國立交通大學

物理研究所

碩士論文

Bianchi Type I 空間宇宙模型的動態系統解析

Analysis of Universe Model in the Bianchi Type I Space by Dynamical Systems

研究生:邱韋嵐

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研 究 生:邱韋嵐　　Student:Wei-Lan Chiu

指導教授:高文芳　　Advisor:Wen-Fang Kao

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學生:邱韋嵐　　　　　　　　　　　　　　　　　　指導教授:高文芳 博士

國立交通大學物理研究所

摘　　要

從 WMAP 的微波背景輻射實驗觀察數據中,可知宇宙在大尺度下是均質均向的時空,但微小區域還是存在些許的不均向,而這些微小的不均勻顯示,宇宙演化起始於不均向膨脹。因此,我們將利用 Bianchi type I 空間的特性,加入不同的宇宙"尺度因子",並且在系統中加入高階修正項求解,隨後將運用動力學系統的方法,分析這個解的穩定性。

Analysis of Universe Model in the Bianchi Type I Space by Dynamical Systems

Student:Wei-Lan Chiu **Advisor:Dr. W. F. Kao**

Institute of Physics
National Chiao Tung University

ABSTRACT

Form the observation of the cosmic microwave background radiation done by WMAP, the physical universe is highly homogeneous and isotropic with tiny anisotropic distribution reflecting the initial anisotropy of the expanding universe. The Bianchi type I space will be adopted along with higher order correction terms as the physical models in charge of the evolution of the anisotropic ally expanding universe. Expanding solutions can be solved accordingly. The evolution of these solutions can be treated as a dynamical system that provides detailed stability analysis of these solutions.

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目　　　　　錄

圖 目 錄

Chapter 1
Introduction

1.1 Background

在1964年，紐澤西州的貝爾實驗室工作兩位年輕的天文學家Penzias潘齊爾斯與Robert Wilson羅伯特．威爾森發現了宇宙背景輻射(Cosmology Microwave Background)，是一個波長七公分的電波換算成溫度大約是2.7K，從兩位科學家觀察到的數據可以發現宇宙的物質能量與動量分布上有很高度的均勻性(Homogeneity)與各向同性(Anisotropy)，接著在1989年11月升空的宇宙背景探測者，（COBE，Cosmic Background Explorer）測量到的結果，宇宙微波背景輻射光譜非常精確地符合溫度為 2.726±0.010K 的黑體輻射，也證實潘齊爾斯與羅伯特．威爾森的觀察是對的，從觀察而言，宇宙需要準確的理論來解釋宇宙發展，因而發展出General Relativity(G.R)，但三十多年的觀察，我們宇宙事實上並非如之前假設的各向同性(Anisotropy)，而是有著些微的差異，約 $\frac{\Delta T}{T} \approx 10^{-5}$ 的差異，這些微的差異對應到早期的宇宙，我們宇宙早期並非各向同性，為了證明宇宙的非各向同性(Anisotropy)，利用Bianchi Type 空間特性來表示些微的差異確實為非各向同性所引起的，再利用動態力學系統來找出Bianchi type 尺度因子的實際解，再由尺度因子來發展非各向同性(Anisotropy) 的宇宙，進而微擾找出宇宙是否穩定，也就是增加時間的演化是否會演變成現今的宇宙(de-Sitter Space)。

愛因斯坦(Albert-Einstein) 在 1905 寫出了狹義相對論，接著在 1915 年發展出廣義相對論，這兩大論可說是曠世鉅作，在廣義相對論下愛因斯坦解出了場方程式解，因此建立了宇宙模型基礎，接著前人的結果，John Barrow 等人導入高階修正項[1][2][3]，並且解出其解，而我們的研究工作就是在修正項下去解出不同宇宙空間的解，並了解其穩定性，下面內容將會使用 Bianchi Type I 宇宙空間並導入動態力學的運動方程式，接著再討論 Bianchi Type I 宇宙性質。

1.2 Einstein Universe

愛因斯坦宇宙是由時間，空間，重力所組成，因此進而發展出相對論，內容可分為狹義相對論和廣義相對論，內容包含了同時相對性、四維時空、彎曲時空等，因此相對論和量子力學的發展帶給物理學家革命性的變化，這兩種理論共同奠定近代物理學的基礎。

1.3 Special Relativity

愛因斯坦在 1905 年的論文中介紹狹義相對論，在狹義相對論提出以前，人們認為時間和空間是各自獨立且絕對存在，但愛因斯坦結合古典物理的時空觀和自己的想法發表了狹義相對論，此理論改變了當時的時空觀。

狹義相對論的兩個基本假設:

狹義相對性原理：物理定律在所有慣性參考座標不變。

光速不變原理：真空中的光速在任何參考系坐標下是不變的。

狹義相對論的時空間表現形式為

$$dx^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{1.1}$$

亦稱做 Minkowski 空間

Proper time 微分式為

$$d\tau = \frac{\sqrt{-dx^2}}{c} \tag{1.2}$$

羅倫茲轉換為(如果只對 x 軸作運動)

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.3}$$

1.4 General Relativity[4]

愛因斯坦在 1915 年發表關於廣義相對論的最初形式,廣義相對論是重力的度規理論，其核心是愛因斯坦場方程式。愛因斯坦場方程式的描述是用四維黎曼流形所表示的幾何空間，也可以說空間的幾何變化是由重力所產生的，在時空之中有能量-動量的張量關係，然而愛因斯坦場方程式是非線性的偏微分方程式，因此想要求得其精確解十分困難，需要一些技巧。

廣義相對論的兩個基本假設

等效原理：重力質量與慣性質量是相等的。

廣義協變性原理：任何物理規律與任何參考座標無關。

廣義相對論的時空間表現形式為

$$ds^2 = g_{ab}dx^a dx^b \tag{1.4}$$

metric tensor $g_{ab}(x) \in T(0\ ,2)$ 是一個用來量測長度的工具

運動在 Lagrangian 路徑上 $S_G = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}R$ 可以求出重力場方程式

愛因斯坦重力場方程式

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.5}$$

$G_{\mu\nu}$ 愛因斯坦張量

$R_{\mu\nu}$ 黎曼張量縮併而成的里奇張量，代表曲率項

$g_{\mu\nu}$ 四維時空的度規張量

$T_{\mu\nu}$ 能量-動量-應力張量

其中 $T_{\mu\nu}$ 的常數項是利用牛頓位能場的 Poisson equation $\nabla^2\Phi = 4\pi G\rho$ 得出

添加宇宙常數項

愛因斯坦為了使宇宙為靜態宇宙（無動態變化的宇宙，既不膨脹也不收縮），故嘗試加入一個常數 Λ 相關的項 $\Lambda g_{\mu\nu}$ 在場方程式中，使得場方程式形式變為：

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.6}$$

注意 $\Lambda g_{\mu\nu}$ 這一項為度規張量，以維持住守恆律，但事實是宇宙會膨脹，所以可以假設為零，近來，發現可能有暗能量與暗物質，所以又保留此項。

愛因斯坦在重力場方程式設想宇宙常數為獨立相關，但常數項可以在等號兩邊移動，因此

stress-energy tensor 可寫成:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda g_{\mu\nu}}{8\pi G} \right) \tag{1.7}$$

$$T_{\mu\nu}^{(vac)} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \tag{1.8}$$

此項為真空能量

Vacuum energy 寫成:

$$\rho_{vac} = \frac{\Lambda}{8\pi G} \tag{1.9}$$

宇宙常數的存在為非零真空能量，也就可以把$T_{\mu\nu}^{(vac)} = \frac{\Lambda}{8\pi G} g_{\mu\nu}$和$T_{\mu\nu}$看成是一樣類型的量，只是$T_{\mu\nu}$的來源是物質與輻射，而$T_{\mu\nu}^{(vac)} = \frac{\Lambda}{8\pi G} g_{\mu\nu}$的來源是真空能量。

真空場方程式解

我們假設宇宙常數為零

若能量-動量張量$T_{\mu\nu} = 0$為零，則場方程式稱作真空場方程式。

真空場方程式可寫為：

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \tag{1.10}$$

使用 Trace，得

$$R = \frac{1}{2} 4R = 2R$$

$$R = 0 \tag{1.11}$$

$$R_{\mu\nu} = 0 \tag{1.12}$$

若宇宙常數不為零下，真空場方程式為：

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} \tag{1.13}$$

相同使用 Trace

$$R = 4\Lambda \tag{1.14}$$

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.15}$$

1.5 FRW metrics

RW metric 是 H.P.羅伯森和沃克分別於 1935 年和 1936 年證明，由於俄國數學家弗里德曼也作出了重要的貢獻，因此也稱作(Friedmann Robertson-Walker metric，縮寫為 FRW metrics）。[5]

在二維均勻且無向的三種曲面幾何可寫為:

$$ds^2 = a^2\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\theta^2\right) \tag{1.16}$$

其中 k=0，1，-1 分別為平面，二微球面，與假球面

在四維時空均勻且無向的 FRW metric 可寫為:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1.17}$$

$a(t)$ 被稱宇宙的「尺度因子」，它能透過愛因斯坦方程式與宇宙間物質能量對作用力相連結，其中 k=0，1，-1 分別為:

k=1 時， 三維空間是球狀的，總體積是有限的，其值為 $2a(t)$

k=-1 時，三維空間是雙曲空間，總體積是無限的

k=0 時， 三維空間是平直的，總體積也是無限的

描述一個均勻且同向性的膨脹宇宙模型需要兩個獨立的弗里德曼方程式，考慮在理想流體下的 FRW metrics 利用愛因斯坦方程式其中(0，0)分量可算出:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.18}$$

另外利用愛因斯坦方程式 trace 可算出:

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.19}$$

a，H ，ρ ，p 是隨時間變化的函數，$H = \dfrac{\dot{a}(t)}{a(t)}$ 是哈柏參數表示宇宙膨脹速率

利用第一個方程式，第二個方程式可變形成

$$\dot{\rho} = -3H\left(\rho + p\right) \tag{1.20}$$

此式消除了宇宙常數並且符合質能守恆定律

因此可重新定義兩個獨立的弗里德曼方程式

$$\rho \to \rho + \frac{\Lambda c^2}{8\pi G} \tag{1.21}$$

$$p \to p - \frac{\Lambda c^2}{8\pi G} \tag{1.22}$$

兩個方程式轉換而得

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.23}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) \tag{1.24}$$

我們可稱第一個方程式為弗里德曼方程式，第二個為弗里德曼加速方程式

密度參數(density parameter) [6]

把第一弗里德曼方程式(1.23)重新整理成

$$\frac{k}{H^2 a^2} = \frac{\rho}{3H^2/8\pi G} - 1 = \Omega - 1 \tag{1.25}$$

我們在此是令 c=1，其中密度參數 Ω 定義作物質密度(matter density) ρ 與臨界密度(critical density) ρ_C 的比，當中 $\rho_C = \frac{3H^2}{8\pi G}$，如此一來我們就把物質分布與宇宙的物理幾何相連起來，在臨界密度時宇宙的形狀是平直的。在真空能量密度為零的假設下，如果密度參數大於一，宇宙在空間上是閉合的，宇宙最終會停止膨脹並開始塌縮；如果密度參數小於一，宇宙在空間上是開放的，宇宙會一直保持膨脹下去。也就是:

$$\Omega > 1 \Rightarrow k = +1 \quad \text{closed universe}$$

$$\Omega = 1 \Rightarrow k = 0 \quad \text{flat universe}$$

$$\Omega < 1 \Rightarrow k = -1 \quad \text{open universe}$$

1.6 De Sitter universe

我們現在的宇宙比較像德希特宇宙(De Sitter universe)，De Sitter Space 是愛因斯坦場方程式的一個解，由德國物理學家 Willem de Sitter 所解出，這個模型是忽略物質的膨脹宇宙模型，而且宇宙的膨脹取決於宇宙常數項(現今認為此宇宙常數為暗能量所導致)，所以較大的宇宙常數會有大的膨脹速率:

$$H \propto \sqrt{\Lambda} \tag{1.26}$$

我們可以令 FRW metric 的宇宙尺度因子 $a(t)$ 為:

$$a(t) = e^{Ht} \text{，且令} H = \sqrt{\frac{\Lambda}{3}} \tag{1.27}$$

帶入 FRW metric 可寫成:

$$ds^2 = dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t}\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1.28}$$

亦可換作笛卡爾座標的 De Sitter space(k=0)為:

$$ds^2 = dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t}\left(dx^2 + dy^2 + dz^2\right) \tag{1.29}$$

(1.29)是一個均質且同向性的膨脹宇宙，與我們現在所觀察到的宇宙相似，因而此說。

Chapter 2
Bianchi geometric type

2.1 Mathematics conception

Bianchi 模型其實是數學幾何，我們先從黎曼幾何流形，介紹共變導數，接著是李導數中的李括號，然後是用 Killing equation 來定義李群中的空間形式，就可整理出所有的 Bianchi Type，在此模型上，我們還須討論它的穩定性，所以需要用到 **Theory of stability** 的概念來找出穩定點與幾何的關係，以方便我了解宇宙模型的發展，並藉由時間演化來看會形成何種宇宙模型。

2.2 Riemannian manifold

黎曼流形(Riemannian manifold) [7]是彎曲的，是種微分流形，在每點 P 的切空間都可定義向量積，在 P 的附近會呈現平滑的改變，因此可定義弧線長度，曲率，面積，體積，角度，函數梯度等。

定義: $\gamma:[a,b]\to M$ 是黎曼流形 M 中的一段連續可微的弧線，可定義長度為:

$$L(\gamma)=\int_a^b\left\|\gamma'(t)\right\|dt \tag{2.1}$$

我們定義黎曼流形和 R^n 的平滑子流形是等距同構的度規空間，等距是指內蘊度規與 R^n 都可以導出相同的度規，而每個 R^n 的平滑子流形都可以導出黎曼度規。

度規空間是一個集合，可以用來定義集合元間的距離，**度規張量**是用來衡量度規空間中距離及角度的二階張量，選定一個座標系統 x_i 其度規張量可用矩陣表示記為 g_{ij}，一段弧線長度定義為:

$$L(\gamma)=\int_a^b\sqrt{g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}}dt \tag{2.2}$$

列舉坐標的歐氏度規分別為:

直角坐標: $(x_0,x_1,x_2,x_3)=(t,x,y,z)$ 又稱平面閔可夫斯基空間

$$g_{ij}=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{2.3}$$

圓柱坐標: $(x_0,x_1,x_2,x_3)=(t,r,\theta,z)$

$$g_{ij}=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&r^2&0\\0&0&0&1\end{bmatrix} \tag{2.4}$$

球坐標: $(x_0,x_1,x_2,x_3)=(t,r,\theta,\varphi)$

$$g_{ij}=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&r^2&0\\0&0&0&r^2\sin^2\varphi\end{bmatrix} \tag{2.5}$$

2.3 共變導數

共變導數(微分)(covariant differentiation)是在流形 M 上沿著向量場導數的方法，向量 u 沿著向量 v 的共變導數寫作 $\nabla_v u$ 或 $D_v u$，也是向量叢沿著空間的一個切向量截面的導數，共變導數有下列性質:

$$\text{加法法則: } \nabla_v(u+k)=\nabla_v u+\nabla_v k \tag{2.6}$$

$$\text{乘法法則: } \nabla_v fu = f\nabla_v u + u\nabla_v f \tag{2.7}$$

$$\text{線性代數法則: } \nabla_{fv+gk} u = f\nabla_v u + g\nabla_k u \tag{2.8}$$

u,v,k 是向量，f,g 是純量函數。

共變微分的定義:

$$D_a A_b \equiv \partial_a A_b - \Gamma_{ab}\left(A_a, g_{ab}\right) \tag{2.9}$$

第一項代表了向量場 A 的分量變化，第二項稱為坐標系對於共變微分的"扭轉"以便線性化，因為對一般作微分會多出項，為了消掉而減掉 $\Gamma_{ab}\left(A_a, g_{ab}\right)$ 此項.

我們對 A 做坐標轉換:

$$\partial'_a A'_b\left(x'\right) = \frac{\partial x^c}{\partial x'^a}\partial_c\left[\frac{\partial x^d}{\partial x'^b}A_d\left(x\right)\right] = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\partial_c A_d\left(x\right) + \frac{\partial^2 x^d}{\partial x'^a \partial x'^b}A_d\left(x\right) \tag{2.10}$$

因為期望能為 $D'_a A'_b = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}D_c A_d$ 故須有 $\Gamma_{ab}\left(A_a, g_{ab}\right)$ 來消掉多餘的項。

Γ^c_{ab} (Christoffel symbols notation)的導法

因為度規張量的共變微分為 0 推導

$$D_b g_{ca} = \frac{\partial g_{ca}}{\partial x^b} - g_{da}\Gamma^d_{cb} - g_{cd}\Gamma^d_{ab} = 0 \tag{2.11}$$

利用交換指標(index)，再求和可得，

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}\right) \tag{2.12}$$

2.4 Lie Derivation

李導數(Lie Derivative)是在流形 M 上的平滑函數所組成在代數上求導數的運算，所有的李導數組成的向量空間對應於李括號會組織成無限維的李代數。[8]

李括號:

$$\left[A, B\right] = L_A B = -L_B A \tag{2.13}$$

我們可以把切空間的基向量寫作 $\frac{\partial}{\partial x^a}$，所以一個向量場可表示成

$$X = X^a \frac{\partial}{\partial x^a} \tag{2.14}$$

重新定義李括號:

$$[X,Y]=X^a\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b}-Y^a\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial x^b}$$
$$=L_X Y \tag{2.15}$$

對一個可微的函數 f 的李導數為:

$$[X,Y]f=X^a\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b}f-Y^a\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial x^b}f$$
$$=X(Yf)-Y(Xf) \tag{2.16}$$

2.5 Killing equation

首先我們先定義 N 維的度規空間[9]

$$ds^2=\sum_{i,k}^{1..n}g_{ik}dx_i dx_k \tag{2.17}$$

Killing equation 為了保持任意兩點間的距離固定，用了一微小的轉換運算符"X"，也就是空間作微小的運動，任兩點的距離不變

定義運算符:

$$Xf=\sum_{r}^{1..n}\xi_r\frac{\partial f}{\partial x_r} \tag{2.18}$$

我們對 N 維的度規空間作轉換

$$X(ds^2)=\sum_{i,k}X(g_{ik})dx_i dx_k+\sum_{r,k}g_{rk}dX(x_r)dx_k+\sum_{i,r}g_{ir}dX(x_r)dx_k \tag{2.19}$$

整理(2.18)，且要為 0

$$X(ds^2)=\sum_{i,k}\left[\sum_r\left(\xi_r\frac{\partial(g_{ik})}{\partial x_r}+g_{rk}\frac{\partial\xi_r}{\partial x_i}+g_{ik}\frac{\partial\xi_r}{\partial x_k}\right)\right]dx_i dx_k=0 \tag{2.20}$$

其中

$$\left[\sum_r\left(\xi_r\frac{\partial(g_{ik})}{\partial x_r}+g_{rk}\frac{\partial\xi_r}{\partial x_i}+g_{ir}\frac{\partial\xi_r}{\partial x_k}\right)\right]dx_i dx_k=0, i,k=1,2,....,n \tag{2.21}$$

故，

$$\sum_r \left(\xi_r \frac{\partial \left(g_{ik} \right)}{\partial x_r} + g_{rk} \frac{\partial \xi_r}{\partial x_i} + g_{ir} \frac{\partial \xi_r}{\partial x_k} \right) = 0 \tag{2.22}$$

Killing equation:

$$g_{ik,r}\xi_r + g_{rk}\xi_{r,i} + g_{ik}\xi_{r,k} = 0 \tag{2.23}$$

2.6 在二維的的度規空間計算

我們二維的度規空間 G_2 為

$$ds^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2 \quad , \quad g_{12} = g_{21} \tag{2.24}$$

考慮李括號$\left[X_1, X_2\right] f = 0$，與$\left[X_1, X_2\right] f = X_1 f$ 的空間可能性

先從$\left[X_1, X_2\right] f = 0$ 算起

$$\left[X_1, X_2\right] f = X_1X_2f - X_2X_1f = \xi_1 \frac{\partial \xi_2}{\partial x_1}\frac{\partial}{\partial x_2} f - \xi_2 \frac{\partial \xi_1}{\partial x_2}\frac{\partial}{\partial x_1} f = 0 \tag{2.25}$$

