國 立 交 通 大 學

統計學研究所

碩 士 論 文

反向抵押貸款定價:
使用 GARCH 模型和 NIG 分配
Pricing Reverse Mortgage Using GARCH Models and the Normal Inverse Gaussian Distribution

研 究 生: 朱鈺錦
指導教授: 楊曉文 博士
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中華民國九十九年六月

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摘　　要

全球人口已經邁入高齡化社會，老年人口比例逐年攀升之已經是不可擋的趨勢，不少退休的老人無子女奉養，身邊只有房子而無錢過生活。因此在英美等國市場上近幾年發展出一種為老年人退休生活融資的工具—反向抵押貸款(Reverse Mortgage)。反向抵押貸款是指房屋所有權人達到一定年齡以後，將房屋抵押給銀行或保險公司而仍保有房屋之所有權，再由銀行或保險公司支付一定金額的養老金給申請人，直到申請人去世，其抵押的房屋就歸銀行或保險公司所有，而在貸款期間，並不需償還任何金錢予貸款人，也就是一種「以房養老」的觀念。反向抵押貸款的貸款價值取決於借款人的年齡，預期貸款利率，和房屋的價值。房屋價值是影響借貸總金額的最大因素，我們所使用的美國房價資料顯示房價的報酬率並非常態且有波動聚集的現象。在本文中，我們將利用 GARCH 模型和 Normal Inverse Gaussian 分配建構房價的隨機過程，並經由 Duan(1995)所提出的局部風險中立評價法(LRNVR)和 Conditional Esscher Transform 轉換到風險中立測度的 NIG-GARCH 模型，最後再利用 Lee-Carter(1992)隨機死亡率模型計算反向抵押貸款之公平保費和最大可貸乘數。

關鍵字:反向抵押貸款、非常態、NIG-GARCH、LRNVR、 Esscher Transform

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ABSTRACT

Today Life expectancy is getting longer globally. How to fund the retirement asset has become an important issue for the individual. To solve the problem of cash poor and equity rich for older persons, reverse mortgage has been introduced in developing countries such the U.S., the U.K. and Australia. Reverse mortgage is a new financial product that allows retirees to convert a proportion of the equity in their home into cash until they die. The loan value is determined by the borrower's age, the interest rate, and the home's value. The earlier models also assume house price returns follow normal distribution. Unfortunately, house price returns in our study are potentially non-normal. In this paper, we want to construct the house price model via GARCH with Normal Inverse Gaussian distribution (NIG-GARCH) option pricing model via local risk-neutral valuation relationship (LRNVR), conditional Esscher transform and mortality rates follows Lee-Carter (1992) model.

Keyword : reverse mortgage、NIG-GARCH、non-normal、LRNVR、Esscher transform

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朱 鈺 錦　謹至于

國 立 交 通 大 學 統 計 學 研 究 所

中 華 民 國 九十九 年 六 月

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1 Introduction

Today in many countries around the globe, life expectancy from birth is well over 80 years. In 1989, after the Department of Housing and Urban Development (HUD) introduced the Home Equity Conversion Mortgage (HECM) program, reverse mortgages became widely available in the United States. A reverse mortgage is a new financial product that allows retirees to convert a proportion of the equity in their home into cash until they die. Homeowners 62 and older who have paid off their mortgages or have only small mortgage balances remaining are eligible to participate in HUD's reverse mortgage program.

A reverse mortgage is a loan against the equity in your home that you don't need to pay back for as long as you live in the home. Thus, the reverse mortgage program enables seniors that may be "real estate rich and cash poor" to unlock the financial potential in their homes, and their homes work for them. In general, the reverse mortgage does not become payable until the senior homeowner no longer occupies the property as his or her primary residence. Homeowners can receive payments in a lump sum, annuity income, on a monthly basis (for a fixed term or for as long as they live in the home), or on an occasional basis as a line of credit. Homeowners whose circumstances change can restructure their payment options. The size of reverse mortgage loans is determined by the borrower's age, the interest rate, and the home's value.

In order to protect lenders of reverse mortgages from possible losses, the Federal Housing Administration (FHA), which is one part of HUD's Office of Housing, which is charges insurances premiums from borrowers, and pays insurance claims to lenders if the loan balance exceeds the home equity value. That is, according to the actuarial equivalence principle, the present value of expected premiums should be equal to the present value of expected losses.

Over the past two decades, most of the existing literature on risk modeling in the HECM program for pricing insurance premium. Szymanoski (1994) provided the method of building actuarial model of HECM. Tse (1995b) incorporate the different risks when modeling interest rates via Cox-Ingersoll-Ross (1985) model. Rodda, Lam, and Youn (2004) analyzed the HECM program using stochastic models of interest rates and housing prices and a new model of termination rates. Chia and Tsui (2004) conducted research of reverse mortgages for Singaporeans. Ma, Kim and Lew (2006; 2007) using stochastic models to estimating reverse mortgage insurer's risk and confirmed that the present values of expected losses were very sensitive to the processes of housing prices and interest rates.

However, the loan balance depends not only on the age and sex of the homeowners, but also on the appraised value of the property, the projected rate of house price appreciation and the levels of interest rates. The most important factor in the size of reverse mortgage loan is the value of home. As a result, it's important to understand how the fluctuation of home prices affect reverse mortgage financing, so that borrowers can make an optimal decision. Empirical work by Kutty (1998) indicates that the use of home equity conversion mortgage products could possibly raise about 29% of the poor elderly homeowners in the US above the poverty line. In order to model the house price risk, Szymanoski (1994), Ma, et al. (2007) and Valdez, et al. (2007) assume house prices are driven by a geometric Brownian motion. However, the dynamics of house prices, like those of any asset, are vital to understand for proper risk and portfolio management. Volatility is a key aspect of such dynamics. Crawford and Fratantoni (2003), Dolde and Tirtiroglu (1997), Miller and Peng (2006) have investigated whether house price volatility is time-varying; that is, house prices exhibit the volatility clustering or GARCH (generalized autoregressive conditional heteroskedasticity) effects, such as those for stocks and bonds. Chen, et al. (2010) propose a generalized Lee-Carter model with permanent jump effects to fit the actually mortality data, and model the house price index via an ARMA-GARCH (Autoregressive moving average - generalized autoregressive conditional heteroskedasticity) process and employ the conditional Esscher transform to price the non-recourse provision of reverse mortgages. Therefore, we follow Chen, et al. (2010) works by using an ARMA-GARCH model to fit house price returns.

A major difference between the cash flows of a traditional home purchase, or forward, mortgage and a reverse mortgage is in the pattern of equity and debt over time. Reverse mortgages loan balance grows due to principal advances, interest accruals, and other loan charges over the life of the loan. The key risk factors affecting the cash flows and pricing of HECM insurance, are (1) borrower mortality rates and voluntary loan terminations, which determine the timing of lump sum or other types repayments; If a borrower lives a longer time than the expected lifespan that may lead the loan balance above the sale proceed of the property. In other words, lenders of reverse mortgages are faced with longevity risk. (2) Interest rate changes, which affect the rate at which the debt rises; Reverse mortgages can generally be categorized as either fixed-rate or variable-rate. Fixed-rate reverse mortgages accrue interest at the same (fixed) rate for their entire duration, whereas the rate associated with variable rate mortgages rises and falls in accordance with a stated benchmark rate, such as the 1 year T-bill, 1-year LIBOR, and other obscure rate indexes. The rise of interest rates increases the possibility of non-repayment when the loan eventually terminates. In this study, we choose a fixed interest rate with a risk adjustment. (3) The future property values, which affects the net proceeds from a sale. If the house price grows at a lower rate than expected, the loan balance may exceed the home value. Lenders may suffer from the losses.

The difference between interest rate risk and house price risk are that the interest rate risk could not be diversified, while the house price risk can be partially diversified by holding a large portfolio of loans across areas. Therefore, we will focus on house price risk.

In general, almost all of previous studies ignore longevity risk. We employ Lee-Carter (1992) model to model the mortality rates for pricing the present value of claim losses and calculating mortgage insurance premiums. Moreover, it is well known that house price returns (quarterly) empirical distributions are closer to the Gaussian case. Unfortunately, house price returns (monthly) in our study are potentially non-Gaussian.

In this paper, we want to construct the house price model via ARMA-GARCH with Normal Inverse Gaussian distribution (ARMA-GARCH-NIG) option pricing model via local risk-neutral valuation relationship (LRNVR) and conditional Esscher transform. Therefore, the proposed method proves to play an important role in pricing reverse mortgage.

The remainder of this paper is organized as follows. In Section 2, we introduce the reverse mortgage pricing framework. Section 0 model the longevity risk via Lee-Carter model in the reverse mortgage. Modeling the house price risk and introduce the normal inverse Gaussian distribution in section 3. In section 5, we introduce the LRNVR and the conditional Esscher transform to in order to valuation. Section 6 use empirical data to show the result of this analysis. The final section concludes this study.

2 Pricing Framework for Reverse Mortgage

This section introduces the discussion of how to price a lump sum reverse mortgage. Chen, et al. (2010) provided a pricing framework and show the HECM loan is a non-recourse debt and Li, et al. (2009) also show the reverse mortgage is a no-negative-equity-guarantee.

2.1 Non-Recourse Provision

In Chen, et al. (2010), when the loan terminates, if the net proceed from the sale of the property is sufficient to pay the outstanding loan balance, the remaining cash usually belongs to the borrower or his/her beneficiaries. If the sale process is not enough to cover the loan balance, the non-recourse provision prevents the lender from pursuing other assets belonging to the borrower, apart from the house.

Denote L_t is the outstanding balance of the loan and H_t is the house price at time t. If the loan is due at time *t*, the borrower pays L_t if $H_t \geq L_t$, and H_t if $H_t < L_t$, under the non-recourse provision. Therefore, we can define the insurance company claim loss function at the time t as follows:

$$V_t = max(L_t - (1-k)H_t, 0) \qquad (2\text{-}1)$$

where k is the transaction cost of percent of the property value. The claim loss function can be expressed as a European exchange options which can changes the loan outstanding balance L_t for the sale property value $(1-k)H_t$. This non-recourse provision can be replicated through a series of European exchange options with different times to maturity. At maturity, the house value is

$$H_t = H_0 e^{\sum_{s=1}^{12t} Y_s} \qquad (2\text{-}2)$$

and the loan balance is

$$L_t = (UP_0 + L_0)e^{ct} \tag{2-3}$$

where H_0 is current house price, Y_s is the monthly log return of house price index, UP_0 is the upfront mortgage insurance premium, L_0 is initial loan amount, and c is a fixed annual contract rate[1] charged on the mortgage loan. Moreover, the symbol (x) is used to denote a life-age-x and the future lifetime of (x) is denoted by $T(x)$. To make probability statements about $T(x)$, we use the notations

$$_tq_x = \Pr[T(x) \le t] \quad t \ge 0, \tag{2-4}$$

$$_tp_x = 1 - {}_tq_x = \Pr[T(x) > t] \quad t \ge 0 \tag{2-5}$$

The symbol $_tq_x$ can be interpreted as the probability that an individual (x) will die within t years. On the other hand, $_tp_x$ can be interpreted as the probability that (x) will attain age $x + t$. If $t = 1$, convention permits us to omit the prefix in the symbol defined in (2-4) and (2-5), and we have

$$q_x = \Pr[(x) \text{ will die within 1 year}]$$

$$p_x = \Pr[(x) \text{ will attain age } x + 1 \text{ year}]$$

Note that if (x) will survive t years and die within the following year, the probability is

$$_tp_x \cdot q_{x+t} = \Pr[t < T(x) \le t + 1] \tag{2-6}$$

Following the Chen, et al. (2010), we also assume all home exits occur in the mid-year and the average delay in time from the point of home exit until the actual sale of the property which we set half-year. Let ω be the highest age, we could determine the present value of total expected claim losses on a cohort group aged x as follows:

$$PVECL = \sum_{t=0}^{\omega-x-1} E_Q\left[{}_tp_x \cdot q_{x+t} \cdot e^{-r(t+1)} V_{t+1} \right] \tag{2-7}$$

where E_Q denotes the expectation under the risk-adjusted measure Q, $_tp_x \cdot q_{x+t}$ is

[1] Total Interest Rate charged to a reverse mortgage is the Margin + Index + Mortgage Insurance of .50%.

defined in (2-6), $e^{-r(t+1)}$ is the discount factor with risk-free interest rate r and V_{t+1} is defined in (2-1).

