國立交通大學

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混合二項分配之容許界限

Tolerance Limits for a Binomial Mixture Model

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中華民國九十九年六月

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混合二項分配之容許界限

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摘要

建立容許區間衡量製程特性在製造業與醫藥等領域廣泛地被使用於制定品質控制系統。我們在本論文中分別提出有母數與無母數的兩種方法，用以建立混合二項分配的容忍界限，而這兩種方法亦可以被使用於許多應用問題之中。本文將其進行模擬並比較兩種方法，而無母數的方法在此表現較佳因為它的覆蓋率可以達到我們給定的標準且容許界限也有令人滿意的結果。

關鍵詞：容許界限、混合二項分配、信賴區間、覆蓋率

Tolerance Limits for a Binomial Mixture Model

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Abstract

The construction of tolerance intervals (TIs) to measure discrete quality characteristics has been one of the major tasks in developing quality control systems used in the manufacturing and pharmaceutical sectors. In this study, we propose two methods based on parametric and nonparametric approaches to construct tolerance limits for a mixture binomial distribution, which can be adopted in many applications. A simulation study is conducted to compare both methods. The nonparametric method has a better performance than that of the parametric method. The simulation study also shows that the proposed tolerance bounds lead to a satisfactory result.

Keyword: Binomial mixture model, Confidence interval, Coverage probability, Tolerance limit.

誌謝

首先感謝指導老師王秀瑛教授細心地教導，在跟著老師做研究的這段時間，老師剛好接下所長的職務，雖然很忙，但是還是像媽媽帶著小朋友一樣仔細地指導著我寫論文，每當我做的結果不如預期時，老師總是笑著說沒關係並且一直幫我找別的辦法來處理，若是結果不錯，老師也是不吝惜稱讚。謝謝老師能讓我如此愉快的完成我的論文；同時也謝謝所上的老師兩年來的照顧。

接著是研究所與我一同打拼的夥伴們，這兩年不論是寫作業、跑程式、趕論文或是玩遊戲喝謎樣液體我們都混在一起，做研究遇到低潮時互相打氣，有什麼資訊也會一起分享，希望未來我們一樣能保持像現在這樣的情感。最後特別感謝從大學就很麻吉的烏龜，有你在怎樣都不會無聊；還有一起深夜算機率的好朋友們，以後有機會再來一較高下吧。

吳尚剛　筆于 408

2010.06

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1 Introduction

The construction of tolerance intervals (TIs) to measure discrete quality characteristics has been one of the major tasks in developing quality control systems used in the manufacturing and pharmaceutical sectors. For example, a manufacture can use a tolerance bound to inspect whether the number of defective units is in a tolerance region.

The investigation of tolerance interval construction has been extensively studied for continuous distributions (e.g., Wald and Wolfowitz 1946 [12]; Odeh and Owen 1980 [10]; Wang and Iyer 1994 [13]; Wolfinger 1998 [15]; Fernholz and Gillespie 2001 [2]; Hamada et al. 2004 [6]; Krishnamoorthy and Mathew 2004 [7]; Liao, Lin, and Iyer 2005 [8]; Van Der Merwe, Pretorius, and Meyer 2006 [11]).

Most recently, Wang and Tsung (2009) and Cai and Wang (2009) have explored the discrete distribution tolerance interval construction, which dates back to Zack (1970) . The tolerance interval for discrete distributions is a useful tool which can be used for quality control testing. Wang and Tsung (2009) and Cai and Wang (2009) constructed improved tolerance intervals for binomial and Poisson distributions, which can be used for defective rate investigation.

The number of defective units in n units in a production process can be assumed to follow a binomial distribution $B(n, p)$, where p represents the defective rate if the defective rate is assumed to be a constant in the production processes. However, in real applications, it is likely that there are several production lines operating simultaneously. The total n units are produced from several production lines with different defective rates. In this case, for an unit, we may not know which production line it comes from. In this case, the number of defective units follows a mixture binomial model. Assume that X is the defective number of n units manufactured from k production lines, and the defective number among n units of the ith line follows a binomial distribution B(n,p_i). Then X follows a mixture binomial distribution. In this study, we focus on exploring tolerance bounds for a mixture binomial distribution.

An interval $(L(X), U(X))$ is said to be a $1-\alpha$ confidence tolerance interval with $\beta - contant$, denoted as $(\beta, 1-\alpha)TI$ for a density function F if

$$Pr_\theta\{[F(U(X)) - F(L(X))] \geq \beta\} = 1 - \alpha.$$

$L(X)$ is defined a $(\beta, 1-\alpha)$ low tolerance bound if $Pr_\theta\{1-F(L(X)) \geq \beta\} = 1-\alpha$ and a bound $U(X)$ is defined a $(\beta, 1-\alpha)$ upper tolerance bound if $Pr_\theta\{F(U(X)) \geq \beta\} = 1-\alpha$.

Some applications require one-sided lower or upper tolerance bounds for the distribution of Y, the number of defective units in future samples of m product units. For example, assume that a production process packing in groups of size m. Suppose that it is desired to have a specified level of confidence, then at least $100\beta\%$ of such packages are less than or greater than the bound.

In this study, we propose two tolerance limits based on parametric and nonparametric approaches for a mixture binomial distribution.

This thesis is organized as follows. The mixture model and mixture binomial distribution are introduce in Section 2. The method for estimating the defective rates is given in Section 3. The method for computing the confidence interval and tolerance limit for binomial distribution are given in Section 4 and 5. New methods to calculate the binomial distribution's tolerance limit are proposed in Section 6. A simulation study is given in Section 7. Finally, we summarize a conclusion in Section 8.

2 Mixture Model

In statistics, a mixture density is a probability density function expressed as a convex combination of other probability density functions. It can be got by following approach. First, pick a probability density function from known probability distributions, then simulate a sample from the chosen probability distribution. Because this approach is a two-step process, they are also called hierarchical models. It is important to distinguish between a random variable whose density is the sum of a set of component densities (i.e a mixture) and a random variable whose value is the sum of the values of two or more random variables, in which case the distribution is given by the convolution operator.
Finite mixtures of distributions have provided a mathematical-based approach to the sta-

tistical modeling of a wide variety of random phenomena. Because of their usefulness as an extremely flexible method of modeling, finite mixture models have continued to receive increasing attention over the years, from both a practical and theoretical point of view. Indeed, in the past decade the extent and the potential of the applications of finite mixture models have widened considerably. Fields in which mixture models have been successfully applied including astronomy, biology, genetics, medicine, psychiatry, economics, engineering, marketing, physical, and social sciences. In addition to their more direct role in data analysis and inference of providing descriptive models for distributions, finite mixture models underpin a variety of techniques in major areas of statistics, including cluster and latent class analyses, discriminant analysis, image analysis, and survival analysis.

