Parameter analysis for frequency-modulation reticle design

Zu Wen Chao
National Chiao Tung University
Institute of Electronics
45 Po-Ai Street
Hsin-Chu 30039
Taiwan

Jih Liang Chu
Chung San Institute of Science and Technology
P.O. Box 1-3
Lung-Tang, Tao-Yuan 32526
Taiwan

Abstract. Two models for the analysis of the parameters essential for the design of a frequency-modulation reticle are introduced. One is the point model, which is suitable for estimating the center frequency and the frequency bandwidth. The other is the spot model, which is suitable for calculating the degree of contrast. Using the analytical results derived from these two models, examples are discussed showing the process of reticle design, e.g., determination of the total number of spokes and design of a modification zone by means of a graphical method.

Subject terms: optical tracking; frequency-modulation reticles; demodulation; amplitude distortion; signal contrast; modification zones.

Optical Engineering 27(6), 443-451 (June 1988).

CONTENTS

1. INTRODUCTION

The application of a tracking system as a guidance tool to aim at a target emitting infrared radiation is important in many fields such as astronomical observation and industrial machinery. One technique on which the design of a tracking system is usually based is the use of reticles[1,2] to modulate the incident light flux. This technique is simple in both design and operation. For years both the amplitude-modulation (AM) reticle[3,4] and the frequency-modulation (FM) reticle[5,6] have been studied. These reticles are so named because they modulate the light flux and give at the detector output end a resulting waveform of voltage versus time similar to the AM or FM waveform usually treated in radio engineering. Because an FM signal is known to be superior to an AM signal with regard to signal quality, that is, it suffers less noise interference, the FM reticle is of greater interest and therefore is considered in this paper.

A common way to construct an FM reticle system is to let the light beam nutate around the axis, with the reticle stationary. Figure 1 shows the configuration of such a system. The incident light from a distant target is collected by an objective, for example, one of Cassegrain type. It is then slightly deflected by the center mirror, which is mounted with a slight tilt, and focused to form a spot in the image plane, where a stationary reticle is put. The rotation of the mirror causes the beam, and hence the spot, to nutate. The loop of nutation is concentric with the reticle center when the target is on the axis; it is eccentric when the target is off the axis.

The light then travels through the reticle, becoming frequency-modulated, and falls onto the detector. The output signal u of the detector, being the voltage proportional to light intensity, is also frequency-modulated and is sent to a discriminator, which electronically demodulates the signal to yield a final output signal v, which, being the voltage proportional to the instantaneous frequency of u, is used to control the orientation of the tracking system to perform the aiming function.

In the signal processing of the electronic circuit in an FM system, there are three important parameters: (1) center frequency, (2) frequency bandwidth, and (3) degree of contrast. All three can be understood through the analysis of the behavior of the image spot nutating in the reticle plane. Different mathematical models must be established so that analytical results for individual parameters can be obtained efficiently. In this paper we present two models for analysis: the point model and the spot model. Examples of the application of the analytical results to reticle design are also presented.

Paper 2429 received June 8, 1987; revised manuscript received Feb. 4, 1988; accepted for publication Feb. 4, 1988; received by Managing Editor Feb. 8, 1988.

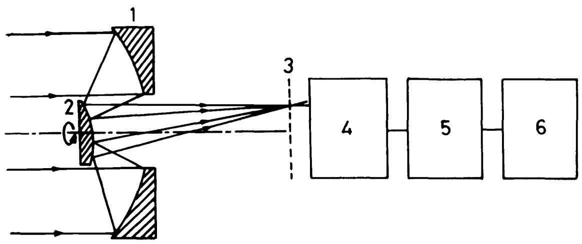

Fig. 1. An FM reticle tracking system. (1) Cassegrain objective, (2) rotating mirror, (3) reticle, (4) detector, (5) discriminator, and (6) controller.

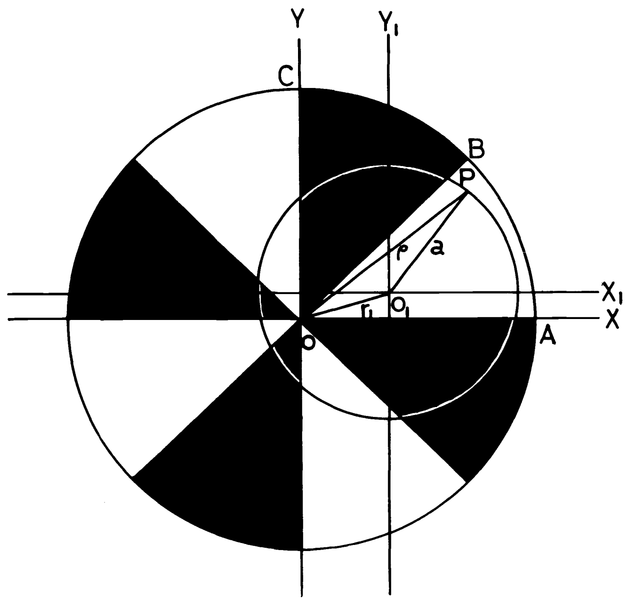

Fig. 2. FM reticle—point model.

2. ANALYSIS

2.1. Point model

We begin by assuming that the image spot is an ideal point. As shown in Fig. 2, the image point P nutates with a constant angular velocity Ω along a circular loop of radius a around the center O_1, which in general does not coincide with the reticle center O. The reticle comprises 2m (m is a positive integer) equally spaced spokes of alternating transmittance 0 (opaque) and 1 (transparent). In system O-X-Y, the polar coordinates of O_1 and P are (r_1, θ_1) and (ρ, θ), respectively.

2.1.1. Derivation of u and v

The detector output voltage is proportional to the light intensity behind the reticle, which is

$$u(t) = \iint_{-\infty}^{\infty} r(\mathbf{x}, t)p(\mathbf{x}, t)d^2\mathbf{x} \ , \tag{1}$$

where $r(\mathbf{x}, t)$ is the intensity transmittance function of the reticle and $p(\mathbf{x}, t)$ is the intensity distribution function of the image. Now that the image is a point, the function p can be written as

$$p(\mathbf{x}, t) = \delta[x - x'(t)]\ \delta[y - y'(t)] \ , \tag{2}$$

where $x'(t)$ and $y'(t)$ are the instantaneous positional coordinates of the image point.