2.2 Mortgage Insurance Premiums

In order to protect lenders from possible losses if non-repayment occurs, as well as to guarantee borrowers receiving monthly payments if lenders default on the loans, HUD provides mortgage insurance for the HECM program. Two insurance options are available for lenders to choose from: the assignment option and the shared premium option. However, none of the lenders chooses the shared premium option because Fannie Mae does not purchase these loans. With the assignment option, FHA collects all the insurance premiums and the lender is allowed to assign the loan to FHA when the loan balance equals the adjusted property value. FHA takes over the HECM loan and pays an insurance claim to the lender covering her losses. By choosing this option, lenders are effectively shifting the collateral risk to HUD.

The mortgage insurance premiums (MIP) are paid by borrowers and include an upfront mortgage insurance premium (UP_0) and a periodic mortgage insurance premium according to the annual rate of $m\%$ of the outstanding loan balances. Mathematically, the present value of mortgage insurance premium of reverse mortgages can be calculated as

$$PVMIP = UP_0 + \sum_{t=1}^{\omega-x-1} e^{-rt} \ {}_tp_x \left(m\% \times L_t\right) \quad (2\text{-}8)$$

2.3 Loan to Value Ratio

The loan-to-value ratio (LTV ratio) is a common number that is used in calculating mortgage loan eligibility. In fact, the loan to value ratio is one of the most important things that a lender is going to look at before approving borrower for a mortgage. The loan-to-value ratio is a number that lenders use to compare how much the property is worth to the loan value against it. They are going to look at how much money is owed against a property and how much money that property is actually worth. By comparing these two numbers against each other, they can accurately gauge the amount of risk that will be involved with doing a loan. The lower the loan-to-value ratio, the more the lender is going to like the loan. This means that if borrowers need to borrow significantly less money than what a property is worth, they odds of getting approved increase dramatically.

The LTV ratio is calculated by dividing the loan amount of a property by the market value and is expressed as a percentage. Mathematically, the initial loan amount can be expressed

$$L_0 = LTV \times H_0 \tag{ 2-9}$$

Substituting (2-9) into (2-7) and (2-8), we can obtain

$$PVECL = \sum_{t=0}^{\omega-x-1} E_Q[\ {}_tp_x \cdot q_{x+t} \cdot e^{-r(t+1)} \max((UP_0 + LTV \times \\ H0ect+1-(1-k)H0es=112t+1Ys, 0)] \tag{2-10}$$

$$PVMIP = UP_0 + \sum_{t=1}^{\omega-x-1} e^{-rt}\ {}_tp_x\,(m\% \times (UP_0 + LTV \times H_0)e^{ct}) \tag{2-11}$$

Assume that the macro longevity risk is independent from the financial risks and the change of measure from real world measure to risk-neutral measure does not affect the marginal distribution of the remaining lifetime. Applying the actuarial equivalence principle (APV), we could determine the fair LTV ratio via (2-10) = (2-11).

3 Modeling Longevity Dynamics

The earlier HECM models use static mortality tables and therefore fail to capture the dynamics of mortality over time. In addition, they do not model longevity risks. On the other hand, an unexpected mortality improvement will increase the life expectancy, and thereby increase both the term and the amount of the outstanding loan balance. Recently a number of approaches have been developed for forecasting mortality. These methods are taken into account to describe the betterments in the mortality trend and to project survival tables. A recent paper, Cairns, et al. (2007), examined the empirical fits of eight different stochastic mortality models. Note that models M1 to M3 can be described as belonging to the family of generalized Lee-Carter (1992) models and models M5 to M8 can be described as members of the family of generalized CBD (2006) models. Cairns, et al. (2008a), then examined the 'goodness of fit' of the remaining six models by analyzing the statistical properties of their various residual series. Therefore, in this section we incorporate the classic Lee-Carter (1992) model in order to model the longevity and adverse mortality risks more accurately.

Stochastic mortality models either model the central mortality rate or the initial mortality rate (Coughlan, Epstein, Sinha, & Honig, 2007). The central mortality rate $m_{x,t}$ is defined as:

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}} \tag{3-1}$$

where $D_{x,t}$ is the actual total number of deaths at time t, and $E_{x,t}$ is the population in age group x at time t. The initial mortality rate q_x is the probability that a person aged x dies within the next year. The different mortality measures are linked by the following approximation:

$$q_x \approx 1 - e^{-m_{x,t}} \tag{3-2}$$

3.1 The Lee-Carter Model

Lee and Carter (1992) base their model on the insight that age-specific death rates in the United States quite accurately follow a common exponential trend over the last decades, and propose the following parsimonious parameterization:

$$ln\, m_{x,t} = a_x + b_x k_t + \varepsilon_{x,t} \qquad (3\text{-}3)$$

where $m_{x,t}$ is the central death rate at age *x* in year *t*, a_x coefficients describe the average shape of the age profile, and the b_x coefficients describe the pattern of deviations from this age profile when the parameter k_t varies. Note that all variables on the right side of the model are unobservable. Therefore, the Lee-Carter model cannot be fitted by the ordinary least square (OLS) approach. In addition, this model is obviously over-parameterized. Lee and Carter (1992) impose the following normalization conditions to obtain a unique solution:

$$\sum_x b_x = 1 \text{ and } \sum_t k_t = 0 \qquad (3\text{-}4)$$

Then a_x becomes the average value of $ln\, m_{x,t}$ over time, i.e.,

$$a_x = \frac{1}{T}\sum_{t=1}^{T} ln\ m_{x,t} \qquad (3\text{-}5)$$

where *T* is the length of the time series of mortality data. An important aspect of stochastic mortality models is the quality of the fit of the model to historical mortality data. We use the U.S. mortality rate data from the data of 1950 to 2006 and generate yearly mortality rate of age 60 and above to age 99, both male and female. The data can be obtained from human mortality database[2].

[2] Source: http://www.mortality.org/

3.2 Fitting the Lee-Carter model

The estimation is completed in two steps according to Lee and Carter (1992). In the first step singular value decomposition (SVD) of the matrix is used to obtain estimates for b_x, (x = 1, 2,…, ω), and k_t (t = 1, 2,..., T). In the second step the time-series evolution of k_t is recalculated based on the actual number of deaths in year t. Brouhns et al. (2002) described a fitting methodology for the Lee-Carter model based on a Poisson model. The main advantage of this is that it accounts for heteroskedasticity of the mortality data for different ages. Therefore, the number of deaths is modeled using the Poisson model, implying:

$$D_{x,t} \sim Poisson(E_{x,t} m_{x,t}) \quad (3\text{-}6)$$

The parameter set $\emptyset$ is fitted with maximum likelihood estimation, where the log-likelihood function of model (3-6) is given by:

$$L(\emptyset; D, E) = \sum_{x,t} \{D_{x,t} ln[E_{x,t} m_{x,t}(\emptyset)]\} - E_{x,t} m_{x,t}(\emptyset) - ln(D_{x,t}!) \quad (3\text{-}7)$$

We used the R-code of the software package "Lifemetrics" as a basis for fitting (3-7).[3] Therefore, we call the corresponding R function passing in vectors and arrays. The fitting procedure will print in the R console the values of the log likelihood which is being maximized during the fitting process. The result of parameter estimates for the Lee-Carter model is plotted in Figure 3-1 and the fitted death rates for different sex and age group is plotted in Figure 3-2. The a_x coefficients, as noted, are just the average values of the logs of the death rates. Not surprisingly, the male coefficients lie above the female at all ages, reflecting the fact that mortality was higher, on average, from 1950 to

[3] Lifemetrics is an (open source) toolkit for measuring and managing longevity and mortality risk, designed by J.P. Morgan. www.lifemetrics.com

2006. The b_x coefficients describe the relative sensitivity of death rates to variation in the k_t parameter. The male coefficients above the female before age 75, reflecting the fact that the male mortality improvement will increase than female before age 75. On the other hand, the female coefficients above the male after age 75. It can be seen that the younger the age, the greater its sensitivity to variation in the k_t parameters. The exponential rate of change of an age group's mortality is proportional to the b_x values: $dln(m_{x,t}) = b_x(dk_t/dt)$. If k_t declines linearly with time, then dk_t/dt will be constant and each $m_{x,t}$ will decline at its own constant exponential rate.

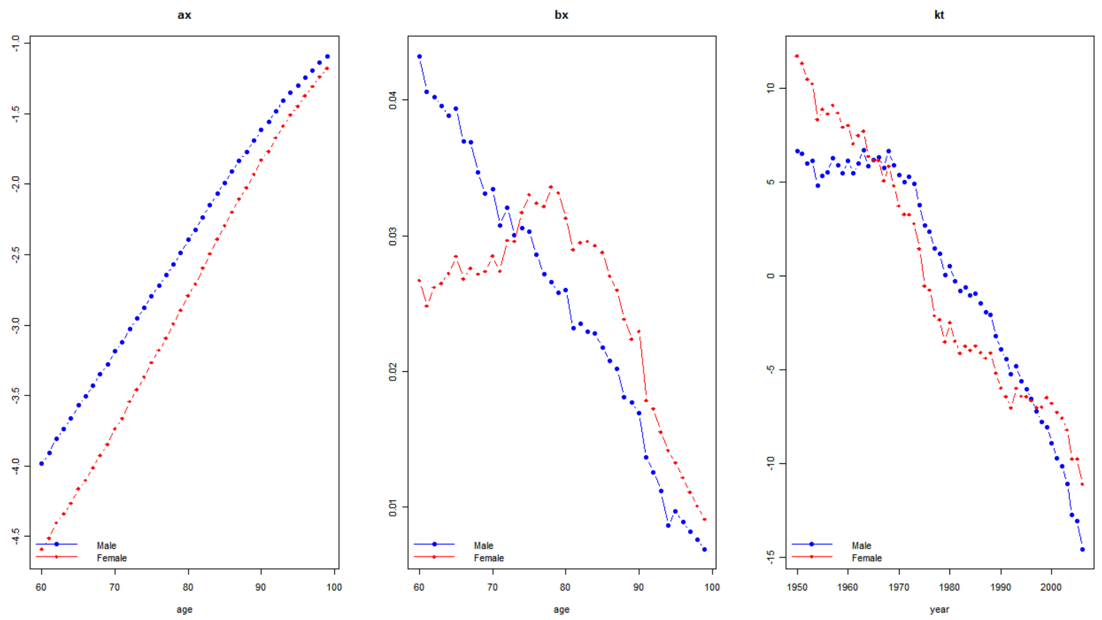

Figure 3-1 Parameters Estimated by Lee-Carter Model

The next step is to model k_t as a stochastic time series process. This is done using standard Box-Jenkins procedures. In most applications so far, k_t is well-modeled as a random walk with drift:

$$k_t = k_{t-1} + d + e_t \quad (3\text{-}8)$$

where d is a constant and e_t is a normally distributed error term with zero mean.

