.96582	26.39101	6.95746	26.37346	6.96164	26.35062	100	0.43067
37	6.96066	26.38155	6.95231	26.39213	6.95649	26.35686	100	0.16538
38	6.96456	26.38834	6.95621	26.37667	6.96038	26.35384	100	0.22870
39	6.96133	26.38242	6.95298	26.38836	6.95716	26.35775	100	0.11004
40	6.96377	26.38668	6.95542	26.37868	6.95960	26.35585	100	0.28817
41	6.96175	26.38297	6.95340	26.38600	6.95758	26.35830	100	0.17164
42	6.96328	26.38563	6.95493	26.37994	6.95910	26.35712	100	0.11981
43	6.96201	26.38332	6.95366	26.38452	6.95784	26.35865	100	0.15914
44	6.96297	26.38498	6.95462	26.38073	6.95879	26.35791	100	0.08908
45	6.96218	26.38353	6.95383	26.38359	6.95800	26.35887	100	0.16307
46	6.96278	26.38457	6.95443	26.38122	6.95860	26.35840	100	0.10665
47	6.96228	26.38366	6.95393	26.38301	6.95811	26.35900	100	0.08033
48	6.96266	26.38432	6.95431	26.38153	6.95848	26.35871	100	0.03428
49	6.96235	26.38375	6.95400	26.38265	6.95817	26.35909	100	0.05589
50	6.96258	26.38416	6.95423	26.38172	6.95841	26.35890	100	0.02990
51	6.96239	26.38380	6.95404	26.38242	6.95821	26.35914	100	0.04676
52	6.96253	26.38406	6.95418	26.38184	6.95836	26.35903	100	0.02317
53	6.96241	26.38383	6.95406	26.38228	6.95824	26.35917	100	0.05311
54	6.96250	26.38399	6.95415	26.38192	6.95833	26.35910	100	0.05229
55	6.96243	26.38385	6.95408	26.38219	6.95825	26.35919	100	0.03982
56	6.96248	26.38395	6.95413	26.38197	6.95831	26.35915	100	0.03940
57	6.96244	26.38387	6.95409	26.38214	6.95826	26.35921	100	0.02231
58	6.96247	26.38393	6.95412	26.38200	6.95830	26.35918	100	0.04210
59	6.96244	26.38388	6.95409	26.38210	6.95827	26.35922	100	0.02407

Appendix K

$(x, y, z) = (1/3 - 0.0002, 1/3 + 0, 0002, 1/3)$. Total number comparison of $P_4^{(i)} * X_b$ and $\beta_k(8)$.

Level	$\beta_k(8)$	$P_4^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
0	1	1	0			
1	1	1	0	1	1	0
2	2	2	0	2	2	0
3	2	2	0	1	1	0
4	4	3.3	-0.7	2	1.66667	0.33333
5	4	3.7	-0.3	1	1.09994	0.09994
6	6	5.4	-0.6	1.5	1.48487	0.01513
7	6	6.3	0.3	1	1.16317	0.16317
8	8	8.8	0.8	1.33333	1.39184	0.05850
9	10	10.6	0.6	1.25	1.20158	0.048420
10	14	14.2	0.2	1.4	1.34034	0.05966
11	18	17.4	-0.6	1.28571	1.22512	0.0606
12	24	22.8	-1.2	1.33333	1.31038	0.02295
13	29	28.2	-0.8	1.20833	1.23966	0.03133
14	36	36.5	0.5	1.24138	1.29245	0.05107
15	44	45.6	1.6	1.22222	1.24869	0.02647
16	58	58.4	0.4	1.31818	1.28152	0.03666
17	72	73.3	1.3	1.24138	1.25432	0.01294
18	91	93.4	2.4	1.26389	1.27479	0.01090
19	113	117.5	4.5	1.24176	1.25783	0.01608
20	143	149.3	6.3	1.26549	1.27062	0.00513
21	179	188.1	9.1	1.25175	1.26003	0.00828
22	227	238.6	11.6	1.26816	1.26802	0.00013
23	287	300.9	13.9	1.26432	1.26140	0.00291
24	366	381.1	15.1	1.27526	1.26641	0.00886
25	460	481.0	21.0	1.25683	1.26226	0.00543
26	578	608.7	30.7	1.25652	1.26539	0.00887
27	732	768.7	36.7	1.26644	1.26280	0.00363
28	926	972.2	46.2	1.26503	1.26476	0.00026