For the function $r(\mathbf{x}, t)$, notice first that the reticle is stationary, so $r(\mathbf{x}, t) = r(\mathbf{x})$. Second, from Fig. 2 we see that

$$\frac{\theta}{2\pi} = \frac{b}{2m} \ , \tag{3}$$

where b stands for a noninteger parameter that takes different values in different spokes. For example, in the spoke OAB, we have $0 \le b < 1$; in OBC, $1 \le b < 2$; ... . Thus, we have

$$r(\mathbf{x}) = \mathrm{STP}[\sin(m\theta)] \ , \tag{4}$$

where

$$\mathrm{STP}(w) = \begin{cases} 1, & w \ge 0 \ , \\ 0, & w < 0 \ . \end{cases}$$

Substituting Eqs. (2) and (4) into Eq. (1), we have

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{STP}[\sin(m\theta)]\delta(x - x')\delta(y - y')dxdy \\ &= \mathrm{STP}[\sin(m\theta)] \\ &= \mathrm{STP}[\sin(\text{argument})] \ . \end{aligned} \tag{5}$$

According to the definition of FM demodulation, the signal v is

$$v(t) = \frac{d}{dt}(\text{argument}) = m\frac{d\theta}{dt} \ . \tag{6}$$

A straightforward way to work out Eq. (6) is to differentiate θ with respect to t, the time.[5] Another method, however, is also available. Here we present a method of differentiation with respect to arc length. In the coordinate system O_1-X_1-Y_1, the arc length s along the circular loop of radius a is

$$s = \int a(\Omega dt) = a\Omega t \propto t \ . \tag{7}$$

Since s is proportional to t, the differentiation with respect to t in Eq. (6) can be replaced by the differentiation with respect to s. Transforming to system O-X-Y, we have the expression for arc length in another form:

$$\begin{aligned} s &= \int_0^\theta \rho d\theta - C \\ &= \int_0^\theta \left[r_1\cos(\theta - \theta_1) + \sqrt{a^2 - r_1^2\sin^2(\theta - \theta_1)} \right] d\theta - C \\ &= r_1\sin(\theta - \theta_1) + r_1\sin\theta_1 \\ &\quad + a\left[\frac{2}{\pi}(\theta - \theta_1)E + \sin(\theta - \theta_1)\cos(\theta - \theta_1) \right. \\ &\quad \left. \times \left(\frac{1}{4}k^2 + \frac{1}{8}k^4A_4 + \frac{1}{16}k^6A_6 + \ldots \right) \right] - C \ , \end{aligned} \tag{8}$$

where

$$C = a\sin^{-1}\left(\frac{r_1}{a}\sin\theta_1\right) ,$$

$$k = \frac{r_1}{a} ,$$

$$E \equiv \int_0^{\pi/2} \sqrt{1 - k^2\sin^2(\theta)}\, d\theta$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \dots\right) ,$$

$$A_4 = \frac{1}{4}\sin^2(\theta-\theta_1) + \frac{3}{8} ,$$

$$A_6 = \frac{1}{6}\sin^4(\theta-\theta_1) + \frac{5}{24}\sin^2(\theta-\theta_1) + \frac{5}{16} ,$$

$$\vdots$$

Differentiation of s with respect to θ gives

$$\begin{aligned}\frac{ds}{d\theta} &= r_1\cos(\theta-\theta_1) \\ &+ a\left\{\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 + \dots\right)\right. \\ &+ \cos2(\theta-\theta_1)\left(\frac{1}{4}k^2 + \frac{3}{64}k^4 + \frac{5}{256}k^6 + \dots\right) \\ &+ [3\sin^2(\theta-\theta_1)\cos^2(\theta-\theta_1) - \sin^4(\theta-\theta_1)] \\ &\times\left(\frac{1}{32}k^4 + \frac{5}{384}k^6 + \dots\right) \\ &+ [5\sin^4(\theta-\theta_1)\cos^2(\theta-\theta_1) - \sin^6(\theta-\theta_1)] \\ &\left.\times\left(\frac{1}{96}k^6 + \dots\right)\right\} . \end{aligned} \tag{9}$$

Substituting Eqs. (7) and (9) into Eq. (6), we have

$$\begin{aligned} v(\theta) &= ma\Omega\frac{1}{\frac{ds}{d\theta}} \\ &\approx ma\Omega\left\{r_1\cos(\theta-\theta_1)\right. \\ &+ a\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6\right) \\ &+ a\cos2(\theta-\theta_1)\left(\frac{1}{4}k^2 + \frac{3}{64}k^4 + \frac{5}{256}k^6\right) \\ &+ a[3\sin^2(\theta-\theta_1)\cos^2(\theta-\theta_1) - \sin^4(\theta-\theta_1)] \\ &\times\left(\frac{1}{32}k^4 + \frac{5}{384}k^6\right) \\ &\left.+ a[5\sin^4(\theta-\theta_1)\cos^2(\theta-\theta_1) - \sin^6(\theta-\theta_1)]\frac{1}{96}k^6\right\}^{-1} . \end{aligned} \tag{10}$$

Corresponding to this equation, a typical relation of v versus θ is plotted in Fig. 3.

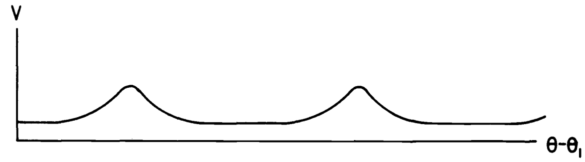

Fig. 3. Signal output after demodulation.

To verify that the above formula really represents a demodulated FM signal, let us consider the simplified case of $k^2 << 1$. We neglect terms higher than k^2 to obtain

$$\begin{aligned} v(\theta) &\approx ma\Omega\frac{1}{a + r_1\cos(\theta-\theta_1)} \\ &\approx m\Omega - m\Omega\frac{r_1}{a}\cos(\theta-\theta_1) . \end{aligned} \tag{11}$$

Because θ is a function of t, the equation is of the form

$$v(t) = v_0 - f(t) , \tag{12}$$

where $v_0 = m\Omega$ and $f(t) = m\Omega(r_1/a)\cos(\theta-\theta_1)$. Comparison of this expression with the definition for the instantaneous frequency of an FM signal, i.e., $\omega = \omega_0 + f(t)$, reveals that v behaves like a demodulated FM signal.