Here are some examples of mixture model and their histograms.

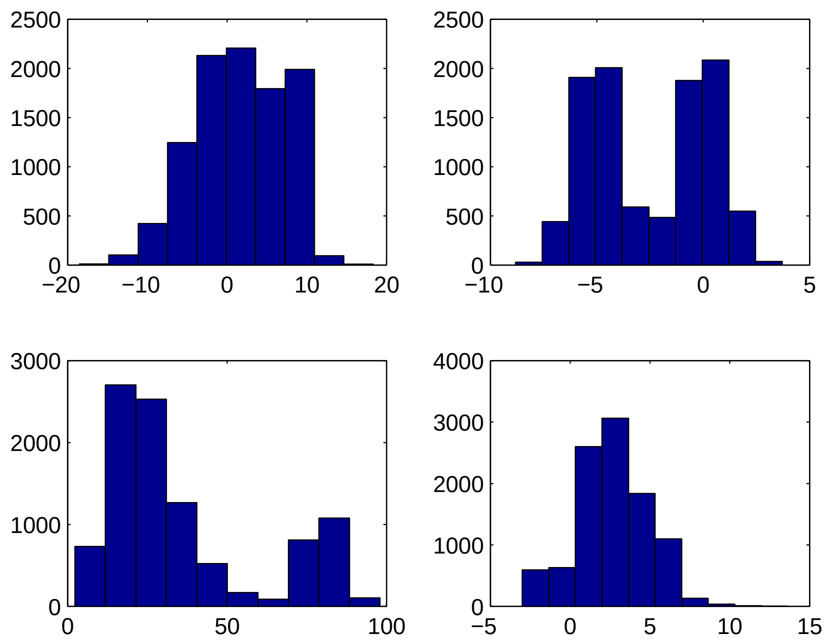

Figure 1: The mixture distribution is composed of two distributions, and the proportion of each distribution is different.

The usefulness of mixture distributions in the modeling of heterogeneity in a cluster analysis context is obvious. In another example where is group structure, they have a very useful role in assessing the error rates of diagnostic and screening procedures in the absence of a gold standard. But as any continuous distribution can be approximated, mixture models provide a convenient semiparametric framework in which to model unknown distribution shapes, whatever the objective, whether it be, say, density estimation or the flexible construction of Bayesian priors.

2.1 Binomial Mixture Model

Suppose the distribution of data is shown to be a single elliptical curve, we can fit it with one distribution. However, if it is shown to be a complex curve, using a single distribution is not suitable to describe the probability density function of the data. In the manufacture process with several production lines discussed in Section 1, we can consider to fit the data with a binomial mixture model. Suppose the number of defective units X is a random variable following a binomial mixture model with k components. The probability function of X is

$$f(x|\Theta) = \sum_{i=1}^{k} \phi_i \binom{n_i}{x} p_i^x (1 - p_i)^{n-x}. \qquad (2.1.1)$$

$\Theta = \{p_1, \ldots, p_k, \phi_1, \ldots, \phi_k\}$ where $n_1, \ldots, n_k$ are the sample sizes of the k binomial distributions, $p_1, \ldots, p_k$ are the corresponding successful probabilities of the probability functions, and $\phi_1, \ldots, \phi_k$ are the weights of k components (i.e. $\phi_i \geq 0 \quad \forall 1 \leq i \leq k$ and $\sum_{i=1}^{k} \phi_i = 1$).

In this study, we assume that $\phi_1, \ldots, \phi_k$ are known and have equal weights $1/k$. To derive a tolerance interval for a random variable following a binomial mixture model, a feasible way is to estimate the unknown parameters $p_1, \ldots, p_k$ first. In estimating parameters in a mixture model, EM algorithm is a widely-used method adopted for estimation in hierarchical model. An alternative estimating method is using Gibbs sampling approach. We develop an algorithm based on Gibbs sampling approach for estimating $p_1, ..., p_k$ in a mixture binomial model.

3 Gibbs sampler

In this mixture model (2.1.1), we don't derive estimators for parameters straightforward. Gibbs sampling is one of the feasible ways to estimate parameters in a mixture model, which is an algorithm to generate a sequence of samples from the joint probability distribution of two or more random variables. One of the purposes of such a sequence is to approximate the joint distribution. Gibbs sampling is a special case of the Metropolis-Hastings algorithm, and thus is an example of a Markov chain Monte Carlo algorithm. The algorithm is named after the physicist J. W. Gibbs, an analogy between the sampling algorithm and statistical physics (Geman S., and Geman D. 1984 [3]). The algorithm was devised by brothers Stuart and Donald Geman.

Gibbs sampling is applicable when the joint distribution is not known explicitly, but the conditional distribution of each variable is known. Gibbs sampling algorithm generates an instance from the distribution of each variable in turn, conditional on the current values of the other variables. It can be shown that the sequence of samples constitutes a Markov chain, and the stationary distribution of that Markov chain is just the sought-after joint distribution.

Gibbs sampling is particularly well-adapted to sample the posterior distribution of a Bayesian network, since Bayesian networks are typically specified as a collection of conditional distributions.

3.1 The algorithm for GIBBS sampling

In this section, we develop an algorithm based on the Gibbs sampling to derive the estimators of $p_1, ..., p_k$ in (2.1.1), which are the defective rates with respect to each of k production lines. The proposed algorithm is as follows. Suppose $x_1, \ldots, x_m$ are i.i.d from (2.1.1) with $n_1 = n_2 = ... = n_k = n$.

Step 1: Let $Z_i, i = 1, ..., m$ be random variables with possible values $1, ..., k$, which associate with $x_i, i = 1, .., m$ by assigning each x_i to one group among the k groups. Let the initial values of $Z_i = 1, 2, \ldots, k$ be $1/k$ for each i.

($Z_i's$ role is to guess which binomial distribution generated X_i.)