$X_1 f = \xi_1 \frac{\partial}{\partial x_1}$ ，$X_2 f = \xi_2 \frac{\partial}{\partial x_2}$ ，ξ_1, ξ_2, ξ_3 為 x_1, x_2, x_3 的函數

由上式(2.23)可得知

$$\frac{\partial \xi_1}{\partial x_2} = \frac{\partial \xi_2}{\partial x_1} = 0 \quad \text{，因} \ \frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f \neq 0 \tag{2.26}$$

故可令，

$$\xi_1 = \alpha = \xi_2 = \beta \qquad \alpha \text{ ，} \beta = constant \tag{2.27}$$

接著算，$\left[X_1, X_2\right] f = X_1 f$

$$= X_1X_2f - X_2X_1f = \xi_1 \frac{\partial \xi_2}{\partial x_1}\frac{\partial}{\partial x_2} f - \xi_2 \frac{\partial \xi_1}{\partial x_2}\frac{\partial}{\partial x_1} f = \xi_1 \frac{\partial}{\partial x_1} f \tag{2.28}$$

相同的對照，可得出

$$-\xi_2 \frac{\partial \xi_1}{\partial x_2} = \xi_1 \quad , \qquad \frac{\partial \xi_2}{\partial x_1} = 0 \tag{2.29}$$

(2.29)故可得

$$\xi_1 = e^{-x_2} \quad , \qquad \xi_2 = \beta \tag{2.30}$$

再帶入(2.23)

Killing equation 來保持任兩點距離不變

二維 Killing equation

$$\sum_r \left(\xi_r \frac{\partial (g_{ik})}{\partial x_r} + g_{rk} \frac{\partial \xi_r}{\partial x_i} + g_{ir} \frac{\partial \xi_r}{\partial x_k} \right)$$

$$= \xi_1 \left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{21}}{\partial x_1} + \frac{\partial g_{22}}{\partial x_1} \right) + \left(\begin{array}{l} 2g_{11}\frac{\partial \xi_1}{\partial x_1} + 2g_{11}\frac{\partial \xi_1}{\partial x_2} + g_{12}\frac{\partial \xi_1}{\partial x_1} + g_{21}\frac{\partial \xi_1}{\partial x_1} \\ + g_{12}\frac{\partial \xi_1}{\partial x_2} + g_{21}\frac{\partial \xi_1}{\partial x_2} \end{array} \right) \tag{2.31}$$

$$+ \xi_2 \left(\frac{\partial g_{11}}{\partial x_2} + \frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} \right) + \left(\begin{array}{l} 2g_{22}\frac{\partial \xi_2}{\partial x_1} + 2g_{22}\frac{\partial \xi_2}{\partial x_2} + g_{12}\frac{\partial \xi_2}{\partial x_1} + g_{21}\frac{\partial \xi_2}{\partial x_1} \\ + g_{12}\frac{\partial \xi_2}{\partial x_2} + g_{21}\frac{\partial \xi_2}{\partial x_2} \end{array} \right) = 0$$

條件 $[X_1, X_2] f = 0$，令 $\xi_1 = \alpha = \xi_2 = \beta = 1$

(2.31)帶入(2.30)做整理得:

$$\left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{21}}{\partial x_1} + \frac{\partial g_{22}}{\partial x_1} \right) + \left(\frac{\partial g_{11}}{\partial x_2} + \frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} \right) = 0 \tag{2.32}$$

得 $g_{11} = g_{12} = g_{22} = constant$ ， 度規空間為

$$ds^2 = \varepsilon_1 dx_1^2 + 2\varepsilon_2 dx_1 dx_2 + \varepsilon_3 dx_2^2 \text{，} \varepsilon_i = \ constant \tag{2.33}$$

再經空間旋轉且令係數為 1，整理得

$$ds^2 = dx_1^2 + dx_2^2 \tag{2.34}$$

接著空間條件(2.26) (2.28) $[X_1, X_2] f = X_1 f$ ，$\xi_1 = e^{-x_2}$ ，$\xi_2 = \beta = 1$

帶入整理(2.31)式

$$e^{-x_2}\left(\frac{\partial g_{11}}{\partial x_1}+\frac{\partial g_{12}}{\partial x_1}+\frac{\partial g_{21}}{\partial x_1}+\frac{\partial g_{22}}{\partial x_1}\right)-e^{-x_2}\left(2g_{11}+g_{12}+g_{21}\right)=0 \tag{2.35}$$

$$e^{-x_2}\frac{\partial g_{11}}{\partial x_1}-e^{-x_2}2g_{11}=0\Rightarrow\frac{\partial g_{11}}{\partial x_1}=2g_{11}\Rightarrow g_{11}=e^{2x_1} \tag{2.36}$$

$$e^{-x_2}\frac{\partial g_{12}}{\partial x_1}-e^{-x_2}2g_{12}=0\Rightarrow\frac{\partial g_{12}}{\partial x_1}=g_{12}\Rightarrow g_{12}=e^{x_1} \tag{2.37}$$

$$\frac{\partial g_{22}}{\partial x_1}=0\Rightarrow g_{22}=constant \tag{2.38}$$

(2.36)~ (2.38)帶回(2.24)的度規空間為

$$ds^2=e^{2x_1}dx_1^2+e^{x_1}dx_1dx_2+dx_2^2 \tag{2.39}$$

2.7 九種三維運動的類型 [10] [11]

I. $[X_1,X_2]f=[X_1,X_3]f=0$，$[X_2,X_3]f=0$

II. $[X_1,X_2]f=[X_1,X_3]f=0$，$[X_2,X_3]f=X_1f$

III. $[X_1,X_2]f=0$，$[X_1,X_3]f=X_1f$，$[X_2,X_3]f=0$

IV. $[X_1,X_2]f=0$，$[X_1,X_3]f=X_1f$，$[X_2,X_3]f=X_1f+X_2f$

V. $[X_1,X_2]f=0$，$[X_1,X_3]f=X_1f$，$[X_2,X_3]f=X_2f$

VI. $[X_1,X_2]f=0$，$[X_1,X_3]f=X_1f$，$[X_2,X_3]f=hX_2f\,(h\neq 0,1)$

VII. $[X_1,X_2]f=0$，$[X_1,X_3]f=X_2f$，$[X_2,X_3]f=-X_1f+hX_2f\,(0\le h<2)$

VIII. $[X_1,X_2]f=X_1f$，$[X_1,X_3]f=2X_2f$，$[X_2,X_3]f=X_3f$

IX. $[X_1,X_2]f=X_3f$，$[X_1,X_3]f=X_1f$，$[X_2,X_3]f=X_2f$

2.8 Bianchi Type I 度規空間

Bianchi 把李群在歐基理德空間類型分類成九個，就是上頁的空間形式，我們在此討論 **Bianchi Type I** 的度規空間，依上頁的 **Bianchi Type I** 空間條件

$$[X_1, X_2]f = [X_1, X_3]f = 0, [X_2, X_3]f = 0 \tag{2.40}$$

可以利用二維的算法對應到三維，可得

$$\xi_1 = \alpha, \xi_2 = \beta, \xi_3 = \gamma \text{ ， } \alpha, \beta, \gamma = constant \tag{2.41}$$

我們在此令為 1

使用三維 Killing equation(2.22)

$$\begin{aligned}
&\sum_r \left(\xi_r \frac{\partial (g_{ik})}{\partial x_r} + g_{rk} \frac{\partial \xi_r}{\partial x_i} + g_{ir} \frac{\partial \xi_r}{\partial x_k} \right) \\
&= \xi_1 \left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{13}}{\partial x_1} + \frac{\partial g_{21}}{\partial x_1} + \frac{\partial g_{22}}{\partial x_1} + \frac{\partial g_{23}}{\partial x_1} + \frac{\partial g_{31}}{\partial x_1} + \frac{\partial g_{32}}{\partial x_1} + \frac{\partial g_{33}}{\partial x_1} \right) \\
&+ \left(\begin{aligned} &2g_{11} \frac{\partial \xi_1}{\partial x_1} + 2g_{11} \frac{\partial \xi_1}{\partial x_2} + 2g_{11} \frac{\partial \xi_1}{\partial x_3} + g_{12} \frac{\partial \xi_1}{\partial x_1} + g_{13} \frac{\partial \xi_1}{\partial x_1} + g_{21} \frac{\partial \xi_1}{\partial x_1} + g_{31} \frac{\partial \xi_1}{\partial x_1} \\ &+ g_{12} \frac{\partial \xi_1}{\partial x_2} + g_{13} \frac{\partial \xi_1}{\partial x_2} + g_{21} \frac{\partial \xi_1}{\partial x_2} + g_{31} \frac{\partial \xi_1}{\partial x_2} + g_{12} \frac{\partial \xi_1}{\partial x_3} + g_{13} \frac{\partial \xi_1}{\partial x_3} + g_{21} \frac{\partial \xi_1}{\partial x_3} + g_{31} \frac{\partial \xi_1}{\partial x_3} \end{aligned} \right) \\
&+ \xi_2 \left(\frac{\partial g_{11}}{\partial x_2} + \frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{13}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} + \frac{\partial g_{23}}{\partial x_2} + \frac{\partial g_{31}}{\partial x_2} + \frac{\partial g_{32}}{\partial x_2} + \frac{\partial g_{33}}{\partial x_2} \right) \\
&+ \left(\begin{aligned} &2g_{22} \frac{\partial \xi_2}{\partial x_1} + 2g_{22} \frac{\partial \xi_2}{\partial x_2} + 2g_{22} \frac{\partial \xi_2}{\partial x_3} + g_{12} \frac{\partial \xi_2}{\partial x_1} + g_{32} \frac{\partial \xi_2}{\partial x_1} + g_{21} \frac{\partial \xi_2}{\partial x_1} + g_{23} \frac{\partial \xi_2}{\partial x_1} \\ &+ g_{12} \frac{\partial \xi_2}{\partial x_2} + g_{32} \frac{\partial \xi_2}{\partial x_2} + g_{21} \frac{\partial \xi_2}{\partial x_2} + g_{23} \frac{\partial \xi_2}{\partial x_2} + g_{12} \frac{\partial \xi_2}{\partial x_3} + g_{32} \frac{\partial \xi_2}{\partial x_3} + g_{21} \frac{\partial \xi_2}{\partial x_3} + g_{23} \frac{\partial \xi_2}{\partial x_3} \end{aligned} \right) \\
&+ \xi_3 \left(\frac{\partial g_{11}}{\partial x_3} + \frac{\partial g_{12}}{\partial x_3} + \frac{\partial g_{13}}{\partial x_3} + \frac{\partial g_{21}}{\partial x_3} + \frac{\partial g_{22}}{\partial x_3} + \frac{\partial g_{23}}{\partial x_3} + \frac{\partial g_{31}}{\partial x_3} + \frac{\partial g_{32}}{\partial x_3} + \frac{\partial g_{33}}{\partial x_3} \right) \\
&+ \left(\begin{aligned} &2g_{33} \frac{\partial \xi_3}{\partial x_1} + 2g_{33} \frac{\partial \xi_3}{\partial x_2} + 2g_{33} \frac{\partial \xi_3}{\partial x_3} + g_{13} \frac{\partial \xi_3}{\partial x_1} + g_{23} \frac{\partial \xi_3}{\partial x_1} + g_{31} \frac{\partial \xi_3}{\partial x_1} + g_{32} \frac{\partial \xi_3}{\partial x_1} \\ &+ g_{13} \frac{\partial \xi_3}{\partial x_2} + g_{32} \frac{\partial \xi_3}{\partial x_2} + g_{31} \frac{\partial \xi_3}{\partial x_2} + g_{23} \frac{\partial \xi_3}{\partial x_2} + g_{13} \frac{\partial \xi_3}{\partial x_3} + g_{32} \frac{\partial \xi_3}{\partial x_3} + g_{31} \frac{\partial \xi_3}{\partial x_3} + g_{23} \frac{\partial \xi_3}{\partial x_3} \end{aligned} \right) \\
&= 0
\end{aligned} \tag{2.42}$$

(2.41)帶入(2.42)得

$$\left(\frac{\partial g_{11}}{\partial x_1}+\frac{\partial g_{12}}{\partial x_1}+\frac{\partial g_{13}}{\partial x_1}+\frac{\partial g_{21}}{\partial x_1}+\frac{\partial g_{22}}{\partial x_1}+\frac{\partial g_{23}}{\partial x_1}+\frac{\partial g_{31}}{\partial x_1}+\frac{\partial g_{32}}{\partial x_1}+\frac{\partial g_{33}}{\partial x_1}\right)$$

$$+\left(\frac{\partial g_{11}}{\partial x_2}+\frac{\partial g_{12}}{\partial x_2}+\frac{\partial g_{13}}{\partial x_2}+\frac{\partial g_{21}}{\partial x_2}+\frac{\partial g_{22}}{\partial x_2}+\frac{\partial g_{23}}{\partial x_2}+\frac{\partial g_{31}}{\partial x_2}+\frac{\partial g_{32}}{\partial x_2}+\frac{\partial g_{33}}{\partial x_2}\right)$$

$$+\left(\frac{\partial g_{11}}{\partial x_3}+\frac{\partial g_{12}}{\partial x_3}+\frac{\partial g_{13}}{\partial x_3}+\frac{\partial g_{21}}{\partial x_3}+\frac{\partial g_{22}}{\partial x_3}+\frac{\partial g_{23}}{\partial x_3}+\frac{\partial g_{31}}{\partial x_3}+\frac{\partial g_{32}}{\partial x_3}+\frac{\partial g_{33}}{\partial x_3}\right)=0 \qquad (2.43)$$

$$g_{11}=g_{22}=g_{33}=g_{12}=g_{13}=g_{21}=g_{23}=g_{31}=g_{32}=constant \qquad (2.44)$$

相同的利用坐標旋轉，再令係數為 $a_i^2(t)$ 可得出，

Bianchi Type I 的度規空間為:

$$ds^2=-dt^2+a_1^2(t)dx^2+a_2^2(t)dy^2+a_3^2(t)dz^2 \qquad (2.45)$$

這也是我們接下來要研究的宇宙空間度規

2.9 Bianchi Type I different form

Bianchi Type I 的形式有很多不同種的形式表現，這裡呈現有四種:

$$ds^2=-dt^2+A(t)^2dx^2+B(t)^2dy^2+C(t)^2dz^2 \qquad (2.46)$$

$$ds^2=-dt^2+e^{2at}dx^2+e^{2bt}dy^2+e^{2ct}dz^2 \qquad (2.47)$$

$$ds^2=-dt^2+(\alpha_1 t+\alpha_2)^{2n_1}dx^2+(\alpha_1 t+\alpha_2)^{2n_2}dy^2+(\alpha_1 t+\alpha_2)^{2n_3}dz^2 \qquad (2.48)$$

$$ds^2=-dt^2+e^{2bt}\left[e^{-4\sigma_+ t}dx^2+e^{2\left(\sigma_++\sqrt{3}\sigma_-\right)t}dy^2+e^{2\left(\sigma_+-\sqrt{3}\sigma_-\right)t}dz^2\right] \qquad (2.49)$$

上述四個為不同的空間表示方式，基本算法會不盡相同，但最後的結論會一樣，然而它們之間可以互相轉換，我們此研究會用(2.46) (2.49)兩種不同式子去換算，當然可以算出相同的解，在穩定性上也求出相同的結果。

2.10 All Bianchi Type 度規空間

I. $ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2$

II. $ds^2 = -dt^2 + a_1^2(t)\left(dx + zdy\right)^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2$

III. $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)\left(\cosh zdx + \sinh zdy\right)^2 + a_2^2\left(\sinh zdx + \cosh zdy\right)^2\right) + a_3^2(t)dz^2$

IV. $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)\left(dx + zdy\right)^2 + a_2^2(t)dy^2\right) + a_3^2(t)dz^2$

V. $ds^2 = -dt^2 + e^{-2z}\left(a_1^2(t)dx^2 + a_2^2(t)dy^2\right) + a_3^2(t)dz^2$

VI. $ds^2 = -dt^2 + a_1^2(t)\left(\cosh zdx + \sinh zdy\right)^2 + a_2^2\left(\sinh zdx + \cosh zdy\right)^2 + a_3^2(t)dz^2$

VII. $ds^2 = -dt^2 + a_1^2(t)\left(\cos zdx + \sin zdy\right)^2 + a_2^2\left(\sin zdx - \cos zdy\right)^2 + a_3^2(t)dz^2$

VIII. $ds^2 = -dt^2 + a_1^2(t)\left(\cosh zdx - \sinh z \sinh xdy\right)^2 + a_2^2\left(\sinh zdx - \cosh z \sinh xdy\right)^2$

$+a_3^2(t)\left(dz + \cosh xdy\right)^2$

IX. $ds^2 =$

$-dt^2 + a_1^2(t)\left(\cos zdx + \sin z \sin xdy\right)^2 + a_2^2 - \left(\sin zdx + \cos z \sin xdy\right)^2 + a_3^2(t)\left(dz + \cos xdy\right)^2$

2.11 Theory of stability

在一個力學系統函數的時間微分表示此系統的時間演化，也就是在某個時間點的系統狀態可以被一個狀態空間中的一個基數來描述，因此我們可以考慮一個應力系統滿足[12][13]，

$$\dot{x} = f(x\ ，t)\text{，}x(t_0) = x_0 \qquad x \in R^n \tag{2.50}$$

我們假設 $f(x\ ，t)$ 滿足存在唯一解且對 x 具連續性在時間 t 是均勻的。

如果 $f(x^*\ ，t)=0$ 那麼就會有一點 $x^* \in R^n$ 為一平衡點，我們就可大膽假設此平衡點為 locally stable(假如起始條件都在 x^* 附近)，如果所有靠近 x^* 的解經過 $t \to \infty$ 時都會朝向 x^*，那麼我們會稱此為 locally asymptotically stable，相反的如果所有靠近 x^* 的解經過 $t \to \infty$ 時都遠離朝向 x^*，則為 unstable，因此我們可拿此微分系統的解來討論平衡點附近的性質，用一般線性化 DE 來討論其穩定性，考慮一個線性 DE $\dot{x} = Ax$，A 與 eigenvalues 和 eigenvectors 有關聯，可解出 $x = e^{At} = e^{\lambda t}$，線性 DE 為 $\dot{x} = \lambda x$，所以只要能符合此式，即可知道 eigenvalues。

我們定義 R^n 的三個子空間:

The stable subspace $\qquad E^s = span(s_1, s_2, ..., s_n) \qquad$ (2.51)

The unstable subspace $\qquad E^u = span(u_1, u_2, ..., u_n) \qquad$ (2.52)

The centre subspace $\qquad E^c = span(c_1, c_2, ..., c_n) \qquad$ (2.53)

這裡，$s_1, s_2, ..., s_n$ 有 eigenvalues 實數部為負的 eigenvectors，就是 $\lambda<0$

$u_1, u_2, ..., u_n$ 有 eigenvalues 實數部為正的 eigenvectors，就是 $\lambda>0$

$c_1, c_2, ..., c_n$ 有 eigenvalues 實數部為零的 eigenvectors ，就是 $\lambda = 0$

也就是，$x \in E^s$ 暗示 $\lim_{t \to +\infty} e^{At} x = 0$ (2.54)

$x \in E^u$ 暗示 $\lim_{t \to -\infty} e^{At} x = 0$ (2.55)

