國 立 交 通 大 學

財務金融研究所

碩 士 論 文

常數彈性變異數過程下的最佳資本結構模型

Optimal Capital Structure Model under the CEV Process

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中 華 民 國 九 十 九 年 六 月

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摘要

在資本結構模型中，最具代表性的 Leland (1994) 以及 Leland and Toft (1996)模型提供了一些關於資本結構問題的深入分析。然而，為了求公司證券的解析解，學者必須加入一些不符合實際的假設，以排除公司證券價值與時間的相關性。評價單一具到期日公司債或者是包含複雜破產法規之公司債時，並無法導出封閉解，必須使用數值方法處理。本文延伸 Broadie and Kaya (2007)的研究，使用二元樹模型，評價具到期日與第十一章破產法規架構下之公司債。為了使模型更有彈性，並更貼近實際，我們允許標的資產價格波動度變動，亦即發展一個常數彈性變異數(CEV)過程下，考量破產程序的資本結構模型。本研究數值分析結果指出，當重整的期限越長，或是 CEV 過程彈性係數 β 越小時，公司債價值越低。

關鍵字：資本結構，第十一章破產保護法，二元樹評價模型，常數彈性變異數過程

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Abstract

The well-known Leland (1994) and Leland and Toft (1996) models provide some insights of the capital structure issues. However, in order to obtain analytical solutions of corporate securities, researchers need to impose some unrealistic assumptions to avoid time and path dependency. While evaluating a single corporate debt with finite maturity or complex bankruptcy proceedings, no analytical solution is available and one needs to resort to numerical methods. In this study, we extend the binomial lattice method by Broadie and Kaya (2007) to develop a capital structure model, which incorporates finite maturity as well as the feature of Chapter 11 bankruptcy proceedings. To make the model more realistic, we assume that the underlying asset value follows the constant elasticity of variance (CEV) process. Our numerical results show that when the reorganization period is longer or the elasticity constant β is smaller, the value of corporate risky debt will be lower.

Keywords: Capital Structure, Chapter 11, Binomial Lattice Method, Constant Elasticity of Variance (CEV) Process

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1. Introduction

A model for pricing risky debt is very important for determining optimal capital structure. The Black-Scholes (1973) model which corporate liabilities can be viewed as a covered call – debt holders own the asset but short the call option. Black and Scholes assumed that the company only issued one zero-coupon bond and equity. When debt is mature, debt holders either receive the face value of debt, or take over the company. The Black-Scholes (1973) model also provides a close-form solution for corporate securities as an option. Merton (1974) used the Black-Scholes (1973) structural model to value risky zero coupon bond and risky consol bond. The result is, the structural model tied default risk to the firm value process and the optimal structure was formed by using the capital structure to value corporate securities.

According to the U.S. bankruptcy code, which includes a liquidation process (Chapter 7) and reorganization process (Chapter 11), the code is used to solve the problems when the firm is under financial distress. Contractual agreements and bankruptcy laws may cause different outcome when the firm fails to make debt payments and declares bankruptcy. For example, bankruptcy may lead to liquidation under Chapter 7 of the bankruptcy code, reorganization under Chapter 11, or the debt may be renegotiated between debt holders and equity holders. Because of the bankruptcy code, we can view corporate securities as a consecutive down-and-out Parisian option. A consecutive down-and-out Parisian option calculates the time price of the underlying asset staying below the barrier level consecutively, not like a barrier option which can knock in or out when the price of underlying asset touches the barrier.

According to the Leland (1994) model, financial distress is triggered when shareholders no longer find a running a company is profitable, even when the cash flow produced by the assets continue servicing the debt. Therefore, bankruptcy is determined endogenously rather than by a certain level of net asset or cash flow constraint.

Merton (1974) and Leland's (1994) model assumes that the underlying asset follows geometric Brownian motion. This implies that prices of underlying asset are lognormally distributed. However, empirical studies have shown just the opposite. That is the reason why we price corporate debt when underlying asset follows constant elasticity of variance (CEV) diffusion process. This process has the advantage that the volatility of the underlying asset is linked to its price level, which is consistent with the empirical observation that prices of underlying asset volatility tend to change as those of underlying asset move up and down.

In this paper, we used binomial lattice numerical evaluation of structure models to price corporate debt because numerical method can be use to solve complex models when analytical pricing techniques are not available. To work in a time-independent setting, these models usually price infinite maturity bonds although these bonds are almost never used in practice. Due to Chapter 11 of the bankruptcy code, we needed to consider the property of Parisian option to price corporate debt. Therefore, it was difficult to obtain analytical solutions in models of bankruptcy proceedings that included automatic stay provisions and grace periods since these features introduced path dependency. Recently, most researches of structural models are under the assumption of geometric Brownian motion that the volatility of underlying asset is constant. As a result, we added the constant elasticity of variance (CEV) to make the volatility more flexible and our structural model more realistic.

The remainder of this paper is organized as follows. Section 2 introduces the literature we use in this paper or the methods we recommend. In Section 3, we describe our model how to implement with more complexity. Section 4 illustrates the result of the model and some computational analysis. Section 5 is the conclusions of our paper.

2. Literature Review

2.1 Structural Model

In the capital structure models, the most famous are the Leland (1994) and Leland and Toft (1996) models. They extend the endogenous default approach by Black and Cox (1976) to include the bankruptcy costs and tax shield of debt, and analyze the static tradeoff theory of capital structure. Leland and Toft (1996) point out that equity in a capital structure model is not similar to a plain vanilla call option or an ordinary down-and-out barrier option.

In Leland model (1994), financial distress is happened when equity holders find that running company is unprofitable, given the debt is still serviced. Bankruptcy is determined endogenously rather than given a certain level of asset or other constraint exogenously. There are two major assumptions as follows: (1) the firm's activities such as financial structure and the capital structure decisions do not change when the decisions was made; (2) the face of debt remains static through time when it was issued. First, Leland (1994) provides the solution of the perpetual debt. Next, Leland (1994) derives the total value of the firm, which contains three terms: the firm's asset value, plus the value of the tax shield of coupon payments, less the value of bankruptcy costs. Then Leland (1994) provides the bankruptcy asset level V_B which is determined endogenously by maximizing the value of equity. Finally, V_B is independent of time and it confirms the assumption of the constant bankruptcy-triggering asset level V_B. However, optimal capital structure relates not only to leverage ratio but also the maturity of debt.

In Leland and Toft (1996) model, the assumption of financial distressed follows Leland (1994) model when shareholders find that running company is unprofitable. In this model, they can choose both maturity and amount of the debt. Their stationary debt structure is that they issue new finite maturity debt with continuous coupon and the same amount of principle will be retired at the same time. The result extends Leland's (1994) closed-form solution to much more alternatives of possible debt structures and dividends payment and develops the method to measure optimal maturity of debt as well as optimal amount of debt. The Leland and Toft (1996) model has some different implications contrast to Leland (1994) model. First, they allow firm to choose its debt maturity, also explain why a firm issue short-term debt even that long-term debt provide more firm value. They find that short-term debt can reduces agency cost. Although long-term debt produces more tax benefit, short-term debt can balance bankruptcy and agency cost in determining the optimal maturity of the capital structure. Second, they find that Macaulay duration overstates the true duration of risky debt for bond portfolio management.

The Fan and Sundaresan (2000) model propose a game-theoretic setting that incorporates bargaining powers to the equity holders and the debt holders. If the firm is liquidated at the bankruptcy point, then debt holders receive liquidation value and equity holders receive nothing. However, if the firm is not liquidated, the firm value will be shared between equity holders and debt holders. The Fan and Sundaresan model shows that debt renegotiation encourages early default and increases credit spreads on corporate debt, given that shareholders can renegotiation in distress to avoid inefficient and costly liquidation. If shareholders do not have bargaining power, no strategic debt service occurs and the model converges to the Leland model.

The Francois and Morellec (2004) model extends the Fan and Sundaresan (2000) model to add the possibility of Chapter 11 proceeding. Shareholders hold a Parisian down-and-out option on the firm's asset. Francois and Morellec solve the endogenous default barrier by maximizing equity value and providing closed-form solution for corporate debt and equity values. The sharing rule of cash flows during bankruptcy has a large impact on optimal leverage, while credit spread on corporate debt shows little sensitivity to the varying bargaining power.

2.2 The CEV Process and Option Pricing

Merton (1973) provided a closed-form solution for down-and-out options. Most of path-dependent options assume that the underlying asset follows geometric Brownian motion. This implies that price of the underlying asset is lognormally distributed. It has some drawbacks when the underlying assumption under Black-Scholes (1973) model. Empirical studies have shown that stock prices are not lognormally distributed. If we use Black-Scholes model to price stock options, there exist well-known strike price bias (volatility smile). Therefore, we price path-dependent options when price of underlying asset follows the constant elasticity of variance (CEV) diffusion process. The CEV option pricing model was developed by Cox (1975). This process let the volatility of underlying asset linked to its price level, which is consist with the empirical evidence that the stock volatility tend to change as stock prices move up and down. The origin of the "volatility smile" is the negative correlation between stock price changes and volatility changes.