2.1.2. Center frequency

If the target is tracked correctly, that is, when the target is on the axis, there should be no lateral deviation, i.e., $r_1 = 0$. Then from Eq. (10) we see that

$$v = v_0 = m\Omega = \text{a constant} . \tag{13}$$

This implies that the tracking system should work in the sense that v approaches a constant. The constant value $m\Omega$ is the center frequency.

2.1.3. Frequency bandwidth

From Fig. 3 we see that there are v_{max} and v_{min} on the curve. The difference between them is the frequency bandwidth. To estimate these quantities, we again refer to Eq. (10). An approximate estimation can be made by again assuming that $k^2 << 1$. Neglecting terms higher than k^2, we have, at $\theta - \theta_1 = \pi$,

$$v_{max} = m\Omega\left(1 + \frac{r_1}{a}\right) \tag{14}$$

and, at $\theta - \theta_1 = 0$,

$$v_{min} = m\Omega\left(1 - \frac{r_1}{a}\right) . \tag{15}$$

Thus, the frequency bandwidth is

$$\Delta v = 2m\Omega\frac{r_1}{a} . \tag{16}$$

Here we can see the usefulness of the method of arc length differentiation. The result that Eq. (10) shows is the dependence of v upon θ, not upon t. In other words, instead of being time

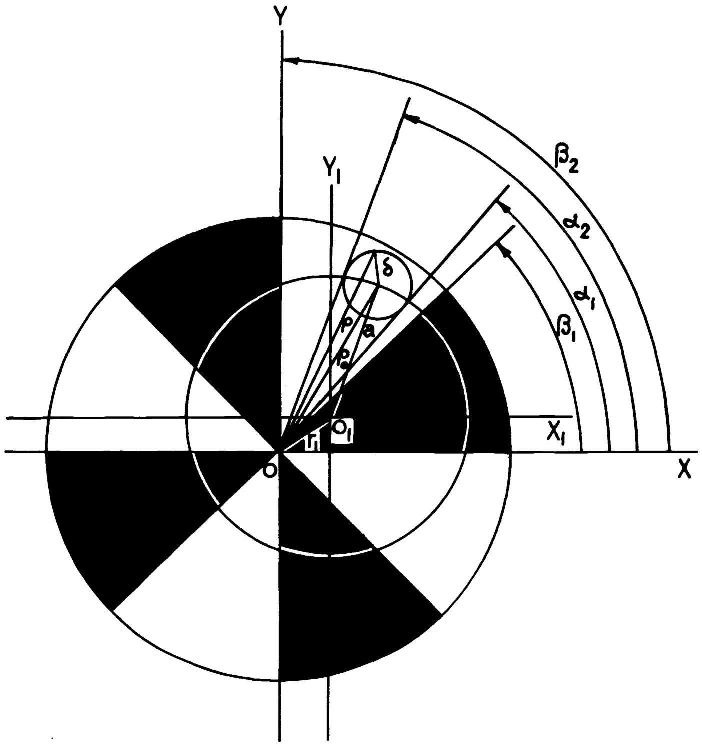

Fig. 4. FM reticle—spot model.

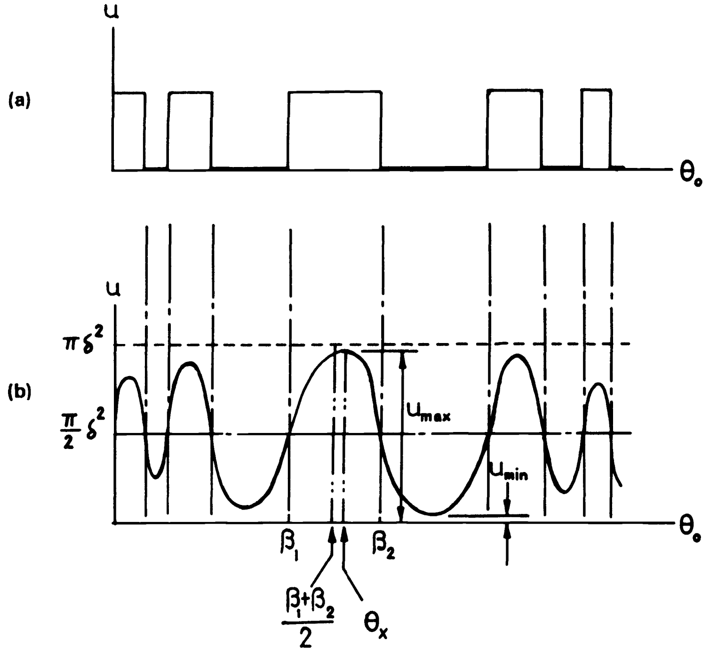

Fig. 5. Detector output. (a) For a small spot, the amplitude is either saturated or cut off. (b) For a large spot, amplitude distortion occurs.

dependent, the problem has been changed to be geometry dependent. Therefore, one can understand the behavior of v anywhere on the reticle, e.g., v_{max} and v_{min}, simply through geometrical interpretation, as we did in deriving Eqs. (14) and (15). For design work, this would be a useful theoretical support to simplify the design process.

2.2. Spot model

We now consider the case in which the image is a circular spot that is the core of the Airy disk, which is closer to the usual situation. The signal behind the reticle produced by such a spot bears not only FM features but also amplitude distortion.

For the reticle system depicted in Fig. 4, the center of the spot of radius δ nutates with a constant angular velocity Ω along a circular loop of radius a centered at O_1, which in general does not coincide with the reticle center O. The stationary reticle comprises 2m equally spaced spokes of alternating transmittance 0 (opaque) and 1 (transparent). In system O-X-Y, the polar coordinates of the spot center, the point on the spot edge, and the loop center are (ρ_0, θ_0), (ρ, θ), and (r_1, θ_1), respectively.

When the whole spot is within the region of one spoke of one kind of transmittance, the light flux along with the corresponding voltage amplitude of signal u is either saturated or cut off. There is no amplitude distortion [see Fig. 5(a)]. When part of the spot is in one spoke of one kind of transmittance while its other part is in the neighboring spoke(s) of a different kind of transmittance, amplitude distortion occurs. The light flux along with the corresponding amplitude of u is neither saturated nor cut off. It is some value in between. Although u is basically of an FM waveform, its amplitude varies. We have u_{max} values in transparent spokes and u_{min} values in opaque spokes, as shown in Fig. 5(b). When the difference between the u_{max} of a transparent spoke and the u_{min} of an adjacent opaque spoke becomes small, the contrast of the signal becomes less significant, and hence the detectivity to the signal becomes poor. This will influence the electronic circuit operation and the system performance. Therefore, it should be studied thoroughly in the analysis stage and handled carefully in the design stage.

