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碩 士 論 文

不對稱跳躍與不完全資訊對於信用風險模型之影響

What Are the Impacts of Asymmetric Jumps and Noisy

Information on the Credit Risk Model?

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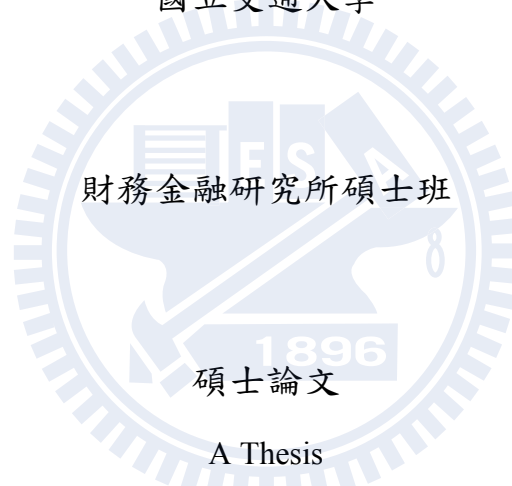
Information on the Credit Risk Model?

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摘要

此篇論文的主旨為在資訊不完全的市場下，建立一個以雙邊指數型跳躍過程為基礎的信用風險模型、推導此模型下之破產機率，並探討不對稱跳躍與不完全資訊對於破產機率、破產強度、以及信用利差之衝擊。研究結果顯示，相較於未考慮跳躍過程之模型，在此含有跳躍過程之一般化模型的假設下，所求得之信用利差的時間結構更為豐富且多元。亦即，此篇論文建構之模型將對於真實市場有較佳之解釋能力。

關鍵字：信用風險；不完全資訊；雙邊指數型跳躍過程；破產機率；信用利差；破產強度；時間結構。

What Are the Impacts of Asymmetric Jumps and Noisy Information on

the Credit Risk Model?

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ABSTRACT

The pure diffusion approach for credit risk model with noisy information is generalized in this paper by including jumps in the firm-value processes. The explicit solution of the default probability for this credit risk model is derived, and the impacts of asymmetric jumps and noisy information on the credit risk are illustrated with numerical results on the default probability, the default intensity, and the credit spread. With the term structure of credit spreads being enriched, our approach is potential to interpret empirical data in real world.

Keywords: credit risk; noisy information; double exponential jump; default probability; default intensity; credit spread; term structure.

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1. Introduction

Default risk analysis plays a critical role in managing the credit risk of bank loan portfolios. In recent research, there are two types of default risk that have been widely used for modeling credit risk. One is the jump risk. For instance, Zhou (1997) and Scherer(2005) incorporates jump risk into the default process and suggests that a firm, the assets of which are perfectly observable, can default instantaneously because of a sudden drop in its value, and the short credit spread of this firm is also bounded away from zero. Another is the default risk due to noisy information. Duffie and Lando(2001) indicates the impacts of the noisy information on the default intensity and the yield spread, and shows that a firm, the asset value of which follows a pure diffusion process but can not be observed directly, can default instantaneously either. However, how do the two types of default risk affect the credit risk model in the same time remains open, and this paper intends to focus on the simultaneous impacts of the asymmetric jumps and noisy information on credit risk.

According to Kou(2002), the double exponential jump-diffusion can capture the empirical phenomenon, which is the asymmetric leptokurtic features that the return distribution is skewed to the left and has a higher peak and two heavier tails than those of pure Brownian motion. With the mean sizes and the probabilities of upward and downward being set respectively, the double exponential jump-diffusion is considered asymmetric and therefore appropriate to capture more realistic jump.

In this paper, a double jump-diffusion process is chosen as the basic model for the asset value of a firm and the noisy information assumption proposed by Duffie and Lando (2001) is incorporated for incomplete information market. The impacts of asymmetric jumps and noisy information on the credit risk are illustrated with numerical results on the default probability, the default intensity, and the credit spread. The result of our analysis shows that the term structure of credit spreads is enriched by simultaneously incorporating the asymmetric jump and the noisy information into the credit risk model. This generalization could be more potential to interpret empirical data in future.

The remainder of this paper is organized as follows. The incorporation of the double exponential jump-process for the asset value of the firm into the credit risk model is given, and then the default probability of the modeled firm is analytically derived in section 2. In section 3, the analysis of the credit risk model is provided for several cases with different arrival rates of jump and information quality to show how asymmetric jumps and noisy information simultaneously influence the default probability, default intensity, and credit spreads. Section 4 concludes this paper.

2. The Basic Model

This section presents our basic model of a firm with noisy information about the credit quality of its debt. We derive the conditional distribution of firm's assets, given noisy information,

along with the associated default probability.

According to Duffie and Lando(2003), after issuance, bond investors are not kept informed of the status of the firm. Although they do understand that optimizing equity owners will force liquidation when assets fall to $\underline{v}$, they can not observe the actual asset level of the company (V_t). Instead, they observe the noisy report of assets ($\hat{V}_t$). We suppose that the value of a company follow a stochastic process $V = \{V_t\}_{t \geq 0}$, and $Z_t = log\left(\dfrac{\hat{V}_t}{V_0}\right) = X_t + U_t$, where U_t follows a normal distribution with mean $\bar{\mu}$ and variance a^2 and is independent of X_t.

The return process $X_t = log\left(\dfrac{V_t}{V_0}\right)$ is supposed to follow a double exponential jump-diffusion. Under the physical probability measure P, the process is given by:

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i , \quad X_0 = 0 \tag{1}$$

where W_t is a standard Brownian motion, N_t is a Poisson process with rate λ, and $\{Y_i\}$ is a sequence of independent identically distributed (i.i.d.) nonnegative random variables that has an asymmetric double exponential distribution with the density $f_Y(y) = p\eta_1 e^{-\eta_1 y} I_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} I_{\{y < 0\}}$. The constants $\eta_1 > 1, \eta_2 > 0; p, q \geq 0, p + q = 1$. All randomness, N_t, W_t, and Y_t are assumed to be independent; the drift μ and the volatility σ are assumed to be constant, and the Brownian motion and jumps are assumed to be one dimensional. Furthermore, the moment generating function of X_t is $\phi(\theta, t) := E\left[e^{\theta X_t}\right] = exp\{G(\theta)t\}$, where the function $G(\cdot)$ is defined as

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$$G(\beta) := \beta\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\beta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \beta} + \frac{q\eta_2}{\eta_2 + \beta} - 1\right). \tag{2}$$

The information filtration $\left(\mathscr{H}_t\right)$ available to the secondary market is defined

by $\mathscr{H}_t = \sigma\left(\{Z(t_1),\ldots,Z(t_n), I_{\{\tau \le s\}} : 0 \le s \le t\}\right)$, for the largest n such that $t_n \le t$, where

$\tau = \tau(\underline{v}) := \inf\{t : V_t \le \underline{v}\}$, the first time that the asset falls below $\underline{v}$.

The main objective in this section is to compute the default probability. Firstly, define

$\psi(x_0, x, t)$ as a probability of event that $\min_{0 \le s \le t}\{X_s : 0 \le s \le t\} > 0$, conditional on $V_0 = v_0$ and

$X_t = x$; that is

$$\psi(x_0, x, t) = P\left\{\min_{0 \le s \le t} X_s > 0 \,\Big|\, X_0 = x_0, X_t = x > 0\right\}, \qquad V_0 = v_0$$

$$= \frac{P\left\{\max_{0 \le s \le t} X_s^* < 0, X_t^* \in dx^*\right\}}{P\left\{X_t^* \in dx^*\right\}} \tag{3}$$

where $X_s^* = -X_s$. The distribution of X_t^* and the joint distribution of the process X_t^* and

its running maxima $\max_{0 \le s \le t} X_s^*$, used in (3) can be computed by the differentiations of the

cumulative distribution function of X_t^* and the joint cumulative distribution of X_t^* and