三個子空間集合起來為 R^n

$$E^s \oplus E^u \oplus E^c = R^n \tag{2.56}$$

如下圖，在 stable subspace 起始點都會趨向穩定點 0

在 unstable subspace 起始點都會遠離穩定點 0

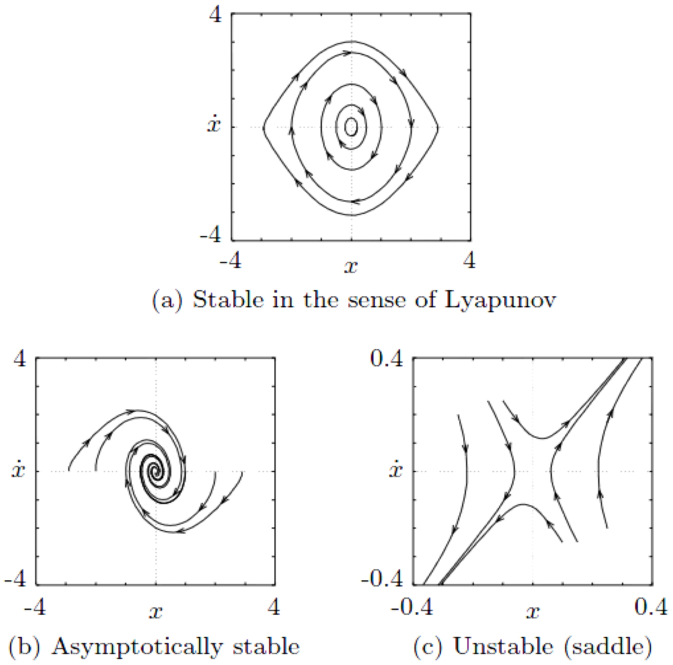

圖 2. 1 Phase portraits for stable and unstable equilibrium points[13]

Chapter 3
Biachi Type I research

3.1 Gravitational action

我們會使用力學系統來研究 Bianchi type I universe，因為運動方程式過於複雜，所以會專注在實際的穩定性解和探討宇宙空間發展的模式，從均向性到非均向性的膨脹形式，由於一些人找到新的膨脹模型與一般宇宙模型不同，所以顯示膨脹模型可以依空間幾何的不同而產生不同的模型[14]。

首先，我們延伸廣義相對論的路徑形式參考 1.4 節，嚐試多加[15] αR^2，$\beta R_{\mu\nu}R^{\mu\nu}$，加入此項能更完整的描述重力理論，而這些項允許我們在奇異點處的空間形式的解釋，也可以得知早期宇宙的膨脹方式，因為可以把早期宇宙看成一奇異點，然後依力學膨脹到現在的宇宙，所以我們考慮重力路徑方程式為[16]

$$S_G = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda) \tag{3.1}$$

加 R^2 項的效應是影響膨脹的速率與範圍也表示能量的大小次方愈大能量愈大，而加入 $R_{\mu\nu}R^{\mu\nu}$ 項可以使非均向的膨脹更加顯著，但我們很難總結單純的修正項是怎樣的宇宙形態，所以通常會合併使用，然後再調整 α，β 常數項就可得到修正的宇宙模型。一般的 Einstein equation 為(1.5) $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$，因為有了修正項所以 Einstein equation 也會跟著改變，在此我們多加 stress tensor term $\Phi_{\mu\nu}$ 項，我們對(3.1)式做路徑最短變分，然後把把多出來的方程式項都丟到 $\Phi_{\mu\nu}$ 裡，因此我們可以得到 Einstein 修正方程式。

修正 Einstein equation:

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.2}$$

如果 $T_{\mu\nu}=0$ 則我們稱為真空解，Einstein equation 可以表示成下列式子方法，推導參照 Appendix A，高階修正項 Einstein equation 表示式:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+2\alpha\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)R+(2\alpha+\beta)\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)R+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)$$
$$+2\beta\left(R_{\mu\sigma\nu\rho}R^{\sigma\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}\right)+g_{\mu\nu}\Lambda=0 \tag{3.3}$$

這裡 $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ 則 stress tensor 可表示成

$$\Phi_{\mu\nu}=2\alpha\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)R+(2\alpha+\beta)\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)R+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)$$
$$+2\beta\left(R_{\mu\sigma\nu\rho}R^{\sigma\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}\right) \tag{3.4}$$

在此我們得到高階修正項 Einstein equation 用 R，$R_{\mu\nu}$，$R_{\mu\sigma\nu\rho}$ 這些空間曲率來表示，更能描述空間形態，而 α ，β 為可調參數能調控空間表現形式。

例如: $\alpha=\beta=0$ 則 $\Phi_{\mu\nu}=0$ 使宇宙模型變回一般的廣義相對論，宇宙膨脹是均勻均向的，也就是 Einstein 所解出來的解，但在此我們研究更廣義的宇宙模型，所以我們令 $\alpha,\beta\neq 0$ 繼續接下來的研究。

3.2 Equation of Motion

John Barrow 等人利用與膨脹速率有關的變數當算符，進而簡化運動方程式並列出一般運動方程式，我們也依此方法做動態系統分析，但其中的化簡合併的過程複雜，我們一一詳述其過程，列出所有方程式之後，就可以依其宇宙模型特性來解出不同的解與特徵值，並可了解宇宙是否為穩定狀態，在此使用的是 Bianchi Type I，其度規為(2.49)式:

$$ds^2=-dt^2+e^{2bt}\left[e^{-4\sigma_+ t}dx^2+e^{2\left(\sigma_+ +\sqrt{3}\sigma_-\right)t}dy^2+e^{2\left(\sigma_+ -\sqrt{3}\sigma_-\right)t}dz^2\right]$$

因此在此度規中限制宇宙模型的剪向量為對角線化，這樣的假設是因為剪向量的對角分量為 $\sigma_{\alpha\beta}=diag\left(\sigma_{11},\sigma_{22},\sigma_{33}\right)$，而 $\sigma_{\alpha\beta}$ 是 trace-free，所以只要兩個獨立分量就可表示，方便起見用

$$\sigma_+=\frac{1}{2}\left(\sigma_{22}+\sigma_{33}\right),\quad \sigma_-=\frac{1}{2\sqrt{3}}\left(\sigma_{22}-\sigma_{33}\right) \tag{3.10}$$

所以 shear tensor 即可寫成

$$\sigma_{\alpha\beta}=diag\left(-2\sigma_+,\sigma_+ +\sqrt{3}\sigma_-,\sigma_+ -\sqrt{3}\sigma_-\right) \tag{3.11}$$

在 Bianchi type I 的度規空間下 $n_{\alpha\beta}=diag\left(0，0，0\right)$

我們要在 Bianchi type I 的度規空間下討論穩定性，所以找出有關的物理空間函數，在動態系統下可以列出以下函數，以空間膨脹速率 H 的冪次來表示子空間，模型的物理態可以用向量 $\left(H,x\right)$ 表示，這裡的 $x=\left(\sigma_+,\sigma_-,n_1,n_2,n_3\right)$，我們以下會列出十一個不同的子空間，並且都經過 normalization，這也是與空間有關的物理態參數，我們的參數是 dimensionless，參數有加速度參數，宇宙常數參數，各向性膨脹參數，空間度規參數:

Inflation universe parameters [17]

$$B=\frac{1}{(3\alpha+\beta)H^2}\quad \chi=\frac{\beta}{(3\alpha+\beta)}\quad Q=\frac{\dot{H}}{H^2}$$
$$Q_2=\frac{\ddot{H}}{H^3}\quad \Omega_\Lambda=\frac{\Lambda}{3H^2}\quad N=\frac{n_{11}}{\sqrt{3}H} \tag{3.12}$$
$$\Sigma_\pm=\frac{\sigma_\pm}{H}\quad \Sigma_{\pm1}=\frac{\dot{\sigma}_\pm}{H^2}\quad \Sigma_{\pm2}=\frac{\ddot{\sigma}_\pm}{H^3}$$

dynamical time variable τ 為 dimensionless，我們定義尺度因子函數為ℓ，則 $\ell=\ell_0 e^{\tau}$

這裡的ℓ_0是ℓ在參考時間上的值，所以$\frac{d\tau}{dt}=H=\frac{\dot{\ell}}{\ell}$ [14] ，$\Sigma^2=\frac{\sigma_+^2+\sigma_-^2}{H^2}$ ，並且假設$\Omega_\Lambda>0$，(3.12)式都已做了膨脹尺度規一化，我們可以看作有十一個子空間，在這些子空間下，我們需要計算每個子空間是否穩定，則須利用章節 2.11 的方式討論，我們的目標是找出在 anisotropic inflation 的空間膨脹是不穩定的，所以只要找到其中一子空間是不穩定的即可證明 anisotropic inflation 的空間是不穩定。

Equation of Motion:對(3.12)作時間微分並找出恆等式

$$B'=-2QB \tag{3.13}$$

$$\Omega_\Lambda'=-2Q\Omega_\Lambda \tag{3.14}$$

$$N'=-\left(Q+1+4\Sigma_+\right)N \tag{3.15}$$

$$Q'=-2Q^2+Q_2 \tag{3.16}$$

$$\Sigma_\pm'=-Q\Sigma_\pm+\Sigma_{\pm1} \tag{3.17}$$

$$\Sigma_{\pm1}'=-2Q\Sigma_{\pm1}+\Sigma_{\pm2} \tag{3.18}$$

$$\begin{aligned}\Sigma_{+2}'=&-3(Q+2)\Sigma_{+2}+\frac{\Sigma_{+1}}{\chi}\left[B-(11\chi-8)+4Q(1-\chi)+4\Sigma^2(1+2\chi)\right]\\&+\frac{\Sigma_+}{\chi}\left[3B+(4-\chi)(6+Q_2+7Q)+4(1+2\chi)\left(3\Sigma^2+2\Sigma\Sigma_1\right)\right]\\&-\frac{4}{\chi}N^2\left[B+8+4Q-4(1+8\chi)N^2\right]-\frac{4}{\chi}N^2\left[(1+15\chi)\left(\Sigma_++\Sigma_{+1}-\Sigma_+{}^2\right)+4(1-\chi)\Sigma_{-1}{}^2\right]\end{aligned} \tag{3.19}$$

$$\Sigma'_{-2} = -3(Q+2)\Sigma_{-2} + \frac{\Sigma_{-1}}{\chi}\left[B-(11\chi-8)+4Q(1-\chi)+4\Sigma^2(1+2\chi)\right]$$
$$+\frac{\Sigma_{-}}{\chi}\left[3B+(4-\chi)(6+Q_2+7Q)+4(1+2\chi)(3\Sigma^2+2\Sigma\Sigma_1)\right] \qquad (3.20)$$
$$-\frac{4(1-\chi)}{\chi}N^2\left[(\Sigma_{-}+\Sigma_{-1}-8\Sigma_{-}\Sigma_{+})\right]$$

$$Q_2' = -3(Q+2)Q_2 - \frac{9}{2}(Q+2)Q - \frac{3}{4}B\left(1+\Sigma^2-\Omega_\Lambda+\frac{2}{3}Q-\frac{1}{3}N^2\right)-\frac{3}{2}(1+2\chi)\Sigma^4$$
$$-\frac{1}{4}(8+\chi)\Sigma_1^2-(4-\chi)\Sigma\Sigma_1-\frac{1}{4}(4-\chi)(3\Sigma^2+2\Sigma\Sigma_2+2Q\Sigma^2)-(1+2Q)N^2$$
$$+N^2\left[\frac{1}{2}(1+8\chi)N^2+5(13+3\chi)\Sigma_+^2+8(2\Sigma_+-\Sigma_{+1})+(1-\chi)\Sigma_-^2\right]$$

(3.21)

以上的運動方程式都要遵守下面的條件(Friedmann-like) constraint

$$0 = B(1-\Omega_\Lambda-\Sigma^2)+12Q-2Q^2+4Q_2-(4-\chi)(3+2Q)\Sigma^2$$
$$-6(1+2\chi)\Sigma^4-\chi(\Sigma_1^2-2\Sigma\Sigma_2)+4(2+\chi)\Sigma\Sigma_1$$
$$+4N^2\left[\frac{1}{2}(1+8\chi)N^2+1+(1+15\chi)\Sigma_+^2+8\Sigma_+ +(1-\chi)\Sigma_-^2\right]$$

(3.22)

(3.13)~ (3.21)的詳細導法會在下一節解釋，所以我們總共會得出十一個微分方程式，接著我們需要找出方程式的零點就是平衡點，然後再找出特徵方程式的特徵值，即可得出空間是否穩定

3.3 導出 Equation of Motion

我們為了看此物理系統在時間變化下是否為穩定，所以會分別對時間作微分，並且檢查是否為線性方程式，如果不是則需要線性化，才能解出特徵值來討論穩定性，因此先從簡單的 DE(differential equation)推導:

$$B' = \frac{dB}{dt}\frac{dt}{d\tau} = \frac{-2\dot{H}}{\left(3\alpha+\beta\right)H^4} = -2QB \tag{3.23}$$

$$\Omega'_{\Lambda} = \frac{d\Omega_{\Lambda}}{dt}\frac{dt}{d\tau} = \frac{-2\dot{H}}{3H^4} = -2Q\Omega_{\Lambda} \tag{3.24}$$

$$Q' = \frac{dQ}{dt}\frac{dt}{d\tau} = \frac{-2\dot{H}^2 H + \ddot{H}H^2}{H^5} = -2Q^2 + Q_2 \tag{3.25}$$

$$\Sigma'_{\pm} = \frac{d\Sigma_{\pm}}{dt}\frac{dt}{d\tau} = \frac{-\dot{H}\sigma_{\pm} + H\dot{\sigma}_{\pm}}{H^3} = -Q\Sigma_{\pm} + \Sigma_{\pm 1} \tag{3.26}$$

$$\Sigma'_{\pm 1} = \frac{d\Sigma_{\pm 1}}{dt}\frac{dt}{d\tau} = \frac{-2H\dot{H}\dot{\sigma}_{\pm} + H^2\ddot{\sigma}_{\pm}}{H^4} = -2Q\Sigma_{\pm 1} + \Sigma_{\pm 2} \tag{3.27}$$

$$\begin{aligned} N' &= \frac{dN}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\frac{n_{11}}{\sqrt{3}H}\frac{1}{H} = \left(\frac{\dot{n}_{11}}{\sqrt{3}H} - \frac{\dot{H}}{\sqrt{3}H^2}\right)\frac{1}{H} \\ &= \frac{1}{\sqrt{3}H^2}\left(-Hn_{11} - 4\sigma_{+}n_{11}\right) - \frac{\dot{H}}{\sqrt{3}H^3} = -\left(Q + 1 + 4\Sigma_{+}\right)N \end{aligned} \tag{3.28}$$

其他(3.19)~ (3.21)無法直接用一般 DE 推導，所以須從基本的運動方程式去推導出來，從對 Lagrange equation 的變分 **Appendix B** 可得出下列式子，令 $\ell = R + \alpha R^2 + \beta R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} - 2\Lambda$:

$$0 = \frac{\ell}{a_i} + \frac{\partial \ell}{\partial a_i} - \left(\frac{d}{dt} + 3H\right)\frac{\partial \ell}{\partial \dot{a}_i} + \left(\frac{d}{dt} + 3H\right)^2 \frac{\partial \ell}{\partial \ddot{a}_i} \qquad \ell = \ell\left(a_i, \dot{a}_i, \ddot{a}_i\right) \tag{3.29}$$

把 $\dot{a}_i, \ddot{a}_i$ 代換成 $H_i, \dot{H}_i$，by $H_i = \frac{\dot{a}_i}{a_i}$，得

$$0 = \ell - \left(\frac{d}{dt} + 3H\right)\frac{\partial \ell}{\partial H_i} + \left(3\dot{H} + 9H^2 + 6H\frac{d}{dt} + \frac{d^2}{dt^2}\right)\frac{\partial \ell}{\partial \dot{H}_i} + a_i \frac{\partial \ell}{\partial a_i} \quad \ell = \ell\left(a_i, H_i, \dot{H}_i\right)$$

$$\text{且 } H_i = \frac{\dot{a}_i}{a_i} \tag{3.30}$$

$$\frac{\partial H}{\partial H_i} = \frac{\partial \dot{H}}{\partial \dot{H}_i} = \frac{1}{3} \qquad \frac{\partial \sigma^i{}_i}{\partial H_i} = \frac{\partial \dot{\sigma}^i{}_i}{\partial \dot{H}_i} = \delta^i{}_j - \frac{1}{3}$$

我們把 $\ell = R + \alpha R^2 + \beta R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} - 2\Lambda$ 分成三項 ℓ_0，ℓ_α，ℓ_β 作 Lagrange equation 的變分，且設 $N = 0$，R，$R^{\mu}{}_{\nu}$ 的方程式在(3.61) (3.62) (3.63)

假設 $\ell_0 = \mathrm{R} - 2\Lambda$

$$f_i = \ell_0 - \left(\frac{d}{dt} + 3H\right)\frac{\partial \ell_0}{\partial H_i} + \left(3\dot{H} + 9H^2 + 6H\frac{d}{dt} + \frac{d^2}{dt^2}\right)\frac{\partial \ell_0}{\partial \dot{H}_i} + a_i\frac{\partial \ell_0}{\partial a_i} \tag{3.31}$$

則，我們的函數為

$$f_i = -2\left(G^i{}_i + \Lambda g^i{}_i\right) \tag{3.32}$$

相同的寫出

$$\ell_\alpha = \mathrm{R}^2$$

$$A_i = \ell_\alpha - \left(\frac{d}{dt} + 3H\right)\frac{\partial \ell_\alpha}{\partial H_i} + \left(3\dot{H} + 9H^2 + 6H\frac{d}{dt} + \frac{d^2}{dt^2}\right)\frac{\partial \ell_\alpha}{\partial \dot{H}_i} + a_i\frac{\partial \ell_\alpha}{\partial a_i} \tag{3.33}$$

$$A_i = R^2 - R\left(16\dot{H} + 4\dot{\sigma}^i_i\right) - \left(16\dot{H} + 4\dot{\sigma}^i_i\right)\left(\dot{R} + 3HR\right) + R\left(12\dot{H} + 36H^2\right) + 24H\dot{R} + 4\ddot{R} \tag{3.34}$$

$$\ell_\beta = R^\mu{}_\nu R^\nu{}_\mu$$

$$B_i = \ell_\beta - \left(\frac{d}{dt} + 3H\right)\frac{\partial \ell_\beta}{\partial H_i} + \left(3\dot{H} + 9H^2 + 6H\frac{d}{dt} + \frac{d^2}{dt^2}\right)\frac{\partial \ell_\beta}{\partial \dot{H}_i} + a_i\frac{\partial \ell_\beta}{\partial a_i} \tag{3.35}$$

$$\begin{aligned} B_i = {} & R^\mu{}_\nu R^\nu{}_\mu - R^0{}_0\left(4\dot{H} + 4\dot{\sigma}^i{}_i\right) - \left(4H + 4\sigma^i{}_i\right)\left(\dot{R}^0{}_0 + 3HR^0{}_0\right) + R^0{}_0\left(6\dot{H} + 18H^2\right) \\ & + 12H\dot{R}^0{}_0 + 2\ddot{R}^0{}_0 + \left[\begin{array}{l} \left(2\dot{H} + 2\dot{\sigma}^j{}_j + 6\dot{H}\delta^i{}_j\right)R^j{}_j - \left(2H + 2\sigma^j{}_j + 6H\delta^i{}_j\right)\left(\dot{R}^j{}_j + 3HR^j{}_j\right) \\ + R^j{}_j\delta^i{}_j\left(6\dot{H} + 18H^2\right) + 12H\dot{R}^j{}_j\delta^i{}_j + 2\ddot{R}^j{}_j\delta^i{}_j \end{array}\right] \end{aligned} \tag{3.36}$$

我們綜合上式運動方程式為

$$\left(EOM\right)^i_i = -\frac{1}{2}\left(f_i + \alpha A_i + \beta B_i\right) = G^i{}_i + \Phi^i{}_i + \Lambda g^i{}_i \tag{3.37}$$

接著我們做線性組合:

$$\frac{1}{H^3}\left[\left(EOM\right)_2^2-\left(EOM\right)_3^3\right]=0$$

$$=2\sqrt{3}\left(\Sigma_{-1}+3\Sigma_{-}\right)+2\sqrt{3}\alpha\left[\Sigma_{-}\left(12Q_2+84Q+72+24\Sigma\Sigma_1+36\Sigma^2\right)+\Sigma_{-1}\left(12Q+24+12\Sigma^2\right)\right]$$

$$+2\sqrt{3}\beta\left[\Sigma_{-}\left(3Q_2+21Q+18+24\Sigma\Sigma_1+36\Sigma^2\right)+\Sigma_{-1}\left(-3+12\Sigma^2\right)+\Sigma_{-2}\left(-6-3Q\right)-\Sigma'_{-2}\right]$$

$$\Sigma'_{-2}=B\left(\Sigma_{-1}+3\Sigma_{-}\right)+\frac{1}{\chi}\left[\Sigma_{-}\left(4Q_2+28Q+24+8\Sigma\Sigma_1+12\Sigma^2\right)+\Sigma_{-1}\left(4Q+8+4\Sigma^2\right)\right]$$
$$+\left[\Sigma_{-}\left(-Q_2-7Q-6+16\Sigma\Sigma_1+24\Sigma^2\right)+\Sigma_{-1}\left(-4Q-11+8\Sigma^2\right)+\Sigma_{-2}\left(-6-3Q\right)\right] \quad (3.38)$$