Level	$\beta_k(8)$	$P_4^{(i)} * X_b$	$D_i(N)$	$L_i(N)/L_{i-1}(N)$	$P_i(N)/P_{i-1}(N)$	$K_i(N)$
29	1174	1228.0	54.0	1.26782	1.26314	0.00468
30	1489	1552.7	63.7	1.26831	1.26437	0.00395
31	1879	1961.6	82.6	1.26192	1.26335	0.00143
32	2365	2479.6	114.6	1.25865	1.26412	0.00547
33	2988	3133.0	145.0	1.26342	1.26348	0.00006
34	3780	3960.0	180.0	1.26506	1.26396	0.00110
35	4788	5003.7	215.7	1.26667	1.26357	0.00310
36	6049	6324.0	275.0	1.26337	1.26387	0.00050
37	7628	7991.1	363.1	1.26103	1.26362	0.00258
38	9635	10099.2	464.2	1.26311	1.26381	0.00070
39	12190	12762.0	572.0	1.26518	1.26365	0.00153
40	15409	16128.1	719.1	1.26407	1.26377	0.00030
41	19452	20380.6	928.6	1.26238	1.26367	0.00129
42	24561	25755.9	1194.9	1.26265	1.26374	0.00110
43	31025	32547.3	1522.3	1.26318	1.26368	0.00050
44	39229	41131.0	1902.0	1.26443	1.26373	0.00070
45	49580	51976.8	2396.8	1.26386	1.26369	0.00017
46	62680	65684.2	3004.2	1.26422	1.26372	0.00050
47	79255	83004.9	3749.9	1.26444	1.26370	0.00074
48	100144	104894.5	4750.5	1.26357	1.26371	0.00015
49	126542	132555.1	6013.1	1.26360	1.26370	0.00010
50	159930	167511.4	7581.4	1.26385	1.26371	0.00014
51	202085	211684.3	9599.3	1.26358	1.26370	0.00012
52	255455	267507.3	12052.3	1.26410	1.26371	0.00039
53	322869	338049.7	15180.7	1.26390	1.26370	0.00020
54	408002	427195.9	19193.9	1.26368	1.26371	0.00003
55	515542	539848.9	24306.9	1.26358	1.26370	0.00013
56	651407	682210.5	30803.5	1.26354	1.26371	0.00017
57	823238	862112.1	38874.1	1.26378	1.26371	0.00008
58	1040490	1089456.0	48966.0	1.26390	1.26370	0.00019
59	1315036	1376750.2	61714.2	1.26386	1.26371	0.00016

Appendix L

$(x, y, z) = (1/3 - 0.0002, 1/3 + 0.0002, 1/3)$. The distribution of $P_4^{(i)} * X_b$.

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
0	0	100	0	0	0	0	100	0
1	0	0	0	100	0	0	100	0
2	16.6567	50	16.6767	0	16.6667	0	100	33.33333
3	0	16.65667	0	66.66667	0	16.67667	100	33.33333
4	13.3253	40	13.3413	9.994	13.3333	10.006	100	36.65867
5	3.02683	21.20055	3.03047	48.48749	3.02865	21.22601	100	28.77340
6	10.8782	34.6927	10.8913	16.31733	10.8848	16.3357	100	22.45511
7	4.67329	23.38048	4.6789	39.18372	4.6761	23.40751	100	9.95285
8	9.37854	31.51016	9.3898	20.15795	9.38417	20.17938	100	15.61583
9	5.58871	24.58139	5.59542	34.03381	5.59207	24.60859	100	14.40793
10	8.45891	29.56159	8.46907	22.51182	8.46399	22.53463	100	8.45891
11	6.12140	25.27979	6.12875	31.03829	6.12508	25.30669	100	10.53792
12	7.89074	28.35789	7.90021	23.96617	7.89548	23.98951	100	5.17716
13	6.44041	25.69807	6.44814	29.24456	6.44428	25.72454	100	7.35269
14	7.53791	27.61043	7.54696	24.86940	7.54244	24.89287	100	2.94513
15	6.63479	25.95296	6.64276	28.15170	6.63878	25.97901	100	4.36207
16	7.31810	27.14478	7.32689	25.43216	7.32249	25.45559	100	1.30259
17	6.75449	26.10994	6.76260	27.47882	6.75854	26.13562	100	2.59188
18	7.18089	26.85411	7.18951	25.78348	7.18520	25.80681	100	1.69069
19	6.82867	26.20723	6.83687	27.06183	6.83277	26.23263	100	2.08595
20	7.09513	26.67245	7.10365	26.00309	7.09939	26.02631	100	2.20854
21	6.87483	26.26778	6.88308	26.80238	6.87896	26.29297	100	1.10652
22	7.04148	26.55882	7.04994	26.14047	7.04571	26.16359	100	1.75514
23	6.90362	26.30556	6.91191	26.64056	6.90777	26.33059	100	1.21845
24	7.00791	26.48771	7.01633	26.22645	7.01212	26.24949	100	1.54764
25	6.92161	26.32916	6.92992	26.53947	6.92577	26.35408	100	0.88730
26	6.98690	26.4432	6.99529	26.28028	6.99109	26.30325	100	0.76691
27	6.93286	26.34392	6.94119	26.47625	6.93702	26.36876	100	0.44003
28	6.97374	26.41533	6.98211	26.31399	6.97792	26.33691	100	0.68946