$\max_{0 \le s \le t} X_s^*$ respectively. The cumulative distribution function of X_t^* and the joint cumulative

distribution of X_t^* and $\max_{0 \le s \le t} X_s^*$ are given by

$$P\left\{X_t^* \ge x^*\right\} = \lim_{mn \to \infty} \sum_{n=0}^{mn} \pi_n P\left(\theta't + \sigma'W_t + \sum_{j=1}^{n} Y_j' \ge x^*\right)$$

$$= \pi_0 \Phi\left(-\frac{x^* - \theta't}{\sigma'\sqrt{t}}\right) + L(\lambda, p', q', \eta_1', \eta_2') + M(\lambda, p', q', \eta_1', \eta_2') \tag{4}$$

and

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$$P\left\{\max_{0\le s\le t} X_s^* < 0, X_t^* < x^*\right\} = P\left\{X_t^* < x^*\right\} - P\left\{\max_{0\le s\le t} X_s^* > 0, X_t^* < x^*\right\}$$

$$= 1 - P\left\{X_t^* \ge x^*\right\} - P\left\{\max_{0\le s\le t} X_s^* > 0\right\} + P\left\{\max_{0\le s\le t} X_s^* > 0, X_t^* \ge x^*\right\} \tag{5}$$

where $\theta' = -\left(\mu - \dfrac{\sigma^2}{2}\right)$, $\sigma' = -\sigma$, $Y' = -Y$ which have the distribution as

$f_{Y'}(y') = p'\eta_1' e^{\eta_1' y'} I_{\{y'>0\}} + q'\eta_2' e^{-\eta_2' y'} I_{\{y'\le 0\}}$ where $\eta_1' = \eta_2 > 0, \eta_2' = \eta_1 > 1; p' = q, q' = p$, and

π_n , $L(\lambda, p', q', \eta_1', \eta_2')$, and $M(\lambda, p', q', \eta_1', \eta_2')$ are provided in appendix A. $P\left\{\max\limits_{0\le s\le t} X_s^* > b\right\}$

and $P\left\{\max\limits_{0\le s\le t} X_s^* > b, X_t^* \ge x\right\}$ can be computed explicitly from their Laplace transforms. The

Laplace transforms of the two cumulative distributions can be written as

$$\int_0^\infty e^{-\alpha t} P\left\{\max_{0\le s\le t} X_s^* > b\right\} dt = \frac{1}{\alpha}\left(\frac{\eta_1' - \beta_{1,\alpha}}{\eta_1'} \frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1'}{\eta_1'} \frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}\right), \tag{6}$$

and

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} P\left\{\max_{0\le s\le t} X_s^* > b, X_t^* \ge x\right\} dt &= \big(A(\lambda, p, q, \eta_1, \eta_2, b) + B(\lambda, p, q, \eta_1, \eta_2, b)\big) C(\lambda, b, x) \\
&\quad + e^{h\sigma'\eta_1'} \big(D(\lambda, p, q, \eta_1, \eta_2, b, x) + E(\lambda, p, q, \eta_1, \eta_2, b, x)\big) \\
&\quad + e^{h\sigma'\eta_1'} B(\lambda, p, q, \eta_1, \eta_2, b) H_0(h, \gamma_\alpha, c_+; 0) \\
&\quad - e^{-h\sigma'\eta_2'} F(\lambda, p, q, \eta_1, \eta_2, b, x)
\end{aligned} \tag{7}$$

where $\beta_{1,\alpha}, \beta_{2,\alpha}$ are the only two positive roots of the equation $\alpha = G(\beta)$ defined in (2), and

A, B, C, D, E, F are given in appendix A. Since the cumulative distributions of the first

passage times are given in terms of Laplace transforms, numerical inversion of Laplace

transforms becomes necessary. To do this, the Gaver-Stehfest algorithm, which does the

inversion on the real line, is used in this paper. For any bounded and continuous function $f(\cdot)$

defined on $[0,\infty)$, $f(t) = \lim\limits_{n\to\infty} \widetilde{f}_n(t)$,

where

$$\widetilde{f}_n(t) = \lim_{n \to \infty} \frac{\ln(2)}{t} \sum_{k=1}^{n} \left\{ (-1)^{k+\frac{n}{2}} \left[\sum_{j=\left\lfloor \frac{1}{2}(k+1) \right\rfloor}^{min(k,\frac{n}{2})} \frac{j^{\frac{n}{2}}(2j)!}{\left(\frac{n}{2}-j\right)!(j)!(j-1)!(k-j)!(2j-k)!} \right] \hat{f}\left(\frac{k\,ln(2)}{t}\right) \right\} \quad (8)$$

and $\widehat{f}$ is the Laplace transform of f, i.e. $\widetilde{f}(\alpha) = \int_0^\infty e^{-\alpha t} f(t)dt$.

Next, the density $b(\cdot|Z_t, x_0, t)$ of X_t, given $V_0 = v_0$ and "killed" at $\tau = inf\{t : X_t \le \underline{x}\}$

where $\underline{x}$ is set to be $log\left(\frac{\underline{v}}{v_0}\right)$, conditional on the observation $Z_t = X_t + U_t$, can be

calculated. By definition,

$$b(x|z, x_0, t)dx = P\{\tau > t, X_t \in dx | Z_t \in dz\}, \quad x \ge \underline{x} \text{ and } V_0 = v_0$$

$$= \frac{\psi(-\underline{x}, x - \underline{x}, t)P\{X_t \in dx\}\varphi_U(z - x)}{P\{Z_t \in dz\}} \quad (9)$$

where φ_U denotes the density of U_t and

$$P\{Z_t \ge z\} = \sum_{n=0}^{\infty} \pi_n P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + U_t + \sum_{j=1}^{n} Y_j \ge z \right)$$

$$= \pi_0 \Phi\left(-\frac{z - \left(\mu - \frac{\sigma^2}{2}\right)t - \overline{\mu}}{\overline{\sigma}} \right)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_1)^2 + \overline{\mu}\eta_1}}{\overline{\sigma}\sqrt{2\pi}} L'(\lambda, p, q, \eta_1, \eta_2) \quad (10)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_2)^2 - \overline{\mu}\eta_2}}{\overline{\sigma}\sqrt{2\pi}} M'(\lambda, p, q, \eta_1, \eta_2)$$

where $\overline{\sigma}^2 = \sigma^2 t + a^2$. $L'(\lambda, p, q, \eta_1, \eta_2)$, $M'(\lambda, p, q, \eta_1, \eta_2)$, and the proof of (12) are given in

appendix B.

Thirdly, according to the result of Duffie and Lando (2001), the density $g(\cdot|Z_t, x_0, t)$ of

X_t, conditional on the noisy observation Z_t and on $\tau > t$, is given by

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$$g(x|z,x_0,t) = \frac{b(x|z,x_0,t)dx}{\int_{\underline{v}}^{\infty} b(x|z,x_0,t)dx} . \tag{11}$$

Finally, the $\mathcal{H}_t$–conditional default probability is given by

$$p\{\tau < s | \tau > t\} = \int_{\underline{v}}^{\infty} \pi(s-t,x-\underline{x})g(x|z,x_0,t)dx \tag{12}$$

where

$$\pi(s-t,x-\underline{x}) = P\left\{\min_{t \le h \le s} X_h < \underline{x} | X_t = x\right\} = P\left\{\max_{t \le h \le s} X_h^* > 0 | X_0^* = 0\right\}, \quad V_0 = v_0 \tag{13}$$

where $X_s^* = -X_s, \ X_0^* = 0$.

3. An Analysis of the Credit Risk Model

In order to analyze how the combination of the noisy information and the asymmetric jump brings the impacts on the credit risk model, we provide several cases for the default probability, the default intensity, and the credit spread with different arrival rates of jump (λ) and standard deviations of noise (a). In these case, the initial asset level given one year ago V_0=86.3 and the default boundary $\underline{v} = 78$. We also suppose in all case that U_t has expectation $\bar{\mu} = -\frac{a^2}{2}$, so that $E(e^{U_t}) = 1$ implies an unbiased asset report. For the jump-diffusion process, r=0.05, $\sigma = 0.05$, $\frac{1}{\eta_1} = 0.02$, and $\frac{1}{\eta_{12}} = 0.03$. Our basic case is $a = 10\%$. All parameters refer to Duffie and Lando(2001) and Kou and Wang(2003).