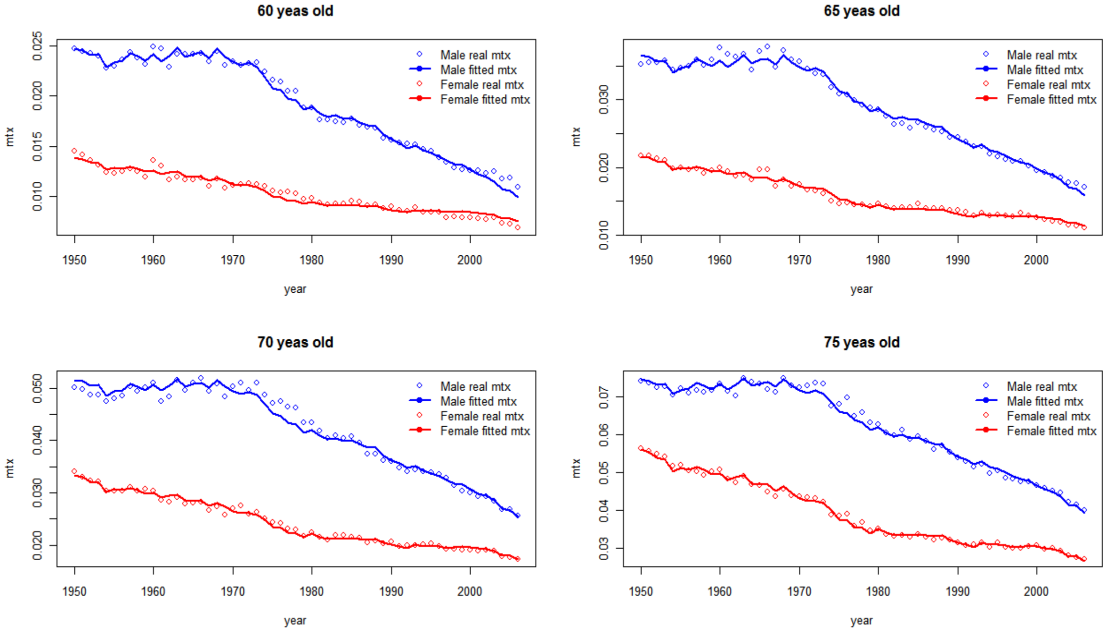

Figure 3-2 Fitted death rates for age group 60, 65, 70 and 75.

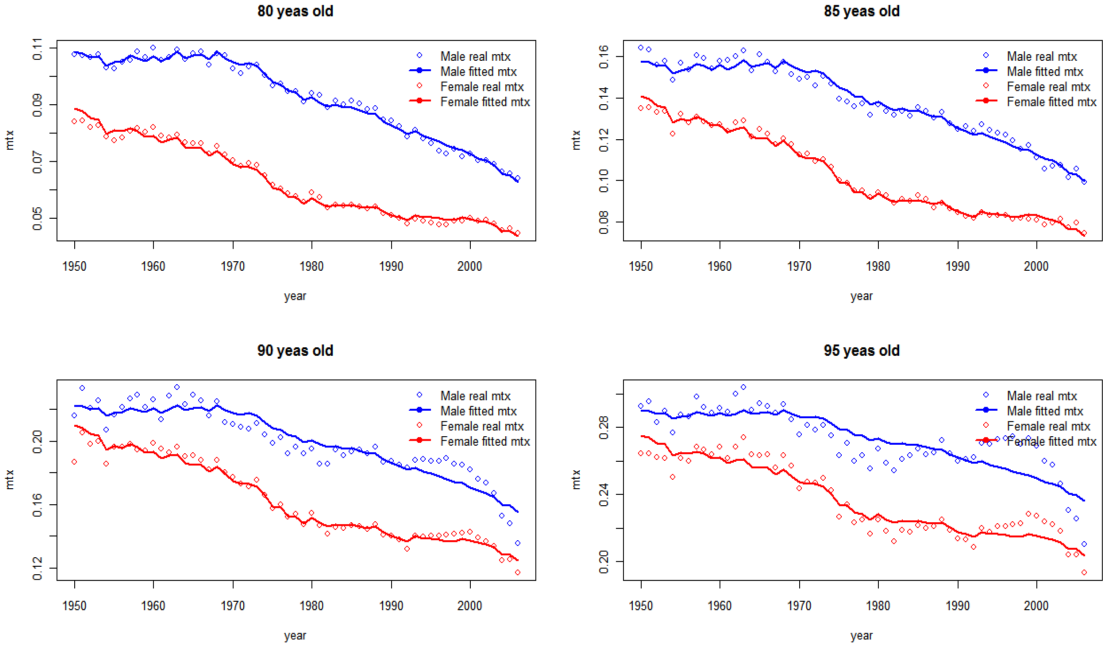

Figure 3-3 Fitted death rates for age group 80, 85, 90 and 95.

4 Modeling Housing Price Dynamics

The future property values which affects the net proceeds from a sale. If the house price grows at a lower rate than expected, the loan balance may exceed the home value. Lenders may suffer from the losses.

4.1 Empirical Investigation for Housing Price Data

The house price data is published by Fannie Mae (Federal National Mortgage Association) and Freddie Mac (Federal Home Loan Mortgage Corporation). The indexes are built repeat-sales weighted averages. This methodology collects data on single-family home re-sales by capturing repeat sale prices to calculate price changes. The repeat sales method was first proposed by Bailey, Muth, and Nourse (1963), and successful applications by Case and Shiller (1989; 1987). Case and Shiller (1987) extended the basic approach by proposing the use of generalized least squares to account for heteroscedastic sampling errors whose variances were assumed to be proportional to the length of time between repeat transactions.

The key difference between FHFA Home Price Index and S&P Case-Shiller house price indices is the S&P Case-Shiller indices are not limited to conforming loans (they include higher-priced homes). Also, they use actual sale prices instead of appraisals when calculating price changes. For this reason, we compare data from the FHFA purchase-only home price index with data from the S&P Case-Shiller Indices. The FHFA monthly House Price Index is calculated using purchase-only houses backed by Fannie Mae or Freddie Mac guaranteed mortgages. We have included data from the FHFA Purchase-Only Home Price Index in Figure 4-1 for relative comparison. In order to compare the Case-Shiller and FHFA data, we rescaled the FHFA purchase-only index to 100 in January 2000, so that both the FHFA and S&P Case-Shiller would have the same scale in this chart. In particular, the Case-Shiller data show a more decline in house prices from their peak that appears to better reflect current housing market conditions.

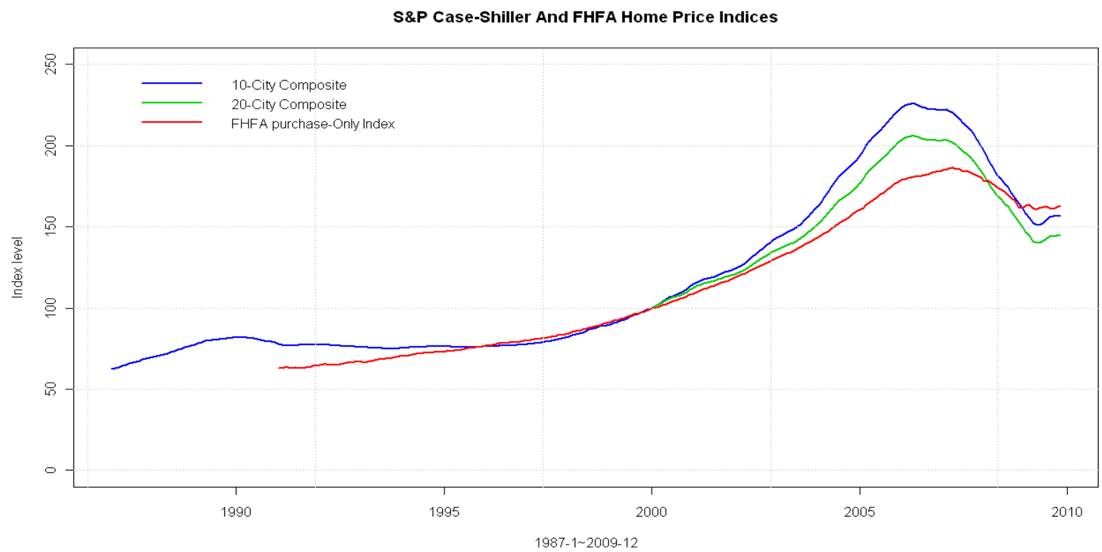

Figure 4-1 S&P Case-Shiller And FHFA Home Price Indices

In the analysis below, we choose the S&P Case-Shiller 10-City Composite Home Price Index[4] (CSXR) rather than use the OFHEO data. In particular, the Case and Shiller data show a more decline in house prices from their peak that appears to better reflect current housing market conditions. As well, the Case and Shiller data are updated more frequently than the OFHEO data and thus give a better picture of the state of the housing market today. The data range from January 1987 through December 2009.

Let H_t denote the monthly house price index. Assume that the one-period log-return Y_t for house price index which is defined as $Y_t = log(H_t/H_{t-1})$. From Figure 4-2 the log return of house price index is obviously not stationary[5]. Unit root tests can be used to determine if trending data should be first differenced on deterministic

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[4] The S&P/Case-Shiller Home Price Indices measures the residential housing market, tracking changes in the value of the residential real estate market in 10 metropolitan regions across the United States. Source: http://www.standardandpoors.com/home/en/us/

[5] The time series $\{Y_t\}$ is said stationary if $E[Y_t] = c$ and $Cov(Y_t, Y_{t-j}) = \gamma_j$ for all t and any j. A non stationary process has time dependent moments, such as deterministic trend terms.

functions of time to render the data stationary. Two standard procedures, the Augmented Dickey-Fuller (ADF) (1984)[6] test and Phillips-Perron (PP) (1988) tests were employed in this study. The results are exhibited in Table 4-3. Both the ADF statistic and the PP statistic in CSXR are higher than the critical values at the significance level of 5%, which means that the log return series are not stationary. Therefore, similar tests were also performed to first difference of CSXR log return ($DY_t = Y_t - Y_{t-1}$). The result shows that the first difference of CSXR log return is stationary.

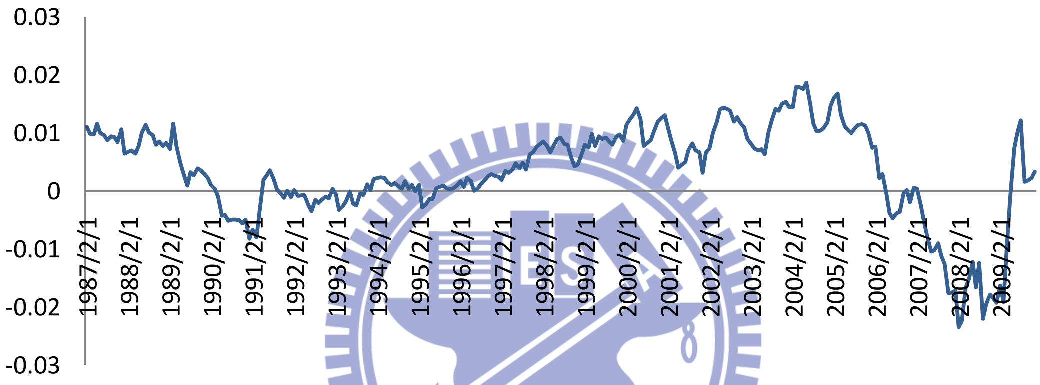

Figure 4-2 Log return of S&P Case-Shiller 10-City Composite Home Price Index

The summary statistics for DY_t are given in the Table 4-1. Over the whole sample we can easy see the kurtosis and skewness of the house price log returns are far from normal and it generally appears to follow a non-normal distribution as the Jarque-Bera[7] statistic indicates.