Step 2: $p_j^{(h)} = \sum_{l=j}^{m} x_l I(z_l = i)/(m)\ \forall 1 \le j \le k$

Step 3: $q_j = \frac{\binom{n}{x_i} p_j^{x_i} (1-p_j)^{n-x_i}}{\sum_{e=1}^{k} \binom{n}{x_i} p_e^{x_i} (1-p_e)^{n-x_i}}$, for $j = 1, 2, \ldots, k.\quad \forall 1 \le i \le m$

Step 4: $P(Z_i = j) = q_j$, for $j = 1, 2, \ldots, k.\quad \forall 1 \le i \le m$

Step 5: Repeat Steps 2-4 until convergence.

4 Confidence Interval for a Binomial Distribution

In this section, we introduce confidence intervals for a binomial distribution. Based on the confidence interval, a proposed tolerance interval is constructed in the next section. For a binomial distribution $B(n, p)$, the sample proportion $\hat{\text{p}}=x/n$ is a point estimate for p, the true population (or process) proportion. However, $\hat{\text{p}}$ differs due to sampling fluctuations. Thus, one frequently desires to compute a two-sided confidence interval or a one-sided confidence bound for p from the sample data.

If X is a random variable following a binomial distribution $B(n, p)$ with unknown p, a conservative two-sided 100(1-α)% confidence interval for p is

$$[\underline{p}, \overline{p}] = [\{1 + \frac{(n-x+1)F_{(1-\alpha/2,2n-2x,2x)}}{x}\}^{-1}, \{1 + \frac{x}{(x+1)F_{(1-\alpha/2,2x+2,2n-2x)}}\}^{-1}] \quad (4.0.1)$$

where F_{a,r_1,r_2} is the $100\alpha th$ percentile of the F distribution with r_1 and r_2 degrees of freedom. Similarly, one-sided upper 100(1-α)% confidence bound is

$$\overline{p} = \{1 + \frac{x}{(x+1)F_{(1-\alpha,2x+2,2n-2x)}}\}^{-1} \quad (4.0.2)$$

and lower 100(1-α)% confidence bound is

$$\underline{p} = \{1 + \frac{(n-x+1)F_{(1-\alpha/2,2n-2x,2x)}}{x}\}^{-1}$$

The upper limit is defined to be $\overline{p}$=1 if $x = n$ and the lower limit $\underline{p}$=0 if $x = 0$.

Sometimes the use of (4.0.1) may not be convenient. For example, commonly percentiles tables of the F distribution only for a limited number of values of γ_1 and γ_2, and interpolation may be necessary. Fortunately, an approximate expression which uses only tabulations of the normal distribution percentiles provides adequate accuracy when both $n\hat{p}$ and $n(1-\hat{p})$ exceed 10. The approximate two-sided $100(1-\alpha)\%$ confidence interval for p is

$$[\underline{p}, \overline{p}] = \hat{p} \pm z_{(1-\alpha/2)}[\frac{\hat{p}(1-\hat{p})}{n}] \quad (4.0.3)$$

where z_γ is the 100γth percentile of the standard normal distribution. Similarly, one-side lower and upper $100(1-\alpha)\%$ confidence bounds are obtained by replacing $\alpha/2$ by α in the lower and upper parts, respectively, of this expression. In this paper we will choose to use the former because it is more in line with this model.

5 Tolerance limit for a Binomial Distribution

The following one-sided binomial tolerance bound was first given by Hahn and Chandra (1981).

The function

$$B(x, n, p) = Pr(Y \leq x) = \sum_{j=0}^{x} \binom{n}{j} p^j (1-p)^{n-j} \tag{5.0.4}$$

is the cumulative distribution function of Y.

If the population proportion β of defect is known, the smallest integer T such that

$$B(T, n, p) \geq \beta \tag{5.0.5}$$

is an upper bound for the number of defective units with 100β% of the future samples from the sampled population, and size is n, but now p is unknown, consisting of X defective units in a random sample of n units. Thus, we can construct an upper tolerance bound for the distribution of Y. An upper tolerance bound for the distribution of Y can be found by the following steps that is named

Procedure 1:

1. Use (4.0.2) to compute $\overline{p}$, upper 100(1-α)% confidence bound for p.
2. Substitute $\overline{p}$ for p, and find the smallest integer T such that satisfying (5.0.5). This integer is an upper tolerance bound $\overline{T}$.

So, one can say with 100(1-α)% confidence that $B(\overline{T}; n; p) \geq \beta$. Thus, we have 100(1-$\alpha$)% confidence that at least 100β% of the future samples of size n will contain $\overline{T}$ or fewer defective units.

6 Tolerance limit for a Binomial Mixture Model

6.1 Tolerance limit based on a mixture model

Based on Procedure 1, we propose a tolerance interval by extending the steps in Procedure 1 to a mixture model using a Gibbs sampling method to estimate the binomial proportions.

$$B'(x,n,\mathbf{p}) = Pr(Y \leq x) = \sum_{j=0}^{x}\sum_{i=1}^{k}\frac{1}{k}\binom{n}{j}p_i^j(1-p_i)^{n-j} \tag{6.1.1}$$

If the population proportion β of defect is known, the smallest integer T such that

$$B'(T,n,\mathbf{p}) \geq \beta \tag{6.1.2}$$

where $\mathbf{p}$=$\{p_1$,p_2,...,$p_k\}$, is an upper bound for the number of defective units with $100\beta\%$ of the future samples from the sampled population, and size is n. But now p_i is unknown and only sample data, consisting of X defective units in a random sample of n units. Thus,we can construct an upper tolerance bound for the distribution of Y. An upper tolerance bound for the distribution of Y can be found by the following steps:

1. Use Gibbs sampling to compute $\hat{p_1}$, $\hat{p_2}$,...,$\hat{p_k}$.

2. Use (4.0.2) to compute $\overline{p_1}$, $\overline{p_2}$,..., $\overline{p_k}$, upper 100(1-α)% confidence bound for p_1, p_2 ,..., p_k.

3. Substitute $\overline{p_i}$ for p_i,and find the smallest integer T such that satisfying (6.1.1). This integer is an upper tolerance limit $\overline{T_\beta}$.

So, one can say with 100(1-α)% confidence that $B'(\overline{T_\beta};n;p) \geq \beta$. Thus, we have 100(1-$\alpha$)% confidence that at least $100\beta\%$ of the future samples size n will contain $\overline{T_\beta}$ or fewer defective units.