3.1 Default Probability

This section presents the numerical illustration for the default probability with varying

standard deviation of noise and different jumps to analyze the cooperative impacts of the asymmetric jump and noisy information on the default probability.

Table 1a and 1b compare the explicit solution computed by (14) with the Monte Carlo simulation in the two cases of $\lambda = 0.01$ and $\lambda = 3$. The simulation is based on 1,000,000 simulation runs.

Figure 1 shows the probability of first passage of a double exponential jump diffusion from an initial condition $V_t = v > 0$ given v_0 to a level below $\underline{v}$ before time s. In order to emphasize the result affected by the jump frequency and the probability of upward jump, the case of $\lambda = 3$ with symmetric jump is compared with the case of $\lambda = 0.01$ with symmetric jump, which is near to without jump and the case of $\lambda = 3$ with asymmetric jump ($p=0.7$). The previous year's asset level is set to be 86.3 arbitrarily. As can be seen from Figure 1, given a jump-condition, the probability of first passage time decreases in the actual asset level v at time t, which is in line with our intuition. Furthermore, given the same actual asset level v at time t, the probability of first passage time increases in jump frequency but decreases in the probability of upward jump, which is also in line with our intuition.

Figure 2a, Figure 2b, and Figure 2c show the conditional density of the current asset level V_t, given $\mathscr{H}_t$, that would be realized in the event that the bond has not yet default and the current asset level reported $\hat{V}_t$ has a outcome equal to the previous year's report. There are three cases of different jump-conditions considered here. One is $\lambda = 0.01$, $p=0.5$ (shown

in Figure 2a); another is $\lambda=3$, *p=0.5* (shown in Figure 2b); still another is $\lambda=3$, *p=0.7*

(shown in Figure 2c). And the standard deviation of noise is set as 5% and 40% for all cases.

The previous year's asset level is set to be 86.3 arbitrarily. It can be shown that in all cases,

the density becomes flattened as the standard deviation of noise (*a*) increases, and the peak

density is shifted to the right with jump, p=0.7, which is asymmetric and favoring upward. It

is worthy to note that the amount of this shift due to asymmetric jump is magnified by the

dubious level of information. It seems that the asymmetric jump cooperates with the

information noise and creates a composite effect on the conditional density.

Figure 3 illustrates outcomes of the conditional default probability for cases of different

jumps and various levels *a* of noise. One is for $\lambda=0.01$ and *p*=0.5; another $\lambda=3$ and *p*=0.5;

and the other $\lambda=3$ and *p*=0.7, given the same previous year asset level of 86.3. For the case

of $\lambda=0.01$, representing a jump of once per hundred year, which is considered rare to

happen, the curve we obtained is quite close to that obtained by Duffie and Lando (2001). It

means that our generalized model can be reduced to Duffie and Lando's model. The default

probability will increase first and then converge to a certain level as the standard deviation of

noise increases. This is because the noise affection will eventually saturate if the standard

deviation of noise is large enough. For $\lambda=3$ and *p*=0.5, the whole curve of the default

probability is raised by the jump, and the point probability of the curve increases first as in the

case of $\lambda=0.01$, reaches the peak value, and then decreases to converge to a saturation level

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as the information become more and more dubious. For the case of $\lambda = 3$ and p=0.7, the whole curve of the default probability is lowered with respect to the case of $\lambda = 3$ and p=0.5, because the asymmetric jump favors upward by assumption; there is also a decrease-to-saturation phenomenon in this case of asymmetric jump. The decrease-to-saturation phenomenon is the result of cooperation of the jump and noisy information. As shown in the equation (12), the default probability is equal to the inner product of the probability of first passage time $\pi(s-t, x-\underline{x})$ and the conditional probability $g(x|z, x_0, t)dx$. From Figure 2a-2c, it can be seen that noisy information makes the conditional density more flattened. At the lower and higher actual asset level, the conditional density increases in the standard deviation of noise, which generates an affection that increases the default probability, whereas the conditional density decreases in the standard deviation of noise at the middle actual asset level, which generates a contrary affection that decreases the default probability. The default probability is determined by these two contrary powers. Because the probability of first passage time decreases and converges to zero more slowly, the affection that decreases the default probability can be displayed more obviously in the case of $\lambda = 3$ than in the case of $\lambda = 0.01$, which result in the decrease-to-saturation phenomenon.

3.2 Default Intensity

We turn now to the implications of asymmetric jumps and noisy information for the default

intensity of the modeled firm.

By the definition, the default intensity is a local default rate, in that

$$\Lambda_t = \lim_{h \to 0} \frac{p\big(t \in (t, t+h)\big|\mathscr{H}_t\big)}{h}. \tag{14}$$

Figure 4a illustrates the different between the default intensity of the modeled firm, the asset

value of which has an asymmetric jump, with and without perfect information; besides, the

special case of one-sided jump is also given. From this figure, we have three findings about

the default intensity. First, the intensity with noise (a=10%) is less sensitive with reported

asset level than the intensity without noise (a=0) as long as the jump is considered in the asset

process. However, in the case that the value of asset follows a pure diffusion (λ=0), the

intensity with noise is always greater than the intensity without noise, which is zero at any

reported asset level (Duffie and Singleton 2001). Second, if only the upward jump is included,

the intensity is close to zero regardless of information quality. Third, at the same probability

of upward jump, the intensity increases over the whole level of the reported asset as the

arrival rate of jump increases, as illustrated in Figure 4b, whereas the intensity decreases for

the low reported asset level but increase for the high reported asset level as information

become more dubious, as illustrated in Figure 4a. It is suggested that there is a neutral point

where the intensity is not affected.

From Figure 4c, it may be concluded that the arrival rate of jump creates greater impact

on the intensity as the standard deviation of noise increases (*a*) from 0 to 25% at the reported

asset level (86) higher than the neutral point. In contrast, from Figure 4d, at the reported asset

level (80) lower than the neutral point, the impact of jump initially increases as *a* increases

from 0 to 5% and then decrease as *a* increases from 5% to 25%. In addition, the impact of

noise on intensities will be saturated as the standard deviation of noise is large enough for

both higher and lower reported asset level.

3.3 Credit Spreads

According to Kou and Wang(2004), under the risk-neutral probability measure P^*, the return

process given by

$$X_t = \left(r - \frac{1}{2}\sigma^2 - \lambda^*\xi^* \right)t + \sigma W_t^* + \sum_{i=1}^{N_t^*} Y_i^*, \quad X_0 = 0. \tag{15}$$

Where W_t^* is a standard Brownian motion under P^*, $\{N_t^*; t \geq 0\}$ is a Poisson process with

intensity λ^*. The log jump size $\{Y_1^*, Y_2^*...\}$ still from a sequence of i.i.d. random variables

with a new double exponential density $f_{Y^*}(y) = p^*\eta_1^* e^{-\eta_1^* y} I_{\{y \geq 0\}} + q^*\eta_2^* e^{\eta_2^* y} I_{\{y < 0\}}$ The constants

$\eta_1^* \geq 1, \eta_2^* > 0; p^*, q^* \geq 0, p^* + q^* = 1$, and $\xi^* := E^*[e^{Y^*}] - 1 = \frac{p^*\eta_1^*}{\eta_1^* - 1} + \frac{q^*\eta_2^*}{\eta_2^* + 1} - 1$. All sources of

randomness, N_t^*, W_t^*, and Y^*s, are still independent under P^*.

For a given time to maturity *T*, the yield spread on a given zero-coupon bond sold at a

price $\varphi > 0$ is the real number ζ such that $\varphi_{t,T} = e^{-(r+\zeta)(T-t)}$. We assume the bond holder

can receive some fraction $R \in [0,1]$ of its market value V_t at default. According to Duffie

and Sinleton(1999), the secondary market price $\varphi(t, T)$ of the bond in the event of no default

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at time *t* is given by

$$\varphi_{t,T} = \mathrm{e}^{-(\mathrm{r}+\zeta)(T-t)} = \left(1 - h_t\right)e^{-r\Delta t}E_t^{P^*}\left(\varphi_{t+\Delta t,T}\right) + h_t e^{-r\Delta t}RE_t^{P^*}\left(\varphi_{t+\Delta t,T}\right) \tag{16}$$

where h_s is the conditional probability at time *s* under a risk-neutral probability measure P^*

of default between *s* and *s+1*, given the information available at time *s* in the event of no

default by *s*. Besides, *r* is the risk-free rate.