2.2.1. Derivation of u

When the spot is nutating in a transparent spoke of angular width $(\beta_2 - \beta_1)$, the light intensity behind the reticle is proportional to the exposure area within the spot, and so is the detector output voltage. From Fig. 4 we have

$$u = \int_{\gamma_1}^{\gamma_2} dA$$

$$= \int_{\gamma_1}^{\gamma_2} \frac{1}{2}(\rho_{up}^2 - \rho_{low}^2)d\theta \ , \qquad (17)$$

where

$$\gamma_1 = \alpha_1 STP_+(\alpha_1 - \beta_1) + \beta_1 STP_-(\beta_1 - \alpha_1)$$

and

$$\gamma_2 = \alpha_2\ STP_+(\beta_2 - \alpha_2) + \beta_2 STP_-(\alpha_2 - \beta_2) \ ,$$

in which

$$\alpha_1 = \theta_0 - \sin^{-1}(\delta/\rho_0) \ ,$$

$$\alpha_2 = \theta_0 + \sin^{-1}(\delta/\rho_0) \ ,$$

$$STP_+(w) = \begin{cases} 1, & w \geq 0 \\ 0, & w < 0 \ , \end{cases}$$

$$STP_-(w) = \begin{cases} 1, & w > 0 \\ 0, & w \leq 0 \ , \end{cases}$$

and

$$\rho_0 = z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1) ,$$

with

$$z_1 = a - \frac{r_1^2}{2a} ,$$

$$z_2 = r_1 ,$$

$$z_3 = \frac{r_1^2}{2a} .$$

From the same figure we see that

$$\rho_{up} = \rho_0\cos(\theta-\theta_0) + \sqrt{\rho_0^2\cos^2(\theta-\theta_0) - \rho_0^2 + \delta^2} \tag{18}$$

and

$$\rho_{low} = \rho_0\cos(\theta-\theta_0) - \sqrt{\rho_0^2\cos^2(\theta-\theta_0) - \rho_0^2 + \delta^2} . \tag{19}$$

Introducing Eqs. (18) and (19) into Eq. (17), the integration can be carried out, giving

$$u(\theta_0) = \delta\rho_0\left[\sin(\gamma_2-\theta_0)\sqrt{1-\left(\frac{\rho_0}{\delta}\right)^2\sin^2(\gamma_2-\theta_0)} - \sin(\gamma_1-\theta_0)\sqrt{1-\left(\frac{\rho_0}{\delta}\right)^2\sin^2(\gamma_1-\theta_0)}\,\right] + \delta^2\left\{\sin^{-1}\left[\frac{\rho_0}{\delta}\sin(\gamma_2-\theta_0)\right] - \sin^{-1}\left[\frac{\rho_0}{\delta}\sin(\gamma_1-\theta_0)\right]\right\} , \tag{20}$$

or, in another form,

$$u(\theta_0) = \left(\left\{\delta\rho_0\sin(\beta_2-\theta_0)\sqrt{1-\left(\frac{\rho_0}{\delta}\right)^2\sin^2(\beta_2-\theta_0)} + \delta^2\sin^{-1}\left[\frac{\rho_0}{\delta}\sin(\beta_2-\theta_0)\right]\right\} STP_+(\alpha_2-\beta_2) + \frac{\pi}{2}\delta^2 STP_-(\beta_2-\alpha_2)\right) - \left(\left\{\delta\rho_0\sin(\beta_1-\theta_0)\sqrt{1-\left(\frac{\rho_0}{\delta}\right)^2\sin^2(\beta_1-\theta_0)} + \delta^2\sin^{-1}\left[\frac{\rho_0}{\delta}\sin(\beta_1-\theta_0)\right]\right\} STP_+(\beta_1-\alpha_1) - \frac{\pi}{2}\delta^2 STP_-(\alpha_1-\beta_1)\right) . \tag{21}$$

2.2.2. Evaluation of v

As soon as the signal u is obtained, the next task is to find the signal v after demodulation. From Eq. (21) this seems to be a tedious task if one intends to apply (d/dt)(argument) directly since the argument of the functions therein is not explicitly shown. However, this problem may be solved from another point of view. At the two edges of the spoke, i.e., when $\theta_0=\beta_1$ and $\theta_0=\beta_2$, we have $u=(\pi/2)\delta^2$ identically. This means that u passes across the level $(\pi/2)\delta^2$, either upward or downward, when the spot center passes across the spoke edge. In other words, the time when the level-crossing happens is uniquely determined by the behavior of the spot center. Since the spot center is equivalent to the ideal point granted in the point model, the level-crossing behavior in the spot model is the same as that in the point model. Since the number of level-crossings of the u signal is equal to the instantaneous frequency[7] and since the instantaneous frequency is proportional to the signal v, we conclude that the signal v in the spot model is the same as that in the point model. Therefore, the formulas for the center frequency and the frequency bandwidth derived in the point model can be applied equally well in the spot model.

2.2.3. Degree of contrast

The degree of contrast is defined as

$$G = \frac{u_{max} - u_{min}}{u_{max} + u_{min}} . \tag{22}$$

The task that follows is to find u_{max} and u_{min}. We first find the unique θ_0 as the solution of the equation $du/d\theta_0=0$ and denote it by θ_x. Then we are able to express u_{max} and u_{min} in terms of θ_x. First, an analytical expression for $du/d\theta_0$ should be derived. Difficulty arises from the complexity involved in applying direct differentiation to the functions on the right-hand side of Eq. (21). It can be seen that only when we expand those functions into a series can we obtain a reasonably simple expression for $du/d\theta_0$. The series should truncate at terms of finite order. Different rules of truncation will suffer different errors. Three candidate models of truncation are as follows:

Model A

$$u \approx \left[2\delta\rho_0\sin(\beta_2-\theta_0)STP_+(\alpha_2-\beta_2) + \frac{\pi}{2}\delta^2 STP_-(\beta_2-\alpha_2)\right] - \left[2\delta\rho_0\sin(\beta_1-\theta_0)STP_+(\beta_1-\alpha_1) - \frac{\pi}{2}\delta^2 STP_-(\alpha_1-\beta_1)\right] , \tag{23}$$