Nelson and Ramaswamy (1990) developed a simple binomial tree under the CEV process. They construct a binomial tree which its number of nodes grows linearly in the number of time intervals. It is shown how to construct computationally simple binomial process that converge weakly to commonly employed diffusion in financial models.

It is not always possible to build a very complex model with realistic features and solve the closed-form solution. As a result, we use a binomial lattice method to price corporate debt and model Chapter 11 proceedings developed by Broadie and Kaya (2007). Their method takes the asset value of the firm as the primitive variable and prices equity and debt value as derivatives of this basic variable. The model generates results that are consistent with the limited liability of equity principle. It uses backward valuation, the continuation value of equity is known at each time step, so it can use to make bankruptcy decision. When the firm has a Chapter 11 bankruptcy alternative, their method can easily extend to the case by adding bankruptcy boundary and increasing the state space on the lattice nodes if necessary. This numerical method (Broadie and Kaya (2007)) described the details of the implementation for three different models: Leland (1994), Francois and Morellec (2004), and Broadie, Chernov and Sundaresan (2005).

2.3 Parisian Options

A down-and-out Parisian option is options that know-out if the underlying asset price remains constantly beyond a given barrier over a pre-specified time interval, the so-called grace period. When a Parisian option is activated, its payoff at maturity is equal to standard European option while the payoff is zero if the option know-out before its maturity. The advantage of Parisian option with respect to standard barrier options is that it is more difficult to influence the option payoff. We use a variant of the lattice-based method, called the forward shooting grid (FSG), it has successfully applied to price path-dependent options, like Parisian option. The FSG approach was developed by Hull and White (1993) and Ritchken, Sankarasubramanian and Vijh (1993) for the pricing of American- and European-style Asian and lookback options. The FSG approach uses auxiliary state vector at each node on the lattice tree. The state vector is used to capture the path-dependent feature of the option contract, like grace period of the asset price.

3. The model

We first denote the asset value of the firm V_t and use it as the primitive variable, and therefore other variables can be view as derivatives with respect to asset value. We assume that the asset value V_t is independent of the capital structure and other financial decisions; its diffusion process under the risk-neutral measure Q and Cox and Ross (1976) constant elasticity of variance (CEV) is given by

$$dV_t = (r-q)V_t dt + \sigma V_t^{\frac{\beta}{2}} dW_t \tag{1}$$

where W_t is a standard Brownian motion under Q, q is the payout ratio of the firm, and

σ is the volatility of asset returns, β is a constant, know as a elasticity factor, and $0 \le \beta < 2$. In the case when $\beta = 2$, equation (1) reduces to geometric Brownian motion, this implies that geometric Brownian motion is a special case of the CEV process. Cox (1975) restricted β between 0 and 2. But empirical study shows that $\beta < 0$.

3.1 A binomial model for the CEV process

We construct a discrete approximation for the CEV process using the binomial method. We assume the asset price dynamics are expressed in terms of the Q-measure. Nelson and Ramaswamy (1990) derived a binomial approximation of the stochastic process described in (1); they built a "computationally simple binomial tree" in order to let the number of nodes in the tree structure grows linearly with number of time intervals. We let

$$y = y(t, V)$$

Applying Ito's Lemma, the stochastic differential equation for y is

$$dy_t = \left(\frac{\partial y}{\partial t} + (r-q)V_t \frac{\partial y}{\partial V} + \frac{1}{2}\sigma^2 V_t^{\beta} \frac{\partial^2 y}{\partial V^2} \right) dt + \frac{\partial y}{\partial V} \sigma V_t^{\frac{\beta}{2}} dW_t \tag{2}$$

In order to have a constant diffusion coefficient for the Y-process we let

$$\frac{\partial y}{\partial V} \sigma V_t^{\frac{\beta}{2}} = \upsilon, \tag{3}$$

for some positive constant υ. Equation (3) equal to

$$\frac{\partial y}{\partial V} = \frac{\upsilon}{\sigma} V_t^{-\frac{\beta}{2}},$$

For $\beta \ne 2$, the transformation is given by

$$y_t = \frac{\upsilon}{\sigma\left(1 - \frac{\beta}{2}\right)} V_t^{1-\frac{\beta}{2}} \tag{4}$$

and for $\beta = 2$, the transformation is given by

$$y_t = \frac{\upsilon}{\sigma} \ln(V_t)$$

Now we can build up a computationally simple binomial tree to approximate the y-process.

Then two-dimensional grid in the (t, y)-space can be build as follows. The value y_t of the process at time t, after one period at time $t+1$, can rise to $y_t + \upsilon\sqrt{\Delta t}$ or decrease to $y_t - \upsilon\sqrt{\Delta t}$. In this way, we can build up the value of the y-process as

$$y_{t+i\Delta t}^{j} = y_t + (2j - i)\upsilon\sqrt{\Delta t}, \qquad i = 0,\ldots,n, \qquad j = 0,\ldots,i$$

where $y_{t+i\Delta t}^{j}$ represents the value on the binomial tree under y-process at time $t + i\Delta t$ after j up steps and $i - j$ down steps. Next step is to build up a binomial tree with V-process on the two-dimensional grid in the (t, V) space; we can use the inverse transformation of (4)

$$V_t = \begin{cases} \left[\dfrac{\sigma\left(1-\frac{\beta}{2}\right)}{\upsilon} y_t \right]^{\frac{1}{1-\frac{\beta}{2}}}, & \text{if } y_t > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Once we have constructed the binomial tree with V-process, and then we define the probability of each up step. First we define $V_{t+i\Delta t}^{\underline{j}}$ to be the greatest $V_{t+i\Delta t}^{j}$, $j = 0,\ldots,i$, make $e^{r\Delta t}V_{t+(i-1)\Delta t}^{j} - V_{t+i\Delta t}^{\underline{j}} \geq 0$ and $V_{t+i\Delta t}^{\bar{j}}$ to be the smallest $V_{t+i\Delta t}^{j}$, $j = 0,\ldots,i$, make $V_{t+i\Delta t}^{\bar{j}} - e^{r\Delta t}V_{t+(i-1)\Delta t}^{j} \geq 0$. The probability with (t, V) space makes an up steps is

$$p_{t+(i-1)\Delta t}^{j} = \begin{cases} \dfrac{e^{r\Delta t}V_{t+(i-1)\Delta t}^{j} - V_{t+i\Delta t}^{\underline{j}}}{V_{t+i\Delta t}^{\bar{j}} - V_{t+i\Delta t}^{\underline{j}}} & \text{if } V_{t+i\Delta t}^{\bar{j}} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The probability with (t, V) space makes a down steps is $q_{t+(i-1)\Delta t}^{j} = 1 - p_{t+(i-1)\Delta t}^{j}$. With the definition given above $p_{t+(i-1)\Delta t}^{j}$ represents a probability for the evolution of the price of underlying asset in the approximation binomial tree. Since the primitive variable has been constructed, we want to compute the value of equity, debt and firm on the lattice. First we denote equity value by E. Second, the value of the debt holder we denote by D. Finally, we want to compute the total firm value we denote by F. At current node, the present value of the equity is given by

$$E = e^{-r\Delta t}\left(pE_u + (1-p)E_d\right) \tag{7}$$

The values of D and F also can be calculated in the same way:

$$D = e^{-r\Delta t}\left(pD_u + (1-p)D_d\right) \tag{8}$$

$$F = e^{-r\Delta t}\left(pF_u + (1-p)F_d\right) \tag{9}$$

These values will be modified based on events such as coupon payment, distress cost and liquidation.

Suppose we are at the current node and we know the equity values in the next step, these are given by E_u and E_d. Assume that at the current node, the firm has to pay coupon payment C and firm cash flow δ_t. The present value of equity which do not consider current coupon payment and current firm cash flow is given by

$$\tilde{E} = e^{-r\Delta t}\left(pE_u + (1-p)E_d\right) \tag{10}$$

We denote the difference between the coupon payment and firm cash flow $\bar{C} = C - \delta_t$. When the coupon payment is less than firm cash flow, $\bar{C}$ is negative and it means that excess firm cash flow over the coupon payment can be receive by equity holders. If $\bar{C}$ is positive, it means equity holders should raise money by equity dilution. So we can show equity value at the current node as follows:

$$E = \begin{cases} 0 & \text{if } \tilde{E} \le \bar{C} \\ \tilde{E} - \bar{C} & \text{if } \tilde{E} > \bar{C} \end{cases} \tag{11}$$

3.2 Bankruptcy with grace period and bargaining

In the real world, the equity holders can liquidate the firm under Chapter 7 of the U.S. bankruptcy code or renegotiate debt payments under Chapter 11. When the firm declares bankruptcy under Chapter 11, the bankruptcy court allows the firm to restructure its debt during a certain grace period. Chapter 11 also prevents debt holders from liquidating the firm's asset. Therefore, a firm may declare bankruptcy under Chapter 11 when it is in financial distress, and it spends some time as a bankruptcy firm which does not make full coupon payment, and then recover to be a healthy firm. In this section, we consider the approach of Francois and Morellec (2004). We assume that equity holders decide to declare bankruptcy at a certain level of the firm asset value V_B, and a grace period G is granted by bankruptcy court. If the firm does not come out from bankruptcy at the end of the grace period, the firm is liquidated. Distress cost ω reduces the net firm cash flow when the firm is in bankruptcy. The liquidation cost is α. When the firm asset value is under default boundary V_B, the debt is serviced strategically. At the time bankruptcy is declared, the debt service is determined by the bargaining game between debt holders and equity holders. We follow Fan and Sundaresan (2000) to determine the debt service using a Nash bargaining game. If the firm is liquidated at the bankruptcy point, the debt holders receive $(1-\alpha)V_B$ and equity holders receive nothing. If the firm is not liquidated, firm asset value will be F_{V_B}, and will be share between debt holders and equity holders.