6.2 Distribution-Free Tolerance Interval

For constructing a tolerance interval for a mixture model, we can adopt a distribution-free approach model, introduced in Hahn and Meeker (1991), to find a one-sided distribution-free conservative upper $100(1-\alpha)\%$ confidence bound for the $100\beta th$ percentile of the sampled population. A distribution-free tolerance limit is an order statistic $\overline{Y} = x_{(u)}$, where u is chosen as the smallest integer satisfied the following inequality

$$B(u-1, n, \beta) \geq 1-\alpha, \qquad 0 < u \leq n+1, \quad 0 < \beta < 1. \tag{6.2.1}$$

where n is the sample size, and the actual confidence level is given by the left-hand side of this inequality.

Similarly, an one-sided distribution-free conservative lower $100(1-\alpha)\%$ confidence bound for for the 100(1-β)th percentile of the sampled population is obtained as $\underline{Y} = x_{(l)}$, where l is chosen as the largest integer satisfied the following inequality:

$$1 - B(l+1, n, \beta) \geq 1-\alpha, \qquad 0 \leq l < n+1, \quad 0 < \beta < 1. \tag{6.2.2}$$

the actual confidence level is given by the left-hand side of this inequality.(Hahn and Meeker 1991, 89-91)

An one-sided distribution-free tolerance limit is equivalent to an one-sided distribution-free confidence limit for a percentile of that population. That is an one-sided distribution-free upper (lower) $100(1-\alpha)\%$ tolerance limit that will exceed (will be exceeded) at least $100\beta\%$ of the population is the same as an one-sided distribution-free conservative upper (lower) $100(1-\alpha)\%$ confidence limit for the $100\beta\%$ percentile of that population.

7 Simulation

A simulation study is conducted to compare the two proposed tolerance intervals. The details are given as follows. The n in (2.1.1) is assumed to be 30.

- Generate $x_1, x_2, ..., x_m$ from $f(x|\theta) = \sum_{i=1}^{3} \phi_i \binom{30}{x} p_i^x (1-p_i)^{30-x}$
- Sampling and estimate $\hat{p_1}, \hat{p_2}$ and $\hat{p_3}$ by the algorithm for Gibbs sampling.
- Method 1: Use step 2 and step 3 in Section 6.1 to produce a (0.9,0.95) upper tolerance bound.
- Method 2: Generate 100 samples $y_1, ...y_{100}$ from this model $f'(x|\theta) = \sum_{i=1}^{3} \phi_i \binom{30}{x} \hat{p_i}^x (1-\hat{p_i})^{30-x}$, by (6.2.1), y_{96} is a one-sided distribution-free (0.9,0.95) upper tolerance bound.

Beside comparing the above two tolerance bounds, we also compare a method when the sample is assumed to follow a binomial distribution. That is, we do not fit the data using a mixture model. Do not estimate $\hat{p_1}, \hat{p_2}$ and $\hat{p_3}$, just estimate $\hat{p}$ by sample mean. That means we assumes only one production line.

At first, we compare the variations between estimator of defective rates by Gibbs sampling for a mixture model and estimator of defective rate by only considering a binomial model.

By the Table 1 and Table 2 we can find estimated defective rates by Gibbs sampling is much better than by means (in addition to the the defective rates is equal). Said by Gibb sampling for a mixture model may be closer to real situation than by only considering a binomial model.

And then we compare the (0.9,0.95) upper tolerance bound's cover probability between method 1 and method 2.

Figure 2 and Figure 3 plots the variations of estimated the defective rates by Gibbs sampling and means. By comparing the lines in Figure 2 and Figure 3 it is clear that the performance of these variations of estimated the defective rates by Gibbs sampling is better than by sample mean, we can see that lines of Gibbs sampling are shock in 0.001 and lines of means are show a quadratic curve and the center is the mean of p_1, p_2 and p_3(actual value), only in this case the variation will be similar between the two ways, otherwise means would be much higher than Gibbs sampling.

Figure 4 to Figure 7 plots the coverage probabilities of one-sided (0.9, 0.95) tolerance bounds built from the Method 1 and Method 2. In these figures we can find that the coverage probabilities of Method 1 almost higher than Method 2 and slightly conservative, sometimes it will near 1, in contrast, the coverage probabilities of Method 2 is better than Method 1.

In Figure 8, we compare the coverage probabilities of Method 2 which estimate the defective rates by Gibbs sampling or means. We can clearly find that estimate by means of instability and most of the coverage probabilities are quite low.

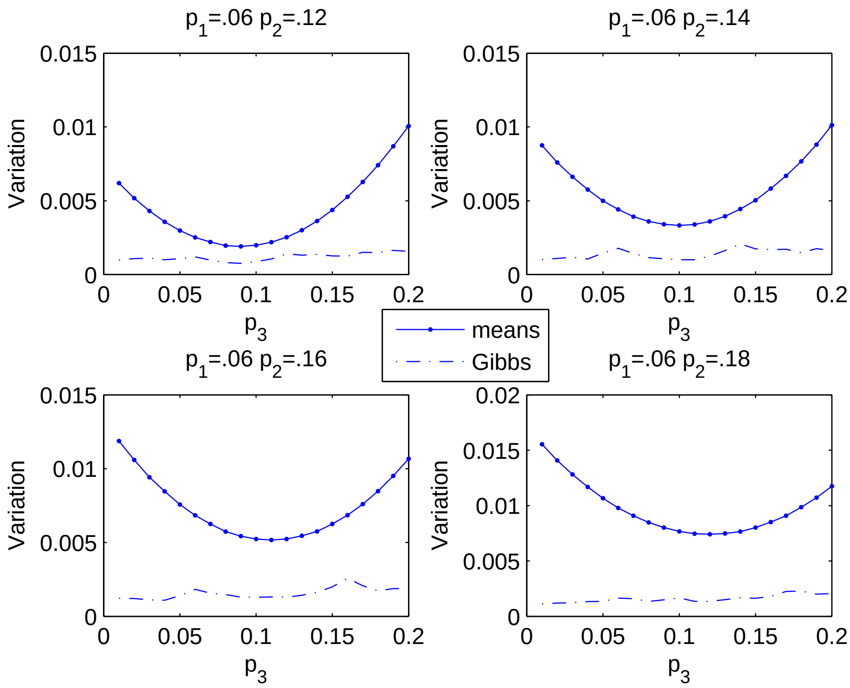

Figure 2: Variation of the estimated the defective rates for p_1=0.06,p_2=0.12-0.18