Model B

$$u \approx \left\{\left[2\delta\rho_0\sin(\beta_2-\theta_0) - \frac{1}{3}\frac{\rho_0^3}{\delta}\sin^3(\beta_2-\theta_0)\right] \times STP_+(\alpha_2-\beta_2) + \frac{\pi}{2}\delta^2 STP_-(\beta_2-\alpha_2)\right\} - \left\{\left[2\delta\rho_0\sin(\beta_1-\theta_0) - \frac{1}{3}\frac{\rho_0^3}{\delta}\sin^3(\beta_1-\theta_0)\right] \times STP_+(\beta_1-\alpha_1) - \frac{\pi}{2}\delta^2 STP_-(\alpha_1-\beta_1)\right\} , \tag{24}$$

Model C

$$u \approx \left\{\left[2\delta\rho_0\sin(\beta_2-\theta_0) - \frac{1}{3}\frac{\rho_0^3}{\delta}\sin^3(\beta_2-\theta_0) - \frac{1}{20}\frac{\rho_0^5}{\delta^3}\sin^5(\beta_2-\theta_0)\right] \times STP_+(\alpha_2-\beta_2) + \frac{\pi}{2}\delta^2 STP_-(\beta_2-\alpha_2)\right\} - \left\{\left[2\delta\rho_0\sin(\beta_1-\theta_0) - \frac{1}{3}\frac{\rho_0^3}{\delta}\sin^3(\beta_1-\theta_0) - \frac{1}{20}\frac{\rho_0^5}{\delta^3}\sin^5(\beta_1-\theta_0)\right] \times STP_+(\beta_1-\alpha_1) - \frac{\pi}{2}\delta^2 STP_-(\alpha_1-\beta_1)\right\} . \tag{25}$$

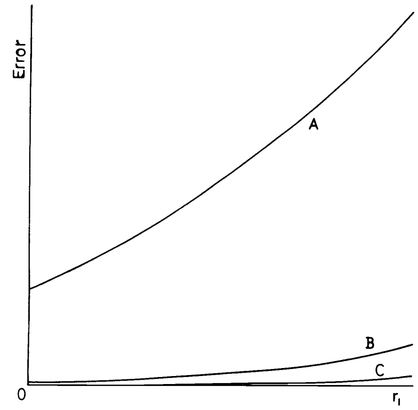

Fig. 6. Error of the series expansion of u. (This is a typical result obtained using $\delta=2$, $a=10$, $\theta_1=15°$, $\beta_1=10°$, $\beta_2=20°$, and $\theta_0=15°$. The abscissa: r_1 from 0 to 8. The ordinate: error from 0 to 14%.)

The choice of one of the three models is done by investigating the error associated with each model when compared with Eq. (21). A typical example is shown in Fig. 6, from which we see that there is a great improvement from model A to model B. This tells us that model B is superior to model A and is a potential candidate. On the other hand, although we can also see improvement from model B to model C, the improvement is so little that the superiority of model C is not obvious. We therefore accept model B as our best choice and use it throughout the following paragraph.

We differentiate the two sides of Eq. (24) with respect to θ_0 to yield

$$
\begin{aligned}
\frac{du}{d\theta_0} = {} & \\
& \Big\{ -2\delta[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]\cos(\beta_2-\theta_0) \\
& -2\delta[z_2\sin(\theta_0-\theta_1) + 2z_3\cos(\theta_0-\theta_1)\sin(\theta_0-\theta_1)]\sin(\beta_2-\theta_0) \\
& +\frac{1}{\delta}[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3\sin^2(\beta_2-\theta_0)\cos(\beta_2-\theta_0) \\
& +\frac{1}{\delta}[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^2[z_2\sin(\theta_0-\theta_1) \\
& +2z_3\cos(\theta_0-\theta_1)\sin(\theta_0-\theta_1)]\sin^3(\beta_2-\theta_0)\Big\}\mathrm{STP}_+(\alpha_2-\beta_2) \\
& +\Big\{2\delta[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]\cos(\beta_1-\theta_0) \\
& +2\delta[z_2\sin(\theta_0-\theta_1) + 2z_3\cos(\theta_0-\theta_1)\sin(\theta_0-\theta_1)]\sin(\beta_1-\theta_0) \\
& -\frac{1}{\delta}[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3\sin^2(\beta_1-\theta_0)\cos(\beta_1-\theta_0) \\
& -\frac{1}{\delta}[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^2[z_2\sin(\theta_0-\theta_1) \\
& +2z_3\cos(\theta_0-\theta_1)\sin(\theta_0-\theta_1)]\sin^3(\beta_1-\theta_0)\Big\}\mathrm{STP}_+(\beta_1-\alpha_1) \\
\frac{du}{d\theta_0} \equiv {} & F(\theta_0) \ . \qquad (26)
\end{aligned}
$$

Denoting the function $du/d\theta_0$ by F, we see that in order to solve $F=0$ we first have to derive F' and then find the solution θ_x from

$$\theta_x = \bar{\theta}_0 - \frac{F(\bar{\theta}_0)}{F'(\bar{\theta}_0)} \ . \qquad (27)$$