From Figure 5, it can be shown how the asymmetric jumps and noisy information create

impacts on the term structure of credit spreads. First of all, noisy information tends to reshape

the term structure from a hump to a monotone; in other words, the short spread will ascend

and the long spread will descend while noise increases. This phenomenon results from the fact

that the uncertainty causes the default probability having a more moderate variation with

maturity, as illustrated in Figure 6. Secondly, as the arrival rate of jump increases, the term

structure is still a humped-like curve, but the spread will be ascended over the whole range of

maturity. Finally, the term structure of credit spreads is monotone if the both the asymmetric

jump and the noisy information are simultaneously considered and large enough. The term

structure of credit spreads therefore can be more enriched by including jumps and noise.

4   Conclusion

The pure diffusion approach for structural model with noisy information is generalized in this

paper by including jumps in the firm-value processes. The explicit solution of the default

probability for this structural model is given, and numerical results show how the combination of the noisy information and the asymmetric jump brings the impacts on the credit risk.

Firstly, the opposite ways of impact of noise on the default intensity depends on whether the lower or higher range of the reported asset level is considered, and the noisy information will hence make the intensity with jump less sensitive with the reported asset level. Secondly, for the term structure of credit spread, the noisy information tends to reshape the term structure from a hump-like curve to a more flattened one; however, as the arrival rate of jump increases, the term structure is still hump-like, but the spread will be ascended over the whole range of maturity. Furthermore, it is found that the term structure of credit spreads is monotone if the two factors are simultaneously considered and large enough.

In short, the term structure of credit spreads is enriched by simultaneously incorporating the asymmetric jump and the noisy information into our credit risk model, and this generalization could be more potential to interpret empirical data in real world where rare events beyond the realm of normal expectations happen.

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Table 1a: Comparison between Monte Carlo simulation and explicit solution of default

probability. (λ=0.01). The simulation is based on 1,000,000 simulation runs.

		5%	10%	25%	40%
0.25 year	Formula	0.0086	0.0148	0.0170	0.0173
	Simulation	0.0085	0.0153	0.0174	0.0178
	Error (%)	1.176	3.27	2.30	2.81
0.5 year	Formula	0.0235	0.0323	0.0352	0.0356
	Simulation	0.0234	0.0323	0.0357	0.0355
	Error (%)	0.427	0	1.40	0.282
1 year	Formula	0.0591	0.0681	0.0706	0.0709
	Simulation	0.0580	0.0682	0.0706	0.0708
	Error (%)	1.90	0.147	0	0.141

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 0.01; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; p = 0.5; V_0 = 86.3; \hat{V}_t = 86.3 .$$

Table 1b: Comparison between Monte Carlo simulation and explicit solution of default

probability. (λ=3). The simulation is based on 1,000,000 simulation runs.

		5%	10%	25%	40%
0.25 year	Formula	0.0491	0.0584	0.0601	0.0602
	Simulation	0.0483	0.0574	0.0598	0.0592
	Error (%)	1.66	1.74	0.50	1.69
0.5 year	Formula	0.1053	0.1131	0.1127	0.1124
	Simulation	0.1027	0.1109	0.1091	0.1083
	Error (%)	2.53	1.98	3.30	3.79
1 year	Formula	0.2064	0.2061	0.2005	0.1994
	Simulation	0.2033	0.2028	0.1958	0.1963
	Error (%)	1.52	1.63	2.40	1.58

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 3; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; p = 0.5; V_0 = 86.3; \hat{V}_t = 86.3 .$$

Figure 1: Probability of first passage time, for different arrival rate of jump and the probability of upward jump.
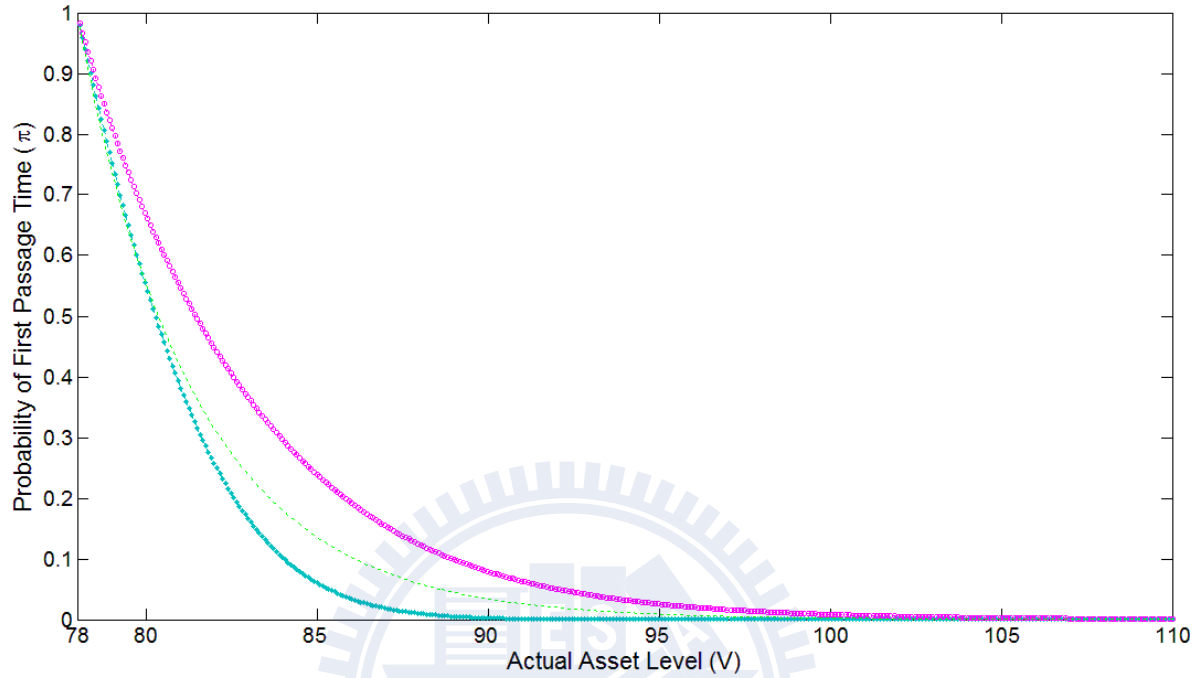
Parameter setting: $\mu - \dfrac{\sigma^2}{2} = 0.01; \sigma = 0.05; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$.
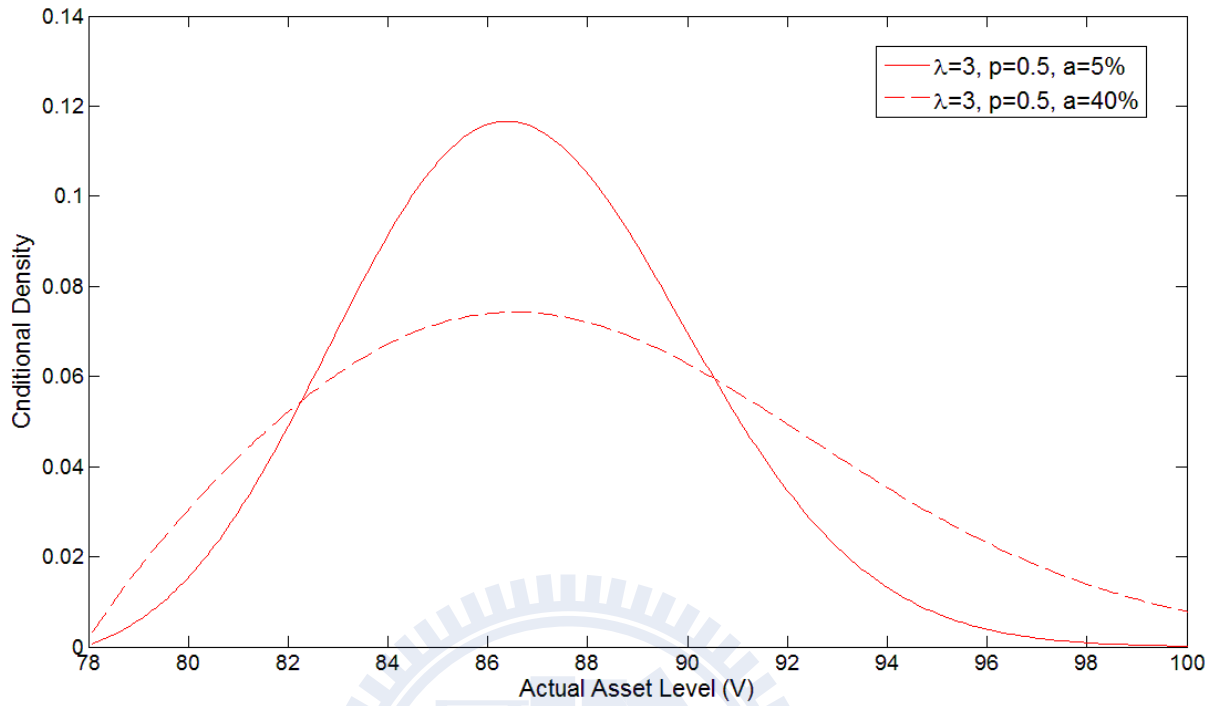
Figure 2a: Conditional density of V_t, for different standard deviation of noise. ($\lambda = 0.01$,