整理(3.42)得

$$\Sigma'_{-2}=\Sigma_{-2}\left(-6-3Q\right)+\frac{\Sigma_{-1}}{\chi}\left[B-11\chi+8+4Q\left(1-\chi\right)+4\Sigma^2\left(1+2\chi\right)\right]$$
$$+\frac{\Sigma_{-}}{\chi}\left[3B+\left(4-\chi\right)\left(Q_2+7Q+6\right)+\left(4+8\chi\right)\left(3\Sigma^2+2\Sigma\Sigma_1\right)\right] \quad (3.39)$$

相同的另一個線性組合:

$$\frac{1}{H^3}\left[2\left(EOM\right)_3^3-\left(EOM\right)_2^2-\left(EOM\right)_3^3\right]=0$$

$$=-6\left(\Sigma_{+1}+3\Sigma_{+}\right)-6\alpha\left[\Sigma_{+}\left(12Q_2+84Q+72+24\Sigma\Sigma_1+36\Sigma^2\right)+\Sigma_{+1}\left(12Q+24+12\Sigma^2\right)\right]$$

$$-6\beta\left[\Sigma_{+}\left(3Q_2+21Q+18+24\Sigma\Sigma_1+36\Sigma^2\right)+\Sigma_{+1}\left(-3+12\Sigma^2\right)+\Sigma_{+2}\left(-6-3Q\right)-\Sigma'_{+2}\right]$$

(3.40)

$$\Sigma'_{+2}=B\left(\Sigma_{+1}+3\Sigma_{+}\right)+\frac{1}{\chi}\left[\Sigma_{+}\left(4Q_2+28Q+24+8\Sigma\Sigma_1+12\Sigma^2\right)+\Sigma_{+1}\left(4Q+8+4\Sigma^2\right)\right]$$
$$+\left[\Sigma_{+}\left(-Q_2-7Q-6+16\Sigma\Sigma_1+24\Sigma^2\right)+\Sigma_{-1}\left(-4Q-11+8\Sigma^2\right)+\Sigma_{-2}\left(-6-3Q\right)\right] \quad (3.41)$$

整理(3.45)得

$$\Sigma'_{+2}=\Sigma_{+2}\left(-6-3Q\right)+\frac{\Sigma_{+1}}{\chi}\left[B-11\chi+8+4Q\left(1-\chi\right)+4\Sigma^2\left(1+2\chi\right)\right]$$
$$+\frac{\Sigma_{+}}{\chi}\left[3B+\left(4-\chi\right)\left(Q_2+7Q+6\right)+\left(4+8\chi\right)\left(3\Sigma^2+2\Sigma\Sigma_1\right)\right] \quad (3.42)$$

.

再利用 *The trace of EOM* (就是 $\sum_{i=1}^{3} G^i{}_i + \Phi^i{}_i + \Lambda g^i{}_i = 0$)

$$
\begin{aligned}
0 &= -R + 4\Lambda + 2(3\alpha+\beta)\Box R \\
&= -R + 4\Lambda - 2(3\alpha+\beta)\left(\ddot{R} + 3H\dot{R}\right) \\
&= -6\dot{H} - 12H^2 - 6\sigma_+{}^2 - 6\sigma_-{}^2 + 4\Lambda - 2(3\alpha+\beta)\left(\ddot{R} + 3H\dot{R}\right) \\
&= -6\dot{H} - 12H^2 - 6\sigma_+{}^2 - 6\sigma_-{}^2 + 4\Lambda \\
&\quad -2(3\alpha+\beta)\begin{bmatrix} 6\dddot{H} + 24H\ddot{H} + 24\dot{H}^2 + 12\dot{\sigma}_+{}^2 12\dot{\sigma}_-{}^2 + 12\sigma_+\ddot{\sigma}_+ + 12\sigma_-\ddot{\sigma}_- \\ +3H\left(6\ddot{H} + 24H\dot{H} + 12\sigma_+\dot{\sigma}_+ + 12\sigma_-\dot{\sigma}_-\right) \end{bmatrix}
\end{aligned}
$$

(3.43)

利用 *Trace* 得 $\frac{\dddot{H}}{H^4}$,

$$\frac{\dddot{H}}{H^4} = -7Q_2 - 4Q^2 - 2\Sigma_1^2 - 2\Sigma\Sigma_2 - 12Q - 3\Sigma\Sigma_1 + B\left(-\frac{1}{2}Q - \frac{1}{2}\Sigma^2 - 1 + \Omega_\Lambda\right) \qquad (3.44)$$

從(3.12)式找出 $Q_2 = \frac{\ddot{H}}{H^3}$ 做微分得 $Q_2' = -3\frac{\ddot{H}}{H^3} + \frac{\dddot{H}}{H^4}$ 再帶入(3.15) (3.48)得到:

$$Q_2' = -3QQ_2 - 7Q_2 - 4Q^2 - 2\Sigma_1^2 - 2\Sigma\Sigma_2 - 12Q - 3\Sigma\Sigma_1 + B\left(-\frac{1}{2}Q - \frac{1}{2}\Sigma^2 - 1 + \Omega_\Lambda\right)$$

(3.45)

以上就是所有 DE 的表示式，此方法以便等下要做的空間動態穩定性分析，來找出穩定點，並且能找出特徵值來看是空間否穩定。

3.4 在 Bianchi type I 空間下運動方程式的解

在正確的 χ 和 B 值，會有真解來描述不均向的宇宙膨脹，而我們就是在 Bianchi type I 下導出真解，在 Bianchi type I 條件下

$$Q = \Sigma_{\pm 1} = \Sigma_{\pm 2} = N = 0 \qquad 因為\ \dot{H} = 0 \qquad (3.46)$$

帶回 Equation of Motion 的 Σ_2' 與 Friedmann-like constraint 得

$$0 = \frac{\Sigma}{\chi}\left[3B + 6(4-\chi) + 4(1+2\chi)(3\Sigma^2)\right] \qquad (3.47)$$

$$0 = B\left(1 - \Omega_\Lambda - \Sigma^2\right) + (4-\chi)3\Sigma^2 - 6(1+2\chi)\Sigma^4 \qquad (3.48)$$

兩式聯立得

$$\Sigma^2 = -\frac{2(2-\chi)+B}{4(2\chi+1)} \qquad \Omega_\Lambda = \frac{18\chi - B}{8(2\chi+1)} \tag{3.49}$$

這裡

$$B = \frac{1}{(3\alpha+\beta)H^2} \qquad \chi = \frac{\beta}{(3\alpha+\beta)}$$

我們的 Bianchi type I 宇宙空間為

$$ds^2 = -dt^2 + e^{2bt}\left[e^{-4\sigma_+ t}dx^2 + e^{2\left(\sigma_+ + \sqrt{3}\sigma_-\right)t}dy^2 + e^{2\left(\sigma_+ - \sqrt{3}\sigma_-\right)t}dz^2\right] \tag{3.50}$$

如果用一般的表示式

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2 \tag{3.51}$$

$$U^\mu = (1,0,0,0) \qquad H_i = \frac{\dot{a}_i}{a_i} \tag{3.52}$$

所以我們可以定義哈伯常數為

$$H = \frac{1}{3}U^\mu_{;\mu} = \frac{1}{3}\nabla_\mu U^\mu = \frac{H_1+H_2+H_3}{3} \tag{3.53}$$

(3.51)的 Ricci tensor 為: $R^0{}_0 = \sum_i \dot{H}_i + H_i^2$

$$R^1{}_1 = \dot{H}_1 + H_1^2 + H_1H_2 + H_1H_3$$

$$R^2{}_2 = \dot{H}_2 + H_2^2 + H_1H_2 + H_2H_3$$

$$R^3{}_3 = \dot{H}_3 + H_3^2 + H_1H_3 + H_2H_3 \tag{3.54}$$

Ricci scalar 為: $R = 2\left(\dot{H}_1 + H_1^2 + \dot{H}_2 + H_2^2 + \dot{H}_3 + H_3^2 + H_1H_2 + H_1H_3 + H_2H_3\right)$

(3.55)

我們的剪向量對哈伯常數的關係為

$$\sigma^a{}_b = U^a_{;b} - H\delta^a{}_b = \nabla_b U^a - H\delta^a{}_b = \begin{pmatrix} H_1 - H & 0 & 0 \\ 0 & H_2 - H & 0 \\ 0 & 0 & H_3 - H \end{pmatrix} \tag{3.56}$$

我們把(3.10)空間的 shear tensor 表示矩陣為

$$\sigma^a{}_b = \begin{pmatrix} 2\sigma_+ & 0 & 0 \\ 0 & \sigma_+ + \sqrt{3}\sigma_- & 0 \\ 0 & 0 & \sigma_+ - \sqrt{3}\sigma_- \end{pmatrix} \tag{3.57}$$

且觀察(3.56) (3.57)的空間度規可得到關係式為

$$\begin{aligned} b - 2\sigma_+ &= H_1 \\ b + \sigma_+ + \sqrt{3}\sigma_- &= H_2 \\ b + \sigma_+ - \sqrt{3}\sigma_- &= H_3 \end{aligned} \tag{3.58}$$

聯立得

$$\begin{aligned} b &= H \\ \sigma_+ &= -\frac{H_1}{3} + \frac{H_2 + H_3}{6} \\ \sigma_- &= \frac{H_2 - H_3}{2\sqrt{3}} \end{aligned} \tag{3.59}$$

$$\sigma^a{}_b \sigma_a{}^b = 6\sigma_+^2 + 6\sigma_-^2 \tag{3.60}$$

利用(3.58) (3.59) (3.60)可改寫(3.54) (3.55)的 Ricci tensor 為:

$$R^0{}_0 = 3\dot{H} + 3H^2 + \sigma^a{}_b \sigma_a{}^b \tag{3.61}$$

$$R^a{}_b = \dot{\sigma}^a{}_b + \dot{H}\delta^a{}_b + 3H\left(\sigma^a{}_b + H\delta^a{}_b\right) \tag{3.62}$$

Ricci scalar 為:

$$R = 6\dot{H} + 12H^2 + \sigma^a{}_b \sigma_a{}^b \tag{3.63}$$

利用(3.12)

$$\Omega_\Lambda = \frac{18\chi - B}{8(2\chi + 1)} = \frac{\Lambda}{3H^2} \qquad \text{與} \qquad B = \frac{1}{(3\alpha + \beta)H^2}$$

且 $b^2 = H^2$，得出:

$$b^2 = \frac{1 + 8\Lambda(\alpha + \beta)}{18\beta} \tag{3.64}$$

利用(3.12)關係式

$$\Sigma^2 = \frac{\sigma_+^2 + \sigma_-^2}{H^2} = -\frac{2(2 - \chi) + B}{4(2\chi + 1)}$$

並且找到關係式 $\Omega_\Lambda = 1 + \frac{1}{2}\Sigma^2$ (3.65)

此解由實驗室團隊合力解出[18]，可得解為:

$$\sigma_+^2+\sigma_-^2=-\frac{1+2\Lambda\left(4\alpha+\beta\right)}{9\beta} \tag{3.66}$$

我們得到(3.64) (3.66)這組解，又因為b^2>0和$\sigma_+^2+\sigma_-^2$>0，所以可以知道此解是非均向的膨脹,也就確實可以解出實際的非均向解,接著就需要討論非均向解是否穩定，如果穩定那可以發展出非均向的膨脹宇宙模型，如果不穩定那非均向宇宙模型就不穩定，也就不能拿來解釋我們實際的膨脹宇宙。

3.5 找出 DE 的 eigenvalue 並且討論穩定性的問題

我們需要利用 theorem of motion stability(2.50)~(2.53) 找出(3.13)~(3.21)的 eigenvalue 並且可得知是否穩定，我首先可以先簡單的看出線性化 DE 的 eigenvalue 的值:

$$B'=-2QB \tag{3.67}$$

$$\Omega_\Lambda'=-2Q\Omega_\Lambda \tag{3.68}$$

$$\Sigma_\pm'=-Q\Sigma_\pm+\Sigma_{\pm1}=-Q\Sigma_\pm-3\Sigma_\pm \tag{3.69}$$

$$\Sigma_{\pm1}'=-2Q\Sigma_{\pm1}-3\Sigma_{\pm1} \tag{3.70}$$

$$N'=-\left(Q+1+4\Sigma_+\right)N \tag{3.71}$$

利用 $\dot{\sigma}_\pm=-3H\sigma_\pm$，$\ddot{\sigma}_\pm=-3\dot{H}\sigma_\pm-3H\dot{\sigma}_\pm$ 可解得上面(3.69) (3.70)式

因為 $Q=0$，所以上面的 eigenvalues 共七個分別為

$$0\ ，\ 0\ ，\ \text{-3}\ ，\text{-3}\ ，\text{-3}\ ，\text{-3}\ ，-\left(1+4\Sigma_+\right) \tag{3.72}$$

以上是簡單的線性化 DE 所算出來的 eigenvalues，接著因為(3.19) (3.20) (3.21)是非線性化 DE，因為這些項與其他空間有關聯所以並非線性而是有交叉項，因此需要先做線性化然後利用特徵矩陣去算 eigenvalues，首先把非線性化 DE 的有關基數做時

間微分的方程式以函數表示:

$$B' = -2QB = A_1 \tag{3.73}$$

$$\Omega'_\Lambda = -2Q\Omega_\Lambda = A_2 \tag{3.74}$$

$$N' = -\left(Q + 1 + 4\Sigma_+\right) N = A_3 \tag{3.75}$$

$$Q' = -2Q^2 + Q_2 = A_4 \tag{3.76}$$

$$\Sigma'_+ = -Q\Sigma_+ + \Sigma_{+1} = A_5 \tag{3.77}$$

$$\Sigma'_- = -Q\Sigma_- + \Sigma_{-1} = A_6 \tag{3.78}$$

$$\Sigma'_{+1} = -2Q\Sigma_{+1} + \Sigma_{+2} = A_7 \tag{3.79}$$

$$\Sigma'_{-1} = -2Q\Sigma_{-1} + \Sigma_{-2} = A_8 \tag{3.80}$$

$$\begin{aligned}\Sigma'_{+2} &= -3\left(Q+2\right)\Sigma_{+2} + \frac{\Sigma_{+1}}{\chi}\left[B - \left(11\chi - 8\right) + 4Q\left(1-\chi\right) + 4\Sigma^2\left(1+2\chi\right)\right] \\ &+ \frac{\Sigma_+}{\chi}\left[3B + \left(4-\chi\right)\left(6 + Q_2 + 7Q\right) + 4\left(1+2\chi\right)\left(3\Sigma^2 + 2\Sigma\Sigma_1\right)\right] \\ &- \frac{4}{\chi}N^2\left[B + 8 + 4Q - 4\left(1+8\chi\right)N^2\right] - \frac{4}{\chi}N^2\left[\left(1+15\chi\right)\left(\Sigma_+ + \Sigma_{+1} - \Sigma_+{}^2\right) + 4\left(1-\chi\right)\Sigma_{-1}{}^2\right] \\ &= A_9\end{aligned} \tag{3.81}$$

$$\begin{aligned}\Sigma'_{-2} &= -3\left(Q+2\right)\Sigma_{-2} + \frac{\Sigma_{-1}}{\chi}\left[B - \left(11\chi - 8\right) + 4Q\left(1-\chi\right) + 4\Sigma^2\left(1+2\chi\right)\right] \\ &+ \frac{\Sigma_-}{\chi}\left[3B + \left(4-\chi\right)\left(6 + Q_2 + 7Q\right) + 4\left(1+2\chi\right)\left(3\Sigma^2 + 2\Sigma\Sigma_1\right)\right] \\ &- \frac{4\left(1-\chi\right)}{\chi}N^2\left[\left(\Sigma_- + \Sigma_{-1} - 8\Sigma_-\Sigma_+\right)\right] \\ &= A_{10}\end{aligned} \tag{3.82}$$

$$
\begin{aligned}
Q_2' &= -3(Q+2)Q_2 - \frac{9}{2}(Q+2)Q - \frac{3}{4}B\left(1+\Sigma^2-\Omega_\Lambda+\frac{2}{3}Q-\frac{1}{3}N^2\right)-\frac{3}{2}(1+2\chi)\Sigma^4 \\
&\quad -\frac{1}{4}(8+\chi)\Sigma_1^2-(4-\chi)\Sigma\Sigma_1-\frac{1}{4}(4-\chi)\left(3\Sigma^2+2\Sigma\Sigma_2+2Q\Sigma^2\right)-(1+2Q)N^2 \\
&\quad +N^2\left[\frac{1}{2}(1+8\chi)N^2+5(13+3\chi)\Sigma_+^2+8(2\Sigma_+-\Sigma_{+1})+(1-\chi)\Sigma_-^2\right] \\
&= A_{11}
\end{aligned}
\tag{3.83}
$$

在(3.73)~ (3.83)的微分方程式作線性化，並且令 DE 為零點可解出平衡點，其中帶入(3.49)的解可求出平衡點位置，所以我們先令 $f_i = 0$ ，然後找出平衡點:

$A_1 = 0$ $A_2 = 0$ $A_3 = 0$ $A_4 = 0$ $A_5 = 0$ $A_6 = 0$ $A_7 = 0$ $A_8 = 0$ $A_9 = 0$ $A_{10} = 0$ $A_{11} = 0$

帶入(3.49)

$$\Sigma^2 = -\frac{2(2-\chi)+B}{4(2\chi+1)},\ \Omega_\Lambda = \frac{18\chi - B}{8(2\chi+1)}$$

可以得出平衡點的解

$B = 2(\chi-4) - 4\Sigma^2(1+2\chi)$，$\Omega_\Lambda = 1+\frac{1}{2}\Sigma^2$，$N=0$，$Q=0$，$\Sigma_+ = \Sigma_+$，$\Sigma_- = \Sigma_-$，$\Sigma_{+1} = 0$，$\Sigma_{-1} = 0$，$\Sigma_{+2} = 0$，$\Sigma_{-2} = 0$，$Q_2 = 0$ (3.84)

所以可以找到十一個子空間的平衡點，我們的一階線性化 DE 為:

$$\dot{x} = Ax \tag{3.85}$$

在此考慮 x 為 $(B,\Omega_\Lambda,N,Q,\Sigma_+,\Sigma_-,\Sigma_{+1},\Sigma_{-1},\Sigma_{+2},\Sigma_{-2},Q_2)$ 的函數，所以我們可以重新把

(3.73)~ (3.81)方程式系統表示為[19]:

$$\frac{dx_i}{dt} = A_i\left(x_1, x_2, \ldots, x_{n+1}\right) \qquad (i = 1,2,\ldots,n+1) \tag{3.86}$$

如果我們的 $A_i(0,0,\ldots,0)=0$，表示 A_i 在 $x_1 = x_1 = \ldots = x_{n+1} = 0$ 為平衡點且在鄰近平衡點的微分方程式具連續性，接著我們需要使用一階線性化系統(3.83)，把(3.84)非線性化系統轉換到線性化需要以下條件，線性化系統方程式:

$$\frac{dx_i}{dt} = a_{i1}x_1 + \ldots + a_{in+1}x_{n+1}$$

$$(a_{ij} = \left[\frac{\partial A_i}{\partial x_j}\right]_{x_j=0} ;i,j=1,2...,n+1) \tag{3.87}$$

然而(3.86)即為一階線性化 DE $\dot{x} = Ax$ ，$A = a_{ij} = \left[\frac{\partial A_i}{\partial x_j}\right]_{x_j=0}$ $;i,j=1,2...,n+1$ 的矩陣，然後須找出 eigenvalues 值所以令 $Ax = \lambda x$ ，且 x 已經有解所以線性化的特徵方程式要為零，即為:

$$\det\begin{bmatrix} a_{11}-\lambda,......,a_{1n+1} \\ .... \\ a_{n+11},...,a_{n+1n+1}-\lambda \end{bmatrix} = 0 \tag{3.88}$$