We assume the bargaining power of the equity holders is η and the bargaining power of debt holders is $1-\eta$. If we denote the sharing rule at the bankruptcy point as θ, the incremental value gained by equity holders is θF_{V_B} and the incremental value gained by debt holders is $(1-\theta)F_{V_B}-(1-\alpha)V_B$.

The optimal sharing rule:

$$\theta^* = \arg\max\left\{\left[\theta F_{V_B}\right]^{\eta}\left[(1-\theta)F_{V_B} - (1-\alpha)F_{V_B}\right]^{1-\eta}\right\} \tag{12}$$

and its solution is

$$\theta^* = \eta\left(1 - \frac{(1-\alpha)V_B}{F_{V_B}}\right) \tag{13}$$

As a result, at the bankruptcy point, the value of the claim of the equity holders is

$$\theta^* F_{V_B} = \eta\left(F_{V_B} - (1-\alpha)V_B\right) \tag{14}$$

and the value of the claim of the debt holders is

$$\left(1-\theta^*\right)F_{V_B} = (1-\eta)\left(F_{V_B} - (1-\alpha)V_B\right) + (1-\alpha)V_B \tag{15}$$

The bargaining game determines the value of equity and debt at the bankruptcy point through equation (14) and (15). So we don't need to know how the debt is serviced when firm is in bankruptcy, just need to know the total firm value F_B.

3.2.1 Binomial lattice computations

We use the binomial lattice as described in section 3.1. If the bond is a consol bond, the bankruptcy boundary will be constant and time independent. However, if the bond has finite maturity, the bankruptcy boundary will be time dependent. In the beginning, we first price infinite maturity debt. We assume that the default boundary V_B fits with the level of nodes on the lattice. If it is not on the lattice, we use the first node level that is higher than V_B to approximate V_B. We assume firm has issued a consol bond with coupon payment C and the effective tax rate is τ and all interest payments are tax deductible. In the binomial lattice, since we use discrete time steps the total firm cash flow at a certain node with asset value is given by:

$$\delta_t = V_t e^{q\Delta t} - V_t \tag{16}$$

The following calculation divides into three types:

- **Nodes with $V > V_B$.** The firm is in healthy state in these nodes, the coupon payments are paid by firm cash flow and equity dilution if firm cash flow is not enough to pay coupon payments. The effective tax rate is τ. Equity, debt, and firm value can update as follows:

$$\text{If } \tilde{E}+\delta_t \geq (1-\tau)C\Delta t: \quad \begin{aligned} E &= \tilde{E}+\delta_t-(1-\tau)C\Delta t, \\ D &= C\Delta t + e^{-r\Delta t}\left(pD_u+(1-p)D_d\right), \\ F &= \delta_t + e^{-r\Delta t}\left(pF_u+(1-p)F_d\right)+\tau C\Delta t. \end{aligned} \tag{17}$$

$$\text{If } \tilde{E}+\delta_t < (1-\tau)C\Delta t: \quad \begin{aligned} E &= 0, \\ D &= (1-\alpha)(V_t+\delta_t), \\ F &= (1-\alpha)(V_t+\delta_t). \end{aligned}$$

where $\tilde{E}$ is given in (10) and δ_t is given in (16).

- **Nodes with $V < V_B$.** The firm is in bankruptcy. The debt is serviced strategically and we do not know how the firm cash flow is shared between debt holders and equity holders. We can use equation (14) and (15) to determine the value of debt and equity at the bankruptcy point, so we only need to track of the firm value F when the firm is in bankruptcy. Also there are no tax benefits while the firm is in bankruptcy. The total time spent below the default boundary V_B needs to be recorded. Let g record the length of time the firm spends in bankruptcy. Because we are working on the binomial lattice, g can only take discrete values. Let $\bar{g}$ denote the maximum number of time steps that the firm can spend in bankruptcy. We have $\bar{g} = G/\Delta t$, where G is the grace period. Assume $\bar{g}$ is an integer, then g will be the value in $[0,1,\ldots,\bar{g}-1,\bar{g}]$. For a given node and a given g, there three possibilities in the next time steps. First, the firm comes out of bankruptcy next time step. Second, if $g=\bar{g}-1$ in the current node, and $V < V_B$ in the next time step, then the grace period will be in expiration and the firm will be liquidated. Finally, the firm can still be in bankruptcy without grace period expires in the next time step. So the value of g will be one higher than the current node. For each node, we need to keep track of the firm value in every possible state of g. Thus, $F[i]$ will represent the firm value at the current node when $g = i$. We can update the firm value as follows:

$$F[i] = \begin{cases} \bar{\delta}_t + e^{-r\Delta t}\left(pF_u[i+1] + (1-p)F_d[i+1]\right) & \text{for } i = 1,\ldots,\bar{g}-1 \\ (1-\alpha)\left(V_t + \bar{\delta}_t\right) & \text{for } i = \bar{g} \end{cases} \tag{18}$$

where

$$\bar{\delta}_t = V_t e^{(q-\omega)\Delta t} - V_t \tag{19}$$

$\bar{\delta}_t$ represents the distress cost adjusted cash flow of the firm.

- **Nodes with $V = V_B$**

This node is the last healthy state before firm goes into bankruptcy or the first healthy state the firm just comes out from bankruptcy. The equity and debt values can be calculate use equation (14) and (15) after firm value is computed. We update equity, debt, and firm values as follows:

$$\begin{aligned} F[0] &= \delta_t + e^{-r\Delta t}\left(pF_u + (1-p)F_d[1]\right), \\ F[i] &= \bar{\delta}_t + e^{-r\Delta t}\left(pF_u + (1-p)F_d[1]\right) \quad \text{for } i = 1,\ldots,\bar{g}, \\ E &= \eta\left(F[0] - (1-\alpha)V_B\right), \\ D &= (1-\eta)\left(F[0] - (1-\alpha)V_B\right) + (1-\alpha)V_B. \end{aligned} \tag{20}$$

$F[0]$ represents the value of the firm at the bankruptcy boundary V_B that have never been in bankruptcy, and it is the value for the node reaching V_B from above. The $F[i]$ is the value of the firm at the bankruptcy boundary V_B just coming out from bankruptcy. As the result, $F[i]$ takes into account the distress cost, while $F[0]$ does not.

- **Price finite maturity debt**

We can use the procedure described above for pricing finite maturity bond with coupon C, face value P and maturity T. At maturity, the face value and the coupon payment should be paid; otherwise the firm will be liquidated. If the firm is still under the bankruptcy boundary V_B when the bond matures, the firm will be liquidated. The terminal values will be calculated as follows:

1. **Nodes with** $V > V_B$

$$\begin{aligned}
&\text{If } V_T+\delta_T \ge (1-\tau)C\Delta t + P: && E = V_T+\delta_T-(1-\tau)C\Delta t - P\\
& && D = C\Delta t + P\\
& && F = V_T+\delta_T+\tau C\Delta t\\
&\text{If } V_T+\delta_T < (1-\tau)C\Delta t + P: && E = 0\\
& && D = (1-\alpha)(V_T+\delta_T)\\
& && F = (1-\alpha)(V_T+\delta_T)
\end{aligned} \tag{21}$$

2. **Nodes with** $V = V_B$

$$F[i] = (1-\alpha)\left(V_T+\bar{\delta}_T\right) \text{ for } i = 1,\ldots,\bar{g} \tag{22}$$

3. **Nodes with** $V < V_B$

$$\begin{aligned}
&\text{If } V_T+\delta_T \ge (1-\tau)C\Delta t + P: && E = V_T+\delta_T-(1-\tau)C\Delta t - P\\
& && D = C\Delta t + P\\
& && F[0] = V_T+\delta_T+\tau C\Delta t\\
& && F[i] = V_T+\bar{\delta}_T+\tau C\Delta t \text{ for } i = 1,\ldots,\bar{g}\\
&\text{If } V_T+\delta_T < (1-\tau)C\Delta t + P: && E = 0\\
& && D = (1-\alpha)(V_T+\delta_T)\\
& && F[0] = (1-\alpha)(V_T+\delta_T)\\
& && F[i] = (1-\alpha)\left(V_T+\bar{\delta}_T\right) \text{ for } i = 1,\ldots,\bar{g}
\end{aligned} \tag{23}$$

- **Bankruptcy boundary**

We assume that V_B is a vector that contains the bankruptcy boundary for each time steps. The optimal V_B in the finite maturity setting will not be constant, but will be time dependent since the remaining value of the bond is changing over time. So we assume a functional form for the bankruptcy boundary and let the equity holders choose a parameter of that function to maximize the equity value. We use this linear function of the riskless bond price.

$$V_B^t = \phi P_t \tag{24}$$

V_B^t is the bankruptcy boundary at an intermediate time t, P_t is the riskless bond price at time t, and ϕ is a positive number that is time independent. Also, if V_B^t is not on the lattice, we use the first node level that is higher than V_B^t to approximate V_B^t.