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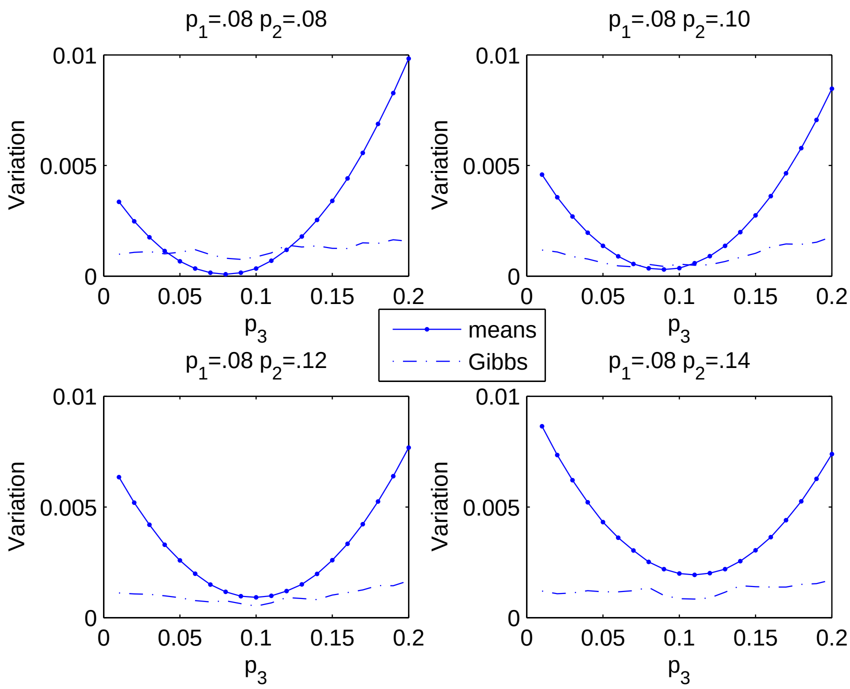

Figure 3: Variation of the estimated the defective rates for p_1=0.08,p_2=0.08-0.14

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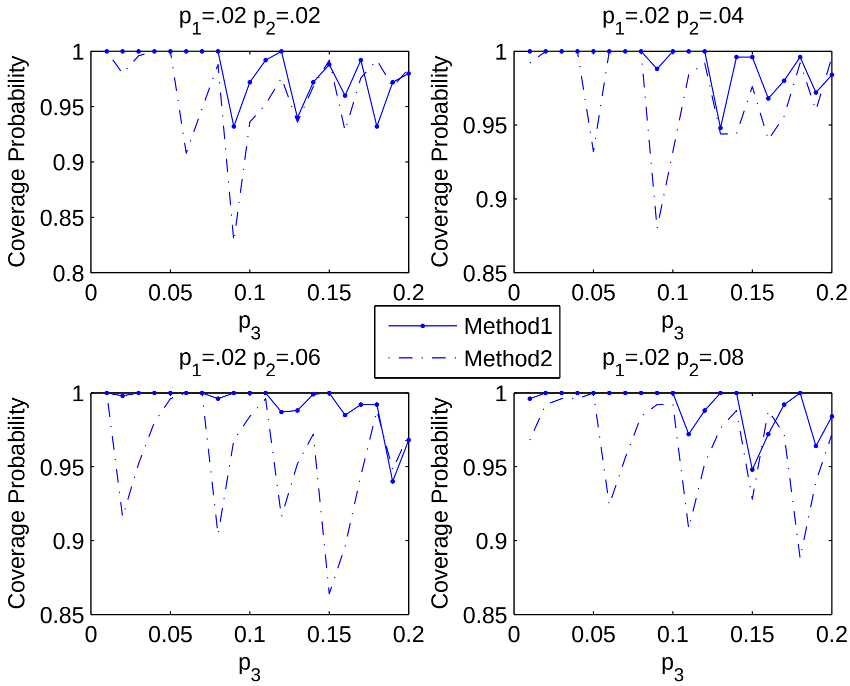

Figure 4: Coverage probabilities of Method 1 and Method 2 for p_1=0.02, p_2=0.02-0.08

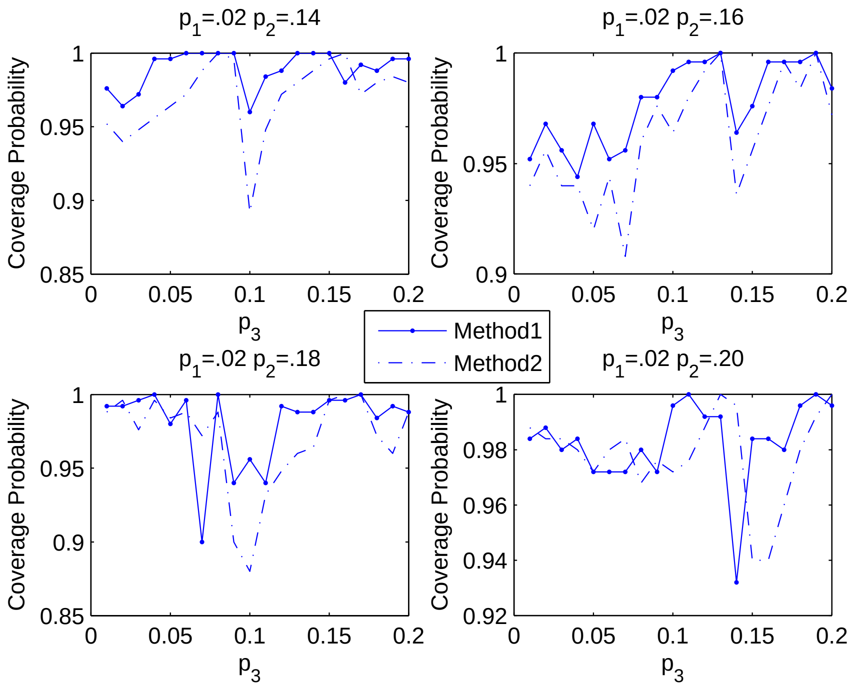

Figure 5: Coverage probabilities of Method 1 and Method 2 for p_1=0.02, p_2=0.14-0.20