為一個 11X11 的矩陣，經團隊運用電腦計算[18]，因此可得到十一組 eigenvalue 解為:

$$\lambda = 0,0,-3,-3,-3,-,3,-\frac{3}{2}\left(1\pm\sqrt{1+8\Sigma^2}\right),-\frac{3}{2}\pm\frac{1}{2}\sqrt{9-2B},-1-4\Sigma_+ \tag{3.89}$$

求 Eigenvalue 的詳細解在 **Appendix D**，其中 0，-3，$-1-4\Sigma_+$ 的 eigenvalues 值和直接用線性 DE 觀察得解的值(3.72)相同，因為這確實是此空間微擾的方程式解，接著我們從(3.89)的 Eigenvalues 討論在 Bianchi Type I 度規空間的 anisotropic inflation universe 是否為穩定，從(3.88)知

$$\lambda = 0,0,-3,-3,-3,-,3,-\frac{3}{2}\pm\frac{1}{2}\sqrt{9-2B},-1-4\Sigma_+ \quad \text{為小於等於 } 0$$

$$\lambda = -\frac{3}{2}\left(1\pm\sqrt{1+8\Sigma^2}\right) \quad \text{，當 } \Sigma^2 = \frac{\sigma_+^2+\sigma_-^2}{H^2} > 0 \text{，} \quad \text{為大於 } 0$$

這裡(3.59) $\sigma_+ = -\frac{H_1}{3} + \frac{H_2+H_3}{6}$ ， $\sigma_- = \frac{H_2-H_3}{2\sqrt{3}}$ (3.90)

所以即證明 Bianchi Type I 度規空間的 anisotropic inflation universe 是為不穩定，也就不是我們現在的宇宙，但當 $H_1 = H_2 = H_3$ 時 $\sum^2 = 0$，我們的 eigenvalues 都會小於等於 0，為穩定的宇宙狀態，這也是 De Sitter 宇宙，是符合我們現在的宇宙的模型，從觀察而言也是如此，所以我們的所有 Bianchi Type I 度規空間的 anisotropic inflation universe 經長時間都會演化到 De Sitter 宇宙，以此得證。

Chapter 4
Conclusions

我們在 Bianchi Tyep I 幾何空間下，導入有高階修正項的 Lagrangian，可以解出兩個不同尺度因子的解，兩個尺度因子分別是同向性與非各向同性，

同向性: $H_1 = H_2 = H_3 = \sqrt{\frac{\Lambda}{3}} = H_0$　　非各向同性 $H_1 \neq H_2 \neq H_3$

然後經動態系統分析後發現，非各向同性是不穩定的解，證明了依時間的推移後，這個系統會趨向 De Sitter Space 的穩定解，也就是起始點雖為不均向的 Bianchi Type I 的度規空間的宇宙膨脹，但最後都會趨向 De Sitter Space，是均質且各向同性的宇宙。

Bianchi Type I 度規空間的 anisotropic inflation universe 為:

$$ds^2 = -dt^2 + e^{2bt}\left[e^{-4\sigma_+ t}dx^2 + e^{2\left(\sigma_+ + \sqrt{3}\sigma_-\right)t}dy^2 + e^{2\left(\sigma_+ - \sqrt{3}\sigma_-\right)t}dz^2\right]$$

我們可以用圖表示之，在此一物理態有兩個平衡點，兩平衡點的穩定性最後會趨向 De Sitter universe 。

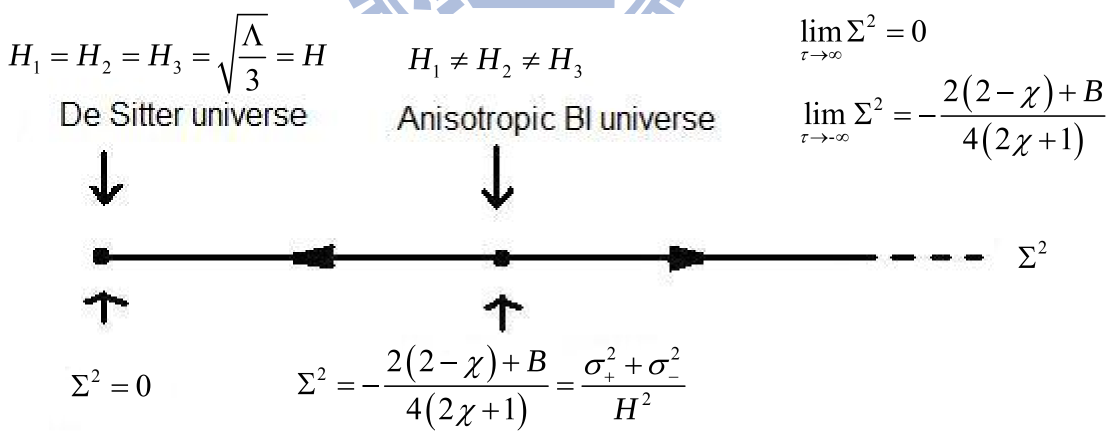

圖 4.1 非各向性的 Bianchi Type I 會趨向 De Sitter Space 的穩定點

Chapter 5
Appendix

Appendix A

高階修正項的 Einstein Equation 推導

首先，我們的高階路徑表示為[20] [21]:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda) \tag{A.1}$$

為了方便我們把路徑分成四個表示式

$$S_1 = \frac{1}{2}\int d^4x\sqrt{-g}(R) = \frac{1}{2}\int d^4x\sqrt{-g}(g^{\mu\nu}R_{\mu\nu}) \tag{A.2}$$

$$S_2 = \frac{1}{2}\int d^4x\sqrt{-g}(\alpha R^2) \tag{A.3}$$

$$S_3 = \frac{1}{2}\int d^4x\sqrt{-g}(\beta R_{\mu\nu}R^{\mu\nu}) \tag{A.4}$$

$$S_4 = \frac{1}{2}\int d^4x\sqrt{-g}(-2\Lambda) \tag{A.5}$$

為了導出場方程式，我們需要做路徑的微擾為零

$$\delta S = \frac{1}{2}\delta\left[\int d^4x\sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda)\right] = 0 \tag{A.6}$$

微擾各項路徑之前，先做細部的微擾形式以方便計算

先從 R 開始

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}{}_{\nu\mu} - \partial_\nu\Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} \tag{A.7}$$

$$\begin{aligned}
\delta\left(R^{\rho}{}_{\mu\lambda\nu}\right) &= \partial_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \partial_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right) + \left(\delta\Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu}\right) - \left(\delta\Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu}\right) \\
&= \partial_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \partial_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right) + \left(\delta\Gamma^{\rho}{}_{\lambda\sigma}\right)\Gamma^{\sigma}{}_{\nu\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) \\
&\quad - \left(\delta\Gamma^{\rho}{}_{\nu\sigma}\right)\Gamma^{\sigma}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\nu\sigma}\left(\delta\Gamma^{\sigma}{}_{\lambda\mu}\right) \\
&= \nabla_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right)
\end{aligned} \tag{A.8}$$

$$\nabla_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) = \partial_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) + \Gamma^{\rho}{}_{\lambda\sigma}\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) - \Gamma^{\sigma}{}_{\lambda\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) - \left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right)\Gamma^{\sigma}{}_{\lambda\mu} \tag{A.9}$$

$$\nabla_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right) = \partial_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right) + \Gamma^{\rho}{}_{\nu\sigma}\left(\delta\Gamma^{\sigma}{}_{\lambda\mu}\right) - \Gamma^{\sigma}{}_{\nu\lambda}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) - \left(\delta\Gamma^{\rho}{}_{\sigma\lambda}\right)\Gamma^{\sigma}{}_{\nu\mu} \tag{A.10}$$

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma} \tag{A.11}$$

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}(\partial_\mu g^{\sigma}{}_{\nu} + \partial_\nu g^{\sigma}{}_{\mu} - \partial^{\sigma} g_{\mu\nu}) = \frac{1}{2}\left(g_{\lambda\nu}\partial_\mu g^{\lambda\sigma} + g_{\lambda\mu}\partial_\nu g^{\lambda\sigma} - g_{\nu\sigma} g_{\mu\rho}\partial^{\sigma} g^{\sigma\rho} \right) \tag{A.12}$$

$$\begin{aligned} \delta\Gamma^{\sigma}{}_{\mu\nu} &= \frac{1}{2}\begin{bmatrix} g_{\lambda\nu}\partial_\mu\left(\delta g^{\lambda\sigma}\right) + \left(\delta g_{\lambda\nu}\right)\partial_\mu g^{\lambda\sigma} + g_{\lambda\mu}\partial_\nu\left(\delta g^{\lambda\sigma}\right) + \left(\delta g_{\lambda\mu}\right)\partial_\nu g^{\lambda\sigma} \\ -\left(\delta g_{\nu\sigma}\right) g_{\mu\rho}\partial^{\sigma} g^{\sigma\rho} - g_{\nu\sigma}\left(\delta g_{\mu\rho}\right)\partial^{\sigma} g^{\sigma\rho} - g_{\nu\sigma} g_{\mu\rho}\partial^{\sigma}\left(\delta g^{\sigma\rho}\right) \end{bmatrix} \\ &= -\frac{1}{2}\left[g_{\lambda\nu}\nabla_\mu\left(\delta g^{\lambda\sigma}\right) + g_{\lambda\mu}\nabla_\nu\left(\delta g^{\lambda\sigma}\right) - g_{\nu\sigma} g_{\mu\rho}\nabla^{\sigma}\left(\delta g^{\sigma\rho}\right) \right] \end{aligned} \tag{A.13}$$

$$\delta\Gamma^{\sigma}{}_{\sigma\mu} = -\frac{1}{2} g_{\lambda\sigma}\nabla_\mu\left(\delta g^{\lambda\sigma}\right) \tag{A.14}$$

$$\delta R_{\mu\nu} = \nabla_\sigma\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) \tag{A.15}$$

$$\left(\delta R_{\mu\nu}\right) X^{\mu\nu} = \left[\nabla_\sigma\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) \right] X^{\mu\nu} \tag{A.16}$$

$$\begin{aligned} \left(\delta R_{\mu\nu}\right) X^{\mu\nu} &= -\frac{1}{2}\begin{bmatrix} \nabla_\sigma\nabla_\mu\left(\delta g^{\lambda\sigma}\right) X^{\mu}{}_{\lambda} + \nabla_\sigma\nabla_\nu\left(\delta g^{\lambda\sigma}\right) X_{\lambda}{}^{\nu} - \nabla_\sigma\nabla^{\sigma}\left(\delta g^{\alpha\beta}\right) X_{\beta\alpha} \\ -g_{\lambda\sigma}\nabla_\nu\nabla_\mu\left(\delta g^{\lambda\sigma}\right) X^{\mu\nu} \end{bmatrix} \\ &= -\frac{1}{2}\begin{bmatrix} \nabla_\nu\nabla_\alpha\left(\delta g^{\mu\nu}\right) X^{\alpha}{}_{\mu} + \nabla_\nu\nabla_\alpha\left(\delta g^{\mu\nu}\right) X_{\mu}{}^{\alpha} - \Box\left(\delta g^{\nu\mu}\right) X_{\nu\mu} \\ -g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(\delta g^{\mu\nu}\right) X^{\beta\alpha} \end{bmatrix} \\ &= -\frac{1}{2}\left(\delta g^{\mu\nu}\right)\left[\nabla_\nu\nabla_\alpha X^{\alpha}{}_{\mu} + \nabla_\nu\nabla_\alpha X_{\mu}{}^{\alpha} - \Box X_{\nu\mu} - g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\beta\alpha} \right] \end{aligned} \tag{A.17}$$

假設 $X^{\mu\nu}$ 為對稱，則 $X^{\mu\nu} = X_{\mu\nu}$

$$\left(\delta R_{\mu\nu}\right) X^{\mu\nu} = \delta g^{\mu\nu}\left[-\nabla_\nu\nabla_\alpha X^{\alpha}{}_{\mu} + \frac{1}{2}\Box X_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta} \right] \tag{A.18}$$

令 $X^{\mu\nu} = fg^{\mu\nu}$

$$\left(\delta R_{\mu\nu}\right) fg^{\mu\nu} = \delta g^{\mu\nu}\left[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu \right] f \tag{A.19}$$

其中 $\Box X = \nabla^{\mu}\nabla_{\mu} X$

$$\delta g = -g\left(g_{\mu\nu}\delta g^{\mu\nu} \right) \tag{A.20}$$

$$\delta\sqrt{-g} = \frac{-\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{A.21}$$

Commutation of $R^{\sigma}_{\ v}$

$$\left[\nabla_{\mu},\ \nabla_{\sigma}\right]R^{\sigma}_{\ v}=R^{\sigma}_{\ \lambda\mu\sigma}R^{\lambda}_{\ v}-R^{\lambda}_{\ v\mu\sigma}R^{\sigma}_{\ \lambda}=-R_{\alpha\mu}R^{\alpha}_{\ v}+R_{\mu\sigma v\rho}R^{\sigma\rho}$$

$$R_{\alpha\mu}R^{\alpha}_{\ v}=R_{\mu\sigma v\rho}R^{\sigma\rho}-\left[\nabla_{\mu},\ \nabla_{\sigma}\right]R^{\sigma}_{\ v} \tag{A.22}$$

$$\begin{aligned}\int dx^{4}\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}&=\int dx^{4}\sqrt{-g}\,g^{\mu\nu}\left[\nabla_{\lambda}\left(\delta\Gamma^{\lambda}_{\nu\mu}\right)-\nabla_{\nu}\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right)\right]\\&=\int dx^{4}\sqrt{-g}\nabla_{\sigma}\left[g^{\mu\nu}\left(\delta\Gamma^{\sigma}_{\nu\mu}\right)-g^{\mu\sigma}\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right)\right]\\&=\int_{\infty}^{\infty}dx^{4}\sqrt{-g}\nabla_{\sigma}\left[g^{\mu\nu}\left(\delta\Gamma^{\sigma}_{\nu\mu}\right)-g^{\mu\sigma}\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right)\right]=0\end{aligned} \tag{A.23}$$

因 $\nabla_{\lambda}g^{\mu\nu}=0$ ，所以 $\nabla_{\lambda}\left(g^{\mu\nu}f^{\alpha}\right)=0+g^{\mu\nu}\nabla_{\lambda}\left(f^{\alpha}\right)$

分別導出各項路徑的微擾形式

$$\begin{aligned}\delta S_{1}&=\frac{1}{2}\int d^{4}x\left[\left(\delta\sqrt{-g}\right)(R)+\left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g}+\left(\delta R_{\mu\nu}\right)g^{\mu\nu}\sqrt{-g}\right]\\&=\frac{1}{2}\int d^{4}x\sqrt{-g}(\delta g^{\mu\nu})\left[-\frac{1}{2}g_{\mu\nu}R+R_{\mu\nu}\right]\end{aligned} \tag{A.24}$$

$$\begin{aligned}\delta S_{2}&=\frac{1}{2}\delta\left[\int d^{4}x\sqrt{-g}(\alpha R^{2})\right]\\&=\frac{1}{2}\int d^{4}x\left[\left(\delta\sqrt{-g}\right)\alpha R^{2}+2\left(\delta g^{\mu\nu}\right)\left(\alpha RR_{\mu\nu}\right)\sqrt{-g}+2\left(\delta R_{\mu\nu}\right)\left(\alpha Rg_{\mu\nu}\right)\sqrt{-g}\right]\\&=\frac{1}{2}\int d^{4}x\sqrt{-g}(\delta g^{\mu\nu})\left[-\frac{1}{2}\alpha g_{\mu\sigma}R^{2}+2\alpha RR_{\mu\nu}+2\alpha\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)R\right]\end{aligned} \tag{A.25}$$

$$\begin{aligned}\delta S_{3}&=\frac{1}{2}\delta\left[\int d^{4}x\sqrt{-g}(\beta R_{\mu\nu}R^{\mu\nu})\right]\\&=\frac{1}{2}\int d^{4}x\left[\left(\delta\sqrt{-g}\right)(\beta R_{\mu\nu}R^{\mu\nu})+\left(\delta g^{\mu\alpha}g^{\nu\beta}\right)(\beta R_{\mu\nu}R^{\mu\nu})\sqrt{-g}+g^{\mu\alpha}g^{\nu\beta}(\delta\beta R_{\mu\nu}R^{\mu\nu})\sqrt{-g}\right]\\&=\frac{1}{2}\int d^{4}x\left[\left(\delta\sqrt{-g}\right)(\beta R_{\mu\nu}R^{\mu\nu})+2\left(\delta g^{\mu\nu}\right)(\beta R^{\alpha}_{\ \nu}R_{\mu\nu})\sqrt{-g}+2\left(\delta R_{\mu\nu}\right)(\delta\beta R^{\mu\nu})\sqrt{-g}\right]\\&=\frac{1}{2}\int d^{4}x\sqrt{-g}\left(\delta g^{\mu\nu}\right)\left[-\frac{1}{2}\beta g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}+2\beta R^{\alpha}_{\ \nu}R_{\mu\nu}+2\beta\left(\begin{aligned}&-\nabla_{\alpha}\nabla_{\nu}R^{\alpha}_{\ \mu}+\frac{1}{2}\Box R_{\mu\nu}+\\&\frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}R^{\alpha\beta}\end{aligned}\right)\right]\end{aligned} \tag{A.26}$$

$$\delta S_4 = \frac{1}{2}\delta\left[\int d^4x\sqrt{-g}(-2\Lambda)\right]$$
$$= -\int d^4x\left(\delta\sqrt{-g}\right)\Lambda = \int d^4x\sqrt{-g}\left(\delta g_{\mu\nu}\right)g_{\mu\nu}\Lambda$$

(A.27)

然後四個路徑為擾項再合併

$$\int d^4x\sqrt{-g}(\delta g^{\mu\nu})\begin{bmatrix} -\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} - \frac{1}{2}\alpha g_{\mu\nu}R^2 + 2\alpha\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R - \frac{1}{2}\beta g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho} \\ +2\beta R^{\alpha}{}_{\nu}R_{\alpha\mu} + 2\beta\left(-\nabla_\alpha\nabla_\nu R^{\alpha}{}_{\mu} + \frac{1}{2}\Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}\right) + g_{\mu\nu}\Lambda \end{bmatrix} = 0$$

(A.28)

因為是 non-trivial 的解所以括號裡的項要為零

$$-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} - \frac{1}{2}\alpha g_{\mu\nu}R^2 + 2\alpha RR_{\mu\nu} + 2\alpha\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R - \frac{1}{2}\beta g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}$$
$$+2\beta R^{\alpha}{}_{\nu}R_{\alpha\mu} + 2\beta\left(-\nabla_\alpha\nabla_\nu R^{\alpha}{}_{\mu} + \frac{1}{2}\Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}\right) + g_{\mu\nu}\Lambda = 0$$

再利用 $\nabla_\mu\nabla_\nu R^{\mu\nu} = \frac{1}{2}\Box R$ 與 $R_{\alpha\mu}R^{\alpha}{}_{\nu} = R_{\mu\sigma\nu\rho}R^{\sigma\rho} - \left[\nabla_\mu, \nabla_\sigma\right]R^{\sigma}_{\nu}$

(A.29)

高階修正項的 Einstein Equation 為

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 2\alpha\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right)R + (2\alpha+\beta)\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R + \beta\Box\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$
$$+2\beta\left(R_{\mu\sigma\nu\rho}R^{\sigma\rho} - \frac{1}{4}\beta g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}\right) + g_{\mu\nu}\Lambda = 0$$

(A.30)

亦可表示成一般的 Einstein 的真空場方程式

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \text{(A.31)}$$

其中

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

所以 stress tensor 為

$$\Phi_{\mu\nu} = 2\alpha\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right)R + (2\alpha+\beta)\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R + \beta\Box\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$
$$+2\beta\left(R_{\mu\sigma\nu\rho}R^{\sigma\rho} - \frac{1}{4}\beta g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}\right)$$

(A.32)