From Eq. (26) we have

$$
\begin{aligned}
F'(\theta_0) \equiv \frac{d^2u}{d\theta_0^2} = {} & \\
& \Big(-2\delta[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1) \\
& + z_2\cos(\theta_0-\theta_1) + 2z_3\cos2(\theta_0-\theta_1)]\sin(\beta_2-\theta_0) \\
& + 4\delta[z_2\sin(\theta_0-\theta_1) + z_3\sin2(\theta_0-\theta_1)]\cos(\beta_2-O_0) \\
& + \frac{1}{\delta}\{[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3 + [z_1 + z_2\cos(\theta_0-\theta_1) \\
& + z_3\cos^2(\theta_0-\theta_1)]^2[z_2\cos(\theta_0-\theta_1) + 2z_3\cos2(\theta_0-\theta_1)] \\
& - 2[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)] \\
& \times[z_2\sin(\theta_0-\theta_1) + z_3\sin2(\theta_0-\theta_1)]^2\}\sin^3(\beta_2-\theta_0) \\
& + \frac{1}{\delta}\{-2[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3\}\sin(\beta_2-\theta_0) \\
& \times\cos^2(\beta_2-\theta_0) + \frac{1}{\delta}\{-6[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^2 \\
& \times[z_2\sin(\theta_0-\theta_1) + z_3\sin2(\theta_0-\theta_1)]\}\sin^2(\beta_2-\theta_0)\cos(\beta_2-\theta_0)\Big) \\
& \times\mathrm{STP}_+(\alpha_2-\beta_2) + \Big(2\delta[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1) \\
& + z_2\cos(\theta_0-\theta_1) + 2z_3\cos2(\theta_0-\theta_1)]\sin(\beta_1-\theta_0) - 4\delta[z_2\sin(\theta_0-\theta_1) \\
& + z_3\sin2(\theta_0-\theta_1)]\cos(\beta_1-\theta_0) \\
& + \frac{1}{\delta}\{-[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3 - [z_1 + z_2 \\
& \times\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^2[z_2\cos(\theta_0-\theta_1) + 2z_3\cos2(\theta_0-\theta_1)] \\
& + 2[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)][z_2\sin(\theta_0-\theta_1) \\
& + z_3\sin2(\theta_0-\theta_1)]^2\}\sin^3(\beta_1-\theta_0) + \frac{1}{\delta}\{6[z_1 + z_2\cos(\theta_0-\theta_1) \\
& + z_3\cos^2(\theta_0-\theta_1)]^2[z_2\sin(\theta_0-\theta_1) + z_3\sin2(\theta_0-\theta_1)]\}\sin^2(\beta_1-\theta_0) \\
& \times\cos(\beta_1-\theta_0) + \frac{1}{\delta}\{2[z_1 + z_2\cos(\theta_0-\theta_1) + z_3\cos^2(\theta_0-\theta_1)]^3\} \\
& \times\sin(\beta_1-\theta_0)\cos^2(\beta_1-\theta_0)\Big)\mathrm{STP}_+(\beta_1-\alpha_1) \ . \qquad (28)
\end{aligned}
$$

From Fig. 5(b) it is reasonable to argue that u_{max} occurs when the spot center is nearly in the middle of the transparent spoke. So we set

$$\bar{\theta}_0 = \frac{\beta_1+\beta_2}{2} \ . \qquad (29)$$

Since both the spoke and the spot are symmetric to the middle line, either of the conditions

$$\mathrm{I}:\begin{cases}\alpha_1>\beta_1\\ \alpha_2<\beta_2\end{cases}$$

$$\mathrm{II}:\begin{cases}\alpha_1<\beta_1\\ \alpha_2>\beta_2\end{cases}$$

will hold. Condition I implies that the whole spot is within the spoke. From Eqs. (21), (26), (28), and (29), we have $u_{max}=\pi\delta^2$, $du/d\theta_0=F=0$, and $d^2u/d\theta_0^2=dF/d\theta_0=0$. Therefore, θ_x becomes indefinite, from Eq. (27), and u_{max} occurs everywhere in the transparent spoke. Actually, this is the case of saturation, and there is no need to specify θ_x. Condition II implies that part of the spot is outside the spoke, so that part of the contribution to exposure disappears and u_{max}, occurring at θ_x, drops from the saturation value to some lower value. From Eqs. (26) through (29) and denoting $\phi=[(\beta_1+\beta_2)/2]-\theta_1$, we have

$$\theta_x=\frac{\beta_1+\beta_2}{2}-\frac{A_1\sin\left(\frac{\beta_2-\beta_1}{2}\right)+A_2\sin^3\left(\frac{\beta_2-\beta_1}{2}\right)}{B_1\sin\left(\frac{\beta_2-\beta_1}{2}\right)+B_2\sin\left(\frac{\beta_2-\beta_1}{2}\right)\cos^2\left(\frac{\beta_2-\beta_1}{2}\right)+B_3\sin^3\left(\frac{\beta_2-\beta_1}{2}\right)}\ , \qquad (30)$$

where

$$A_1=-4\delta(z_2\sin\phi+z_3\sin2\phi)\ ,$$

$$A_2=\frac{2}{\delta}(z_1+z_2\cos\phi+z_3\cos^2\phi)^2(z_2\sin\phi+z_3\sin2\phi)\ ,$$

$$B_1=-4\delta(z_1+2z_2\cos\phi+z_3\cos^2\phi+2z_3\cos2\phi)\ ,$$

$$B_2=-\frac{4}{\delta}(z_1+z_2\cos\phi+z_3\cos^2\phi)^3\ ,$$

$$B_3=\frac{2}{\delta}(z_1+z_2\cos\phi+z_3\cos^2\phi)^3+\frac{2}{\delta}(z_1+z_2\cos\phi+z_3\cos^2\phi)^2\times(z_2\cos\phi+2z_3\cos2\phi)-\frac{4}{\delta}(z_1+z_2\cos\phi+z_3\cos^2\phi)\times(z_2\sin\phi+z_3\sin2\phi)^2\ .$$

Since $\beta_2-\beta_1=\pi/m$, we have from Eq. (30) that

$$\theta_x=\frac{\beta_1+\beta_2}{2}-\frac{C_1+C_2\sin^2\left(\frac{\pi}{2m}\right)}{1+C_3\cos^2\left(\frac{\pi}{2m}\right)+C_4\sin^2\left(\frac{\pi}{2m}\right)}\ , \qquad (31)$$

where

$$C_1=\frac{A_1}{B_1}\ ,\quad C_2=\frac{A_2}{B_1}\ ,$$

$$C_3=\frac{B_2}{B_1}\ ,\quad C_4=\frac{B_3}{B_1}\ .$$

Once this is found, the value of u_{max} when the spot is nutating through the transparent spoke of middle angle $(\beta_1+\beta_2)/2$ is readily found from