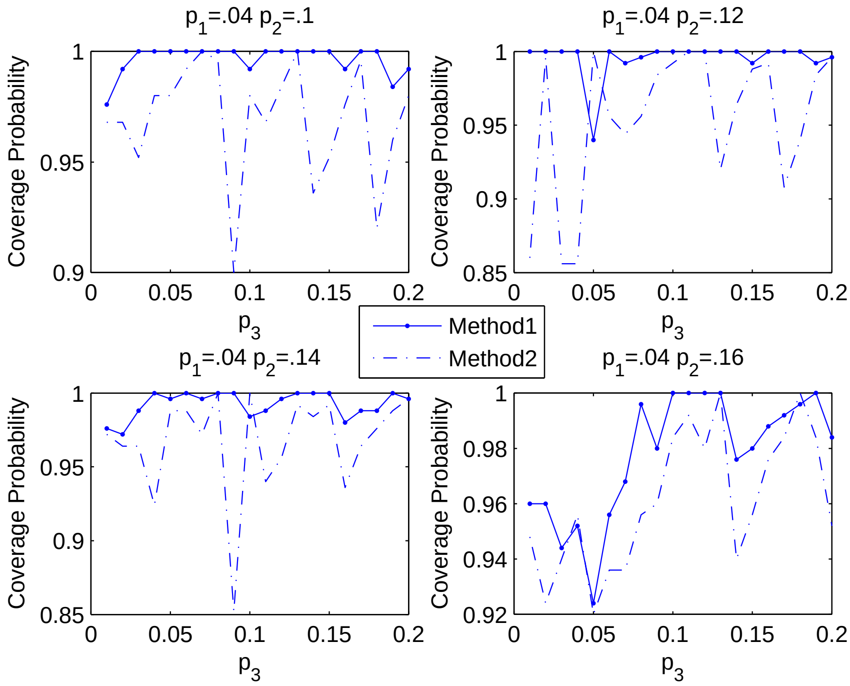

Figure 6: Coverage probabilities of Method 1 and Method 2 for p_1=0.04, p_2=0.10-0.16

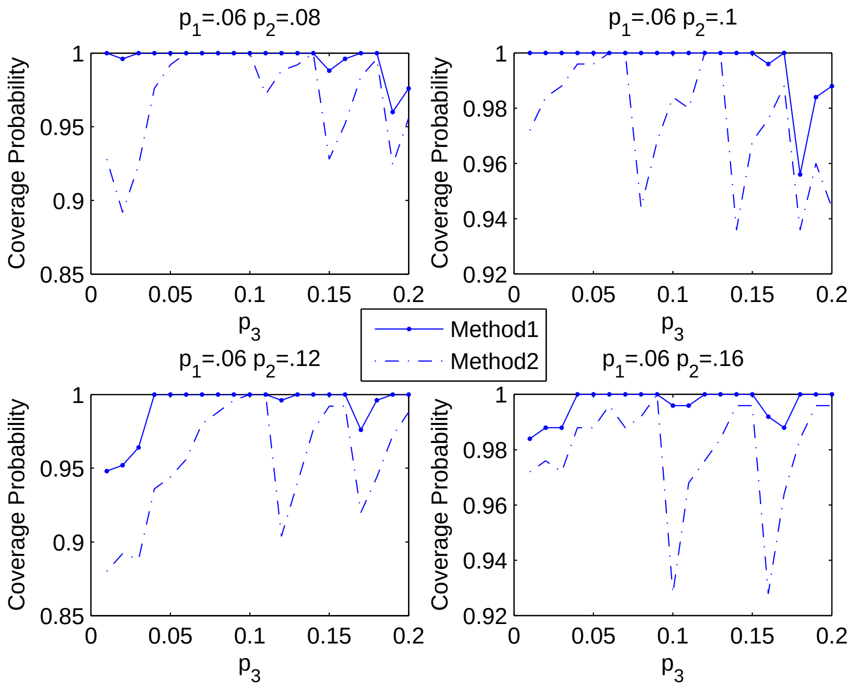

Figure 7: Coverage probabilities of Method 1 and Method 2 for p_1=0.06, p_2=0.08-0.14