Appendix B
空間與時間的運動方程式

我們可導出無論 Einstein-Hilbert action 或是有高階修正項都能通用的運動方程式，首先把 Action 的部分空間積分為 $S=\int d^4xL=\int d^4x\sqrt{-g}\,\ell=\int d^4xV\ell$ **，其中** $L=\sqrt{-g}\,\ell=a_1(t)a_2(t)a_3(t)\ell=V\ell$ **，然後做尺度因子的變分，我們先把 action 對時間作變分，利用 Euler-Lagrange equation 可得方程式:**

$$\begin{aligned}
0&=\frac{\partial L}{\partial B}-\frac{d}{dt}\left(\frac{L}{\dot B}\right)\\
&=\left(-\frac{1}{2}V\ell+V\frac{\partial\ell}{\partial B}\right)-\frac{d}{dt}\left(V\frac{\partial\ell}{\partial\dot B}\right)\\
&=\left(-\frac{1}{2}V\ell+V\frac{\partial\ell}{\partial B}\right)-\left(\dot V\frac{\partial\ell}{\partial\dot B}+\frac{d}{dt}\frac{\partial\ell}{\partial\dot B}\right)\\
&=\left(-\frac{1}{2}V\ell+V\frac{\partial\ell}{\partial B}\right)-\left(H_i\frac{\partial\ell}{\partial\dot B}+\frac{d}{dt}\frac{\partial\ell}{\partial\dot B}\right)
\end{aligned}\tag{B.1}$$

因為因次的關係 B **總會出現在** $B\dot H_i$ **，** BH_iH_j **項中，所以可以用連鎖定律作代換**

$$\frac{\partial\ell}{\partial B}=\frac{H_i}{2}\frac{\partial\ell}{\partial H_i}+\dot H_i\frac{\partial\ell}{\partial\dot H_i}\tag{B.2}$$

$$\frac{\partial\ell}{\partial\dot B}=\frac{H_i}{2}\frac{\partial\ell}{\partial\dot H_i}\tag{B.3}$$

帶回得，

$$-\ell+H_i\frac{\partial\ell}{\partial H_i}+\dot H_i\frac{\partial\ell}{\partial\dot H_i}-3H\frac{H_i\partial\ell}{\partial\dot H_i}-\frac{d}{dt}\frac{H_i\partial\ell}{\partial\dot H_i}=0\tag{B.4}$$

所以可得 Friedmann equation:

$$D_0L=\ell+H_i\left(\frac{d}{dt}+3H\right)\frac{\partial\ell}{\partial\dot H_i}-H_i\frac{\partial\ell}{\partial H_i}-\dot H_i\frac{\partial\ell}{\partial\dot H_i}=0\tag{B.5}$$

接著對空間項空間因子變分

$$
\begin{aligned}
0&=\frac{\partial L}{\partial a_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_i}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{a}_i}\right)\\
&=V\left[\frac{\ell}{a_i}+\frac{\partial \ell}{\partial a_i}-\left(\partial_0+3H\right)\frac{\partial \ell}{\partial \dot{a}_i}+\left(\partial_0^2+2\cdot 3H\partial_0+\dot{H}_i+H_iH_j\right)\frac{\partial \ell}{\partial \ddot{a}_i}\right]\\
&=\frac{\ell}{a_i}+\frac{\partial \ell}{\partial a_i}-\left(\partial_0+3H\right)\frac{\partial \ell}{\partial \dot{a}_i}+\left(\partial_0+3H\right)^2\frac{\partial \ell}{\partial \ddot{a}_i}
\end{aligned}
\tag{B.5}
$$

我們把 **Lagrangian density** 以哈伯常數 H_i ，$\dot{H}_i$ 表示，相同的利用連鎖定律:

$$
\frac{\partial \ell}{\partial a_i}=\frac{\partial \ell}{\partial H_i}\frac{\partial H_i}{\partial a_i}+\frac{\partial \ell}{\partial \dot{H}_i}\frac{\partial \dot{H}_i}{\partial a_i}=-\frac{H_i}{a_i}\frac{\partial \ell}{\partial H_i}-\frac{1}{a_i}\left[\dot{H}_i-H_i^2\right]\frac{\partial \ell}{\partial \dot{H}_i}
\tag{B.6}
$$

$$
\frac{\partial \ell}{\partial \dot{a}_i}=\frac{\partial \ell}{\partial H_i}\frac{\partial H_i}{\partial \dot{a}_i}+\frac{\partial \ell}{\partial \dot{H}_i}\frac{\partial \dot{H}_i}{\partial a_i}=\frac{1}{a_i}\frac{\partial \ell}{\partial \dot{H}_i}-\frac{2}{a_i}H_i\frac{\partial \ell}{\partial \dot{H}_i}
\tag{B.7}
$$

$$
\frac{\partial \ell}{\partial \ddot{a}_i}=\frac{1}{a_i}\frac{\partial \ell}{\partial \dot{H}_i}
\tag{B.8}
$$

又

$$
\begin{aligned}
&\left(\partial_0+H_i\right)^2L=\left(\partial_0+H_i\right)\left(\partial_0+H_j\right)L\\
&=\partial_0^2L+\partial_0\left(H_jL\right)+H_i\partial_0L+H_iH_jL\\
&=\left(\partial_0^2+\dot{H}_j+2H_i\partial_0+H_iH_j\right)L
\end{aligned}
\tag{B.9}
$$

帶入且同乘 a_i 通分

$$
0=\ell-\left(\partial_0+3H\right)\frac{\partial \ell}{\partial H_i}+\left(\partial_0+3H\right)^2\frac{\partial \ell}{\partial \dot{H}_i}+a_i\frac{\partial \ell}{\partial a_i}
\tag{B.10}
$$

又寫作空間運動方程式表示式:

$$
D_i\ell=\ell-\left(\partial_0+3H\right)\frac{\partial \ell}{\partial H_i}+\left(\partial_0+3H\right)^2\frac{\partial \ell}{\partial \dot{H}_i}+a_i\frac{\partial \ell}{\partial a_i}=0
\tag{B.11}
$$

Appendix C
The other method for finding Bianchi Type I solution

依據 Bianchi type 模型來研究 anisotropic 宇宙模型，首要先了解 Bianchi 模型，一般的 Bianchi type I 表示式為:

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2 \quad \text{(C.1)}$$

$a_1(t)$，$a_2(t)$，$a_3(t)$ 為可調參數，且時間為度常數為 1

$a_1(t) = a_2(t) = a_3(t)$，isotropic university

$a_1(t) \neq a_2(t) \neq a_3(t)$，anisotropic university

時空表示式寫成矩陣為

$$ds^2 = -dt^2 + g_{\mu\nu} dx_i dx_j \quad \text{(C.2)}$$

故時空間度規為

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a_1^2(t) & 0 & 0 \\ 0 & 0 & a_2^2(t) & 0 \\ 0 & 0 & 0 & a_3^2(t) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a_1^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a_2^2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{a_3^2(t)} \end{pmatrix} \quad \text{(C.3)}$$

我們需要一些基本方程式

The Christoffe connection

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \quad \text{(C.4)}$$

Riemann tensor

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}{}_{\nu\mu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} \quad \text{(C.5)}$$

The Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \quad \text{(C.6)}$$

The scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \tag{C.7}$$

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{C.8}$$

依據公式的推導可得，

$\Gamma^0_{11} = a_1 \dot{a}_1 \quad \Gamma^0_{22} = a_2 \dot{a}_2 \quad \Gamma^0_{33} = a_3 \dot{a}_3 \quad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}_1}{a_1} \equiv H_1 \quad \Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{a}_2}{a_2} \equiv H_2$

$\Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}_3}{a_3} \equiv H_3$ ，在此我們令 $\frac{\dot{a}_i}{a_i} \equiv H_i = \text{constant}$ 為哈伯常數

Riemann tensor

$$R_{00} = -(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2)$$

$$R_{11} = a_1^2 (\dot{H}_1 + H_1^2 + H_1 H_2 + H_1 H_3)$$

$$R_{22} = a_2^2 (\dot{H}_2 + H_2^2 + H_1 H_2 + H_2 H_3)$$

$$R_{33} = a_3^2 (\dot{H}_3 + H_3^2 + H_1 H_2 + H_2 H_3) \tag{C.9}$$

The Ricci tensor

$$R = R^{\mu}{}_{\mu} = 2(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1 H_2 + H_1 H_3 + H_2 H_3) \tag{C.10}$$

我們的四維重力場路徑為

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} (R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} - 2\Lambda) \quad , \quad \sqrt{-g} = a_1(t) a_2(t) a_3(t)$$

其中令 $L = (R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} - 2\Lambda)$ (C.11)

帶入所有代數

$$\begin{aligned} L &= 2\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1 H_2 + H_1 H_3 + H_2 H_3\right) \\ &+ 4\alpha\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1 H_2 + H_1 H_3 + H_2 H_3)\right)^2 - 2\Lambda \\ &+ \beta\left[\begin{array}{l} \left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2\right)^2 + \left(\dot{H}_1 + H_1^2 + H_1 H_2 + H_1 H_3\right)^2 \\ + \left(\dot{H}_2 + H_2^2 + H_1 H_2 + H_2 H_3\right)^2 + \left(\dot{H}_3 + H_3^2 + H_1 H_3 + H_2 H_3\right)^2 \end{array}\right] \end{aligned} \tag{C.12}$$

且為了方便計算，我們令新代數再重新整理

$X = H_1 + H_2 + H_3$

$Y = H_1^2 + H_2^2 + H_3^2 + \dot{H}_1 + \dot{H}_2 + \dot{H}_3$

$Z = H_1H_2 + H_1H_3 + H_2H_3$

$W = \dot{H}_1H_2^2 + \dot{H}_1H_3^2 + \dot{H}_2H_1^2 + \dot{H}_2H_3^2 + \dot{H}_3H_1^2 + \dot{H}_3H_2^2$

$+\dot{H}_1H_2H_3 + H_1\dot{H}_2H_3 + H_1H_2\dot{H}_3 + \dot{H}_1\dot{H}_2 + \dot{H}_2\dot{H}_3 + \dot{H}_1\dot{H}_3$

(C.13)

可簡化成

$$L = 2(Y+Z) + 4\alpha(Y+Z)^2 + 2\beta Y(Y+Z) - 2\beta W - 2\Lambda \quad \text{(C.14)}$$

我們先算出 L 分別對 H_i，$\dot{H}_i$

$$L_i = \frac{\partial L}{\partial H_i}，L^i = \frac{\delta L}{\delta \dot{H}_i}，L_{ij} = \frac{\partial L}{\partial H_i \partial H_j}，L_i^j = \frac{\partial L}{\partial H_i \partial \dot{H}_j}，L^{ij} = \frac{\partial L}{\partial \dot{H}_i \partial \dot{H}_j} \quad \text{(C.15)}$$

可以把上式公式化

$$L_i = \frac{\partial L}{\partial H_i} = 2\frac{\partial(Y+Z)}{\partial H_i} + 8\alpha(Y+Z)\frac{\partial(Y+Z)}{\partial H_i} + 2\beta\left[(Y+Z)\frac{\partial Y}{\partial H_i} + Y\frac{\partial(Y+Z)}{\partial H_i}\right] - 2\beta\frac{\partial W}{\partial H_i}$$

(B.16)

$$L^i = \frac{\partial L}{\partial \dot{H}_i} = 2\frac{\partial(Y+Z)}{\partial \dot{H}_i} + 8\alpha(Y+Z)\frac{\partial(Y+Z)}{\partial \dot{H}_i} + 2\beta\left[(Y+Z)\frac{\partial Y}{\partial \dot{H}_i} + Y\frac{\partial(Y+Z)}{\partial \dot{H}_i}\right] - 2\beta\frac{\partial W}{\partial \dot{H}_i}$$

(C.17)

$$L_{ij} = \frac{\partial L}{\partial H_i \partial H_j} = 2\frac{\partial^2(Y+Z)}{\partial H_i \partial H_j} + 8\alpha\left[\frac{\partial(Y+Z)}{\partial H_i}\frac{\partial(Y+Z)}{\partial H_j} + (Y+Z)\frac{\partial^2(Y+Z)}{\partial H_i \partial H_j}\right] - 2\beta\frac{\partial^2 W}{\partial H_i \partial H_j}$$

$$+ 2\beta\left[(Y+Z)\frac{\partial^2 Y}{\partial H_i \partial H_j} + Y\frac{\partial^2(Y+Z)}{\partial H_i \partial H_j} + \frac{\partial(Y+Z)}{\partial H_i}\frac{\partial Y}{\partial H_j} + \frac{\partial(Y+Z)}{\partial H_j}\frac{\partial Y}{\partial H_i}\right]$$

(C.18)

$$L_i^j = \frac{\partial L}{\partial H_i \partial \dot{H}_j} = 2\frac{\partial^2 (Y+Z)}{\partial H_i \partial \dot{H}_j} + 8\alpha\left[\frac{\partial (Y+Z)}{\partial H_i}\frac{\partial (Y+Z)}{\partial \dot{H}_j} + (Y+Z)\frac{\partial^2 (Y+Z)}{\partial H_i \partial \dot{H}_j}\right] - 2\beta\frac{\partial^2 W}{\partial H_i \partial \dot{H}_j}$$

$$+2\beta\left[(Y+Z)\frac{\partial^2 Y}{\partial H_i \partial \dot{H}_j} + Y\frac{\partial^2 (Y+Z)}{\partial H_i \partial \dot{H}_j} + \frac{\partial (Y+Z)}{\partial H_i}\frac{\partial Y}{\partial \dot{H}_j} + \frac{\partial (Y+Z)}{\partial \dot{H}_j}\frac{\partial Y}{\partial H_i}\right] \tag{C.19}$$

$$L^{ij} = \frac{\partial L}{\partial \dot{H}_i \partial \dot{H}_j} = 2\frac{\partial^2 (Y+Z)}{\partial \dot{H}_i \partial \dot{H}_j} + 8\alpha\left[\frac{\partial (Y+Z)}{\partial \dot{H}_i}\frac{\partial (Y+Z)}{\partial \dot{H}_j} + (Y+Z)\frac{\partial^2 (Y+Z)}{\partial \dot{H}_i \partial \dot{H}_j}\right] - 2\beta\frac{\partial^2 W}{\partial \dot{H}_i \partial \dot{H}_j}$$

$$+2\beta\left[(Y+Z)\frac{\partial^2 Y}{\partial \dot{H}_i \partial \dot{H}_j} + Y\frac{\partial^2 (Y+Z)}{\partial \dot{H}_i \partial \dot{H}_j} + \frac{\partial (Y+Z)}{\partial \dot{H}_i}\frac{\partial Y}{\partial \dot{H}_j} + \frac{\partial (Y+Z)}{\partial \dot{H}_j}\frac{\partial Y}{\partial \dot{H}_i}\right] \tag{C.20}$$

其中

$$\frac{\partial Y}{\partial H_i} = 2H_i \text{ ， } \frac{\partial Y}{\partial \dot{H}_i} = 1 \text{ ， } \frac{\partial Y}{\partial H_i \partial H_j} = 2\delta_{ij} \text{ ， } \frac{\partial Y}{\partial H_i \partial \dot{H}_j} = \frac{\partial Y}{\partial \dot{H}_i \partial \dot{H}_j} = 0$$

$$\frac{\partial Z}{\partial H_i} = \frac{|\varepsilon_{ijk}|}{2}(H_j + H_k) \text{ ， } \frac{\partial Z}{\partial \dot{H}_i} = 0 \text{ ， } \frac{\partial Z}{\partial H_i \partial H_j} = 1 - \delta_{ij} \text{ ， } \frac{\partial Z}{\partial H_i \partial \dot{H}_j} = \frac{\partial Z}{\partial \dot{H}_i \partial \dot{H}_j} = 0$$

$$\frac{\partial (Y+Z)}{\partial H_i} = 2H_i + \frac{|\varepsilon_{ijk}|}{2}(H_j + H_k) \text{ ， } \frac{\partial (Y+Z)}{\partial \dot{H}_i} = 1 \text{ ， } \frac{\partial (Y+Z)}{\partial H_i \partial \dot{H}_j} = \frac{\partial (Y+Z)}{\partial \dot{H}_i \partial \dot{H}_j} = 0$$

$$\frac{\partial (Y+Z)}{\partial H_i \partial H_j} = \delta_{ij} + 1 \text{ ， } \frac{\partial W}{\partial H_i} = H_i |\varepsilon_{ikl}| (\dot{H}_l + \dot{H}_k) + \frac{|\varepsilon_{ikl}|}{2}(\dot{H}_l H_k + H_l \dot{H}_k)$$

$$\frac{\partial W}{\partial \dot{H}_i} = \frac{|\varepsilon_{ikl}|}{2}(H_l^2 + H_k^2 + H_l H_k + \dot{H}_l + \dot{H}_k) \text{ ， } \frac{\partial^2 W}{\partial \dot{H}_i \partial \dot{H}_j} = 1 - \delta_{ij} \text{ ， } i, j, k, l = 1, 2, 3$$

$$\frac{\partial^2 W}{\partial H_i \partial H_j} = \begin{pmatrix} 2\dot{H}_2 + 2\dot{H}_3 & \dot{H}_3 & \dot{H}_2 \\ \dot{H}_3 & 2\dot{H}_1 + 2\dot{H}_3 & \dot{H}_1 \\ \dot{H}_2 & \dot{H}_1 & 2\dot{H}_1 + 2\dot{H}_2 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$$

$$\frac{\partial^2 W}{\partial H_i \partial \dot{H}_j} = \begin{pmatrix} 0 & 2H_1 + H_3 & 2H_1 + H_2 \\ 2H_2 + H_3 & 0 & 2H_2 + H_1 \\ 2H_3 + H_2 & 2H_3 + H_1 & 0 \end{pmatrix} = \begin{pmatrix} W_1^1 & W_1^2 & W_1^3 \\ W_2^1 & W_2^2 & W_2^3 \\ W_3^1 & W_3^2 & W_3^3 \end{pmatrix} \tag{C.21}$$

上述參數代回原式得

$$L_i = 2\left[2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right] + 8\alpha(Y+Z)\left[2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right]$$

$$+2\beta\left[(Y+Z)(2H_i) + Y\left(2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right)\right]$$

$$-2\beta\left[H_i|\varepsilon_{ikl}|(\dot{H}_l + \dot{H}_k) + \frac{|\varepsilon_{ikl}|}{2}(\dot{H}_l H_k + H_l \dot{H}_k)\right]$$

(C.22)

$$L_i^j = 8\alpha\left[\left(2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right)\right] + 2\beta\left[\left(2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right) + 2H_i\right]$$

$$-2\beta\begin{pmatrix} 0 & 2H_1 + H_3 & 2H_1 + H_2 \\ 2H_2 + H_3 & 0 & 2H_2 + H_1 \\ 2H_3 + H_2 & 2H_3 + H_1 & 0 \end{pmatrix}$$

(C.23)

$$L_{ij} = 2(\delta_{ij} + 1) + 8\alpha\left[\left(2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right)\left(2H_j + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right) + (Y+Z)(\delta_{ij} + 1)\right]$$

$$+2\beta\left[\begin{array}{l}(Y+Z)(2\delta_{ij}) + Y(\delta_{ij}+1) + (2H_i)\left(2H_j + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right) \\ +(2H_j)\left(2H_i + \frac{|\varepsilon_{ikl}|}{2}(H_l + H_k)\right)\end{array}\right]$$

$$-2\beta\begin{pmatrix} 2\dot{H}_2 + 2\dot{H}_3 & \dot{H}_3 & \dot{H}_2 \\ \dot{H}_3 & 2\dot{H}_1 + 2\dot{H}_3 & \dot{H}_1 \\ \dot{H}_2 & \dot{H}_1 & 2\dot{H}_1 + 2\dot{H}_2 \end{pmatrix}$$

(C.24)

$$L^{ij} = 8\alpha + 2\beta[2 - 1 + \delta_{ij}]$$

(C.25)

重力場方程式[22] [23]

$$D_0 L = L + H_i(\frac{d}{dt}+3H)L^i - H_i L_i + \dot{H}_i L^i = 0$$ 又稱 **Friedmann equation** 為時間項方式 (C.26)

$$D_i L = L + (\frac{d}{dt}+3H)^2 L^i - (\frac{d}{dt}+3H)L_i = 0$$ 為空間項運動方式 (C.27)