Level	$W_i(A)$	$W_i(B)$	$W_i(C)$	$W_i(D)$	$W_i(E)$	$W_i(F)$	$W_i(N)$	$Y_i(M)$
29	6.93990	26.35316	6.94823	26.43670	6.94407	26.37794	100	0.88822
30	6.96550	26.39787	6.97386	26.33509	6.96968	26.35799	100	0.58956
31	6.94431	26.35894	6.95265	26.41194	6.94848	26.38368	100	0.54712
32	6.96034	26.38695	6.96870	26.34832	6.96452	26.37119	100	0.71077
33	6.94707	26.36256	6.95541	26.39645	6.95124	26.38728	100	0.34461
34	6.95711	26.38010	6.96546	26.35660	6.96128	26.37945	100	0.47563
35	6.94879	26.36483	6.95714	26.38674	6.95297	26.38953	100	0.43635
36	6.95508	26.37582	6.96343	26.36178	6.95926	26.38463	100	0.41804
37	6.94988	26.36625	6.95822	26.38066	6.95405	26.39094	100	0.18068
38	6.95382	26.37313	6.96217	26.36503	6.95799	26.38787	100	0.21796
39	6.95055	26.36714	6.95890	26.37686	6.95473	26.39182	100	0.12082
40	6.95302	26.37145	6.96137	26.36707	6.95720	26.38990	100	0.28577
41	6.95098	26.36770	6.95933	26.37447	6.95515	26.39237	100	0.16922
42	6.95252	26.37040	6.96087	26.36834	6.95670	26.39117	100	0.08576
43	6.95125	26.36805	6.95959	26.37298	6.95542	26.39272	100	0.16507
44	6.95221	26.36974	6.96056	26.36914	6.95639	26.39196	100	0.12313
45	6.95141	26.36827	6.95976	26.37204	6.95559	26.39293	100	0.15230
46	6.95202	26.36933	6.96037	26.36964	6.95619	26.39246	100	0.09231
47	6.95152	26.36840	6.95986	26.37146	6.95569	26.39307	100	0.07801
48	6.95190	26.36907	6.96024	26.36995	6.95607	26.39277	100	0.01893
49	6.95158	26.36849	6.95993	26.37109	6.95575	26.39315	100	0.06182
50	6.95182	26.36891	6.96017	26.37015	6.95599	26.39297	100	0.04515
51	6.95162	26.36854	6.95997	26.37086	6.95580	26.39321	100	0.04918
52	6.95177	26.36880	6.96012	26.37027	6.95594	26.39309	100	0.04364
53	6.95165	26.36858	6.95999	26.37072	6.95582	26.39324	100	0.04718
54	6.95174	26.36874	6.96009	26.37035	6.95591	26.39317	100	0.03327
55	6.95166	26.36860	6.96001	26.37063	6.95584	26.39326	100	0.01897
56	6.95172	26.36870	6.96007	26.37040	6.95590	26.39321	100	0.04059
57	6.95167	26.36861	6.96002	26.37057	6.95585	26.39327	100	0.02957
58	6.95171	26.36868	6.96006	26.37043	6.95588	26.39324	100	0.04980
59	6.95168	26.36862	6.96003	26.37054	6.95585	26.39328	100	0.01408