4. Result

4.1 Numerical method

4.1.1 The FSG approach

We have already described the binomial lattice computation in section 3.2. The most we care about is how to deal with path-dependence price. We use the Forward Shooting Grid approach which was developed by Hull and White (1993). Let $g(k,j)$ denote the grid function. The binomial tree of the FSG algorithm can be represent by

$$V[m-1,j;k]=\{p_u V[m,j+1;g(k,j+1)]+p_u V[m,j+1;g(k,j+1)]\}e^{-r\Delta t} \tag{25}$$

$$g(k,j)=(k+1)\mathbf{1}_{\{x_j\le V_B\}} \tag{26}$$

As a result, we use the method to cope with nodes which are under the bankruptcy boundary. It is necessary to compute $V[m,j;k]$ for all index value k, $k=\overline{g}-1,\ \overline{g}-2,\ \cdots,\ 0$ before we move to next time level. In order to enhance the compute efficiency and shorten the compute time, we do not compute k in all nodes.

We only need to compute the nodes which are under the bankruptcy boundary. It is not necessary to compute k for the nodes which are below the bankruptcy boundary for more than $\overline{g}$ steps. As a result, we can save a lot of time in computation. Figure 1 gives an example for the binomial lattice. Figure 2 shows how the FSG approach works in the binomial lattice given ϕ =1. To illustrate the rollback procedure, we take node (3,1) as an example to compute the firm value. From Equation (18), firm value

$$\mathrm{F}[0]=57.57\times\left(e^{(q-w)\Delta t}-1\right)e^{-r\Delta t}\left(68.08\times P+22.04\times(1-P)\right)=45.44$$

Where P=0.5205

Figure 1

Binomial Lattice

The parameters are $V_0 = 100,\ \sigma = 20\%,\ C = 3,\ P = 60,\ r = 5\%,\ q = 3\%,\ \alpha = 50\%,$ ω=1%, $\tau = 25\%,\ \ \eta = 50\%,\ \ G = 1,\ \ \beta = 0,\ \ T = 3,\ N = 2,\ \phi = 1,\ \ V_B = 60$

The top number at each node denote the asset value, the middle number is the firm cash flow and the bottom number is the payment due to debt holders

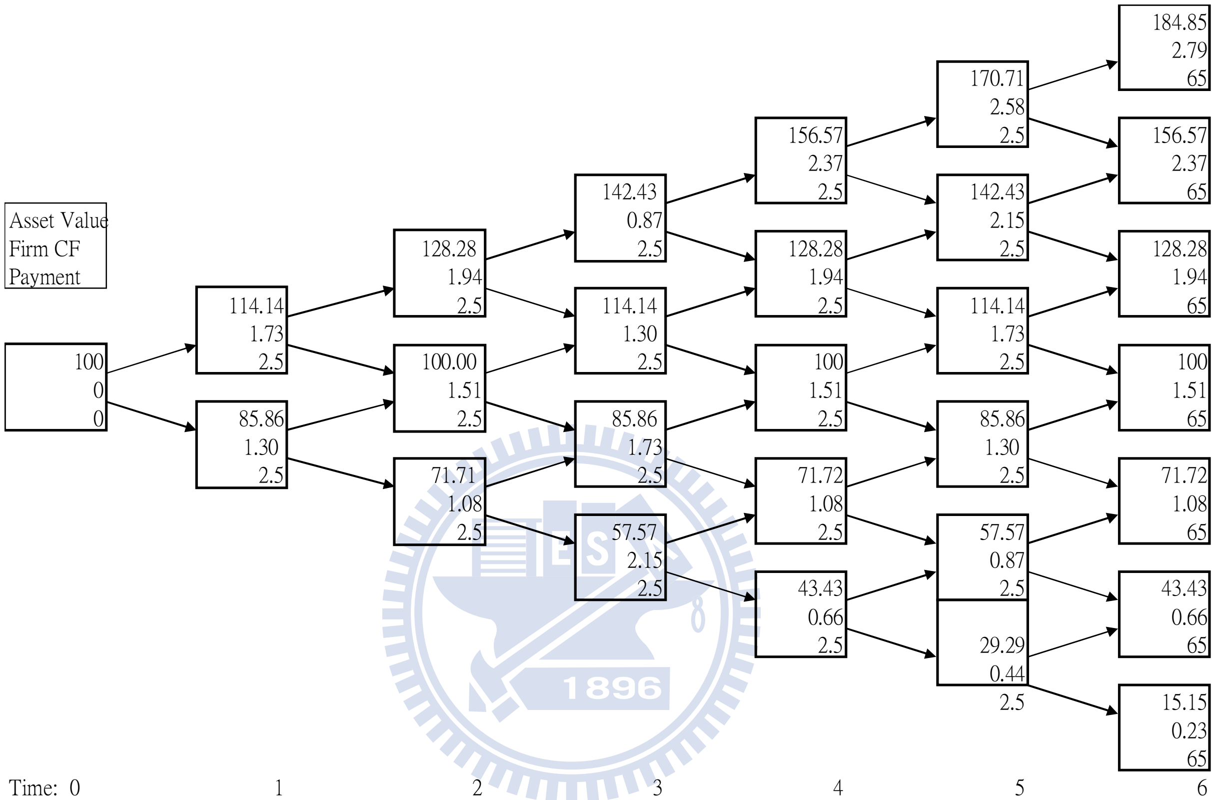
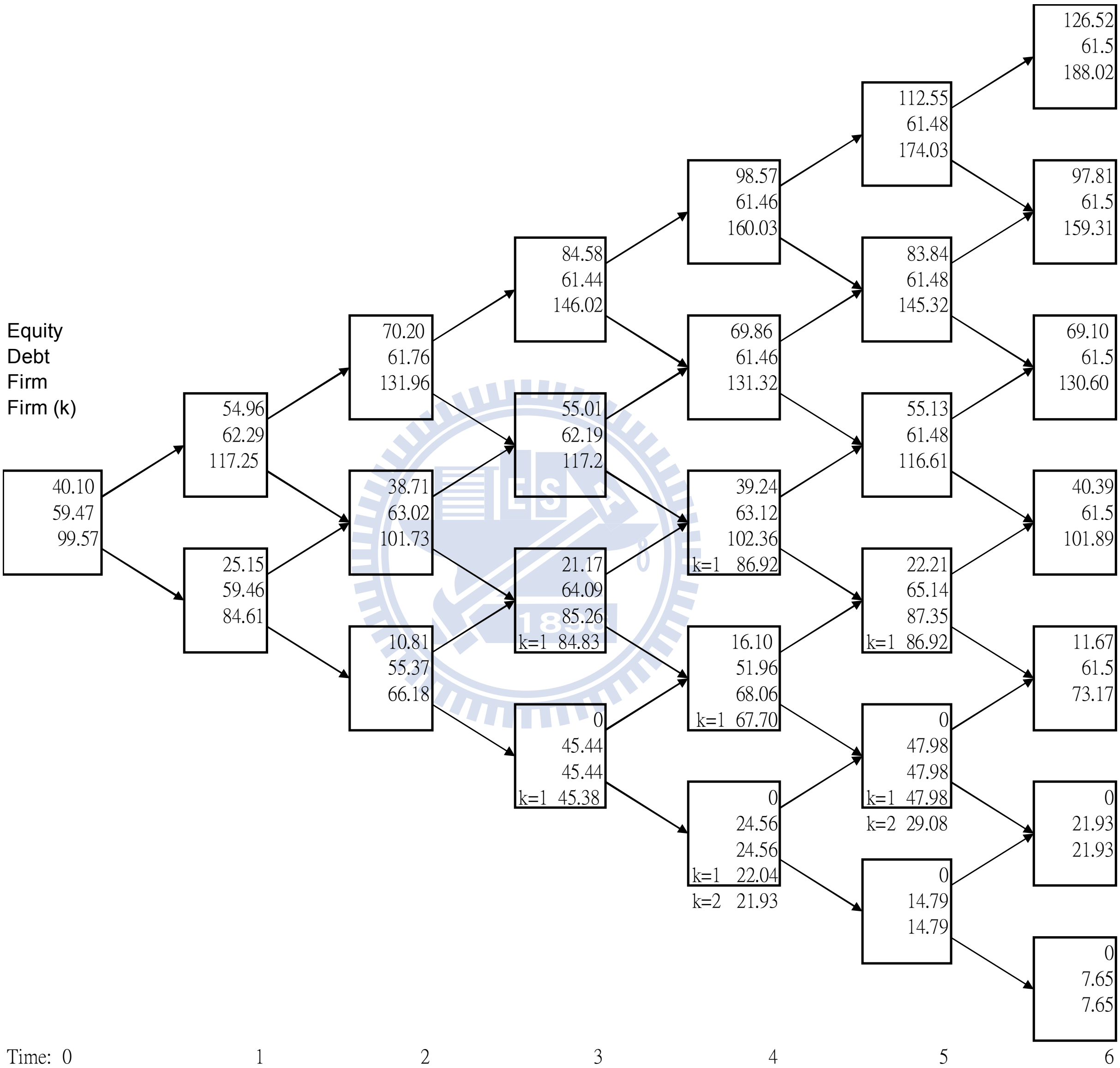