[6] The testing procedure for the ADF test is $\Delta y_t = \alpha + \beta t + \gamma y_{t-1} + \delta_1 \Delta y_{t-1} + \cdots + \delta_p \Delta y_{t-p} + \varepsilon_t$, where α is a constant, β the coefficient on a time trend and p is the lag order. The unit root test is then carried out under the null hypothesis γ = 0 against the alternative hypothesis of γ < 0. Once a value for the test statistic DF = $\hat{\gamma}/\mathrm{SE}(\hat{\gamma})$ is computed it can be compared to the relevant critical value for the Dickey–Fuller Test. If the test statistic is less (this test is non symmetrical so we do not consider an absolute value) than (a larger negative) the critical value, then the null hypothesis of γ = 0 is rejected and no unit root is present.

[7] In statistics, the Jarque–Bera test is a goodness-of-fit measure of departure from normality, based on the sample kurtosis and skewness. The test statistic JB is defined as $JB = n(S^2 + K^2/4)/6$, where n is the number of observations, S is the sample skewness, and K is the sample kurtosis. The statistic JB has an asymptotic chi-square distribution with two degrees of freedom and can be used to test the null hypothesis that the data are from a normal distribution.

Table 4-1 Descriptive statistics of DY_t

Data	Mean	Sd	Min	Max	Skew	Kurtosis
DY_t	-2.80e-05	0.00219	-0.01067	0.01087	-0.06499	5.64095

Table 4-2 Jarque–Bera test for DY_t

Data	Statistic	Degree of freedom	p-value
DY_t	371.6913	2	< 2.2e-16

Table 4-3 ADF and PP test

Data	Augmented Dickey-Fuller test			Phillips–Perron Unit Root Test		
	stat.	lag	p-value	stat.	lag	p-value
Y_t	-2.1799	6	0.5002	-12.0582	5	0.4339
DY_t	-6.1681	6	< 0.01	-252.0607	5	<0.01

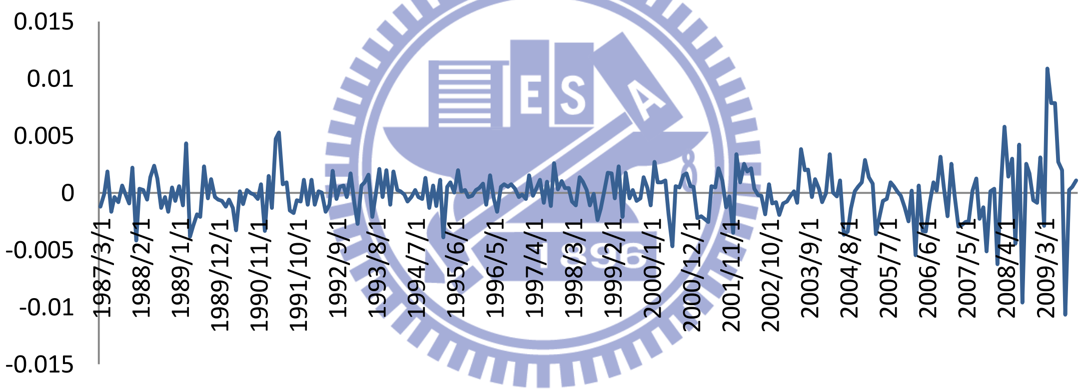

Figure 4-3 First Difference of S&P Case-Shiller Home Price Index Log Return

4.2 ARMA Models with GARCH Errors

Autoregressive moving average (ARMA) models are widely used in all areas of economics and finance, and properties of these models are well understood.

$$DY_t = const + \sum_{i=1}^{m} \phi_i DY_{t-i} + \sum_{j=1}^{n} \theta_j z_{t-j} + z_t \quad (4\text{-}1)$$

where DY_t is the first difference of CSXR log return at time *t*, *const* is the constant, ϕ_i is the *i* th previous log-return of DY_t ,and θ_j is *j* th previous innovation of z_t. Note that $\sum_{i=1}^{m} \phi_i < 1$. The innovations z_{t-j} are assumed to be normally distributed with zero mean and constant variance. We therefore plot the sample autocorrelation coefficients (ACF) and partial autocorrelation coefficients (PACF) of DY_t in Figure 4-4.

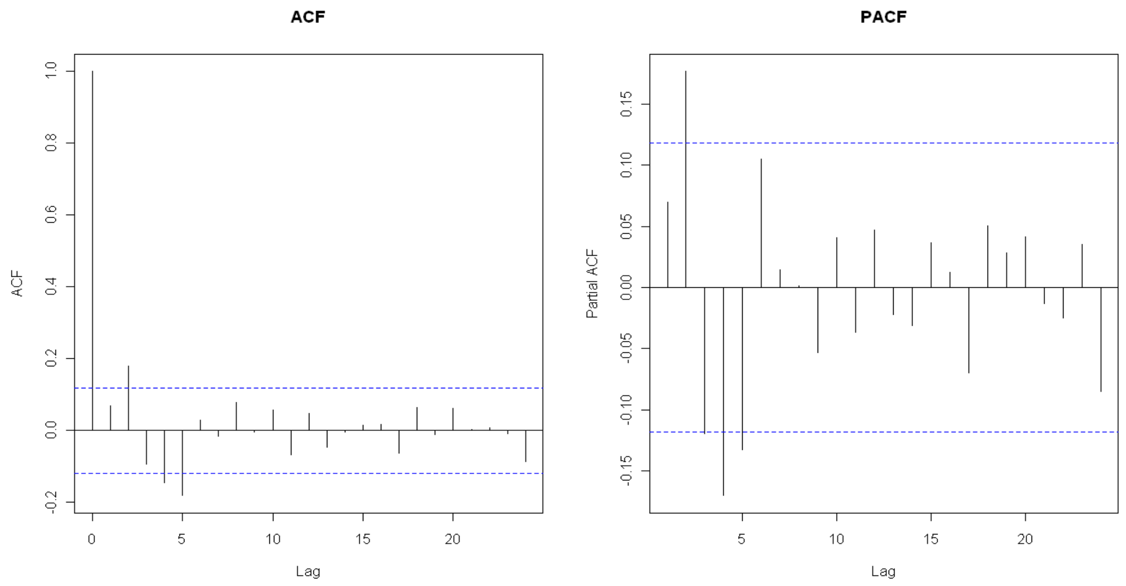

Figure 4-4 ACF and PACF for the First Difference of CSXR log return

The sample ACFs are almost zero after two lag, and the sample PACFs die off after four lags. According to the Box and Jenkins (1976) approach, Akaike (AIC) and Bayesian (BIC) information criterion can be used as a guide for the appropriate lag order selection (See Table 4-4). After several trials, we found that an ARMA (3,2) with zero mean would be appropriate for the (4-4). In addition, the AIC reach the lowest values and the BIC value is the lowest except the ARMA(2,2) .

Table 4-4 Candidate ARMA specification

Model	AIC	BIC
ARIMA(2,0,0) with zero mean	-2583.34	-2572.5
ARIMA(3,0,0) with zero mean	-2585.25	-2570.8
ARIMA(4,0,0) with zero mean	-2592.22	-2574.15
ARIMA(2,0,2) with zero mean	-2593.61	-2575.54
ARIMA(3,0,2) with zero mean	-2596.7	-2575.02
ARIMA(4,0,2) with zero mean	-2595.28	-2569.99

After fitting a candidate ARMA specification, we should check that there are no autocorrelations. Figure 4-5 shows that all the ACFs are within the 95% confidence interval, but the ACFs of the squared innovations are not, which indicates that the existence of AutoRegressive Conditional Heteroskedastic (ARCH) effects.

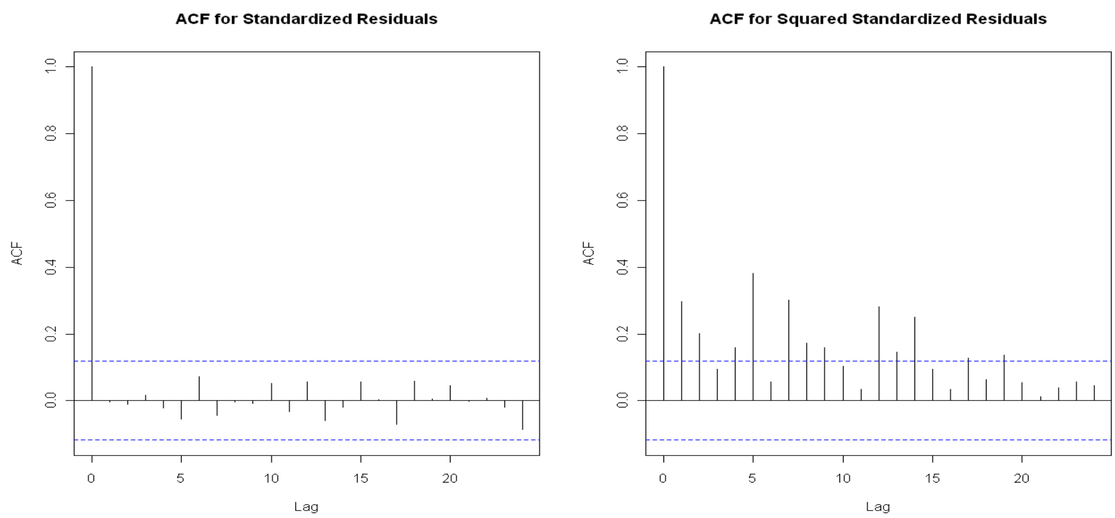

Figure 4-5 ACF for ARIMA(3,0,2) standardized residuals

The linear ARIMA-type models assume homeskedasticity, however, it is not uncommon that many financial time series is still conditionally heteroscedastic after stationary transformation. The AutoRegressive Conditional Heteroskedastic (ARCH) model of Engle (1982) was the first formal model which successful addressed the problem of heteroskedastic. Let z_t denote the error terms and assume that $z_t = \varepsilon_t \sigma_t$, where $\varepsilon_t \sim N(0,1)$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \, z_{t-i}^2 \qquad (4\text{-}2)$$

where $\alpha_0 > 0 \; and \; \alpha_i \geq 0, i > 0$

Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process of an asset return. A more parsimonious model proposed by Bollerslev (1986) called GARCH model. In that case, the GARCH(p, q) model is

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \, z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \, \sigma_{t-j}^2 \tag{4-3}$$

where $\alpha_0 > 0 \;\; \alpha_i, \beta_j \geq 0 \; and \; \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$

The ARCH LM test[8] proposed by Engle (1982) was undertaken to investigate whether there is volatility clustering in the housing price series. It is important to perform a formal LM test prior to employing a GARCH model. If there are no ARCH effects in the residuals, then the ARCH model is unnecessary. The result of LM tests is depicted in Table 4-5.

Table 4-5 ARCH LM test

Data	stat.	lag	p-value
DY_t	126.1215	12	< 2.2e-16

Apparently, positive and statistic significant LM values are observed for both series. This clearly suggests rejecting the null hypothesis of homoskedascity, indicating that volatility clustering effects are evident in these series. The strong evidence of volatility clustering also denotes the appropriateness of employing a GARCH model in analyze the volatility spillover in housing market. Once the ARCH effects are determined, an analysis of the housing volatility determinants is conducted that exhibits volatility clustering.