其中 $L_i = \frac{\delta L}{\delta H_i}$，$L^i = \frac{\delta L}{\delta \dot{H}_i}$，$3H = \sum_{i=1}^{n} H_i$

且因 $\frac{\dot{a}_i}{a_i} \equiv H_i = \text{constant}$，所以 $\dot{H}_i = 0$

代入再簡化得

$$\begin{cases} D_0 L = 2Z + 4\alpha\left(Z^2 - Y^2\right) + 2\beta Y\left(Z - Y\right) - 2\Lambda = 0 \\ D_i L = 4Y + 2Z + 4\alpha\left(Y - Z\right)\left(Y + Z\right) + 2\beta Y\left(Y - Z\right) - 6\Lambda = 0 \end{cases} \tag{C.28}$$

所以可以解出解來

$$\begin{cases} Y = -\frac{1+8\alpha\Lambda}{2\beta} \\ Z = \frac{1+8\alpha\Lambda+4\beta\Lambda}{2\beta} \end{cases} \tag{C.29}$$

在此 $Y = H_1^{\ 2} + H_2^{\ 2} + H_3^{\ 2}$ $Z = H_1 H_2 + H_2 H_3 + H_1 H_3$ 兩者相加為 $Y + Z = 2\Lambda$

$$H_1 = H_2 = H_3 = \sqrt{\frac{\Lambda}{3}} = H_0$$ 為一解，為 **De Sitter** 解 (C.30)

接著我們要證明(C.29)與(3.64)(3.66)的解相同，利用(3.59) 代入(3.64)(3.66)做代數代換，我們可以得:

$$b^2 = H^2 = \left(\frac{H_1+H_2+H_3}{3}\right)^2 = \frac{H_1^{\ 2}+H_2^{\ 2}+H_3^{\ 2}+2\left(H_1H_2+H_2H_3+H_1H_3\right)}{9} = \frac{Y+2Z}{9}$$
$$= \frac{1}{9}\left(-\frac{1+8\alpha\Lambda}{2\beta} + 2\frac{1+8\alpha\Lambda+4\beta\Lambda}{2\beta}\right) = \frac{1+8\Lambda\left(\alpha+\beta\right)}{18\beta} \tag{C.31}$$

$$\sigma_+^2 + \sigma_-^2 = \left(-\frac{H_1}{3} + \frac{H_2+H_3}{6}\right)^2 + \left(\frac{H_2-H_3}{2\sqrt{3}}\right)^2 = \frac{4\left(H_1^{\ 2}+H_2^{\ 2}+H_3^{\ 2}\right) - 4\left(H_1H_2+H_2H_3+H_1H_3\right)}{36}$$
$$= \frac{Y-Z}{9} = \frac{1}{9}\left(-\frac{1+8\alpha\Lambda}{2\beta} - \frac{1+8\alpha\Lambda+4\beta\Lambda}{2\beta}\right) = -\frac{1+2\Lambda\left(4\alpha+\beta\right)}{9\beta} \tag{C.32}$$

由(C.31) (C.32)證明確實與(3.64)(3.66)相同。

微擾穩定性討論

微擾的方法是在測試方程式是否有穩定解或不穩定解，我們通常可以假設多加微擾項 δA，如果 δA 是時間的函數，則可以假設為任一函數 f(t)，可 t^n，sin(nt)，cos(nt)，exp(nt)，等形式微擾，在此我們假設 $\delta A_i(t)=k_i\exp(vt)$，因為微擾時時空因子為 $a_i=\exp(A_i(t))$，微擾後為 $a_{pi}=\exp(A_i(t)+\delta A_i(t))$，我們希望可以找出 v 大於零的解，因此可以證明方程式的微擾為不穩定，隨時間無窮大會膨脹到無窮大，就可證明 **Bianchi Type I** 不為 **de Sitter Space** 解:

對 D_0L ， D_iL 作微擾

$$\begin{aligned}\delta\left(D_0L\right)=&L_i\delta H_i+L^i\delta\dot{H}_i+\delta H_i\left(\partial_0+3H\right)L^i+H_i\left(\dot{L}^i_j\delta H_j+\dot{L}^{ij}\delta\dot{H}_j+L^i_j\delta\dot{H}_j+L^{ij}\delta\ddot{H}_j\right)\\&+H_i\left(\delta 3H\right)L^i+H_i\left(3H\right)\left(L^i_j\delta H_j+L^{ij}\delta\dot{H}_j\right)-\delta H_iL_i-H_i\left(L_{ij}\delta H_j+L^i_j\delta\dot{H}_j\right)\\&-\delta\dot{H}_iL^i-\dot{H}_i\left(L^i_j\delta H_j+L^{ij}\delta\dot{H}_j\right)\end{aligned} \tag{C.33}$$

$$\begin{aligned}\delta\left(D_iL\right)=&3L_i\delta H_i+3L^i\delta\dot{H}_i+\dot{L}^i_j\delta H_j+\ddot{L}^{ij}\delta\dot{H}_j+L^i_j\delta\ddot{H}_j+L^{ij}\delta\dddot{H}_j+2\dot{L}^i_j\delta\dot{H}_j+2\dot{L}^{ij}\delta\ddot{H}_j\\&+3H\left(\dot{L}^i_j\delta H_j+\dot{L}^{ij}\delta\dot{H}_j+L^i_j\delta\dot{H}_j+L^{ij}\delta\ddot{H}_j\right)+\left(\delta 3H\right)\dot{L}^i\\&+\left(\dot{H}_1+\dot{H}_2+\dot{H}_3\right)\left(L^i_j\delta H_j+L^{ij}\delta\dot{H}_j\right)+\left(\delta\dot{H}_1+\delta\dot{H}_2+\delta\dot{H}_3\right)L^i\\&+2\left(H_1+H_2+H_3\right)\left(\delta H_1+\delta H_2+\delta H_3\right)L^i\\&+\left(H_1^2+H_2^2+H_3^2+2H_1H_2+2H_1H_3+2H_2H_3\right)\left(L^i_j\delta H_j+L^{ij}\delta\dot{H}_j\right)\\&-\dot{L}_{ij}\delta H_j-\dot{L}^i_j\delta\dot{H}_j-L_{ij}\delta\dot{H}_j-L^i_j\delta\ddot{H}_j-3H\left(L_{ij}\delta H_j-L^j_i\delta\dot{H}_j\right)-\delta\left(3H\right)L_i\end{aligned} \tag{C.34}$$

因 $a_i=\exp\left(A_i\left(t\right)\right)$ ， $H_i=\dfrac{\dot{a}_i}{a_i}$ 故得

$$\delta H_i=\delta\dot{A}_i\left(t\right) \text{，} \delta H_i=v\delta A_i \text{，} \delta\dot{H}_i=v^2\delta A_i \text{，} \delta\ddot{H}_i=v^3\delta A_i \text{，} \delta\dddot{H}_i=v^4\delta A_i \tag{C.35}$$

$$\delta L=L_i\delta H_i+L^i\delta\dot{H}_i \text{，} \delta L_i=L_{ij}\delta H_j+L^i_j\delta\dot{H}_j \text{，} \delta L^i=L^i_j\delta H_j+L^{ij}\delta\dot{H}_j \tag{C.36}$$

$$\delta\left(D_0 L\right)=$$

$$\delta A_1\begin{bmatrix} L_1+vL^1+H_1(vL^1{}_1+v^2L^{11})+H_2\left(vL^2{}_1+v^2L^{21}\right)+H_3\left(vL^3{}_1+v^2L^{31}\right)+\left(H_1+H_2+H_3\right)L^1+H_1L^1+H_2L^2+H_3L^3 \\ +H_1\left(H_1+H_2+H_3\right)\left(L^1{}_1+vL^{11}\right)+H_2\left(H_1+H_2+H_3\right)\left(L^2{}_1+vL^{21}\right)+H_3\left(H_1+H_2+H_3\right)\left(L^3{}_1+vL^{31}\right)-L_1 \\ -H_1\left(L_{11}+vL_1^1\right)-H_2\left(L_{21}+vL_2^1\right)-H_3\left(L_{31}+vL_3^1\right)-vL^1 \end{bmatrix}$$

$$+\delta A_2\begin{bmatrix} L_2+vL^2+H_1(vL^1{}_2+v^2L^{21})+H_2\left(vL^2{}_2+v^2L^{22}\right)+H_3\left(vL^3{}_2+v^2L^{32}\right)+\left(H_1+H_2+H_3\right)L^2+H_1L^1+H_2L^2+H_3L^3 \\ +H_1\left(H_1+H_2+H_3\right)\left(L^1{}_2+vL^{12}\right)+H_2\left(H_1+H_2+H_3\right)\left(L^2{}_2+vL^{22}\right)+H_3\left(H_1+H_2+H_3\right)\left(L^3{}_2+vL^{32}\right)-L_2 \\ -H_1\left(L_{12}+vL_1^2\right)-H_2\left(L_{22}+vL_2^2\right)-H_3\left(L_{32}+vL_3^2\right)-vL^2 \end{bmatrix}$$

$$+\delta A_3\begin{bmatrix} L_3+vL^3+H_1(vL^1{}_3+v^2L^{31})+H_2\left(vL^2{}_3+v^2L^{23}\right)+H_3\left(vL^3{}_3+v^2L^{33}\right)+\left(H_1+H_2+H_3\right)L^3+H_1L^1+H_2L^2+H_3L^3 \\ +H_1\left(H_1+H_2+H_3\right)\left(L^1{}_3+vL^{13}\right)+H_2\left(H_1+H_2+H_3\right)\left(L^2{}_3+vL^{23}\right)+H_3\left(H_1+H_2+H_3\right)\left(L^3{}_3+vL^{33}\right)-L_3 \\ -H_1\left(L_{13}+vL_1^3\right)-H_2\left(L_{23}+vL_2^3\right)-H_3\left(L_{33}+vL_3^3\right)-vL^3 \end{bmatrix}$$

$$=A\delta A_1+B\delta A_2+C\delta A_3 \tag{C.37}$$

$$\delta\left(D_i L\right)$$

$$=\ \delta A_1 2\begin{bmatrix} 2L_1-L_2-L_3+v\left(4L^1+L^2+L^3\right)+2\left(H_1+H_2+H_3\right)\left(L^1+L^2+L^3\right) \\ \left[v^2+(H_1+H_2+H_3+v)(H_1+H_2+H_3+2v)\right]\left[\left(L^1{}_1+L^2{}_1+L^3{}_1\right)+v\left(L^{11}+L^{21}+L^{31}\right)\right] \\ -(H_1+H_2+H_3+v)\left(L_{11}+L_{21}+L_{31}\right)-(H_1+H_2+H_3+v)v\left(L_1^1+L_2^1+L_3^1\right) \end{bmatrix}$$

$$+\delta A_2 2\begin{bmatrix} 2L_2-L_1-L_3+v\left(L^1+4L^2+L^3\right)+2\left(H_1+H_2+H_3\right)\left(L^1+L^2+L^3\right) \\ \left[v^2+(H_1+H_2+H_3+v)(H_1+H_2+H_3+2v)\right]\left[\left(L^1{}_2+L^2{}_2+L^3{}_2\right)+v\left(L^{12}+L^{22}+L^{32}\right)\right] \\ -(H_1+H_2+H_3+v)\left(L_{12}+L_{22}+L_{32}\right)-(H_1+H_2+H_3+v)v\left(L_1^1+L_2^1+L_3^1\right) \end{bmatrix}$$

$$+\delta A_1 2\begin{bmatrix} 2L_3-L_1-L_3+v\left(L^1+L^2+4L^3\right)+2\left(H_1+H_2+H_3\right)\left(L^1+L^2+L^3\right) \\ \left[v^2+(H_1+H_2+H_3+v)(H_1+H_2+H_3+2v)\right]\left[\left(L^1{}_3+L^2{}_3+L^3{}_3\right)+v\left(L^{13}+L^{23}+L^{33}\right)\right] \\ -(H_1+H_2+H_3+v)\left(L_{13}+L_{23}+L_{33}\right)-(H_1+H_2+H_3+v)v\left(L_1^3+L_2^3+L_3^3\right) \end{bmatrix}$$

$$=D\delta A_1+E\delta A_2+F\delta A_3 \tag{C.38}$$

因為有 H_1，H_2，H_3 三個變數所以需要偶三個獨立方程式才能得解，故找 $\delta R=0$，使得

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix}=\begin{bmatrix} \delta\left(D_0 L\right) \\ \delta\left(D_i L\right) \\ \delta\left(R\right) \end{bmatrix}=0\text{，}\det\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}=0 \quad H_1\text{，}H_2\text{，}H_3\text{有解}$$

$$\begin{aligned} \delta R &= R_i\delta H_i+R^i\delta\dot{H}_i \\ &=\left(R_1+vR^1\right)\delta A_1+\left(R_2+vR^2\right)\delta A_2+\left(R_3+vR^3\right)\delta A_3 \\ &=\left(4H_1+2H_2+2H_3+2v\right)\delta A_1+\left(2H_1+4H_2+2H_3+2v\right)\delta A_2+\left(2H_1+2H_2+4H_3+2v\right)\delta A_3 \\ &=G\delta A_1+H\delta A_2+I\delta A_3 \end{aligned} \tag{C.39}$$

從新整理 A~I

$$A=2\begin{bmatrix} H_2+H_3+4\alpha\left(-2H_1^{\,3}-H_1H_2^{\,2}+H_2^{\,2}H_3-H_1^{\,2}H_3+H_2H_3^{\,2}+2H_1H_2H_3\right) \\ +\beta\left(-4H_1^{\,3}+H_2^{\,3}+H_3^{\,3}+3H_1^{\,2}H_2+3H_1^{\,2}H_3-4H_1H_2^{\,2}+H_2^{\,2}H_3-4H_1H_3^{\,2}+H_2H_3^{\,2}+2H_1H_2H_3\right) \end{bmatrix}$$
$$+2v\left[4\alpha\left(b^2+3H_1H_2+3H_1H_3+2H_2H_3\right)+2\beta\left(H_1^{\,2}+2H_1H_2+2H_1H_3+H_2H_3\right)\right]$$
$$+2v^2\left[4\alpha\left(H_1+H_2+H_3\right)+\beta\left(2H_1+H_2+H_3\right)\right]$$

(C.40)

$$B=2\begin{bmatrix} H_1+H_3+4\alpha\left(-2H_2^{\,3}-H_1^{\,2}H_2+H_1^{\,2}H_3+H_1H_3^{\,2}-H_2H_3^{\,2}+2H_1H_2H_3\right) \\ +\beta\left(H_1^{\,3}-4H_2^{\,3}+H_3^{\,3}-4H_1^{\,2}H_2+H_1^{\,2}H_3+3H_1H_2^{\,2}+H_2^{\,2}H_3+H_1H_3^{\,2}-4H_2H_3^{\,2}+2H_1H_2H_3\right) \end{bmatrix}$$
$$+2v\left[4\alpha\left(H_2^{\,2}+3H_1H_2+2H_1H_3+3H_2H_3\right)+2\beta\left(H_2^{\,2}+2H_1H_2+H_1H_3+2H_2H_3\right)\right]$$
$$+2v^2\left[4\alpha\left(H_1+H_2+H_3\right)+\beta\left(H_1+2H_2+H_3\right)\right]$$

(C.41)

$$C=2\begin{bmatrix} H_1+H_2+4\alpha\left(-2H_3^{\,3}+H_1^{\,2}H_2-H_1^{\,2}H_3+H_1H_2^{\,2}-H_2H_3^{\,2}+2H_1H_2H_3\right) \\ +\beta\begin{pmatrix} H_1^{\,3}+H_2^{\,3}-4H_3^{\,3}+H_1^{\,2}H_2-4H_1^{\,2}H_3+H_1H_2^{\,2}-4H_2^{\,2}H_3 \\ +3H_1H_3^{\,2}+3H_2H_3^{\,2}+2H_1H_2H_3 \end{pmatrix} \end{bmatrix}$$
$$+2v\left[4\alpha\left(H_3^{\,2}+3H_1H_3+3H_2H_3+2H_1H_2\right)+2\beta\left(H_3^{\,2}+H_1H_2+H_1H_3+2H_2H_3\right)\right]$$
$$+2v^2\left[4\alpha\left(H_1+H_2+H_3\right)+\beta\left(H_1+H_2+2H_3\right)\right]$$

(C.42)

$$\mathrm{D}=2\begin{bmatrix} 4H_1+H_2+H_3+4\alpha\left(2H_1^{\,3}+H_1H_2^{\,2}-H_2^{\,2}H_3+H_1H_3^{\,2}-H_2H_3^{\,2}-2H_1H_2H_3\right) \\ +\beta\begin{pmatrix} 4H_1^{\,3}-H_2^{\,3}-H_3^{\,3}-3H_1^{\,2}H_2-3H_1^{\,2}H_3+4H_1H_2^{\,2}-H_2^{\,2}H_3 \\ +4H_1H_3^{\,2}-H_2H_3^{\,2}-2H_1H_2H_3 \end{pmatrix} \end{bmatrix}$$
$$+2v\begin{bmatrix} 2+4\alpha\left(5H_1^{\,2}+3H_2^{\,2}+3H_3^{\,2}+6H_1H_2+6H_1H_3+4H_2H_3\right) \\ +\beta\left(6H_1^{\,2}+4H_2^{\,2}+4H_3^{\,2}+8H_1H_2+8H_1H_3+6H_2H_3\right) \end{bmatrix}$$
$$+2v^2\left[4\alpha\left(8H_1+5H_2+5H_3\right)+\beta\left(10H_1+7H_2+7H_3\right)\right]+v^3\left(12\alpha+4\beta\right)$$

(C.43)

$$
E = 2\begin{bmatrix} H_1 + 4H_2 + H_3 + 4\alpha\left(2H_1^3 + H_1^2H_2 - H_1^2H_3 - H_1H_3^2 + H_2H_3^2 - 2H_1H_2H_3\right) \\ +\beta\begin{pmatrix} -H_1^3 + 4H_2^3 - H_3^3 + 4H_1^2H_2 - H_1^2H_3 - 3H_1H_2^2 - 3H_2^2H_3 - H_1H_3^2 \\ +4H_2H_3^2 - 2H_1H_2H_3 \end{pmatrix} \end{bmatrix}
$$

$$
+2v\begin{bmatrix} 2 + 4\alpha\left(3H_1^2 + 5H_2^2 + 3H_3^2 + 6H_1H_2 + 4H_1H_3 + 6H_2H_3\right) \\ +\beta\left(4H_1^2 + 6H_2^2 + 4H_3^2 + 8H_1H_2 + 6H_1H_3 + 8H_2H_3\right) \end{bmatrix}
$$

$$
+2v^2\left[4\alpha\left(5H_1 + 8H_2 + 5H_3\right) + \beta\left(7H_1 + 10H_2 + 7H_3\right)\right] + v^3\left(12\alpha + 4\beta\right) \tag{C.44}
$$

$$
F = 2\begin{bmatrix} H_1 + H_2 + 4H_3 + 4\alpha\left(2H_3^3 - H_1^2H_2 + H_1^2H_3 - H_1H_2^2 + H_2^2H_3 - 2H_1H_2H_3\right) \\ +\beta\begin{pmatrix} -H_1^3 - H_2^3 + 4H_3^3 - H_1^2H_2 + 4H_1^2H_3 - H_1H_2^2 + 4H_2^2H_3 \\ -3H_1H_3^2 - 3H_2H_3^2 - 2H_1H_2H_3 \end{pmatrix} \end{bmatrix}
$$

$$
+2v\begin{bmatrix} 2 + 4\alpha\left(3H_1^2 + 3H_2^2 + 5H_3^2 + 4H_1H_2 + 6H_1H_3 + 6H_2H_3\right) \\ +\beta\left(4H_1^2 + 4H_2^2 + 6H_3^2 + 6H_1H_2 + 8H_1H_3 + 8H_2H_3\right) \end{bmatrix}
$$

$$
+2v^2\left[4\alpha\left(5H_1 + 5H_2 + 8H_3\right) + \beta\left(7H_1 + 7H_2 + 10H_3\right)\right] + v^3\left(12\alpha + 4\beta\right) \tag{C.45}
$$

$$
G = 4H_1 + 2H_2 + 2H_3 + 2v \tag{C.46}
$$

$$
H = 4H_1 + 4H_2 + 2H_3 + 2v \tag{C.47}
$$

$$
I = 2H_1 + 2H_2 + 4H_3 + 2v \tag{C.48}
$$

算出 determinant

$$
\det\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = -32\alpha H_1H_2\left(H_1 - H_2\right)\left(2H_1 - H_2 - H_3\right)
$$

$$
\begin{bmatrix} 2H_1 + 2H_2 + 2H_3 \\ +\alpha\left(8H_1^3 + 16H_1^2H_2 + 16H_1H_2^2 + 8H_2^3 + 16H_1^2H_3 + 24H_1H_2H_3 + 16H_2^2H_3 + 16H_1H_3^2 + 16H_2H_3^2 + 8H_3^3\right) \\ +\beta\left(6H_1^3 + 4H_1^2H_2 + 4H_1H_2^2 + 6H_2^3 + 4H_1^2H_3 + -6H_1H_2H_3 + 4H_2^2H_3 + 4H_1H_3^2 + 4H_2H_3^2 + 6H_3^3\right) \\ +v\left[1 + \alpha\left(4H_1^2 + 4H_1H_2 + 4H_2^2 + 4H_1H_3 + 4H_2H_3 + 4H_3^2\right) + \beta\left(3H_1^2 - 4H_1H_2 + 3H_2^2 - 4H_1c - 4H_2H_3 + 3H_3^2\right)\right] \\ -2v^2\beta\left(H_1 + H_2 + H_3\right) - v^3\beta \end{bmatrix} = 0
$$