p=0.5)
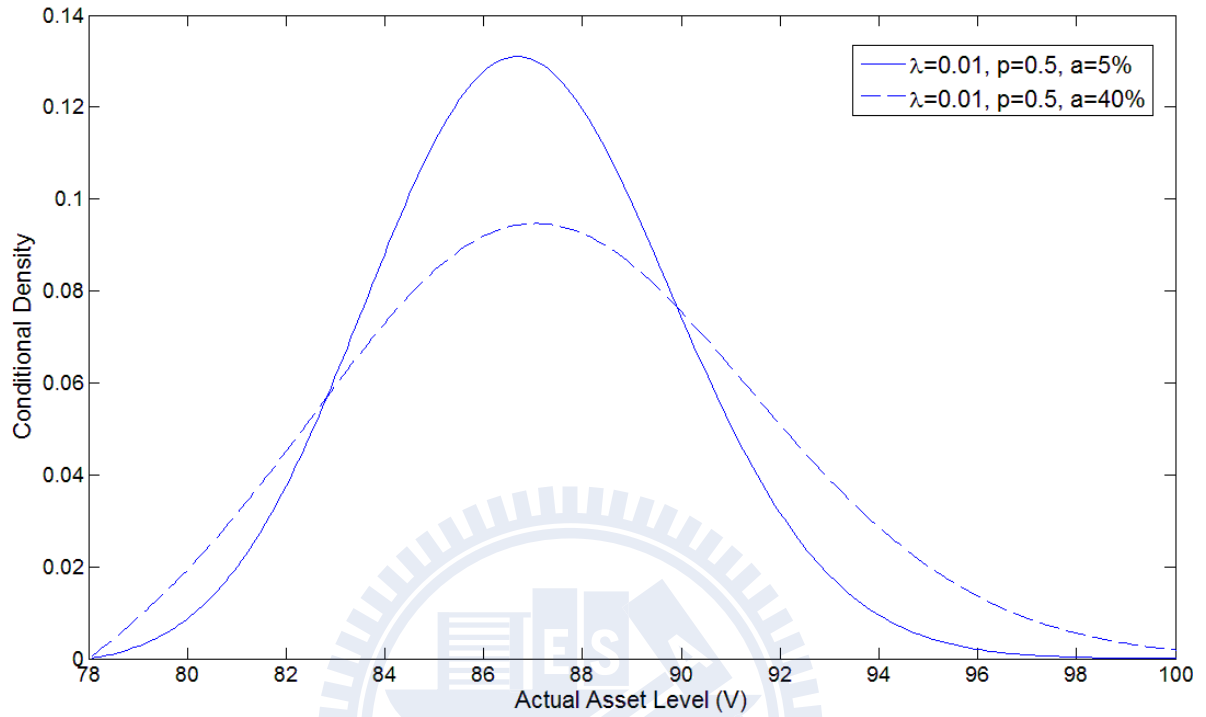
Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 0.01; p = 0.5; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$$
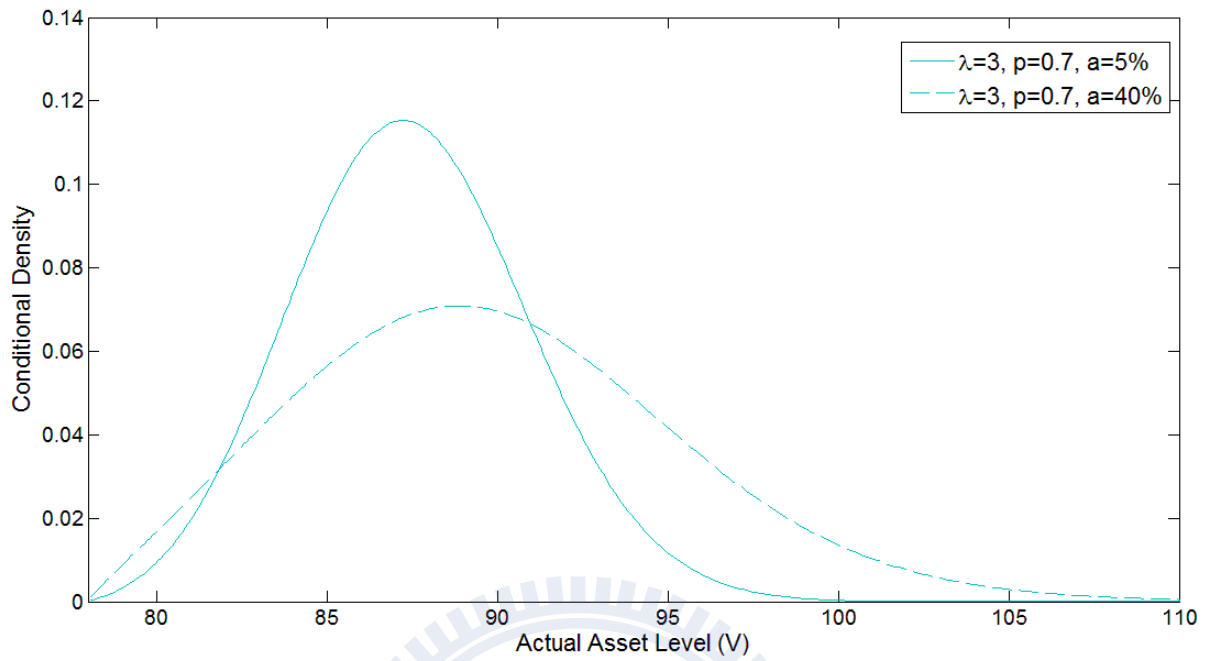
Figure 2b: Conditional density of V_t, for different standard deviation of noise. (λ=3, p=0.5)