The differences between the types of models tested here relate to the specification of the mean and variance of the series. ARMA models have both a constant mean and a constant variance; GARCH models have a constant mean, but time-varying variance. Clearly, these models can be combined, for example, ARMA-GARCH models.

8 This procedure is as follows: Estimate the best fitting AR(q) model and obtain the squares of the error $\hat{z}_t^2 = \alpha_0 + \alpha_1 \hat{z}_{t-1}^2 + \cdots + \alpha_p \hat{z}_{t-p}^2$. Under the null hypothesis that there are no ARCH effects: $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$, the test statistic is LM = $T \cdot R^2 \sim \chi^2(p)$ where T is the sample size and R^2 is computed from the regression $\hat{z}_t^2 = \alpha_0 + \alpha_1 \hat{z}_{t-1}^2 + \cdots + \alpha_p \hat{z}_{t-p}^2$ using estimated residuals.

According to previous investigation, we propose the ARMA-GARCH model to price reverse mortgage. In most applications, GARCH(1,1) is enough to capture the ARCH effects. We keep ARMA without the constant term for the conditional mean, and use GARCH(1,1) to model the conditional variance. Under ARMA-GARCH model setting, the dynamics of house price return process is

$$DY_t = c + \sum_{i=1}^{m} \phi_i DY_{t-i} + \sum_{j=1}^{n} \theta_j z_{t-j} + z_t \tag{4-4}$$

$$z_t = \sigma_t \varepsilon_t, \ \ \varepsilon_t \sim D(0,1;\theta_D) \tag{4-5}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \, z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \, \sigma_{t-j}^2 \tag{4-6}$$

where DY_t is the first difference of Y_t and θ_D is an arbitrary distribution of parameters. While the generalized ARMA-GARCH framework can be used with a host of conditional distributions such as the generalized error distribution or Student's t-distribution.

4.3 The Normal Inverse Gaussian distribution

Traditional HECM model assumes that the house price returns empirical distributions are closer to the normal case (like stock market). Unfortunately, house price log return in our study is potentially non-normal. The recently introduced Generalized Hyperbolic distributions by Barndorff-Nielsen (1977) have been suggested as a model for financial price processes. Barndorff (1995), Eberlein and Prause (2000) shows that their exponentially decreasing tails seem to fit the statistical behavior of asset returns. The generalized hyperbolic distribution is a continuous probability distribution defined as the normal variance-mean mixture where the mixing distribution is the generalized inverse Gaussian distribution. Its probability density function is given in terms of modified Bessel function of the third kind. The generalized hyperbolic distribution is well-used in economics, with particular application in the fields of modeling financial markets and risk management, due to its semi-heavy tails.

The family of NIG distributions is a special case of the generalized hyperbolic distributions. Due to their specific characteristics, NIG distributions are very interesting for applications in finance - they are a generally flexible four parameter distribution family that can produce fat tails and skewness, the class is convolution stable under certain conditions and the cumulative distribution function, density and inverse distribution functions can still be computed sufficiently fast.

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

A non-negative random variable X has an Inverse Gaussian (IG) distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function is of the form:

$$f_{IG}(x;\alpha,\beta) = \frac{\alpha}{\sqrt{2\pi}} x^{-3/2} exp\left(-\frac{(\alpha-\beta y)^2}{2\beta x}\right), if\ x>0 \tag{4-7}$$

We can write $X{\sim}I\mathcal{G}(\alpha,\beta)$.

A random variable Y follows a Normal Inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if

$$Y \mid X = x \sim N(\mu + \beta y, y) \tag{4-8}$$

$$X \sim I\mathcal{G}(\delta\gamma, \gamma^2) \text{ with } \gamma = \sqrt{\alpha^2 - \beta^2} \tag{4-9}$$

with parameters satisfying the following conditions: $0 \leq |\beta| < \alpha$ and $\delta > 0$. We then write $Y{\sim}NIG(\alpha,\beta,\mu,\delta)$, where α is tail heaviness parameter, β asymmetry parameter, $\mu$is a location parameter and δ is a scale parameter.

The density of a random variable $Y{\sim}NIG(\alpha,\beta,\mu,\delta)$

$$f_Y(y) = \frac{\alpha}{\pi} e^{(\delta\gamma-\beta\mu)} \cdot \frac{K_1\left(\delta\alpha\sqrt{1+[(y-\mu)/\delta]^2}\right)}{\left(\sqrt{1+[(y-\mu)/\delta]^2}\right)^2} e^{\beta y} \tag{4-10}$$

where $K_1(\omega) = \frac{1}{2}\int_0^{\infty} exp\left(-\frac{1}{2}\omega(t+t^{-1})\right)dt$ is the modified Bessel function of the third kind. While the density function of NIG distribution is quite complicated, its moment generating function has a simple form. The NIG moment generating function $M_Y(z) = E[e^{Yz}]$ is given by

$$M_Y(z) = e^{\mu z+\delta\left(\gamma-\sqrt{\alpha^2-(\beta+y)^2}\right)} \tag{4-11}$$

We use another parameterization for NIG distribution suggested by Lars Forsberg (2002), which is more intuitive in the context of conditional variance modeling and have scale invariant properties for parameters except the scaling and location parameters. Let $\bar{\alpha} = \alpha\delta,\ \bar{\beta} = \beta\delta, \mu = \mu\ and\ \sigma^2 = \delta/\alpha$, then

$$f_Y(y) = \frac{\sqrt{\bar{\alpha}}}{\pi\sqrt{\sigma^2}} e^{\bar{\alpha}\sqrt{1-\bar{\beta}^2}+\bar{\beta}\sqrt{\bar{\alpha}/\sigma^2}(y-\mu)} \cdot \frac{K_1\left(\bar{\alpha}\sqrt{1+(y-\mu)^2/\bar{\alpha}\sigma^2}\right)}{\sqrt{1+(y-\mu)^2/\bar{\alpha}\sigma^2}} \tag{4-12}$$

where $0 \le \bar{\alpha},\ \mu \in \mathbb{R},\ |\bar{\beta}| < 1\ and\ 0 \le\ \sigma^2$. Under this parameterization the moment generating function becomes

$$M_Y(z) = exp\left\{\mu z + \bar{\alpha}\sqrt{1-\bar{\beta}^2} - \sqrt{\bar{\alpha}^2 - \left(\bar{\alpha}\bar{\beta} + \sqrt{\bar{\alpha}\sigma^2}z\right)^2}\right\} \tag{4-13}$$

The first four central moments can be obtained by the cumulant generating function $\ln M_Y(z)$

$$E(Y) = \mu + \frac{\sqrt{\bar{\alpha}\sigma^2}}{1-\bar{\beta}^2}\bar{\beta} \tag{4-14}$$

$$Var(Y) = \frac{\sigma^2}{\left(1-\bar{\beta}^2\right)^{3/2}} \tag{4-15}$$

$$skewness = \frac{3\bar{\beta}}{\sqrt{\bar{\alpha}}\left(1-\bar{\beta}^2\right)^{1/4}} \tag{4-16}$$

$$kurtosis = \frac{3\left(4\bar{\beta}+1\right)}{\sqrt{\bar{\alpha}}\sqrt{1-\bar{\beta}^2}} \tag{4-17}$$

Note that if $\bar{\beta} = 0$, the variance is represented by parameter, σ^2, which might be more intuitive in the context of volatility modeling.

The main properties of the NIG distribution class are the scaling property

$$Y \sim NIG(\alpha, \beta, \mu, \delta) \;\Rightarrow\; cY + d \sim NIG\left(\bar{\alpha}, \bar{\beta}, c\mu + d, c^2\sigma^2\right) \quad (4\text{-}18)$$

and the closure under convolution for independent random variables Y_1 and Y_2

$$Y_1 \sim NIG(\alpha, \beta, \mu_1, \delta_1), Y_2 \sim NIG(\alpha, \beta, \mu_2, \delta_2)$$

$$\Rightarrow\; Y_1 + Y_2 \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2) \quad (4\text{-}19)$$

This means that if we use this parameterization in a conditional variance modeling framework, we can fit the model, standardize the observed returns using the conditional standard deviation, and the parameters of the distribution for the standardized returns will be constant.

From Figure 4-6, we can see that NIG distribution provides a significant improvement with respect to the normal distribution. Note that the fitted NIG distributions provides skewness parameter near close to 0, thus in order to make the model fitting more tractable, an assigned symmetric NIG distribution ($\beta = 0$).

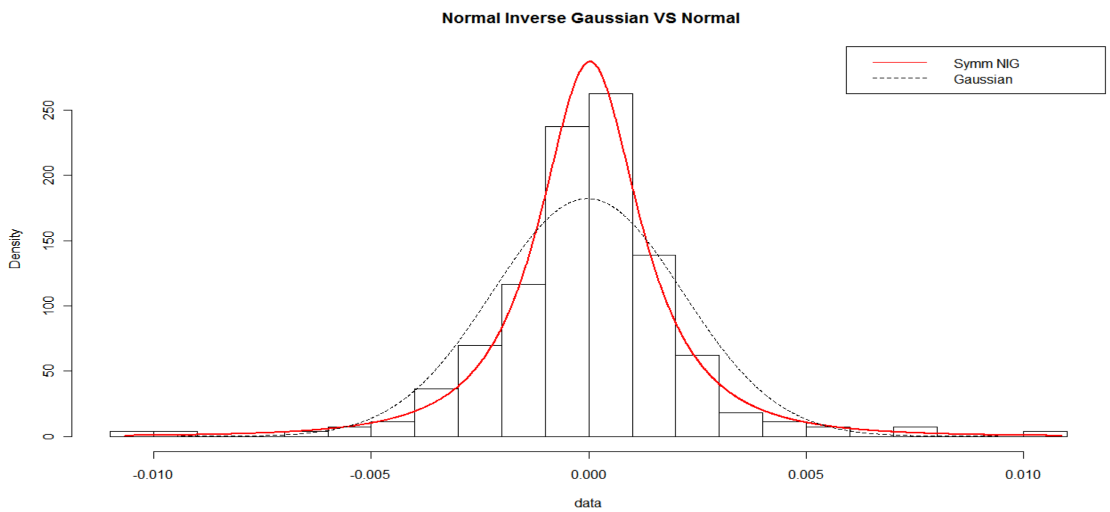

Figure 4-6 Histogram of the first difference of CSXR log return

4.4 ARMA-GARCH Models with NIG Distribution

In section (3.2), we fit DY_t series with a conditional mean model ARMA (3, 2) and a conditional variance model GARCH (1,1); moreover, we suggest that to use NIG distribution rather than use Normal distribution which gives a better fit to our data set in section (3.3). Therefore, the ARMA (3, 2)-GARCH(1,1) model with symmetric NIG distribution is :

$$DY_t = \sum_{i=1}^{3} \phi_i DY_{t-i} + \sum_{j=1}^{2} \theta_j z_{t-j} + z_t \tag{4-20}$$

$$z_t = \sigma_t \varepsilon_t,\ \ \varepsilon_t | \mathcal{F}_{t-1} \sim NIG(\bar{\alpha}, 0,0,1) \tag{4-21}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{4-22}$$

4.5 Parameters Estimation

In order to estimate the ARMA-GARCH model with NIG innovations, we estimate all parameters by two stages. Quasi Maximum Likelihood Estimation (QMLE) is first used to determine the parameters $(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2, \alpha_0, \alpha_1, \beta_1)$ of the GARCH model. At the second stage, since we exactly know the form of the density function of a NIG distribution, we perform use a maximum likelihood estimation to estimate the unknown parameter $\bar{\alpha}$ using the residuals obtained at the previous stage. Note that the parameters must be satisfies following constraints:

$$\bar{\alpha} \geq 0,\ \sum_i \phi_i < 1,\ \alpha_0 > 0,\ \alpha_1 \geq 0,\ \beta_1 \geq 0, \text{ and } \alpha_1 + \beta_1 < 1$$

Parameter estimates and the corresponding statistics are reported in Table 4-6. All the coefficients in both series are highly significant; moreover, Thereafter, we use the likelihood-ratio test (LR test) to test Normal distribution against symmetric Normal Inverse Gaussian distribution, and the NIG distribution is provides a significant improvement with respect to the Normal distribution. We also perform Q-Q Plot in Figure 4-7, and the NIG distribution also shows significant improvement with respect to the Normal distribution. In this empirical work, we suggest that we perform to use NIG distribution rather than use Normal distribution which gives a better fit to our data set. We may thus conclude that the ARMA-GARCH-NIG model provides an important extension to the ARMA-GARCH-Normal model when it comes to option pricing.