Figure 2

Equity, Debt and Firm Value

The parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3$, $P = 60$, $r = 5\%$, $q = 3\%$, $\alpha = 50\%$, ω=1%, $\tau = 25\%$, $\eta = 50\%$, $G = 1$, $\beta = 0$, $T = 3$, $N = 2$, $\phi = 1$, $V_B = 60$

k : number of periods under barrier



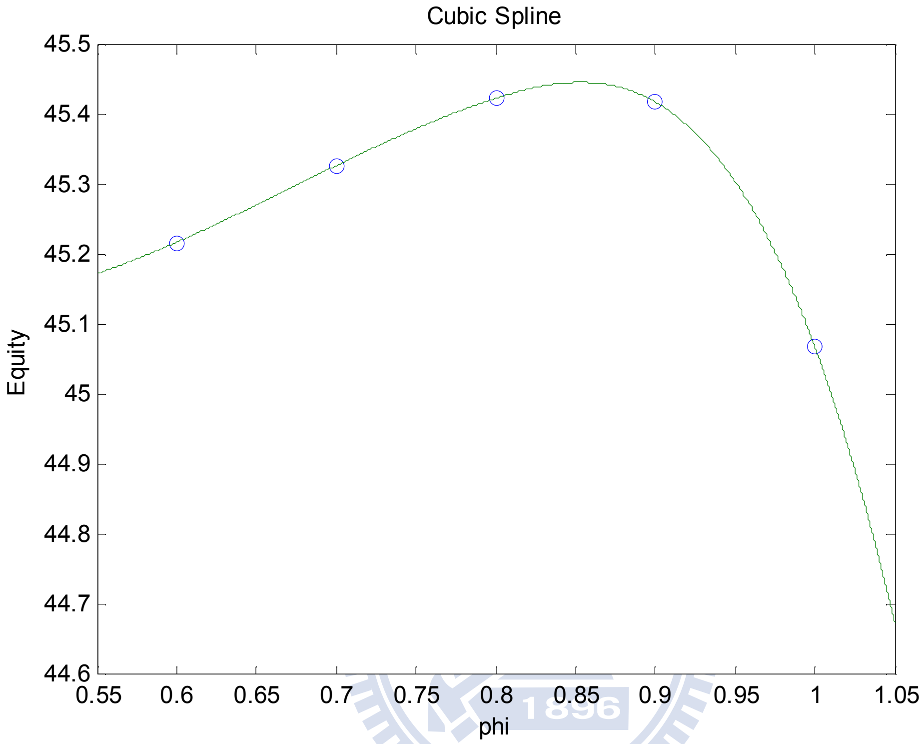
4.1.2 Optimal bankruptcy boundary

The optimal bankruptcy boundary will be chosen to maximize the equity value. In the case of infinite maturity debt, we choose an arbitrary bankruptcy boundary that is lower than optimal boundary. And then we start to increase the boundary on the lattice and reprice the equity value. The equity value first increases and then starts to decrease after it reach the maximum value when we move the boundary up on the boundary. Therefore, we stop moving the boundary when the equity value starts to decrease. As a result, we obtain the discrete observation points and we fit a cubic spline to approximate the exact functional form and use this spline to find the maximum value of equity and the optimal boundary. In the finite maturity debt case, $V_B^t = \phi P_t$ equity holders will choose ϕ to maximize the equity value. We use the same method as describe in infinite maturity debt case, we choose arbitrary ϕ and reprice the equity value. And thus we find the maximum equity value. We need to find out the appropriate ϕ in the search algorithm and fit a cubic spline then find out the maximum equity value.

Figure 3

Cubic Spline

The model parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3$, $P = 60$, $r = 5\%$, $q = 3\%$, $\alpha = 50\%$, ω=1%, $\tau = 25\%$, $\eta = 50\%$, $G = 1$, $\beta = 1$, $T = 5$ $\Delta t = 0.005$ *years* equity is 45.4476



4.1.3 Convergence of the method

In our method, we compute the equity value and debt value in 5000 time steps as the true value. We analyze our numerical method by comparing the results from the method with the true value describe above. Figure 4 and Figure 5 show, under $\beta = 1$ and $\beta = 0.5$, respectively, the convergence of equity and debt pricing errors as the number of time steps increases. We can see the size of the oscillation is relatively small. When we use 1000 time steps, the largest errors value is less than 0.2% of the true value.

Figure 4

Convergence of Equity and Debt Errors (β =1)

The model parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3$, $P = 60$, $r = 5\%$, $q = 3\%$, $\alpha = 50\%$, ω=1%, $\tau = 25\%$, $\eta = 50\%$, $G = 1$, $\beta = 1$, $T = 5$. The true value of equity is 45.4671, and the true value of debt is 55.0929.