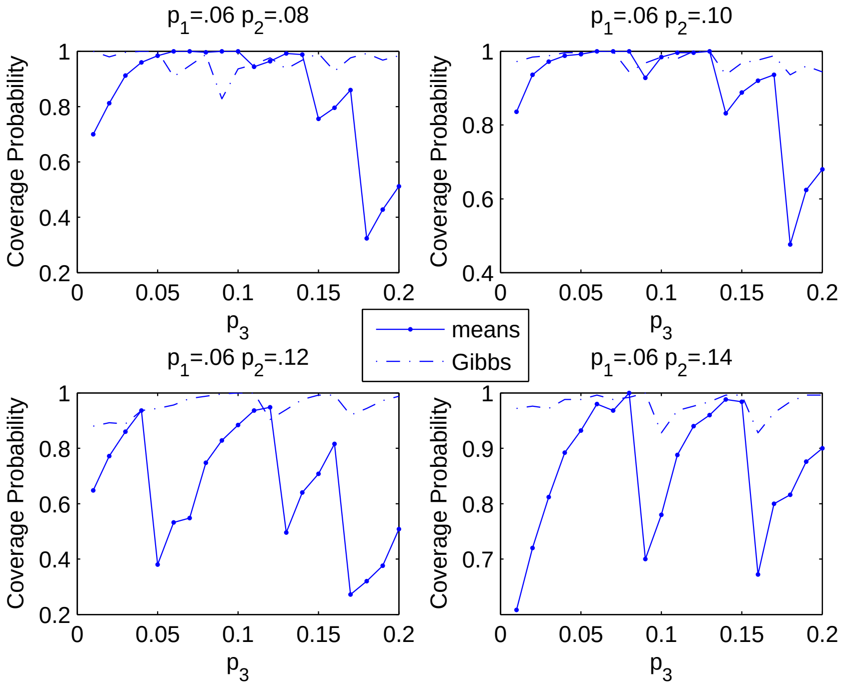

Figure 8: Coverage probabilities of Method 2 for p_1=0.06, p_2=0.08-0.14

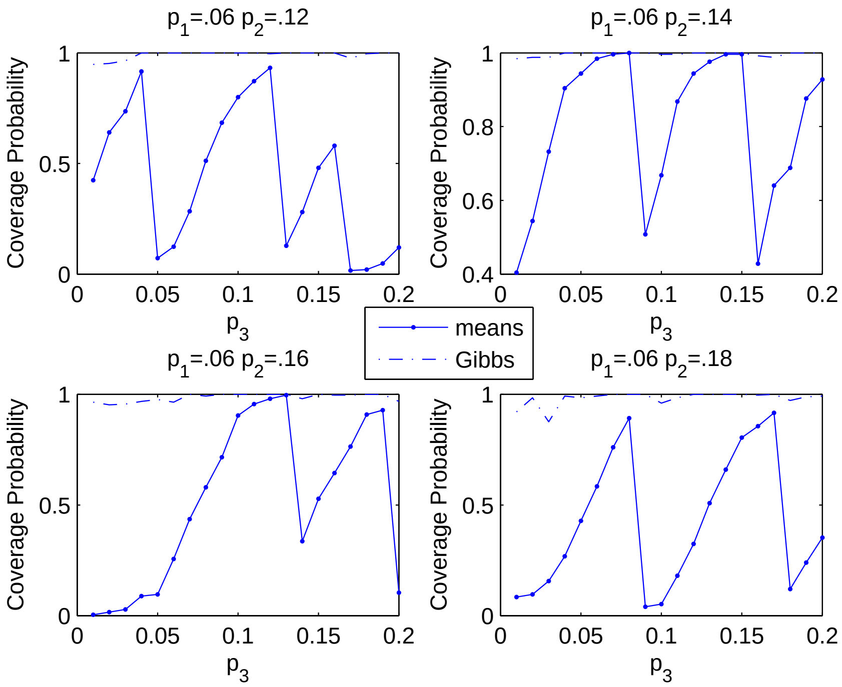

Figure 9: Coverage probabilities of Method 1 for p_1=0.06, p_2=0.12-0.18

8 Conclusions

In this study, we propose two methods in constructing tolerance bounds for a binomial mixture models. One method is based on a parametric method, which used the Gibbs sampling to estimate the unknown parameter first. With these estimated values and the existing approach of the binomial tolerance bounds, we construct the first tolerance bound for a binomial mixture model. The second method is based on the nonparametric method to derive a distribution-free tolerance interval. The endpoint of a distribution-free tolerance interval is an order statistic.

In this study, we assume that the weights of each distribution are equal, but it may not hold for real applications. The algorithm for deriving good estimators for the weight is still under investigation.

The comparison of these two methods shows that the second method has better performance than the first method. Since in real manufacturing process, there are common many production lines of operation, the existing methods can not directly apply to these cases. The methods we proposed here can provide an efficient way to construct tolerance intervals for the defective numbers estimation.

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Table 1 is a variation of the estimated defective rates by Gibbs sampling($\hat{p_1}$, $\hat{p_2}$, $\hat{p_3}$). Data in Table 1 is calculated by following formula

$$P_v = \frac{\sum_{i=1}^{d} (\hat{p_1} - p_1)^2 + (\hat{p_2} - p_2)^2 + (\hat{p_3} - p_3)^2}{d}$$

where d is the number of experiments under the same parameters(p_1,p_2 and p_3).

Table 1: Variation of estimated defective rates by Gibbs sampling for p_1=0.06.

p_2	p_3							
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.12	0.000986	0.001075	0.001107	0.001003	0.001081	0.001190	0.000972	0.000807
0.14	0.001008	0.001088	0.001166	0.001059	0.001449	0.001774	0.001430	0.001137
0.16	0.001225	0.001203	0.001116	0.001089	0.001396	0.001828	0.001570	0.001481
0.18	0.001130	0.001189	0.001236	0.001337	0.001356	0.001646	0.001587	0.001352
0.20	0.001008	0.001488	0.001268	0.001181	0.001382	0.001718	0.001355	0.001244

p_2	p_3							
	0.09	0.10	0.11	0.12	0.13	0.014	0.15	0.16
0.12	0.000756	0.000866	0.001047	0.001396	0.001315	0.001364	0.001257	0.001244
0.14	0.001069	0.001005	0.000999	0.001248	0.001619	0.002071	0.001738	0.001681
0.16	0.001291	0.001298	0.001304	0.001293	0.001427	0.001638	0.001989	0.002580
0.18	0.001489	0.001663	0.001350	0.001343	0.001505	0.001679	0.001634	0.001811
0.20	0.001430	0.001866	0.001557	0.001551	0.001771	0.001785	0.001695	0.001914

p_2	p_3			
	0.17	0.18	0.19	0.20
0.12	0.001502	0.001474	0.001640	0.001570
0.14	0.001706	0.001485	0.001757	0.001628
0.16	0.002074	0.001721	0.001878	0.001882
0.18	0.002247	0.002275	0.001998	0.002065
0.20	0.001843	0.001965	0.002560	0.002735

Table 2 is a variation of the estimated defective rate by means($\hat{p}$). Data in Table 2 is calculated by following formula

$$P_v = \frac{\sum_{i=1}^{d}(\hat{p}-p_1)^2 + (\hat{p}-p_2)^2 + (\hat{p}-p_3)^2}{d}$$

Table 2: Variation between estimated $\hat{p}$ (sample mean) and actual p_i for p_1=0.06

p_2	p_3							
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.12	0.006194	0.005179	0.004305	0.003572	0.002974	0.002515	0.002198	0.001960
0.14	0.008751	0.007588	0.006622	0.005759	0.004990	0.004411	0.003927	0.003593
0.16	0.011866	0.010596	0.009418	0.008463	0.007567	0.006832	0.006248	0.005736
0.18	0.015527	0.014078	0.012822	0.011681	0.010657	0.009788	0.009072	0.008470
0.20	0.019700	0.018113	0.016768	0.015481	0.014329	0.013303	0.012472	0.011716

p_2	p_3							
	0.09	0.10	0.11	0.12	0.13	0.014	0.15	0.16
0.12	0.001910	0.001988	0.002185	0.002528	0.003000	0.003622	0.004375	0.005259
0.14	0.003408	0.003332	0.003393	0.003592	0.003953	0.004438	0.005032	0.005822
0.16	0.005422	0.005234	0.005167	0.005235	0.005453	0.005749	0.006256	0.006853
0.18	0.008015	0.007670	0.007456	0.007408	0.007473	0.007647	0.008004	0.008511
0.20	0.011102	0.010636	0.010328	0.010117	0.010009	0.010082	0.010291	0.010634

p_2	p_3			
	0.17	0.18	0.19	0.20
0.12	0.006269	0.007408	0.008684	0.010064
0.14	0.006689	0.007668	0.008807	0.010125
0.16	0.007602	0.008480	0.009501	0.010669
0.18	0.009071	0.009852	0.010719	0.011754
0.20	0.011127	0.011702	0.012452	0.013317