Table 4-7 show the Ljung–Box test[9] on the standardized residuals fails to reject the null hypothesis that there is no autocorrelation up to 20 lags. The Ljung–Box test on the squared standardized innovations concluded that conditional heteroskedasticity has disappeared.

Table 4-6 Parameter estimates for the ARMA (3, 2)-GARCH (1, 1)-NIG model

	Estimate	Std. Error	t value	Pr(>|t|)
ϕ_1	0.869986	0.120335	7.229687	4.84E-13
ϕ_2	-0.36639	0.179585	-2.04019	0.041332
ϕ_3	-0.25465	0.07354	-3.46277	0.000535
θ_1	-0.87659	0.117124	-7.4843	7.19E-14
θ_2	0.592712	0.172343	3.439147	0.000584
α_0	2.93E-08	1.46E-08	2.002815	0.045197
α_1	0.072242	0.024443	2.955457	0.003122
β_1	0.927758	0.016519	56.16197	0.000000
$\bar{\alpha}$	1.20118	0.245527	4.892252	9.968E-07

Thereafter, we use the likelihood-ratio test (LR test) to test Normal distribution against symmetric Normal Inverse Gaussian distribution, and the NIG distribution is provides a significant improvement with respect to the Normal distribution. We also perform Q-Q Plot in Figure 4-7, and the NIG distribution also shows significant improvement with respect to the Normal distribution. In this empirical work, we suggest

[9] The Ljung–Box test is a type of statistical test of whether any of a group of autocorrelations of a time series are different from zero. The test statistic is $Q = n(n+2)\sum_{j=1}^{h}\hat{\rho}_j^2/(n-j)$, where n is the sample size, $\hat{\rho}_j$ is the sample autocorrelation at lag j, and h is the number of lags being tested. For significance level α, the critical region for rejection of the hypothesis of randomness is rejected if $Q>\chi^2_{1-\alpha,h}$, where $\chi^2_{1-\alpha,h}$ is the α-quantile of the chi-square distribution with h degrees of freedom.

that we perform to use NIG distribution rather than use Normal distribution which gives a better fit to our data set. We may thus conclude that the ARMA-GARCH-NIG model provides an important extension to the ARMA-GARCH-Normal model when it comes to option pricing.

Table 4-7 Ljung–Box test for the ARMA (3, 2)-GARCH (1, 1)-NIG model

Standardized Residuals			Standardized Squared Residuals		
Lag	statistic	p-value	Lag	statistic	p-value
10	0.3802	1.0000	10	7.102	0.7157
15	5.2511	0.9898	15	12.324	0.6544
20	15.7818	0.7301	20	19.340	0.4999

Table 4-8 LR test for ARMA-GARCH-N model against ARMA-GARCH-NIG model

Statistic	Degree of freedom	p-value
0.001456959	1	0.0003012175

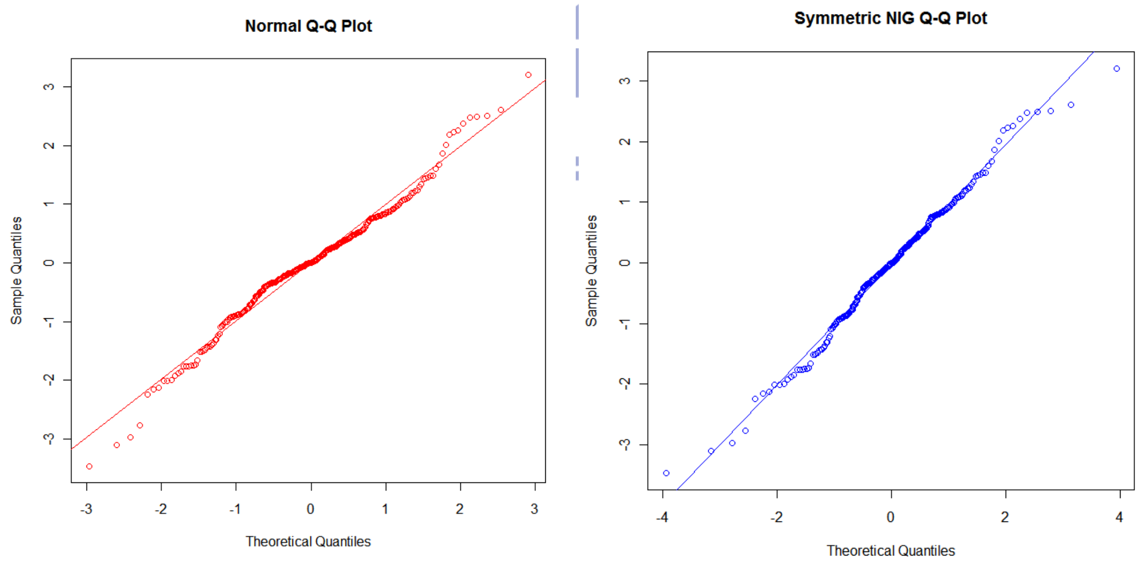

Figure 4-7 Q-Q Plot for the standardized residuals of ARMA-GARCH-type model

5 Risk Neutral Valuation

Recall the reverse mortgage can be writing a series of European exchange options with different times to maturity. For pricing reverse mortgage, we require a risk-neutrality measure Q which is equivalent to the real world measure P, such that the discounted asset price process is a martingale.

5.1 Locally Risk Neutral Valuation Relationship

Duan (1995) provided the first rigorous theoretical foundation for option pricing using this powerful econometric model. He introduced us to the Locally Risk Neutral Valuation Relationship (LRNVR). The generalized LRNVR incorporated the condition that the conditional variances of the log-returns remain unchanged under a change from the real world measure to the risk neutral measure. A pricing measure Q is said to satisfy the LRNVR if measures P and Q are mutually absolutely continuous and measure Q must also satisfy the following requirements: The following equation must hold for all $0 \leq t \leq T$

$$E_Q[H_t/H_{t-1} \,|\mathcal{F}_{t-1}] = e^{r-g} \qquad (5\text{-}1)$$

$$Var_Q[log(H_t/H_{t-1}) \,|\mathcal{F}_{t-1}] \triangleq Var_P[log(H_t/H_{t-1}) \,|\mathcal{F}_{t-1}] \qquad (5\text{-}2)$$

where g is rental yield[10]. Equation (5-2) means conditional variances are unaffected by the change of measure. Under the LRNVR, the one-period ahead conditional variance, is invariant with respect to a change to the risk-neutral measure.

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10 In the risk-neutral world, the expected total return from any asset is the risk-free rate, r. Since the total return on the house price index is the sum of the capital return and rental income (net of insurance and maintenance costs) from the portfolio of properties from which the index is constructed, the expected return on the house price index (i.e., the capital value return) in the risk-neutral world is r less the rental yield g.

5.2 Conditional Esscher Transform

The Esscher transform was introduced by Esscher (1932) and has become a tool in actuarial science. More recently, it has been applied to pricing financial and insurance securities in an incomplete market. Gerber and Shiu (1994) creat an equivalent martingale measure by the Esscher transform is justified by maximizing the expected power utility of an economic agent. Buhlmann, et al. (1996) generalize the Esscher transform to stochastic processes and introduce the concept of the conditional Esscher transform. We start by giving the definition of the conditional Esscher transform with respect to the return process introduced in the last section. Assume $\{\theta_t\}_{t\in\tau\backslash\{0\}}$ is a stochastic process with $\theta_t \in \mathcal{F}_{t-1}$, for all $t \in \tau\backslash\{0\}$, the conditional moment generating function of the return process Y_t evaluated at time t given $\mathcal{F}_{t-1}$ under P is defined as:

$$M^P_{Y_t|\mathcal{F}_{t-1}}(z) := E_P[e^{zY_t}|\mathcal{F}_{t-1}] \tag{5-3}$$

where $z \in R$. Suppose $M^P_{Y_t|\mathcal{F}_{t-1}}(\theta)$ exist, we define a sequence $\{\Lambda_t\}_{t\in\tau}$ with $\Lambda_0 = 1$ and

$$\Lambda_t = \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M^P_{Y_k|\mathcal{F}_{k-1}}(\theta_k)},\ t \in \tau\backslash\{0\} \tag{5-4}$$

Buhlmann, et al. (1996) prove $\{\Lambda_t\}_{t\in\tau}$ is a martingale. Let $P_t = P|\mathcal{F}_{t-1}$, $\forall\, t \in \tau\backslash\{0\}$, and $P_T = P$. We define a family of probability measure $\{P_{t,\Lambda_t}\}_{t\in\tau\backslash\{0\}}$ by the following conditional Esscher transform:

$$P_{t,\Lambda_t}(\mathcal{A}|\mathcal{F}_{t-1}) = E_P\left[I_{\mathcal{A}} \frac{e^{\theta_t Y_t}}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta_t)} |\mathcal{F}_{t-1}\right] \tag{5-5}$$

By the martingale property of $\{\Lambda_t\}_{t\in\tau}$, one can prove that $P_{t,\Lambda_t} = P_{t+1,\Lambda_{t+1}}|\mathcal{F}_{t-1}, \forall\, t \in \tau\backslash\{0\}$. The associated parameter θ_t is called the conditional Esscher parameter given $\mathcal{F}_{t-1}$. Let $F(y;\theta_t|\mathcal{F}_{t-1}) = P_{t,\Lambda_t}(Y_t \le y|\mathcal{F}_{t-1})$, we have

$$F(y;\theta_t|\mathcal{F}_{t-1}) = \frac{\int_{-\infty}^{y} exp(\theta_t y)dF(y)}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta_t)} \tag{5-6}$$

$$M^Q_{Y_t|\mathcal{F}_{t-1}}(z;\theta_t) = \frac{M^P_{Y_t|\mathcal{F}_{t-1}}(z+\theta_t)}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta_t)} \tag{5-7}$$

For pricing an option, we construct a martingale measure Q equivalent to P by adopting Esscher Transforms. First, choose a sequence of conditional Esscher parameters $\{\theta^q_t\}_{t\in\tau\backslash\{0\}}$ by solving the following equation

$$e^{r-g} = M^Q_{Y_t|\mathcal{F}_{t-1}}(1;\theta^q_t), \forall\, t \in \tau\backslash\{0\} \tag{5-8}$$

then we can define a family of probability measure $\{P_{t,\Lambda_t}\}_{t\in\tau\backslash\{0\}}$ associated with $\{\theta^q_t\}_{t\in\tau\backslash\{0\}}$. According to above result, we have

$$P_{t,\Lambda_t} = P_{s,\Lambda_s}|\mathcal{F}_t, s, t \in \tau \; with\; t \le s \tag{5-9}$$

Let $Q = P_{t,\Lambda^q_t}$ and H_t is the house value, we have

$$E_Q[e^{-rt}H_t|\mathcal{F}_{t-1}] = e^{-rt}E_Q[H_t|\mathcal{F}_{t-1}] = e^{-rt}E_P\left[H_{t-1}e^{Y_t}\Lambda^q_t|\mathcal{F}_{t-1}\right]$$

$$= e^{-rt}E_P\left[H_{t-1}e^{Y_t}\frac{e^{\theta^q_t Y_t}}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta^q_t)}|\mathcal{F}_{t-1}\right] = e^{-rt}H_{t-1}\frac{M^P_{Y_t|\mathcal{F}_{t-1}}(1+\theta^q_t)}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta^q_t)}$$

$$= e^{-rt}H_{t-1}M^Q_{Y_t|\mathcal{F}_{t-1}}(1;\theta^q_t) = e^{-rt}H_{t-1}e^{r} = e^{-(r-1)t}H_{t-1}$$

then $e^{-rt}H_t$ is a martingale under Q. Then by risk-neutral pricing formula, the price of the option V at time $t \in T$ is

$$V_t = E_Q\left[e^{-r(T-t)}V_T\right] \tag{5-10}$$

We call Q a conditional risk neutralized Esscher pricing measure. Sheu and Chuang (2006) justify the pricing result justify by solving a dynamic utility maximization problem. Siu, Tong and Yang (2004) employ the conditional Esscher transform to price derivatives, assuming the underlying asset returns follow GARCH processes.