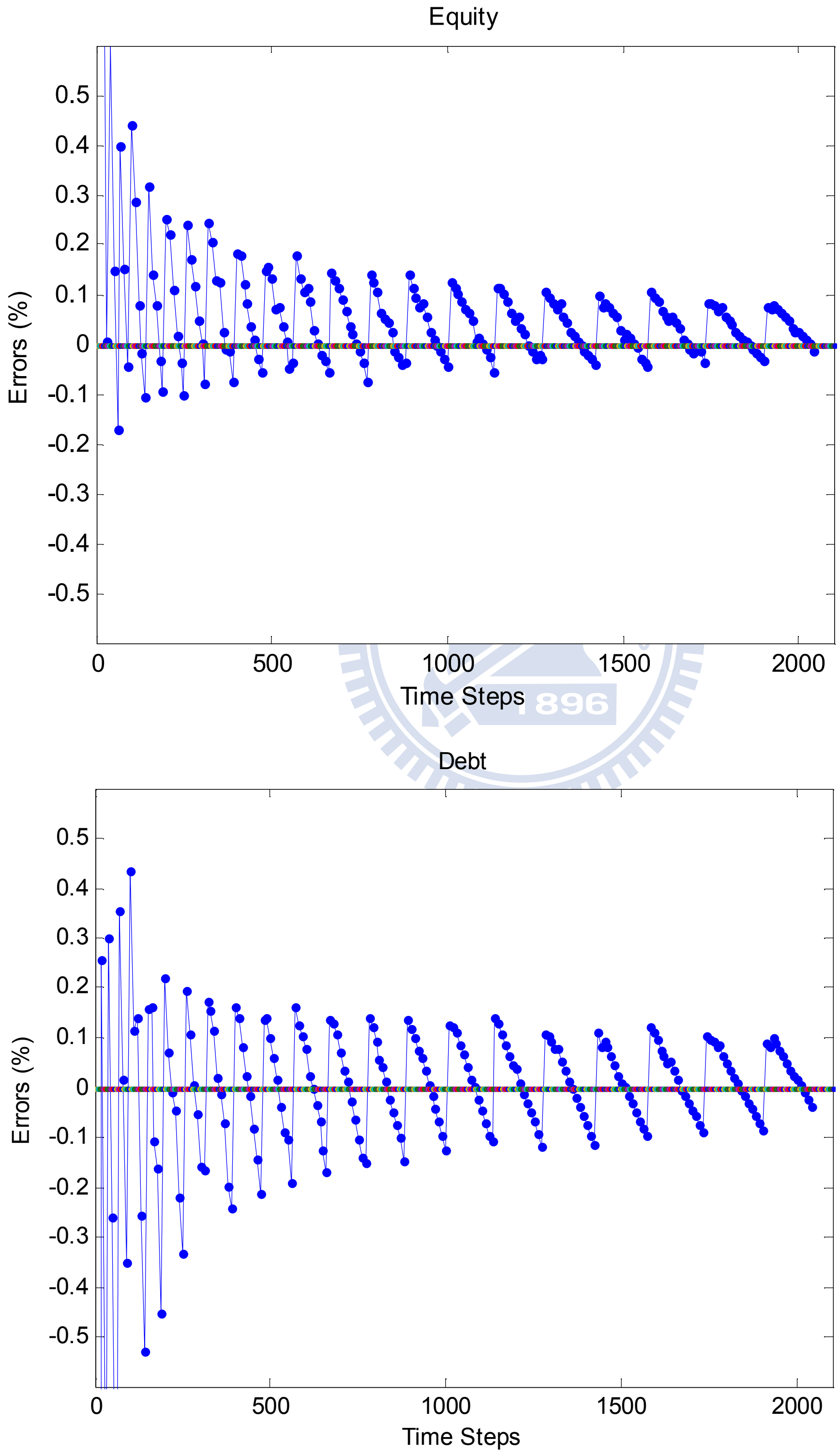
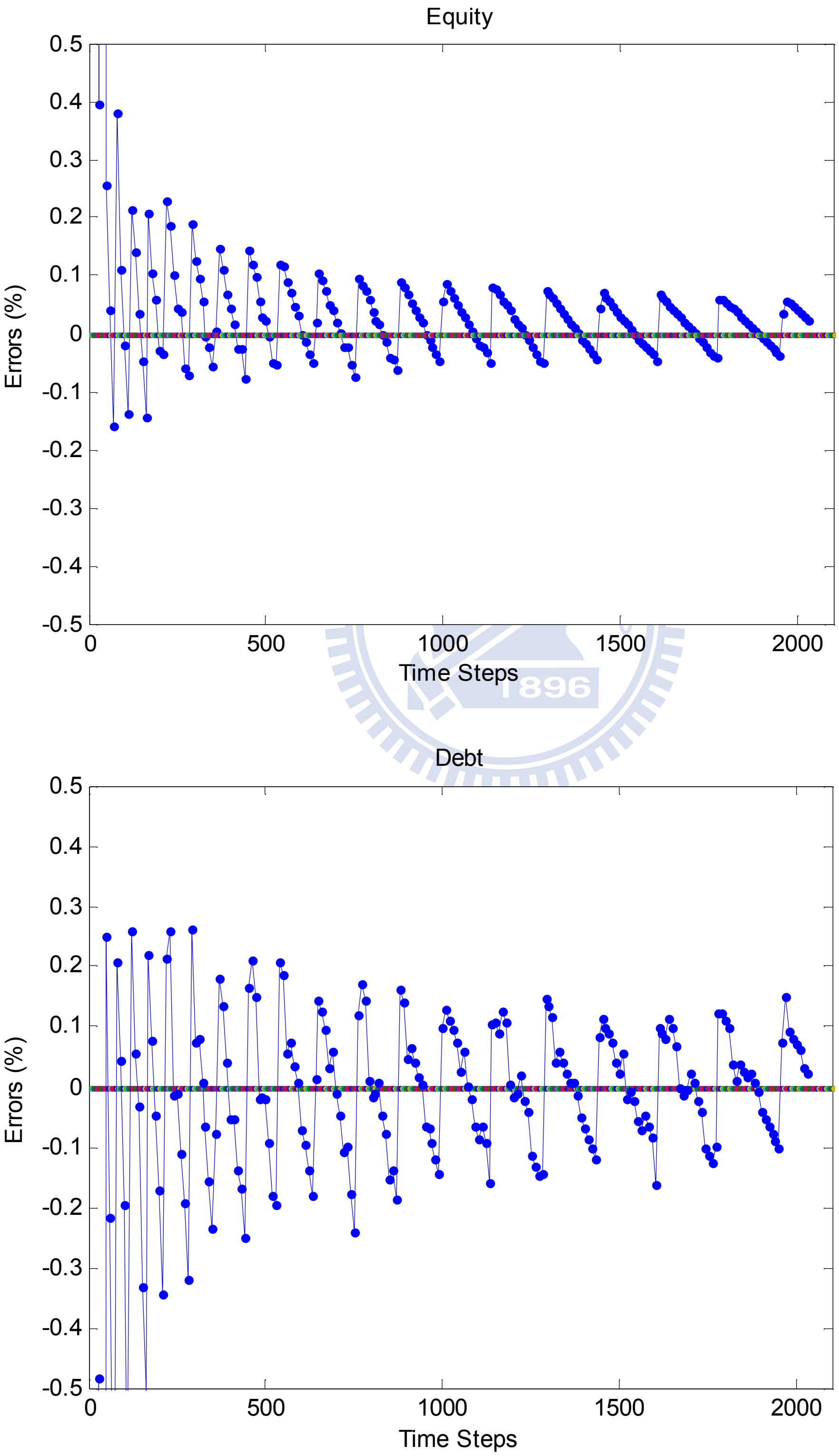


Figure 5

Convergence of Equity and Debt Errors (β =0.5)

The model parameters are $V_0 = 100,\ \sigma = 20\%,\ C = 3,\ P = 60,\ r = 5\%,\ q = 3\%,\ \alpha = 50\%,$ ω=1%, $\tau = 25\%,\ \ \eta = 50\%,\ \ G = 1,\ \ \beta = 0.5,\ \ T = 5$. The true value of equity is 45.8437, and the true value of debt is 54.9405 .



4.2 Numerical Results

In this section, we will study the price and the yield spreads of coupon bond with finite maturity before comparing them with different βs of the CEV process, different grace periods and infinite maturities. We were aware that bankruptcy leads to intermediate liquidation case because the length of the granted grace period in a Chapter 11 setting may be different for bonds according to maturity. Therefore, we chose G=0 to make comparison on effects of maturity on prices. We found as the maturity increases, the price of the finite maturity coupon bond converged to the price of the infinite maturity coupon bond.

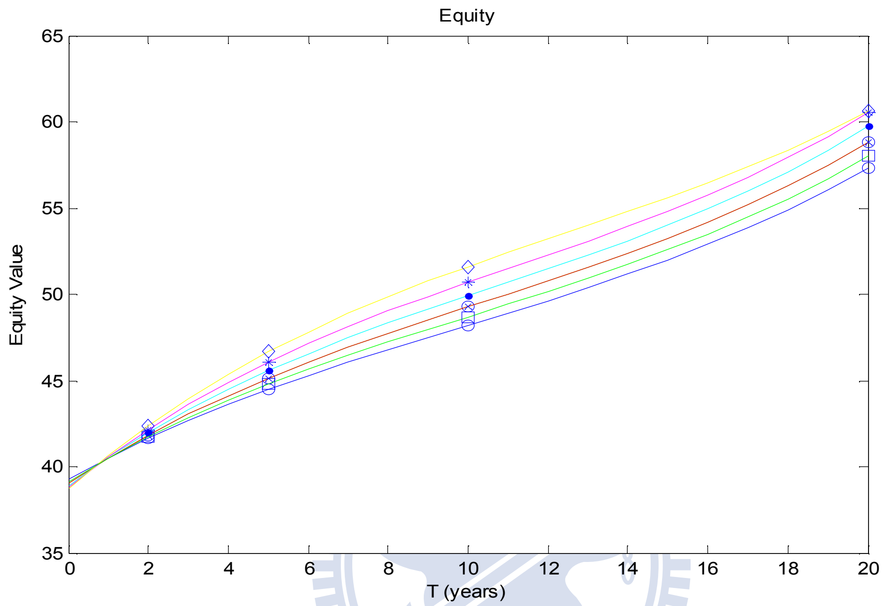
Figure 6 shows the graph of equity, debt and yield spread value as the maturity increases. The yield spread is defined as $Spreads = yield - r$, where r is a risk-free rate. The value of the equity increases and the value of the debt decreases as the β of the CEV process increases. The result makes sense because the decrease in β can be viewed as an increasing volatility. Therefore, equity, also an option, increases value. Debt holders who expect the firm to have steady cash flow thus lose its value. The yield spread first goes up when the maturity is becoming longer and then decreases.

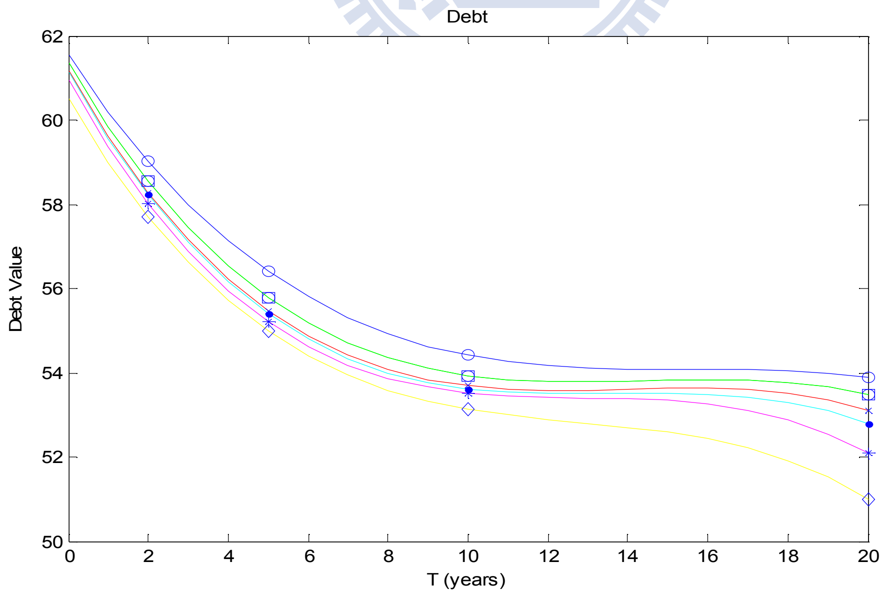
Figure 6

Effect of Maturity on Equity, Debt and Yield Spreads

for a Coupon Bond for Various β

The model parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3$, $P = 60$, $r = 5\%$, $q = 3\%$, $\alpha = 50\%$, ω=1%, $\tau = 25\%$, $\eta = 50\%$, G=0. The time increment in the lattice is $\Delta t = 0.005$ *years*.