$H_1 = H_2 = H_3$ **為一組解，也就是 De Sitter 解** (C.49)

然而我們要找出穩定性就需要令:

$$\begin{bmatrix} 2H_1+2H_2+2H_3 \\ +\alpha\left(8H_1^{\ 3}+16H_1^{\ 2}H_2+16H_1H_2^{\ 2}+8H_2^{\ 3}+16H_1^{\ 2}H_3+24H_1H_2H_3+16H_2^{\ 2}H_3+16H_1H_3^{\ 2}+16H_2H_3^{\ 2}+8H_3^{\ 3}\right) \\ +\beta\left(6H_1^{\ 3}+4H_1^{\ 2}H_2+4H_1H_2^{\ 2}+6H_2^{\ 3}+4H_1^{\ 2}H_3+-6H_1H_2H_3+4H_2^{\ 2}H_3+4H_1H_3^{\ 2}+4H_2H_3^{\ 2}+6H_3^{\ 3}\right) \\ +v\left[1+\alpha\left(4H_1^{\ 2}+4H_1H_2+4H_2^{\ 2}+4H_1H_3+4H_2H_3+4H_3^{\ 2}\right)+\beta\left(3H_1^{\ 2}-4H_1H_2+3H_2^{\ 2}-4H_1c-4H_2H_3+3H_3^{\ 2}\right)\right] \\ -2v^2\beta\left(H_1+H_2+H_3\right)-v^3\beta \end{bmatrix}=0 \tag{C.50}$$

因此我們由(C.50)可以找到三組解

$$v=-\left(H_1+H_2+H_3\right)$$

$$v=-\frac{1}{2}\left(H_1+H_2+H_3\pm\sqrt{9H_1^{\ 2}+9H_2^{\ 2}+9H_3^{\ 2}-6H_1H_2-6H_2H_3-6H_1H_3}\right) \tag{C.51}$$

我要找出 $v>0$ 才能為不穩定性

$v=-\left(H_1+H_2+H_3\right)$ 無法確定大於 **0**

我們來看

$$v=-\frac{1}{2}\left(H_1+H_2+H_3\pm\sqrt{9H_1^{\ 2}+9H_2^{\ 2}+9H_3^{\ 2}-6H_1H_2-6H_2H_3-6H_1H_3}\right)$$

需要

$$\sqrt{9H_1^{\ 2}+9H_2^{\ 2}+9H_3^{\ 2}-6H_1H_2-6H_2H_3-6H_1H_3}>H_1+H_2+H_3 \tag{C.52}$$

才會有負號出現，因平方相減

$$\begin{aligned}&\left(9H_1^{\ 2}+9H_2^{\ 2}+9H_3^{\ 2}-6H_1H_2-6H_2H_3-6H_1H_3\right)-\left(H_1+H_2+H_3\right)^2\\&=4\left(H_1-H_2\right)^2+4\left(H_2-H_3\right)^2+4\left(H_1-H_3\right)^2\geq 0\end{aligned} \tag{C.53}$$

而我們不考慮 $H_1=H_2=H_3$ 這組，則確實有一組微擾 v >0，故得證。

Appendix D Calculate Eigenvalues

在計算 eigenvalues 處，會用矩陣的方式表示，在分別顯示每一列矩陣的數值以方便計算，然再做乘法計算處因為過於複雜，所以會用 mathmetica 計算，由於計算複雜所以用電腦計算，以下是計算 eigenvalues 的過程:

我們已知平衡點為

$$B=2(\chi-4)-4\Sigma^2(1+2\chi)\text{，}\Omega_\Lambda=1+\frac{1}{2}\Sigma^2\text{，}N=0\text{，}Q=0\text{，}\Sigma_+=\Sigma_+\text{，}\Sigma_-=\Sigma_-\text{，}\Sigma_{+1}=0$$

$$\Sigma_{-1}=0\text{，}\Sigma_{+2}=0\text{，}\Sigma_{-2}=0\text{，}Q_2=0 \qquad \text{(D.1)}$$

使用線性化系統

$$\frac{dx_i}{dt}=a_{i1}x_1+\ldots+a_{in+1}x_{n+1}$$

$$(a_{ij}=\left[\frac{\partial A_i}{\partial x_j}\right]_{x_j=0};i,j=1,2\ldots,n+1) \qquad \text{(D.2)}$$

表示成矩陣為:

$$\frac{d}{dt}\begin{pmatrix} B\\ \Omega_\Lambda\\ N\\ Q\\ \Sigma_+\\ \Sigma_-\\ \Sigma_{+1}\\ \Sigma_{-1}\\ \Sigma_{+2}\\ \Sigma_{-2}\\ Q_2 \end{pmatrix}=\begin{pmatrix}
\left(\frac{\partial A_1}{\partial B}\right)_0 & \left(\frac{\partial A_1}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_1}{\partial N}\right)_0 & \left(\frac{\partial A_1}{\partial Q}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_1}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_1}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_2}{\partial B}\right)_0 & \left(\frac{\partial A_2}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_2}{\partial N}\right)_0 & \left(\frac{\partial A_2}{\partial Q}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_2}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_2}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_3}{\partial B}\right)_0 & \left(\frac{\partial A_3}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_3}{\partial N}\right)_0 & \left(\frac{\partial A_3}{\partial Q}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_3}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_3}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_4}{\partial B}\right)_0 & \left(\frac{\partial A_4}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_4}{\partial N}\right)_0 & \left(\frac{\partial A_4}{\partial Q}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_4}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_4}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_5}{\partial B}\right)_0 & \left(\frac{\partial A_5}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_5}{\partial N}\right)_0 & \left(\frac{\partial A_5}{\partial Q}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_5}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_5}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_6}{\partial B}\right)_0 & \left(\frac{\partial A_6}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_6}{\partial N}\right)_0 & \left(\frac{\partial A_6}{\partial Q}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_6}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_6}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_7}{\partial B}\right)_0 & \left(\frac{\partial A_7}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_7}{\partial N}\right)_0 & \left(\frac{\partial A_7}{\partial Q}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_7}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_7}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_8}{\partial B}\right)_0 & \left(\frac{\partial A_8}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_8}{\partial N}\right)_0 & \left(\frac{\partial A_8}{\partial Q}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_8}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_8}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_9}{\partial B}\right)_0 & \left(\frac{\partial A_9}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_9}{\partial N}\right)_0 & \left(\frac{\partial A_9}{\partial Q}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_9}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_9}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_{10}}{\partial B}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_{10}}{\partial N}\right)_0 & \left(\frac{\partial A_{10}}{\partial Q}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_{10}}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_{10}}{\partial Q_2}\right)_0 \\
\left(\frac{\partial A_{11}}{\partial B}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Omega_\Lambda}\right)_0 & \left(\frac{\partial A_{11}}{\partial N}\right)_0 & \left(\frac{\partial A_{11}}{\partial Q}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_+}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_-}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_{+1}}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_{-1}}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_{+2}}\right)_0 & \left(\frac{\partial A_{11}}{\partial \Sigma_{-2}}\right)_0 & \left(\frac{\partial A_{11}}{\partial Q_2}\right)_0
\end{pmatrix}\begin{pmatrix} B\\ \Omega_\Lambda\\ N\\ Q\\ \Sigma_+\\ \Sigma_-\\ \Sigma_{+1}\\ \Sigma_{-1}\\ \Sigma_{+2}\\ \Sigma_{-2}\\ Q_2 \end{pmatrix}$$

(D.3)

其中，

$$\left(\left(\frac{\partial A_1}{\partial B}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_1}{\partial N}\right)_0 \quad \left(\frac{\partial A_1}{\partial Q}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_1}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_1}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad -2B,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0\right) \tag{D.4}$$

$$\left(\left(\frac{\partial A_2}{\partial B}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_2}{\partial N}\right)_0 \quad \left(\frac{\partial A_2}{\partial Q}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_2}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_2}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad -\Sigma^2-2,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0\right) \tag{D.5}$$

$$\left(\left(\frac{\partial A_3}{\partial B}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_3}{\partial N}\right)_0 \quad \left(\frac{\partial A_3}{\partial Q}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_3}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_3}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad -1-4\Sigma_+,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0\right) \tag{D.6}$$

$$\left(\left(\frac{\partial A_4}{\partial B}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_4}{\partial N}\right)_0 \quad \left(\frac{\partial A_4}{\partial Q}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_4}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_4}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 1\right) \tag{D.7}$$

$$\left(\left(\frac{\partial A_5}{\partial B}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_5}{\partial N}\right)_0 \quad \left(\frac{\partial A_5}{\partial Q}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_5}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_5}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad -\Sigma_+,\quad 0,\quad 0,\quad 1,\quad 0,\quad 0,\quad 0,\quad 0\right) \tag{D.8}$$

$$\left(\left(\frac{\partial A_6}{\partial B}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_6}{\partial N}\right)_0 \quad \left(\frac{\partial A_6}{\partial Q}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_6}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_6}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad -\Sigma_-,\quad 0,\quad 0,\quad 1,\quad 0,\quad 0,\quad 0,\quad 0\right) \tag{D.9}$$

$$\left(\left(\frac{\partial A_7}{\partial B}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_7}{\partial N}\right)_0 \quad \left(\frac{\partial A_7}{\partial Q}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_7}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_7}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 1,\quad 0,\quad 0\right) \tag{D.10}$$

$$\left(\left(\frac{\partial A_8}{\partial B}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Omega_\Lambda}\right)_0 \quad \left(\frac{\partial A_8}{\partial N}\right)_0 \quad \left(\frac{\partial A_8}{\partial Q}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_+}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_-}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_{+1}}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_{-1}}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_{+2}}\right)_0 \quad \left(\frac{\partial A_8}{\partial \Sigma_{-2}}\right)_0 \quad \left(\frac{\partial A_8}{\partial Q_2}\right)_0\right)$$
$$=\left(0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 1,\quad 0\right) \tag{D.11}$$

$$\left(\left(\frac{\partial A_9}{\partial B}\right)_0 \left(\frac{\partial A_9}{\partial \Omega_\Lambda}\right)_0 \left(\frac{\partial A_9}{\partial N}\right)_0 \left(\frac{\partial A_9}{\partial Q}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_+}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_-}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_{+1}}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_{-1}}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_{+2}}\right)_0 \left(\frac{\partial A_9}{\partial \Sigma_{-2}}\right)_0 \left(\frac{\partial A_9}{\partial Q_2}\right)_0\right)$$

$$=\begin{pmatrix} -\frac{6\Sigma_-\left(4\Sigma^2-1\right)}{4\Sigma^2+B+8}, \quad 0, \quad 0, \quad -\frac{7\Sigma_-\left(36\Sigma^2+B\right)}{4\Sigma^2+B+8}, \quad \frac{48\Sigma_+^2\left(B+9\right)}{4\Sigma^2+B+8}, \quad \frac{48\Sigma_+\Sigma_-\left(B+9\right)}{4\Sigma^2+B+8}, \\ \frac{16B\Sigma_+^2-9\left(B+8\right)-36\left(\Sigma^2-4\Sigma_+^2\right)}{4\Sigma^2+B+8}, \quad \frac{16\Sigma_+\Sigma_-\left(B+9\right)}{4\Sigma^2+B+8}, \quad -6, \quad 0, \quad -\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2+B+8} \end{pmatrix} \tag{D.12}$$

$$\left(\left(\frac{\partial A_{10}}{\partial B}\right)_0 \left(\frac{\partial A_{10}}{\partial \Omega_\Lambda}\right)_0 \left(\frac{\partial A_{10}}{\partial N}\right)_0 \left(\frac{\partial A_{10}}{\partial Q}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_+}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_-}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_{+1}}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_{-1}}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_{+2}}\right)_0 \left(\frac{\partial A_{10}}{\partial \Sigma_{-2}}\right)_0 \left(\frac{\partial A_{10}}{\partial Q_2}\right)_0\right)$$

$$=\begin{pmatrix} -\frac{6\Sigma_-\left(4\Sigma^2-1\right)}{4\Sigma^2+B+8}, \quad 0, \quad 0, \quad -\frac{7\Sigma_-\left(36\Sigma^2+B\right)}{4\Sigma^2+B+8}, \quad \frac{48\Sigma_+\Sigma_-\left(B+9\right)}{4\Sigma^2+B+8}, \quad \frac{48\Sigma_-^2\left(B+9\right)}{4\Sigma^2+B+8} \\ \frac{16\Sigma_+\Sigma_-\left(B+9\right)}{4\Sigma^2+B+8}, \quad \frac{16B\Sigma_-^2-9\left(B+8\right)-36\left(\Sigma^2-4\Sigma_+^2\right)}{4\Sigma^2+B+8}, \quad 0, \quad -6, \quad -\frac{\Sigma_-\left(36\Sigma^2+B\right)}{4\Sigma^2+B+8} \end{pmatrix} \tag{D.13}$$

$$\left(\left(\frac{\partial A_{11}}{\partial B}\right)_0 \left(\frac{\partial A_{11}}{\partial \Omega_\Lambda}\right)_0 \left(\frac{\partial A_{11}}{\partial N}\right)_0 \left(\frac{\partial A_{11}}{\partial Q}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_+}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_-}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_{+1}}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_{-1}}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_{+2}}\right)_0 \left(\frac{\partial A_{11}}{\partial \Sigma_{-2}}\right)_0 \left(\frac{\partial A_{11}}{\partial Q_2}\right)_0\right)$$

$$=\begin{pmatrix} -\frac{3}{8}\Sigma^2, \quad \frac{3}{4}B, \quad 0, \quad -\frac{1}{4}\frac{9\Sigma^2\left(16+B+4\Sigma^2\right)}{4\Sigma^2-1}, \quad \frac{3}{4}\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2-1} \quad \frac{3}{4}\frac{\Sigma_-\left(36\Sigma^2+B\right)}{4\Sigma^2-1}, \\ -\frac{1}{2}\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2-1}, \quad -\frac{1}{2}\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2-1}, \quad -\frac{1}{4}\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2-1}, \quad -\frac{1}{4}\frac{\Sigma_+\left(36\Sigma^2+B\right)}{4\Sigma^2-1}, \quad -6 \end{pmatrix} \tag{D.14}$$

我們要確實有真解所以矩陣依照(2.50)線性化 **DE** 方程式為 $\dot{x}=Ax=\lambda x$，因此矩陣的 **determinant** 要為 **0:**

$$\det\begin{bmatrix} a_{11}-\lambda,......,a_{1n+1} \\ .... \\ a_{n+11},...,a_{n+1n+1}-\lambda \end{bmatrix}=0 \tag{D.14}$$

接著我們可以算出一元多次方程式，是利用 **mathmetica** 計算程式來做計算，計算顯示在下頁可以看出很難用手算出，所以拜託林益弘學長教我用電腦算，我們用電腦可算出，在此我們可以用因式分解的方式找出 **eigenvalues**

$$
\begin{aligned}
&-\lambda^{11}+\left(-4\Sigma_{+}-19\right)\lambda^{10}+\left(18\Sigma_{+}^{2}+18\Sigma_{-}^{2}-72\Sigma_{+}-153-\frac{B}{2}\right)\lambda^{9}+\left(72\Sigma_{+}^{3}+72\Sigma_{+}\Sigma_{-}^{2}+288\Sigma_{+}^{2}+288\Sigma_{-}^{2}-2B\Sigma_{+}-540\Sigma_{+}-8B-675\right)\lambda^{8}\\
&+\left(1080\Sigma_{+}^{3}+1080\Sigma_{+}\Sigma_{-}^{2}+1890\Sigma_{+}^{2}+9B\Sigma_{+}^{2}+1890\Sigma_{-}^{2}+9B\Sigma_{-}^{2}-30B\Sigma_{+}-2160\Sigma_{+}-\frac{105}{2}B-1755\right)\lambda^{7}\\
&+\left(6480\Sigma_{+}^{3}+36B\Sigma_{+}^{3}+6480\Sigma_{+}\Sigma_{-}^{2}+36B\Sigma_{+}\Sigma_{-}^{2}+6480\Sigma_{+}^{2}+117B\Sigma_{+}^{2}+6480\Sigma_{-}^{2}+117B\Sigma_{-}^{2}-180B\Sigma_{+}-4860\Sigma_{+}-180B-2673\right)\lambda^{6}\\
&+\left(19440\Sigma_{+}^{3}+432B\Sigma_{+}^{3}+19440\Sigma_{+}\Sigma_{-}^{2}+432B\Sigma_{+}\Sigma_{-}^{2}+12150\Sigma_{+}^{2}+594B\Sigma_{+}^{2}+12150\Sigma_{-}^{2}+594B\Sigma_{-}^{2}-540B\Sigma_{+}-5832\Sigma_{+}-\frac{675}{2}B-2187\right)\lambda^{5}\\
&+\left(29160\Sigma_{+}^{3}+1944B\Sigma_{+}^{3}+29160\Sigma_{+}\Sigma_{-}^{2}+1944B\Sigma_{+}\Sigma_{-}^{2}+11664\Sigma_{+}^{2}+1458B\Sigma_{+}^{2}+11664\Sigma_{-}^{2}+1458B\Sigma_{-}^{2}-810B\Sigma_{+}-2916\Sigma_{+}-324B-729\right)\lambda^{4}\\
&+\left(17496\Sigma_{+}^{3}+3888B\Sigma_{+}^{3}+17496\Sigma_{+}\Sigma_{-}^{2}+3888B\Sigma_{+}\Sigma_{-}^{2}+4374\Sigma_{+}^{2}+1701B\Sigma_{+}^{2}+4374\Sigma_{-}^{2}+1701B\Sigma_{-}^{2}-486B\Sigma_{+}-\frac{243}{2}B\right)\lambda^{3}\\
&+\left(2916B\Sigma_{+}^{3}+2916B\Sigma_{+}\Sigma_{-}^{2}+729B\Sigma_{+}^{2}+729B\Sigma_{-}^{2}\right)\lambda^{2}\\
&=-\frac{1}{2}(\lambda+3)^{4}\left(4\Sigma_{+}+\lambda+1\right)\left(-18\Sigma^{2}+\lambda^{2}+3\lambda\right)\left(2\lambda^{2}+6\lambda+B\right)\lambda^{2}=0
\end{aligned}
\tag{D.15}
$$

我們的等號為零，所以可以得出下列 eigenvalues:

$$\lambda=0,0,-3,-3,-3,-,3,-\frac{3}{2}\left(1\pm\sqrt{1+8\Sigma^{2}}\right),-\frac{3}{2}\pm\frac{1}{2}\sqrt{9-2B},-1-4\Sigma_{+} \tag{D.16}$$

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