Recall that in Section 4, the discrete time economy we consider is characterized by the time series dynamic of the house price process $ln\left(\frac{H_t}{H_{t-1}}\right) = Y_t$, and we fit $DY_t = Y_t - Y_{t-1}$ using an ARMA (3, 2)-GARCH(1,1) with innovation NIG distribution process, i.e.,

$$DY_t = \sum_{i=1}^{3} \phi_i DY_{t-i} + \sum_{j=1}^{2} \theta_j z_{t-j} + z_t$$

where $z_t|\mathcal{F}_{t-1} \sim NIG(\bar{\alpha}, 0,0, \sigma_t)$ and $\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

Let $\mu_t = \sum_{i=1}^{3} \phi_i DY_{t-i} + \sum_{j=1}^{2} \theta_j z_{t-j}$, then DY_t is NIG distributed with mean μ_t and variance σ_t^2 given the information set $\mathcal{F}_{t-1}$, i.e.

$$DY_t|\mathcal{F}_{t-1} \sim NIG(\bar{\alpha}, 0, \mu_t, \sigma_t^2) \quad (5\text{-}11)$$

Note that μ_t and σ_t^2 are not random given the information $\mathcal{F}_{t-1}$. Recall that $DY_t = Y_t - Y_{t-1}$, we can obtain $Y_t = DY_t + Y_{t-1}$. Consequently,

$$Y_t|\mathcal{F}_{t-1} \sim NIG(\bar{\alpha}, 0, \hat{\mu}_t, \sigma_t^2) \quad (5\text{-}12)$$

under the physical measure P, where $\hat{\mu}_t = \mu_t + Y_{t-1}$

Under the risk-adjusted measure Q, we can use the condition Esscher transform to obtain

$$Y_t|\mathcal{F}_{t-1} \sim NIG\left(\bar{\alpha}, \sqrt{\sigma_t^2/\bar{\alpha}}\theta_t^q, \hat{\mu}_t, \sigma_t^2\right) \quad (5\text{-}13)$$

where θ_t^q is Esscher parameters. (See Appendix A)

Notice that the Esscher transform moves the skewness parameter by the factor $\sqrt{\sigma_t^2/\bar{\alpha}}\theta_t^q$ while keeping the other parameters constant.

On the other hand, the logarithm of house price return under the ARMA (3, 2)-GARCH(1,1) with innovation normal distribution under the real world measure P is

$$Y_t \mid \mathcal{F}_{t-1} \sim N(\hat{\mu}_t, \sigma_t^2) \qquad (5\text{-}14)$$

Therefore, under the risk-neutral measure Q, we can derive the risk-neutral house price return process as:

$$Y_t \mid \mathcal{F}_{t-1} \sim N(r - g - \tfrac{1}{2}\sigma_t^2, \sigma_t^2) \qquad (5\text{-}15)$$

This property states exactly that the dynamics of Y_t under the risk-adjusted measure is the same as that under the physical measure, except that the mean is shifted by an amount of $-\hat{\mu}_t + r - g - \frac{1}{2}\sigma_t^2$.

6 Numerical Illustrations

Recall that the present value of total expected claim losses can be expressed as

$$PVECL = \sum_{t=0}^{\omega-x-1} {}_tp_x \cdot q_{x+t} \cdot E_Q\left[e^{-r(t+1)}V_{t+1}\right] \quad (6\text{-}1)$$

where

$$V_{t+1} = max(L_{t+1} - (1-k)H_{t+1}, 0) \quad (6\text{-}2)$$

$$H_{t+1} = H_0 e^{\sum_{s=1}^{12(t+1)} Y_s} \quad (6\text{-}3)$$

$$L_{t+1} = (UP_0 + L_0)e^{c(t+1)} \quad (6\text{-}4)$$

We use the following baseline assumptions to illustrate the present value of total expected claim losses:

- The initial house value in here is \$300,000, i.e., $H_0 = \$300{,}000$.
- The risk-free interest rate is 3.78%. It is the 10 year U.S. Treasury rate[11], which is 3.85% per annum at 01/02/2010, i.e., $r = 3.78\%$, compounded continuously.
- We choose one year constant maturity Treasury rate[12] of 0.45% plus lender margin 1.15%[13] plus mortgage insurance premium 0.5%, which is equal to 2.45% per annum, i.e., $c = 2.42\%$, compounded continuously.
- Upfront mortgage insurance premium is 2% of house price, i.e., $UP_0 = 0.02H_0$
- The transaction cost of selling the house is 6 percent, i.e. $k = 5\%$
- The rental yield is 2% per annum, compounded continuously, i.e., $g = 2\%$.
- The borrower highest attained age $\omega = 100$.

Since the primary goal of this study is to examine the effect on house prices when the underlying distribution is skewed and leptokurtic, we will report the pricing results in our

[11] Source: https://www.federalreserve.gov/releases/h15/update/
[12] Source: https://www.federalreserve.gov/releases/h15/update/
[13] Assume lender's margin is 150bp.

model, ARMA-GARCH-NIG, compared to that of the ARMA-GARCH-Normal model and the Geometric Brownian Motion (GBM) models. Recall the reverse mortgage can be writing a series of European exchange options with different times to maturity. When pricing (European) options in a GARCH-style framework, it is common practice to rely upon simulation-based valuation approaches. This is due to the fact that, on the one hand, the final risk-neutral distribution is not known in closed form, but, on the other hand, the discrete nature of the GARCH framework makes simulation based approaches straightforward to implement.

We first simulate the DY_t series for 10,000 paths and transform DY_t to Y_t on each path[14]. By applying the pricing framework discussed above, we change the probability measure from P to Q on each path of Y_t, and then calculate the value of the house value H_t. According to the housing price index, we can also simulate the H_t series based on the GBM model with annual volatility $\sigma = 0.0282$. The simulation results are shown in Table 6-1. The simulation result shows the GBM assumption yields lowest housing price. Moreover, the ARMA-GARCH-N model has lower housing price than ARMA-GARCH-NIG model. This enables us to calculate an estimate of the $E_Q\left[e^{-r(t+1)}V_{t+1}\right]$.

Table 6-1 Simulation of housing price by different model with 95% confidence interval

Time	GBM	ARMA-GARCH-N	ARMA-GARCH-NIG
10 years	\$358,195 (\$309,089, \$412,943)	\$360,592 (\$331,463, \$394,077)	\$361,995 (\$333,836, \$398,414)
20 years	\$428,169 (\$345,564, \$522,669)	\$435,415 (\$386,535, 501,041)	\$438,856 (\$390,797, \$512,439)
30 years	\$511662 (\$390,628, \$653,531)	\$526,734 (\$454,700, \$631,395)	\$533,095 (\$459,856, \$660,094)
40 years	\$611,664 (\$449,382, \$ 804,782)	\$638,169 (\$536,216, \$808,342)	\$648,416 (\$543,617, \$852,714)

[14] In ARMA-GARCH-NIG model, we need to compute θ_t^q by solving (A-1).

Note that the $E_Q\left[e^{-r(t+1)}V_{t+1}\right]$ values based on the GBM assumption with Black-Scholes formulas:

$$L_0 e^{(c-r)(t+1)}N(-d_2) - H_0 e^{-g(t+1)}N(-d_1) \qquad (6\text{-}5)$$

where

$$d_1 = \left[ln(H_0/L_0) + \left(r - g - c + \frac{\sigma^2}{2}\right)(t+1)\right]/\sigma(t+1)$$

and $d_2 = d_1 - \sigma\sqrt{(t+1)}$.

In order to obtain ${}_tp_x \cdot q_{x+t}$, we fitted the Lee-Carter model in section 3.2, and now use the fitted time series model for k_t to forecast it over the desired time period. Figure 6-1 shows past values of k_t for the U.S. from 1950 to 2006 and their forecasts from 2007 to 2050. Note that the estimated values of k_t over the base period change in a linear fashion. The approximate linearity of k_t in the base period is a great advantage from the point of view of forecasting. Long-term extrapolation is always a hazardous undertaking, but it is less so when supported in this way by the regularity of change in a ninety-year empirical series.

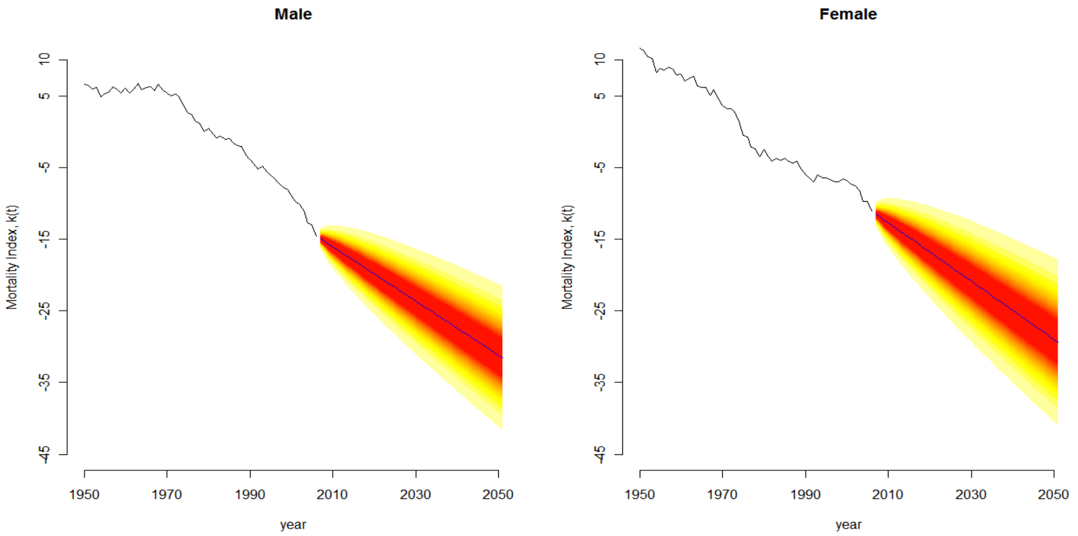

Figure 6-1 Mortality Forecast from 1950–2006 to 2050 with 95% Probability Interval

The next step is to convert the forecasts of k_t into forecasts of life table functions, given the previously estimated age specific coefficients a_x and b_x, using the earlier equation for $\ln m_{x,t}$. Once the implied forecasts of $m_{x,t}$ have been recovered in this way, any desired life table function can be calculated. We then construct the dynamic complete life table for every year. That is, we calculate q_{x+t} the probability that an individual aged $x+t$ will die in one year at time t, and ${}_tp_x$, the probability that an individual aged x will attain another t years at time 0.