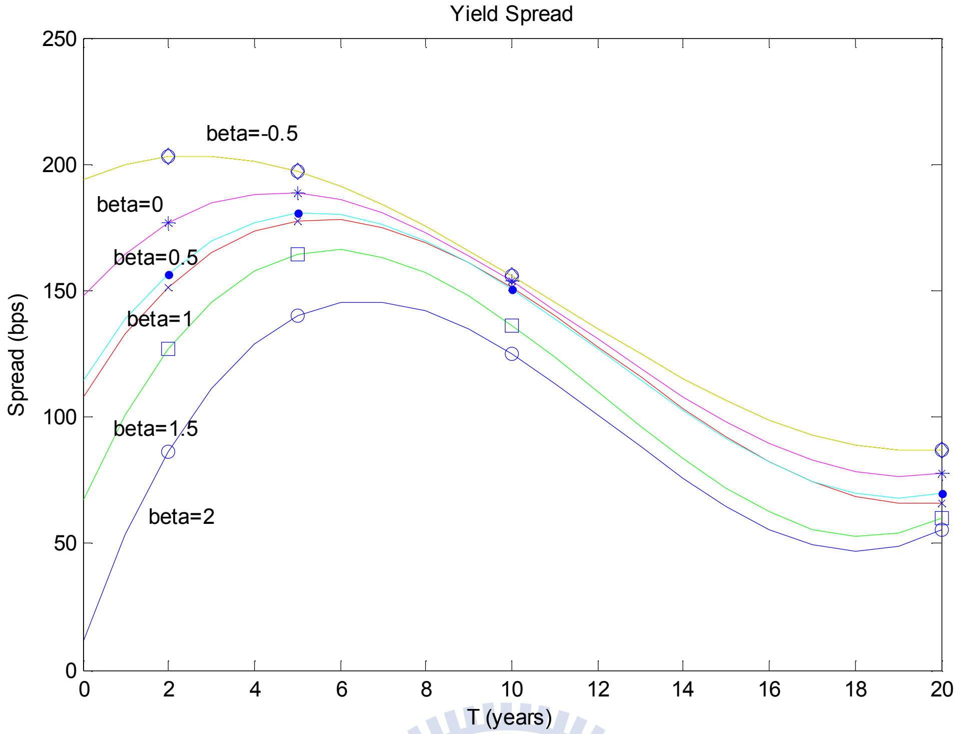


Figure 7 shows the graph of equity and yield spread values as the grace period is prolonged. We can tell that, when the β of the CEV process increases, the value of the equity increases and the value of the debt decreases. The reason is the same as what we have seen in Figure 1. We can see that as grace period increases, the equity and spread increase then converge to a certain level. Debt value goes down as grace period is on the rise because grace period serves as a benefit for equity holders and debt holders do not benefit from it. The results are expected: when maturity increases, the equity value rise accordingly while the debt value decreases.

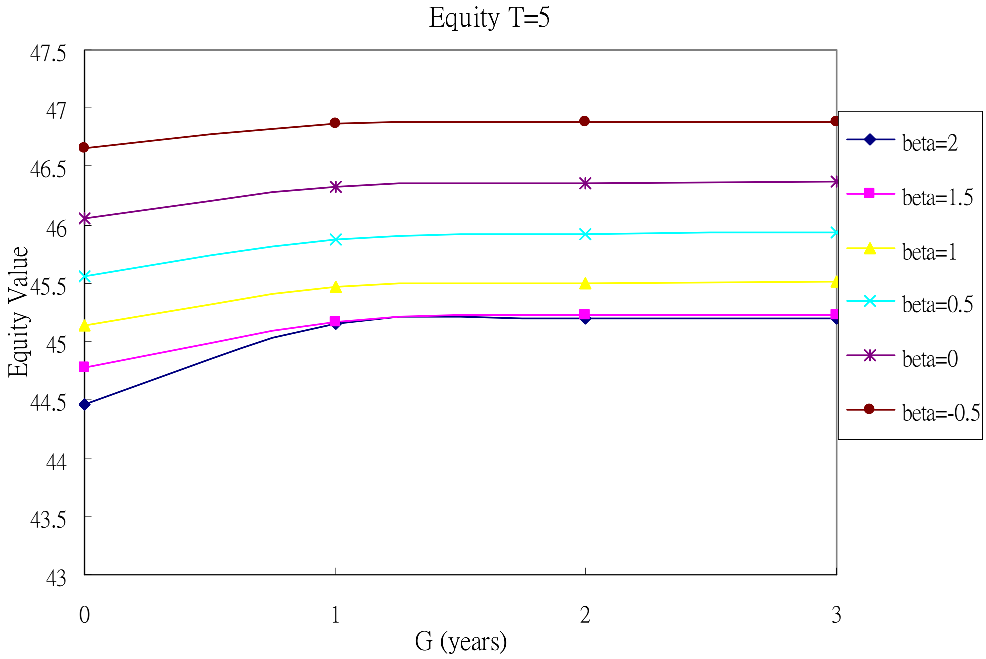
Figure 7

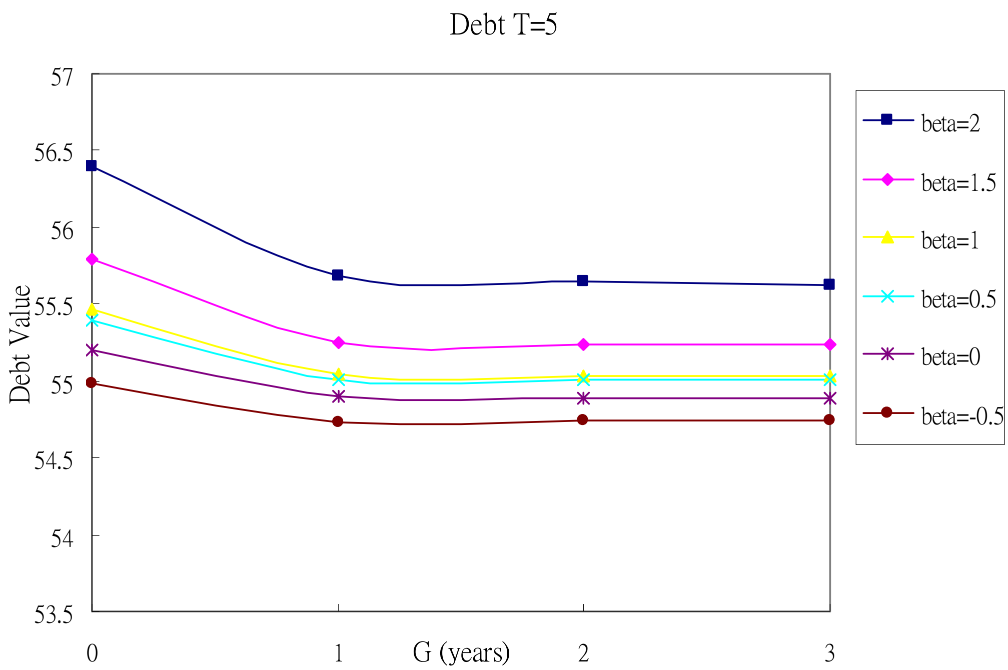
Effect of Grace Period on Equity, Debt and Yield Spread