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 3; p = 0.5; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$$
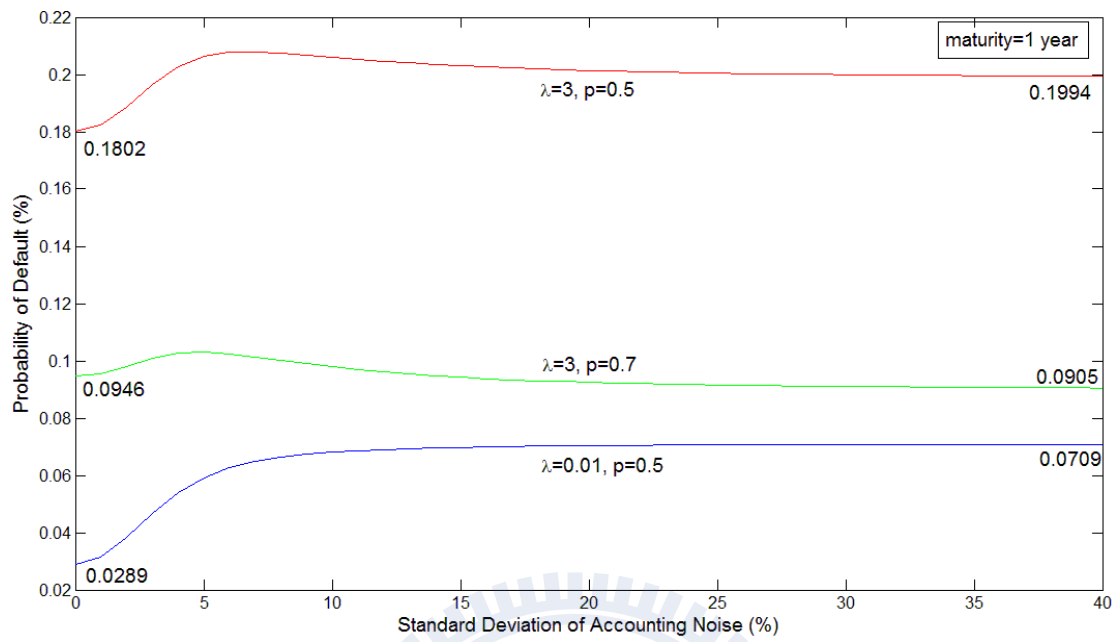
Figure 2c: Conditional density of V_t, for different standard deviation of noise. (λ =3, p=0.7)

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 3; p = 0.7; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3 \,.$$

Figure 3: Default probability for time horizon 1 year, varying standard deviation of noise.

Parameter setting: $\mu - \dfrac{\sigma^2}{2} = 0.01; \sigma = 0.05; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$.
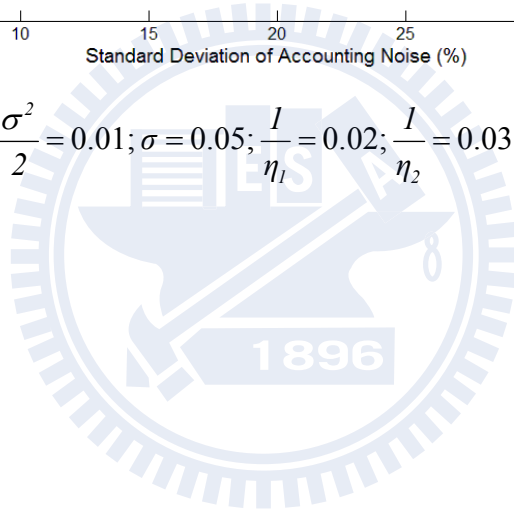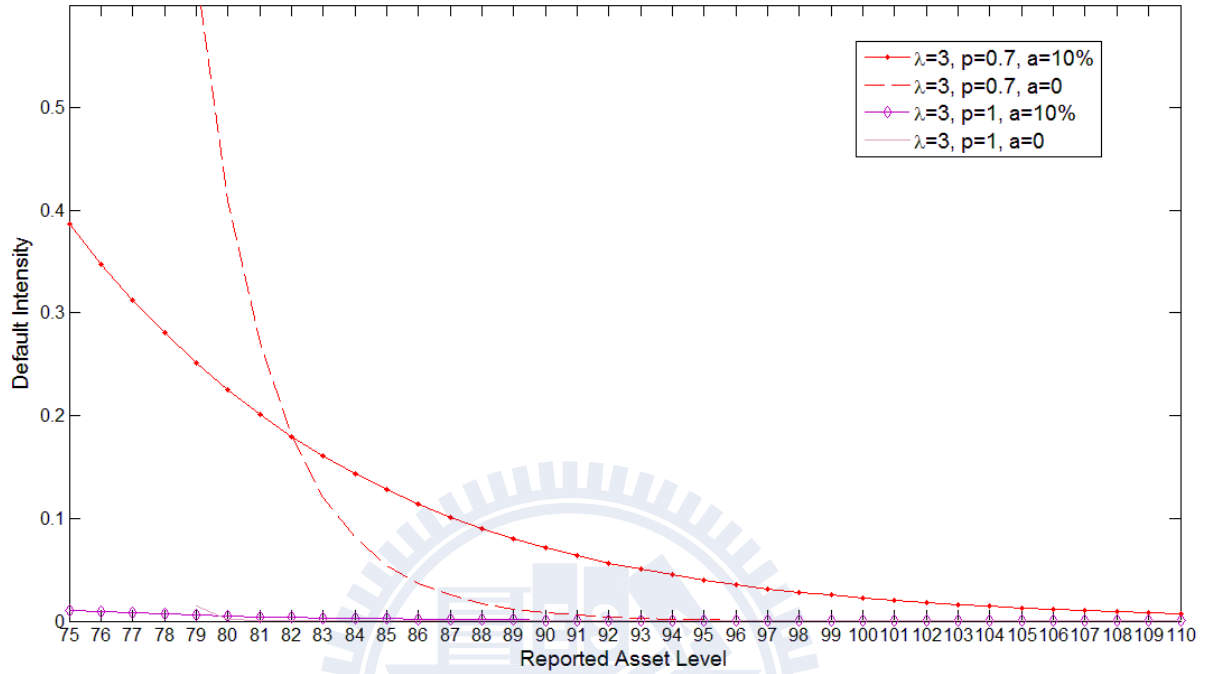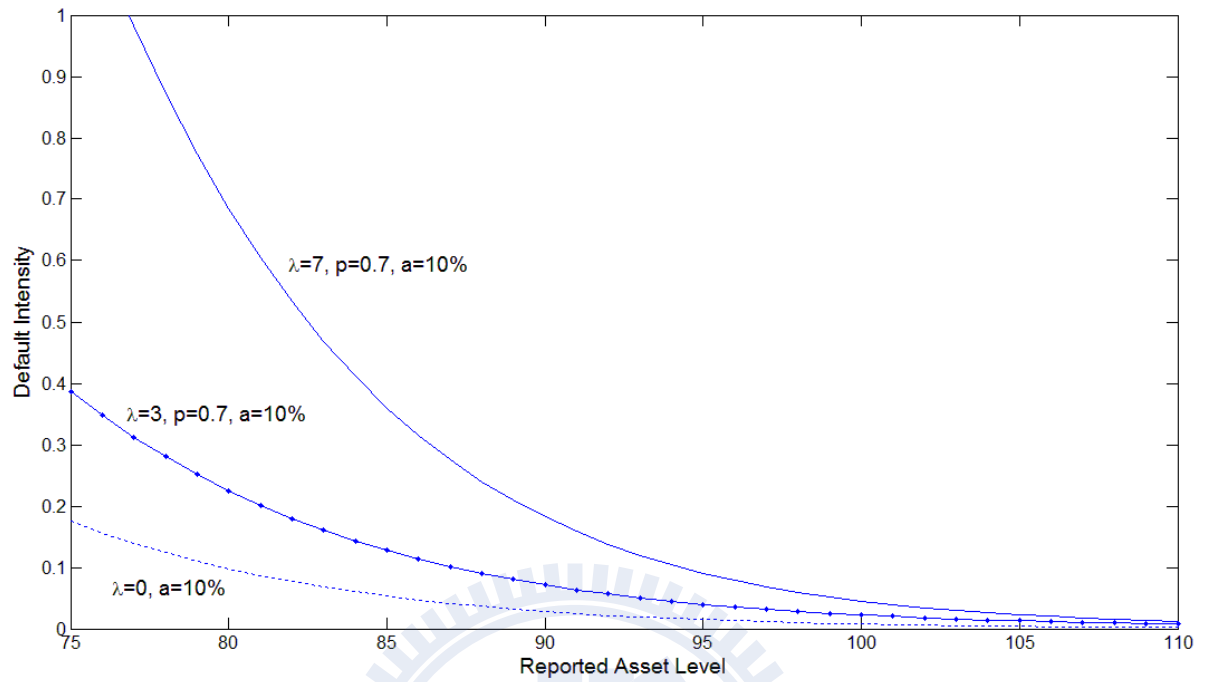
Figure 4a: Default intensity, for different probabilities of upward jump, with noiseless and noisy information.