Finally, the present value of total expected claim losses can be calculated as follows:

$$PVECL = \sum_{t=0}^{\omega-x-1} \frac{1}{M} \sum_{j=1}^{M} {}_tp_x(j) \cdot q_{x+t}(j) \cdot e^{-r(t+1)} max(L_{t+1} - 1-kHt+1(j),0) \quad (6\text{-}6)$$

where M represents the total number of paths which we set 10,000.

In addition, the present value of mortgage insurance premium can be calculated as follows

$$PVMIP = UP_0 + \sum_{t=1}^{\omega-x-1} \frac{1}{M} \sum_{j=1}^{M} e^{-rt} \; {}_tp_x(j)(m\% \times L_t) \quad (6\text{-}7)$$

where $m\%$ is the annual periodic mortgage insurance premium of loan balance, which we set 0.5%.

In theoretical, the present value of total mortgage insurance premiums should be equal to the present value of total loss so that we can obtain the fair loan-to-value (LTV) ratio by bisection method. Table 6-2 presents simulation results using different models.

On the basis of the ARMA-GARCH-NIG assumption, the fair LTV ratio ranges from 75.36 percent to 80.13 percent of the cash advanced, depending on the gender and age of the borrower, and the fair LTV ratio was slightly higher than ARMA-GARCH-N model. For all genders and ages, the GBM assumption yields lower fair LTV ratio, indicating that we may underestimate the fair LTV ratio if the classical GBM approached is used.

Table 6-2 Fair Loan to Value Ratio

Model	sex	62	65	70	75	80	85
GBM	Male	60.02%	61.17%	63.14%	65.32%	68.34%	71.54%
	Female	59.81%	60.88%	62.69%	64.63%	67.53%	70.89%
GARCH-	Male	71.91%	72.67%	73.97%	75.61%	77.12%	78.62%
Normal	Female	72.03%	72.62%	73.68%	75.32%	76.93%	78.53%
GARCH-	Male	75.36%	75.66%	76.24%	77.07%	78.36%	80.13%
NIG	Female	75.87%	76.03%	76.37%	76.97%	78.12%	79.94%

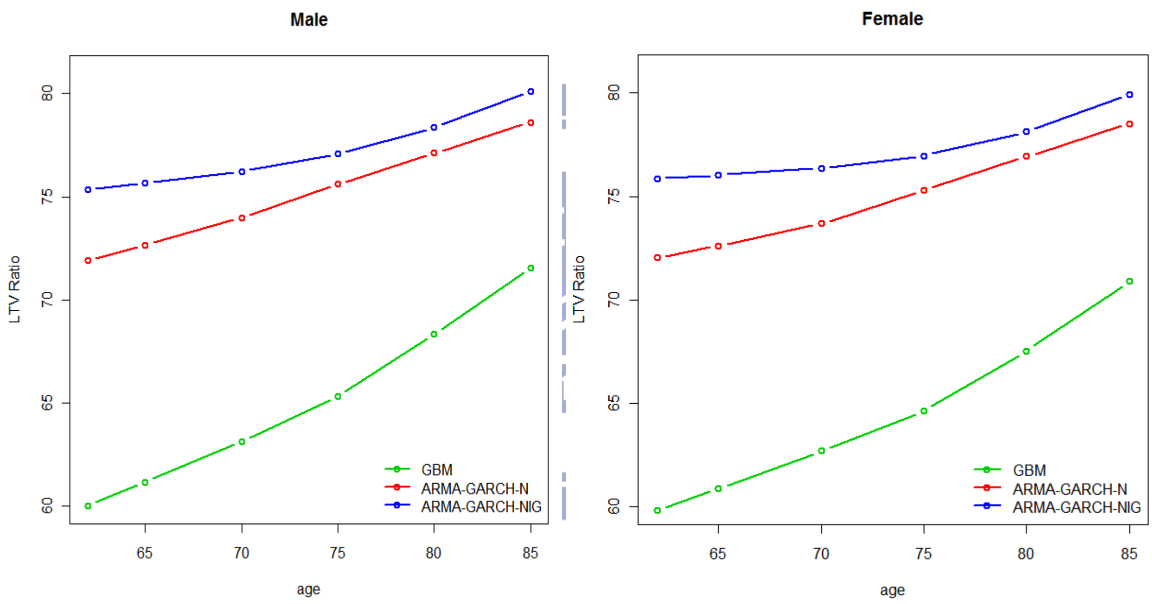

Figure 6-2 Fair Loan-to-Value Ratio

Moreover, the fair LTV ratio increases as the age at loan origination goes up. This is reasonable because other conditions equal, the risk of reverse mortgages ultimately depends on when the termination event occurs and for how long the loan has been accruing. The elder borrower has a shorter life expectancy, so the lender faces less risk and can advance more cash amounts to the borrower. The next step is to calculate the mortgage insurance premiums according to Table 6-2, and the results are show in Table 6-3.

Table 6-3 Fair Mortgage Insurance Premiums

Model	sex	62	65	70	75	80	85
GBM	Male	$22,581	$21,071	$18,561	$16,177	$14,016	$12,072
	Female	$24,129	$22,687	$20,182	$17,641	$15,204	$12,929
GARCH-Normal	Male	$25,759	$23,815	$20,649	$17,732	$15,017	$12,657
	Female	$27,714	$25,803	$22,589	$19,508	$16,447	$13,655
GARCH-NIG	Male	$26,681	$24,528	$21,087	$17,953	$15,157	$12,781
	Female	$28,838	$26,707	$23,180	$19,796	$16,606	$13,789

At the same age group and the same gender, the fair mortgage insurance premiums increase when LTV ratio increase. Although the fair LTV ratio of male slightly higher than female, the mortgage insurance premiums female significantly higher than male in all assumption housing price models. The male borrower has a shorter life expectancy, so the lender faces less risk and can advance charge less premiums to the borrower. For the same reason, the elder can be charged less mortgage insurance premiums. Table 6-4 shows the above comments given LTV ratio = 60% for all ages and genders.

Table 6-4 Mortgage Insurance Premiums given LTV ratio = 60%

sex	62	65	70	75	80	85
Male	$ 22,575	$ 20,793	$17,955	$15,372	$13,066	$11,119
Female	$ 24,184	$ 22,453	$19,591	$16,832	$14,207	$11,894

The mortgage insurance premiums decrease with the age at loan origination. Therefore, when the initial age increases, the borrower gets more cash advances, pays less insurance premiums, and can spend more money to improve he/her living standard.

In addition, we defined lender’s net liability as difference between the present value of expected claims and that of expected insurance premiums. Therefore, when loan terminates at time t, it means lender’s net profit if the value of net liability is negative (-) while it means insurer’s net loss if the value of net liability is positive (+).

In this analysis, we analyzed the case of reverse mortgage male borrower's age was 62, LTV ratio was 60% and other parameters are the same as those we use calculate fair LTV ratio. Table 6-5 shows these results. We could imagine that the reverse mortgage lender would experience tremendous losses if she suffers the losses from operating reverse mortgages due to the features of probability distributions of net liability which have a long tales to the right side. In this analysis, we confirmed the mean and VaR (Value at Risk) at 95% of confidence levels.

Table 6-5 Estimating mean and VaR of Net liability

	GBM	ARMA-GARCH-N	ARMA-GARCH-NIG
Mean	-$25.	-$14,739	-$19,292
VaR 95%	$125	-$14,688	-$19,174

7 Conclusion

We analyze longevity risk, and house price risk in this study. In house price risk, we model the S&P Case-Shiller house price indices via ARMA-GARCH-NIG model. In longevity risk, we consider the famous class model, Lee-Carter. However, there are other risks we need to take into account, such as interest rate risk, and refinancing risk. We assume a fixed interest rate for the life of the HECM loans so that we can determine the conditional Esscher parameters and create the risk-adjusted probability measure for pricing purposes. However, we do not live in a simple world with a flat term structure subject only to additive shifts. Therefore, we do need to model the stochastic interest rates with a more realistic term structure, for example, the Vasicek (1977) model, CIR (1985) model. Moreover, the traditional HECM model assumes that the house price index and the interest rate are independent of each other. However, historical data shows significant correlations between the change in house price index and the change in interest rates (Rodda, et al., 2004). Furthermore, we assume the lender's margin is 150 bps; however, the lender's margin depends on the initial age of the borrowers. Finally, further research can model house price return with interest rate risk via copula approach and determine the maximum level of lender's margin.

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Appendix A

If $Y_t|\mathcal{F}_{t-1}{\sim}NIG(\bar{\alpha},0,\hat{\mu}_t,\sigma_t^2)$ under physical measure P, then $Y_t|\mathcal{F}_{t-1} \sim NIG(\bar{\alpha},\sqrt{\sigma_t^2/\bar{\alpha}}\,\theta_t^q,\hat{\mu}_t,\sigma_t^2)$ under the risk neutral measure Q via the condition Esscher transform, where θ_t^q is Esscher parameters.

Proof:

$$M^Q_{Y_t|\mathcal{F}_{t-1}}(z;\theta_t) = \frac{M^P_{Y_t|\mathcal{F}_{t-1}}(z+\theta_t)}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta_t)}$$

$$= exp\left(\hat{\mu}_t + \sqrt{\bar{\alpha}^2 - \left[\sqrt{\bar{\alpha}\sigma_t^2}\theta_t^q\right]^2}\right) \times exp\left(\sqrt{\bar{\alpha}^2 - \left[\sqrt{\bar{\alpha}\sigma_t^2}\left(z+\theta_t^q\right)\right]^2}\right)$$

$$= exp\left(\hat{\mu}_t + \bar{\alpha}\sqrt{1 - \left[\sqrt{\sigma_t^2/\bar{\alpha}}\,\theta_t^q\right]^2}\right)$$

$$\times\, exp\left(\sqrt{\bar{\alpha}^2 - \left[\bar{\alpha}\left(\sqrt{\sigma_t^2/\bar{\alpha}}\,\theta_t^q\right) + \left(\sqrt{\bar{\alpha}\sigma_t^2 z}\right)\right]^2}\right)$$

In order for Q to be an equivalent martingale measure, we need to have:

$$e^{r-g} = M^Q_{Y_t|\mathcal{F}_{t-1}}(1;\theta_t) = \frac{M^P_{Y_t|\mathcal{F}_{t-1}}(1+\theta_t)}{M^P_{Y_t|\mathcal{F}_{t-1}}(\theta_t)}$$

Therefore,

$$r-g = \hat{\mu}_t + \sqrt{\bar{\alpha}^2 - \left[\sqrt{\bar{\alpha}\sigma_t^2}\theta_t^q\right]^2} - \sqrt{\bar{\alpha}^2 - \left[\sqrt{\bar{\alpha}\sigma_t^2}\left(1+\theta_t^q\right)\right]^2} \qquad \text{(A-1)}$$

This equation can be solve explicitly by a quadratic form, the solution must satisfies:

$$\left|\sqrt{\sigma_t^2/\bar{\alpha}}\,\theta_t^q\right| < 1$$

We can solve this equation by a quadratic form; therefore under Q

$$Y_t|\mathcal{F}_{t-1} \sim NIG\left(\bar{\alpha}, \sqrt{\sigma_t^2/\bar{\alpha}}\theta_t^q, \hat{\mu}_t, \sigma_t^2\right)$$