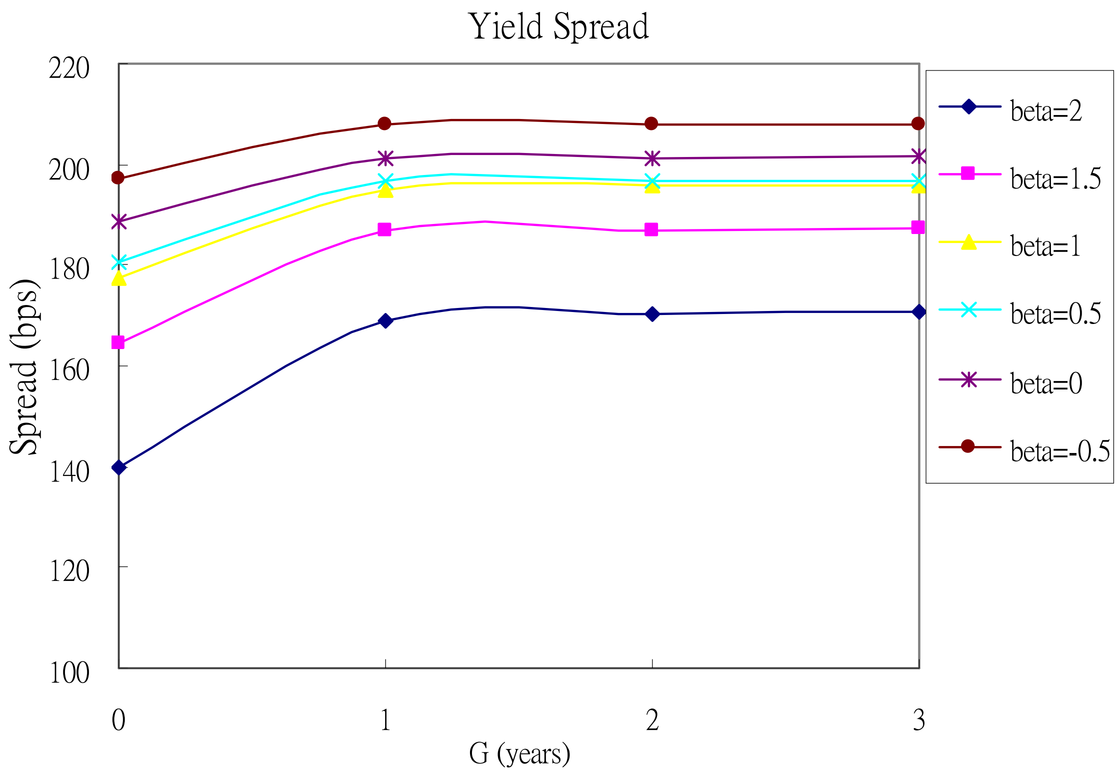
for a Coupon Bond with Various β

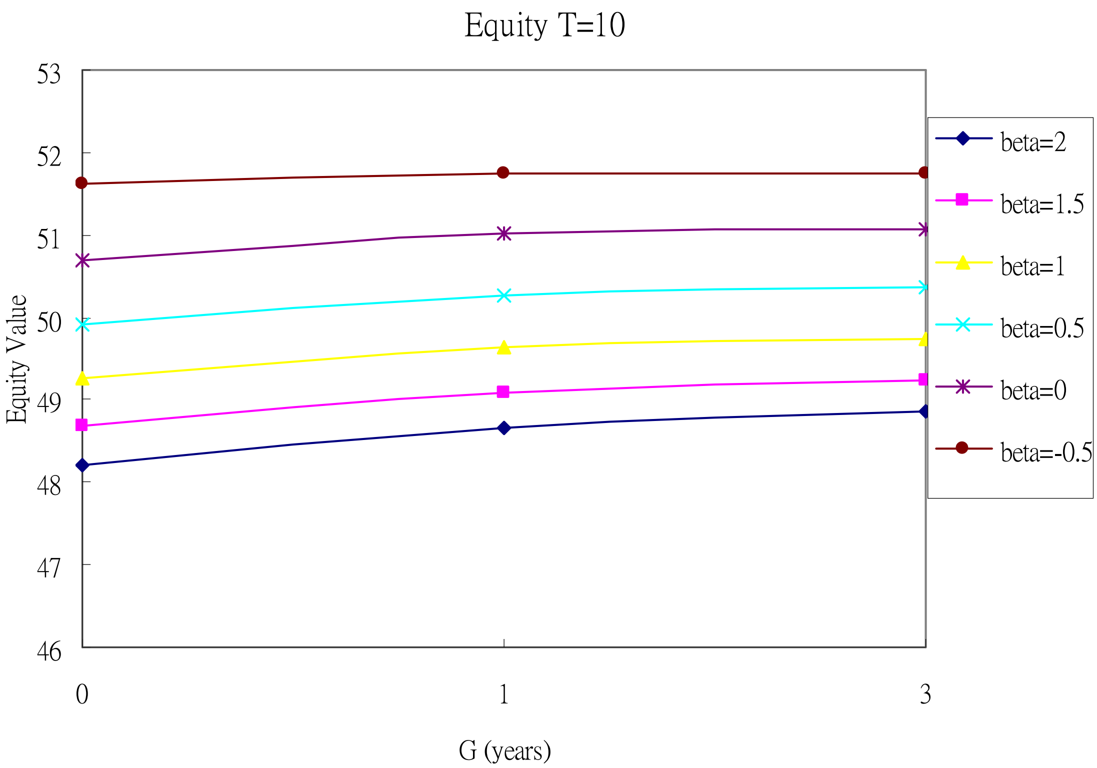
The model parameters are $V_0 = 100,\ \sigma = 20\%,\ C = 3,\ P = 60,\ r = 5\%,\ q = 3\%,\ \alpha = 50\%,$

ω=1%, $\tau = 25\%,\ \ \eta = 50\%,\ \ T = 5$ The time increment in the lattice is $\Delta t = 0.005$ *years*






Yield Spread
220
200
180
160
140
120
100
Spread (bps)
0
1
2
3
G (years)
beta=2
beta=1.5
beta=1
beta=0.5
beta=0
beta=-0.5


Equity T=10
53
52
51
50
49
48
47
46
Equity Value
0
1
3
G (years)
beta=2
beta=1.5
beta=1
beta=0.5
beta=0
beta=-0.5

5. Conclusion

In this paper, we provide a lattice method to price risky corporate debt using structural model. We take the asset value of the firm as the primitive variable and calculate other variables as derivatives of this basic variable. Furthermore, we add constant elasticity of variance (CEV) process into the primitive variable in order to give the risky corporate debt pricing model a more realistic touch. Our method generates results that are consistent with the limited liability of equity principle.

Our method can be beneficial to corporate debt pricing model as well. While many existing models use infinite maturity bonds to obtain a close-form solution, our method can be used to solve these for finite maturity debt.

We analyze the equity value with finite maturity coupon bonds in different grace periods and the result is intuitive: the equity value increases and the debt value decreases when the grace period increases. We can find that increasing in grace period is extra benefit for equity holders. In a viewpoint of equity holders, if there is no grace period the firm is liquidated immediately and equity holders receive nothing (Gilson, John and Lang (1990)). But if there is grace period for equity holders, equity holders can wait for firm recovery from financial distress, not just only liquidation. An infinite maturity case had the same result as in a finite maturity case. We also compared finite maturity debt and infinite maturity debt in different β s of the CEV process. We can find that β of the CEV process decreases in value while the equity value increases. In addition, we analyze the term structure of yield spreads for finite maturity coupon bonds and leave out the Chapter 11 bankruptcy code. We find that term structure of yield spreads first rise up and then converge to a certain level. The future research in our paper may be about shortening the time-computing in the numerical method, taking more complex bankruptcy procedures under discussion and adding empirical study to see why low firm asset value has high volatility(β <0).

References

Avellaneda M. and L. Wu "Pricing Parisian-Style Options with a Lattice Method". *International Journal of Theoretical and Applied Finance* 2, 1-16

Boyle P. P. and Y. "Sam" Tan "Pricing Lookback and Barrier Options under the CEV Process" *Journal of Financial and Quantitative Analysis*. Vol.34, No.2, June1999

Broadie M., M. Chernov, and S. Sundaresan "Optimal Debt and Equity Values in the Presence of Chapter 7 and Chapter 11" *The Journal of Finance*, Vol. LXII, No.3 June 2007

Broadie M. and O. Kaya "A Binomial Lattice Method for Pricing Corporate Debt and Modeling Chapter 11 Proceedings" *Journal of Financial and Quantitative Analysis*. Vol. 42, No. 2, June 2007

Bernard C., O. L. Courtois, and F. Quittard-Pinon. 2005, "A new Procedure for Pricing Parisian Options," *Journal of Derivatives* 12, no. 4: 45–53.

Costabile M. "On Pricing Lookback Options under the CEV Process" *Mathematics Subject Classification* (2000): 05A10, 91B28 .*Journal of Economic Literature Classification* C63

Chesney M., M. Jeanblanc-Picqu´e, and M. Yor, 1997, "Brownian Excursion and Parisian Barrier Options," *Advances in Applied Probability* 29, 165–184

Costabile, M., 2002, "A Combinatorial Approach for Pricing Parisian Options," *Decisions in Economics and Finance*, 25(2), 111–125

Fracois P. and E. Morellec "Capital Structure and Asset Prices: Some Effects of Bankruptcy Procedures" *The Journal of Business;* Apr 2004; 77, 2

Fan H. and S. M. Sundaresan "Debt Valuation, Renegotiation, and Optimal Dividend Policy" *The Review of Financial Studies*, Vol. 13, No. 4. (Winter, 2000), pp. 1057-1099

Gilson S.C., Kose John and Larry H.P. Lang, 1990, " Troubled Debt Restructurings: An Empirical Study of Private Reorganization of Firms in Default**,**" *Journal of Financial Economics*. Vol 27, Issue 2, October 1990, pp.315-353

Kwok Y. K. and K. W. Lau "Pricing Algorithms for Options with Exotic Path-Dependence" The *Journal of Derivatives*. Fall 2001, Vol. 9, No. 1: pp. 28-38

Leland E. H. "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure.", *The Journal of Finance*, Vol. 49, No. 4 (Sep., 1994), pp. 1213-1252.

Leland E. H. and K. B. Toft "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit" *The Journal of Finance*, Vol. 51, No. 3, 1996 , pp. 987-1019

Nelson D. B. and K. Ramaswamy "Simple Binomial Processes as Diffusion Approximations in Financial Models" *The Review of Financial Studies* 1990 Vol. 3, No.3, pp.393-430