$$\begin{aligned}u_{max}=&\Big(\{2\delta[z_1+z_2\cos(\theta_x-\theta_1)+z_3\cos^2(\theta_x-\theta_1)]\sin(\beta_2-\theta_x)\\&-\frac{1}{3\delta}[z_1+z_2\cos(\theta_x-\theta_1)+z_3\cos^2(\theta_x-\theta_1)]^3\sin^3(\beta_2-\theta_x)\}\\&\times STP_+(\alpha_2'-\beta_2)+\frac{\pi}{2}\delta^2STP_-(\beta_2-\alpha_2')\Big)\\&-\Big(\{2\delta[z_1+z_2\cos(\theta_x-\theta_1)+z_3\cos^2(\theta_x-\theta_1)]\sin(\beta_1-\theta_x)\\&-\frac{1}{3\delta}[z_1+z_2\cos(\theta_x-\theta_1)+z_3\cos^2(\theta_x-\theta_1)]^3\sin^3(\beta_1-\theta_x)\}\\&\times STP_+(\beta_1-\alpha_1')-\frac{\pi}{2}\delta^2STP_-(\alpha_1'-\beta_1)\Big)\ ,\end{aligned} \qquad (32)$$

where $\alpha_1'=\theta_x-\sin^{-1}(\delta/\rho_x)$, $\alpha_2'=\theta_x+\sin^{-1}(\delta/\rho_x)$, with $\rho_x=z_1+z_2\cos(\theta_x-\theta_1)+z_3\cos^2(\theta_x-\theta_1)$.

Similarly, we can find the value of u_{min} when the spot is nutating through the opaque spoke, of middle angle $(\beta_1+\beta_2)/2+\pi/m$, adjacent to the transparent spoke just mentioned above. The governing equations are

$$\phi'=\beta_2+\frac{\pi}{2m}-\theta_1\ , \qquad (33)$$

$$A_1'=-4\delta(z_2\sin\phi'+z_3\sin2\phi')\ , \qquad (34)$$

$$A_2'=\frac{2}{\delta}(z_1+z_2\cos\phi'+z_3\cos^2\phi')^2(z_2\sin\phi'+z_3\sin2\phi')\ , \qquad (35)$$

$$B_1'=-4\delta(z_1+2z_2\cos\phi'+z_3\cos^2\phi'+2z_3\cos2\phi')\ , \qquad (36)$$

$$B_2'=-\frac{4}{\delta}(z_1+z_2\cos\phi'+z_3\cos^2\phi')^3\ , \qquad (37)$$

$$B_3'=\frac{2}{\delta}(z_1+z_2\cos\phi'+z_3\cos^2\phi')^3+\frac{2}{\delta}(z_1+z_2\cos\phi'+z_3\cos^2\phi')^2\times(z_2\cos\phi'+2z_3\cos2\phi')-\frac{4}{\delta}(z_1+z_2\cos\phi'+z_3\cos^2\phi')\times(z_2\sin\phi'+z_3\sin2\phi')^2\ , \qquad (38)$$

$$\theta_x'=\beta_2+\frac{\pi}{2m}-\frac{C_1'+C_2'\sin^2\left(\frac{\pi}{2m}\right)}{1+C_3'\cos^2\left(\frac{\pi}{2m}\right)+C_4'\sin^2\left(\frac{\pi}{2m}\right)}\ , \qquad (39)$$

where

$$C_1'=\frac{A_1'}{B_1'}\ ,\quad C_2'=\frac{A_2'}{B_1'}\ ,$$

$$C_3'=\frac{B_2'}{B_1'}\ ,\quad C_4'=\frac{B_3'}{B_1'}\ ,$$

$$\rho_x'=z_1+z_2\cos(\theta_x'-\theta_1)+z_3\cos^2(\theta_x'-\theta_1)\ , \qquad (40)$$

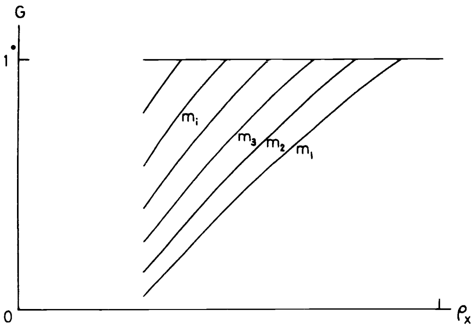

Fig. 7. Degree of contrast G versus radial distance ρ. (This is a typical result obtained using $\delta = 0.1$, $r_1 = 4$, $\theta_1 = 0$, and $a = 10$. The m values: $m_1 = 220$, $m_2 = 200$, $m_3 = 180$, etc.)

$$\alpha_2'' = \theta_x' + \sin^{-1}\left(\frac{\delta}{\rho_x'}\right) , \tag{41}$$

$$\alpha_1'' = \theta_x' - \sin^{-1}\left(\frac{\delta}{\rho_x'}\right) , \tag{42}$$

$$\begin{aligned} u_{min} = \pi\delta^2 - \Bigg(\Bigg(\Big\{ & 2\delta[z_1 + z_2\cos(\theta_x' - \theta_1) + z_3\cos^2(\theta_x' - \theta_1)] \\ & \times \sin\left(\beta_2 + \frac{\pi}{m} - \theta_x'\right) - \frac{1}{3\delta}[z_1 + z_2\cos(\theta_x' - \theta_1) + z_3\cos^2(\theta_x' - \theta_1)]^3 \\ & \times \sin^3\left(\beta_2 + \frac{\pi}{m} - \theta_x'\right)\Big\} STP_+\left(\alpha_2'' - \beta_2 - \frac{\pi}{m}\right) + \frac{\pi}{2}\delta^2 STP_-\left(\beta_2 + \frac{\pi}{m} - \alpha_2''\right)\Bigg) \\ & + \Bigg(\Big\{2\delta[z_1 + z_2\cos(\theta_x' - \theta_1) + z_3\cos^2(\theta_x' - \theta_1)]\sin(\beta_2 - \theta_x') \\ & - \frac{1}{3\delta}[z_1 + z_2\cos(\theta_x' - \theta_1) + z_3\cos^2(\theta_x' - \theta_1)]^3\sin^3(\beta_2 - \theta_x')\Big\} \\ & \times STP_+(\beta_2 - \alpha_1'') - \frac{\pi}{2}\delta^2 STP_-(\alpha_1'' - \beta_2)\Bigg) . \end{aligned} \tag{43}$$

From Eqs. (22), (32), and (43), the value of G can always be obtained within the range $0 \leqslant G \leqslant 1$. $G = 1$ implies a pure FM signal; $G = 0$ implies no FM signal. Also, from these three equations the curves showing the dependence of G upon ρ_x can be plotted, as shown in Fig. 7, with m being a parameter, $m_1 > m_2 > \ldots > m_i \ldots$. Since a G value that is too low is not preferable, one of the jobs of reticle design is to find the best spoke arrangement that can produce a G value that is not less than a lower limit G_1, i.e., $G_1 \leqslant G \leqslant 1$. Such an arrangement usually requires that the reticle be divided into two zones: the inner zone and the outer zone. Each zone has a unique total number of spokes. The design method of the modification zone is discussed in Sec. 3.2.