Table 3: Coverage probability of a (0.9,0.95) upper tolerance bound for p_1=0.02

p_2	p_3									
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.932	0.972
0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.988	1.000
0.06	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.996	1.000	1.000
0.08	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	0.956	0.992	0.988	1.000	0.996	1.000	1.000	1.000	0.956	0.992
0.12	0.992	1.000	1.000	1.000	1.000	0.960	0.976	0.992	0.984	1.000
0.14	0.976	0.964	0.972	0.996	0.996	1.000	1.000	1.000	1.000	0.960
0.16	0.952	0.968	0.956	0.62	0.968	0.952	0.956	0.980	0.980	0.992
0.18	0.992	0.992	0.996	1.000	0.98	1.000	0.996	1.000	0.94	0.956
0.2	0.984	0.998	0.98	0.984	0.972	0.972	0.972	0.98	0.972	0.996

p_2	p_3									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.02	0.992	1.000	0.940	0.972	0.988	0.960	0.992	0.932	0.972	0.980
0.04	1.000	1.000	0.948	0.996	0.996	0.968	0.980	0.996	0.972	0.984
0.06	1.000	0.987	0.988	0.999	1.000	0.985	0.992	0.992	0.940	0.968
0.08	0.972	0.988	1.000	1.000	0.948	0.972	0.992	1.000	0.964	0.984
0.1	1.000	1.000	1.000	0.948	0.968	0.988	0.996	0.916	0.984	0.996
0.12	1.000	0.952	0.984	0.988	0.996	1.000	0.960	0.980	0.984	0.996
0.14	0.984	0.988	1.000	1.000	1.000	0.980	0.992	0.988	0.996	0.996
0.16	0.996	0.996	1.000	0.964	0.976	0.996	0.996	0.996	1.000	0.984
0.18	0.940	0.992	0.988	0.988	0.996	0.996	1.000	0.984	0.992	0.988
0.2	1.000	0.992	0.992	0.932	0.984	0.984	0.980	0.996	1.000	0.996

Table 4: Coverage probability of a distribution-free tolerance interval for p_1=0.02

p_2	p_3									
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.02	1.000	0.980	0.996	1.000	1.000	0.908	0.948	0.988	0.828	0.936
0.04	0.992	1.000	1.000	1.000	0.932	1.000	1.000	1.000	0.880	0.932
0.06	1.000	0.916	0.952	0.980	0.996	1.000	1.000	0.904	0.968	0.984
0.08	0.968	0.992	0.996	0.996	1.000	0.924	0.956	0.984	0.992	0.992
0.10	0.876	0.928	0.920	0.948	0.972	0.968	0.992	1.000	1.000	0.932
0.12	0.988	0.992	0.980	0.888	0.996	0.892	0.956	0.944	0.972	0.980
0.14	0.952	0.940	0.948	0.956	0.964	0.972	0.988	1.000	0.996	0.892
0.16	0.940	0.956	0.940	0.940	0.920	0.944	0.908	0.960	0.976	0.964
0.18	0.988	0.996	0.976	0.996	0.984	0.988	0.972	0.988	0.900	0.880
0.2	0.988	0.984	0.984	0.980	0.972	0.980	0.984	0.968	0.976	0.972

p_2	p_3									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.02	0.952	0.976	0.936	0.968	0.992	0.928	0.976	0.992	0.968	0.984
0.04	0.984	0.992	0.944	0.944	0.976	0.940	0.956	0.992	0.960	0.996
0.06	0.996	0.916	0.952	0.972	0.864	0.896	0.944	0.988	0.948	0.972
0.08	0.908	0.952	0.976	0.988	0.928	0.988	0.972	0.888	0.940	0.972
0.10	0.968	0.992	1.000	0.996	0.976	0.980	0.988	0.876	0.960	0.972
0.12	0.988	0.988	0.932	0.960	0.980	0.996	0.996	0.956	0.960	0.992
0.14	0.948	0.972	0.980	0.988	0.996	1.000	0.972	0.980	0.984	0.980
0.16	0.980	0.992	1.000	0.936	0.956	0.976	0.996	0.984	1.000	0.972
0.18	0.932	0.948	0.960	0.964	0.996	1.000	1.000	0.972	0.960	0.988
0.2	0.976	0.988	1.000	0.996	0.940	0.940	0.960	0.980	0.992	1.000

Table 5: Variation between (0.9,0.95) upper tolerance bound and $90th$ percentile of population for p_1=0.02

p_2	p_3									
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.02	1.208	0.396	0.824	1.004	1.200	0.200	0.552	1.116	0.336	0.684
0.04	0.904	0.984	1.044	1.260	0.276	0.600	0.928	1.376	0.340	0.688
0.06	1.484	0.208	0.364	0.620	0.816	1.064	1.356	0.328	0.548	0.852
0.08	0.880	1.012	1.056	1.316	1.616	0.296	0.536	0.764	1.076	1.500
0.1	0.632	0.580	0.624	0.692	0.832	0.952	1.156	1.536	0.396	0.692
0.12	2.052	1.932	1.760	1.956	2.040	0.548	0.544	0.688	0.896	1.152
0.14	1.492	1.324	1.568	1.240	1.252	1.488	1.380	1.428	1.792	0.516
0.16	1.004	1.048	0.980	1.000	0.876	0.940	0.808	1.020	0.972	1.192
0.18	2.596	2.908	2.344	2.408	2.248	2.324	0.620	2.372	0.608	0.576
0.2	1.836	1.804	1.724	1.764	1.644	1.444	1.620	1.448	1.352	1.472

p_2	p_3									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.02	1.064	2.164	0.844	1.424	2.272	0.940	1.840	0.688	1.116	2.048
0.04	1.276	2.028	0.844	1.528	2.24	0.980	1.508	2.388	1.040	1.684
0.06	1.476	0.460	0.696	1.388	2.244	0.776	1.392	2.284	1.008	1.708
0.08	0.448	0.716	1.180	1.868	0.612	0.896	1.520	2.148	0.864	1.692
0.1	0.848	1.144	1.660	0.48	0.828	1.276	1.760	0.596	0.976	1.732
0.12	1.500	0.436	0.708	0.82	1.168	1.468	0.464	0.840	1.228	1.784
0.14	0.612	0.896	1.136	1.40	1.940	0.656	0.828	1.088	1.716	0.496
0.16	1.396	1.756	2.104	0.544	0.740	0.992	1.424	1.748	2.424	0.768
0.18	0.74	0.840	0.988	1.148	1.348	1.844	2.328	0.660	0.844	1.244
0.2	1.572	1.680	2.120	0.496	0.660	0.768	1.016	1.140	1.648	1.688