Parameter setting: $\mu - \dfrac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 3; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V_t} = 86.3$
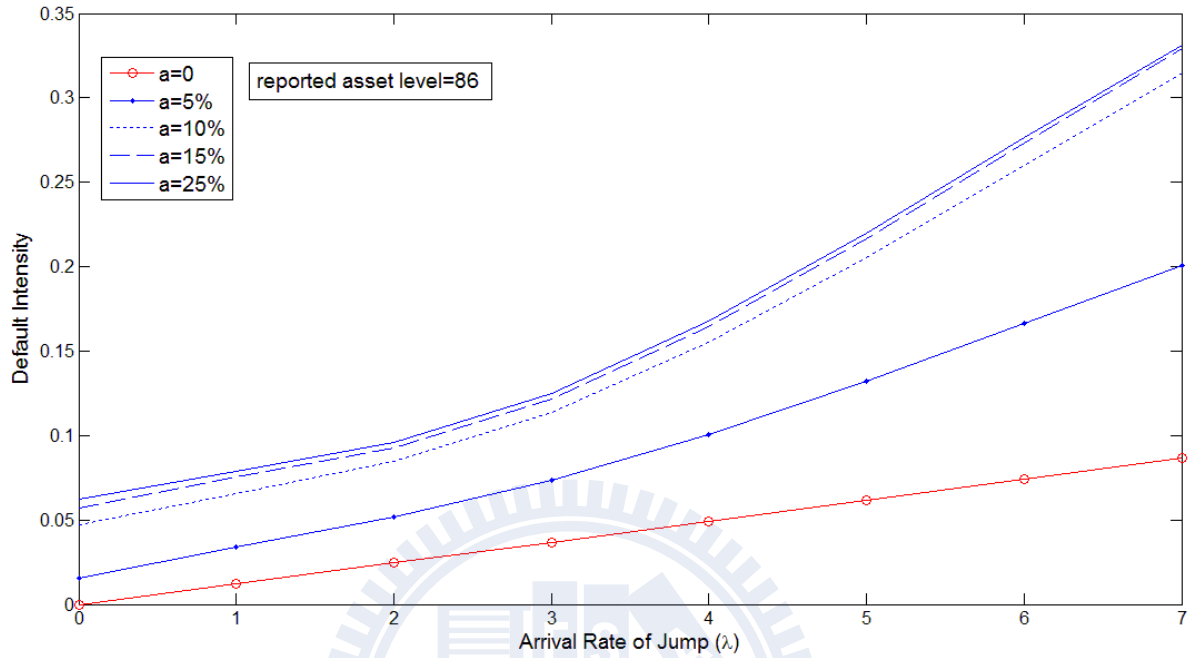
Figure 4b: Default intensity, for different arrival rates of jump, with noisy information.

Parameter setting: $\mu - \dfrac{\sigma^2}{2} = 0.01; \sigma = 0.05; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$

Figure 4c: Default intensity, for different arrival rates of jump and varying standard deviation

of noise. ($\hat{V}_t = 86$)

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01;\ \sigma = 0.05;\ \lambda = 3;\ p = 0.7;\ \frac{1}{\eta_1} = 0.02;\ \frac{1}{\eta_2} = 0.03;\ V_0 = 86.3;\ \hat{V}_t = 86 .$$

Figure 4d: Default intensity, for different arrival rates of jump and varying standard deviation
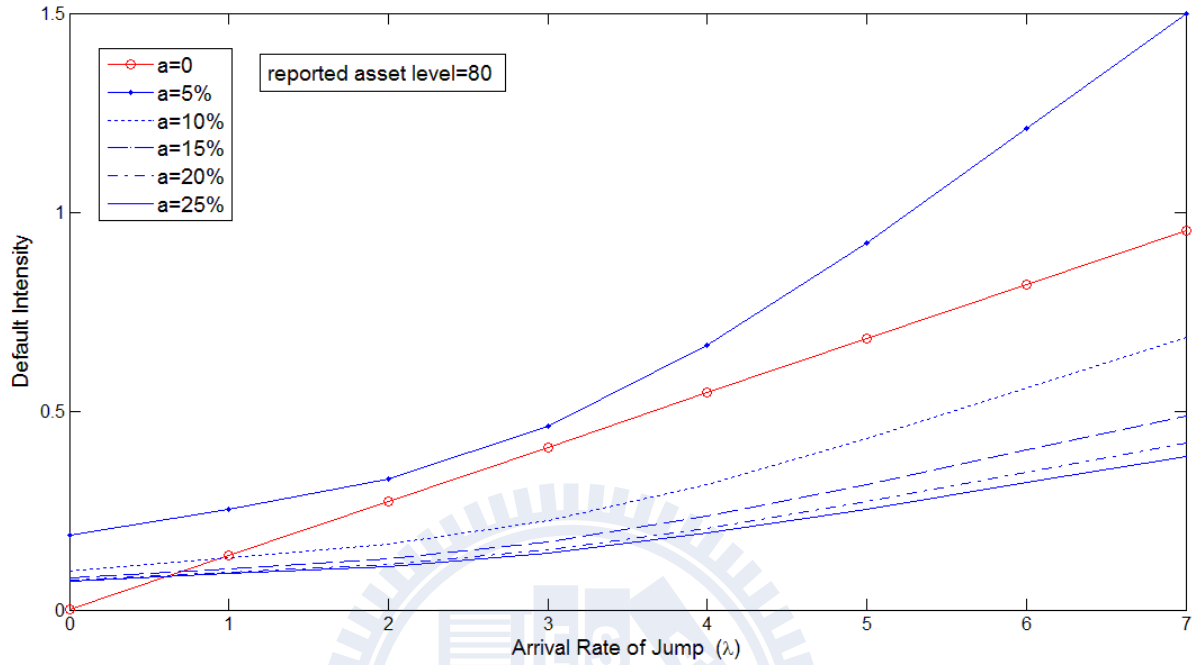
of noise. ($\hat{V}_t = 80$)

Parameter setting:

$$\mu - \frac{\sigma^2}{2} = 0.01; \sigma = 0.05; \lambda = 3; p = 0.7; \frac{1}{\eta_1} = 0.02; \frac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 80.$$

Figure 5: Credit Spread, for different arrival rates of jump, with noiseless and noisy information.
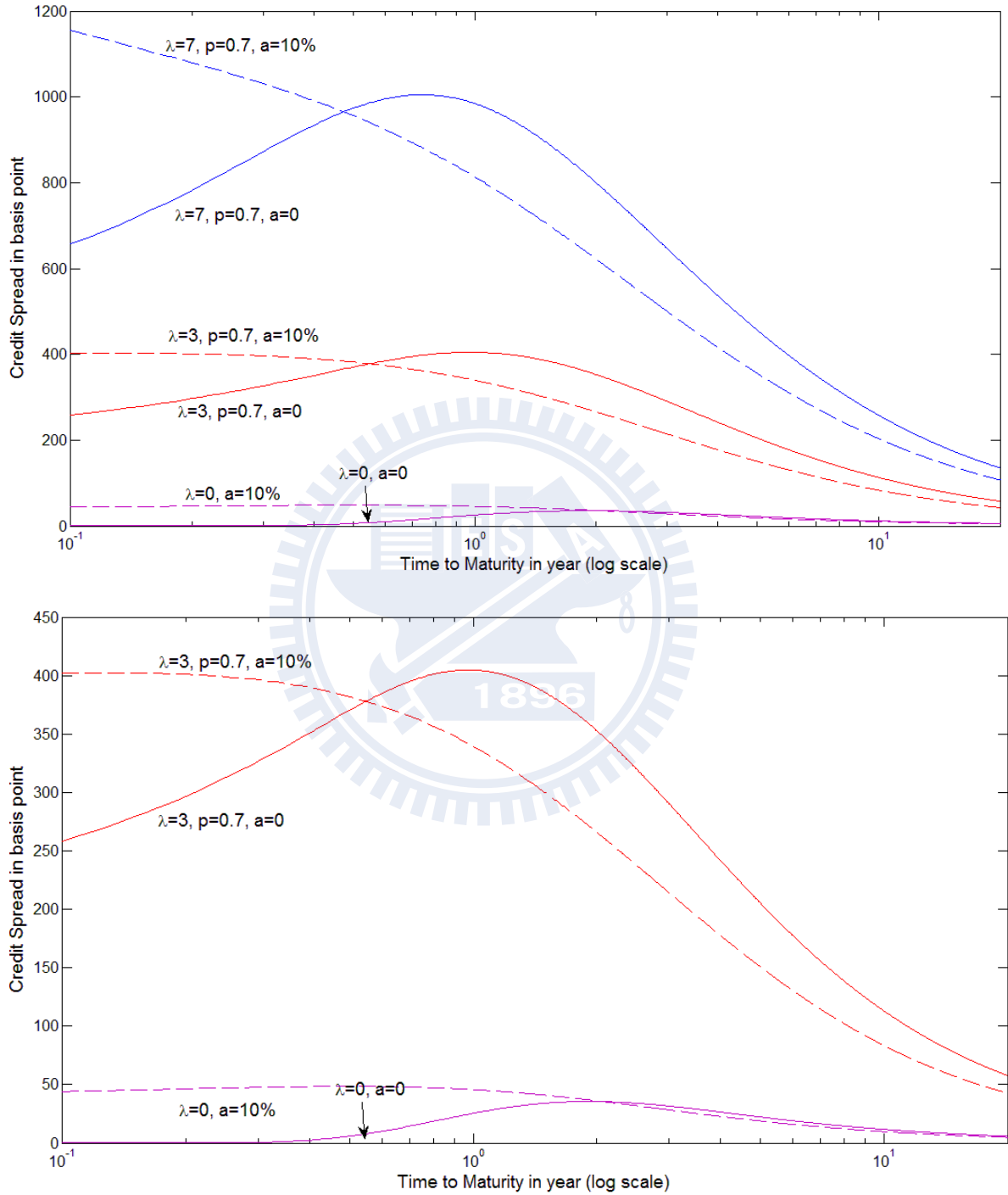
Parameter setting: $r = 0.05; \sigma = 0.05; p = 0.7; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$