3. EXAMPLES

To apply the above results to reticle design, two examples will be discussed.

3.1. Determination of the total number of spokes

When a spot nutates concentrically around O, i.e., $r_1 = 0$, the total number of spokes needed to provide a desired degree of contrast G_0 can be found by first solving y from the following equation, deduced from Eqs. (22), (32), and (43):

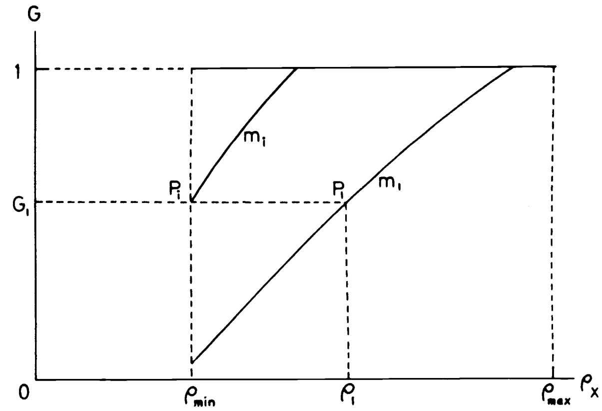

Fig. 8. Graphical method for the design of the modification zone.

$$\frac{4}{3\pi}\left(\frac{a}{\delta}\right)^3 y^3 - \frac{8}{\pi}\left(\frac{a}{\delta}\right) y + (1 + G_0) = 0 . \tag{44}$$

Then the total number of spokes is

$$2m_0 = \frac{\pi}{\sin^{-1} y} . \tag{45}$$

Furthermore, given f_0, the desired center frequency, the angular velocity is determined from Eq. (13) to be

$$\Omega = \frac{f_0}{m_0} . \tag{46}$$

Also, given Δf, the desired frequency bandwidth, we find from Eq. (16) that the amount of eccentricity r_1 that the spot can undergo in the reticle plane is limited to

$$0 \leqslant r_1 \leqslant \frac{a\Delta f}{2m_0\Omega} . \tag{47}$$

3.2. Design of modification zone

When the spot nutates eccentrically with respect to O, ρ_x varies in the range from $\rho_{max} = z_1 + z_2 + z_3$ to $\rho_{min} = z_1 - z_2 + z_3$. Since G varies with ρ_x, it may drop, along the curve of a fixed m, from high values to low values, as shown in Fig. 7. When G is high, there is no problem, but when G is going to become too low, one has to find a way to prevent it from doing so. The way is to let m become unfixed so that G can drop along one curve and then switch to another curve before its value becomes too low. This is the principle of zone modification. We suggest a graphical method, shown in Fig. 8, for the design of a modification zone. The figure is a simplified version of Fig. 7.

Given G_1, the lower contrast limit, we can draw a horizontal line passing through G_1. This line intersects curve m_1 at point P_1 with the coordinates (ρ_1, G_1). From the figure we see that when the spot nutates into the region $\rho_x < \rho_1$, G becomes less than G_1, so we have to modify the m value for the zone inside ρ_1. To do this, we search for another point P_i, where the horizontal line intersects curve m_i $(m_i < m_1)$ and the constraint $\rho_i \leqslant \rho_{min}$ is fulfilled. Then we can determine that the total number of spokes in the inner zone $(0 < \rho_x < \rho_1)$ should be $2m_i$, while the total number of spokes in the outer zone $(\rho_x \geqslant \rho_1)$ should be $2m_1$,

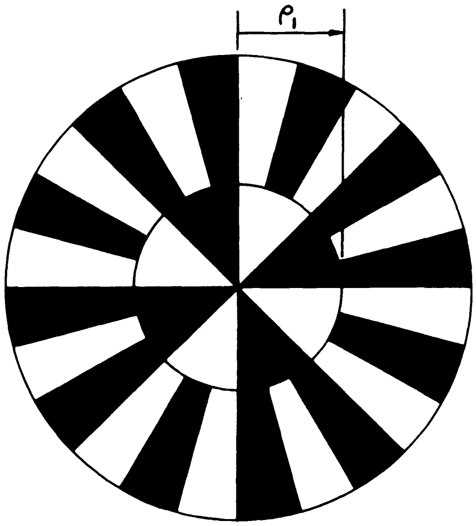

Fig. 9. An FM reticle with a modification zone.

as shown in Fig. 9. In this manner, the design of the modification zone is completed.

4. CONCLUSION

We have presented two models for the analysis of an FM reticle: the point model and the spot model. The former is useful in calculating the center frequency and the frequency bandwidth of the demodulated signal. The latter is useful in calculating the degree of contrast of the modulated signal. Given the desired values of these three parameters, we can make a basic reticle design. From the examples provided, it is seen that the analytical results derived from the two models can be used to conveniently determine the total number of spokes, the angular velocity, the allowed amount of eccentricity, and the zone of modification.

5. REFERENCES

1. L. M. Biberman, *Reticles in Electro-Optical Devices*, Pergamon Press, London (1966).
2. R. D. Hudson, Jr., "Optical modulation," in *Infrared System Engineering*, pp. 235–261, Wiley, New York (1968).
3. S. Craubner, "Digital simulation of reticle systems," Opt. Eng. 20(4), 608–615 (1981).
4. P. E. Mengers and K. B. O'Brien, "Analysis of error response of amplitude modulated reticles," J. Opt. Soc. Am. 54(5), 668–671 (1964).
5. K. Suzuki, "Analysis of rising-sun reticle," Opt. Eng. 18(3), 350–351 (1979).
6. R. C. Anderson and P. R. Callary, "Computer modeling of optical trackers," Opt. Eng. 20(6), 861–865 (1981).
7. F. L. H. M. Stumpers, "Theory of frequency-modulation noise," Proc. IRE 36, 1081 (1948).

Zu Wen Chao and **Jih Liang Chu**: Biographies and photographs appear with the paper "General analysis of frequency-modulation reticles" in this issue.