Figure 6: The default probability, with noiseless and noisy information.
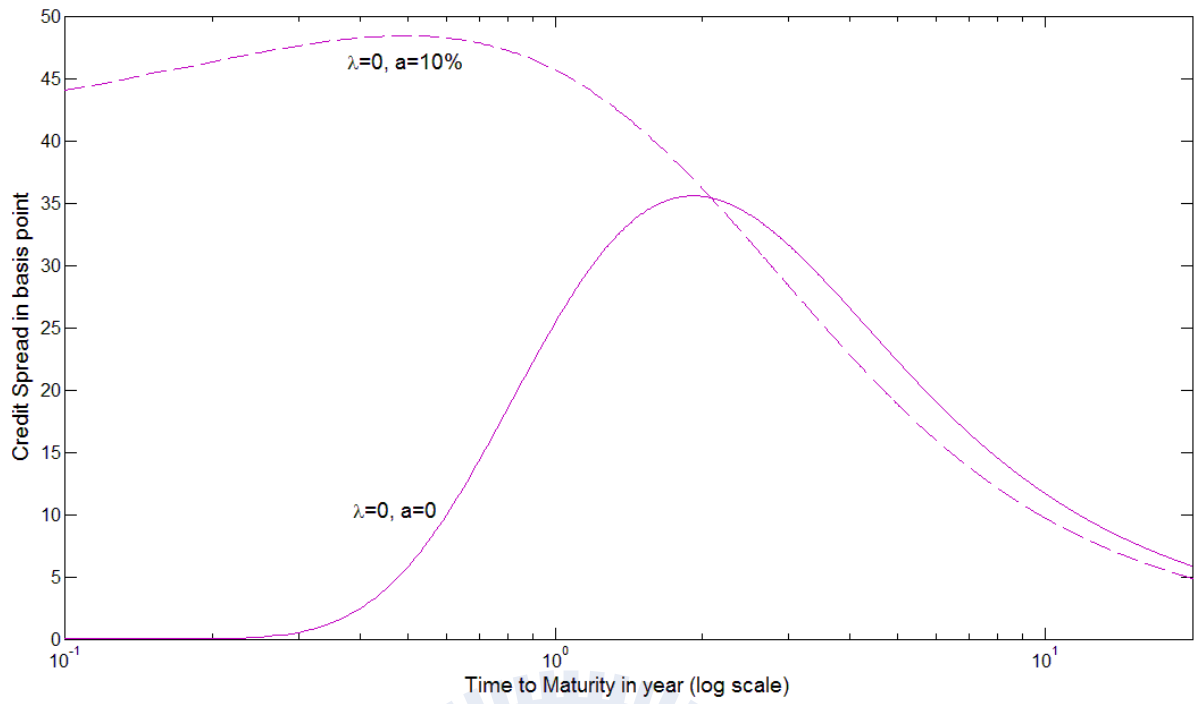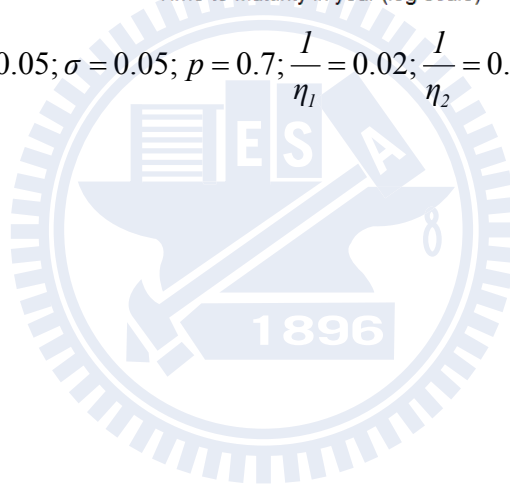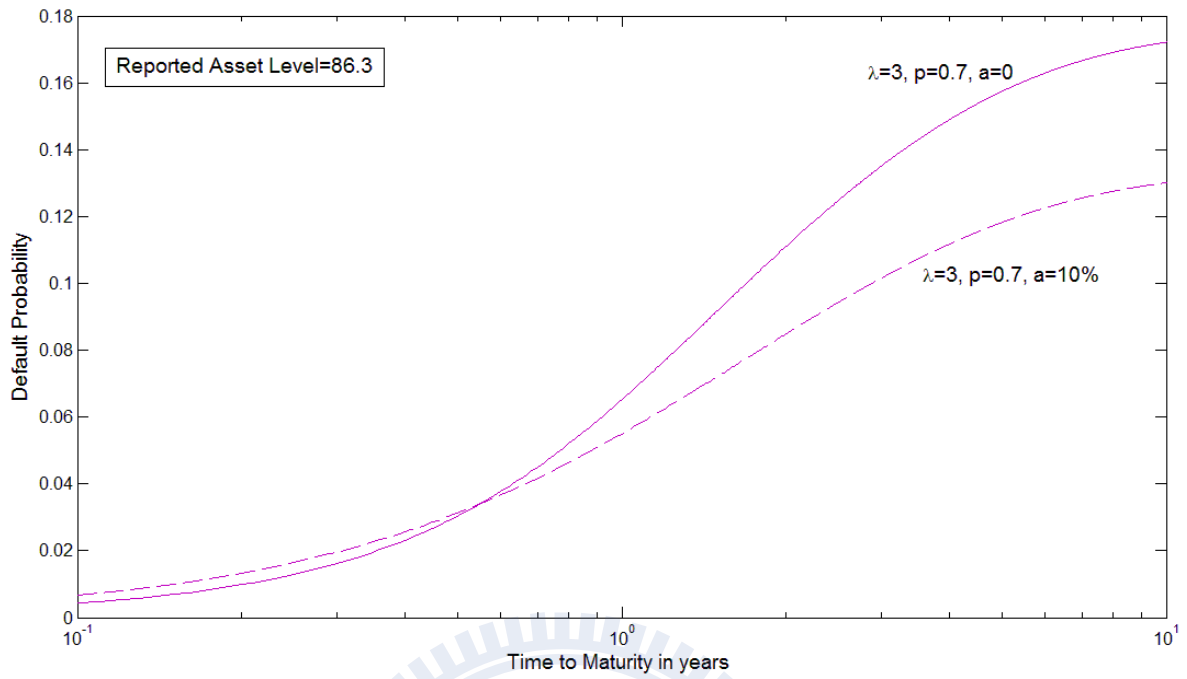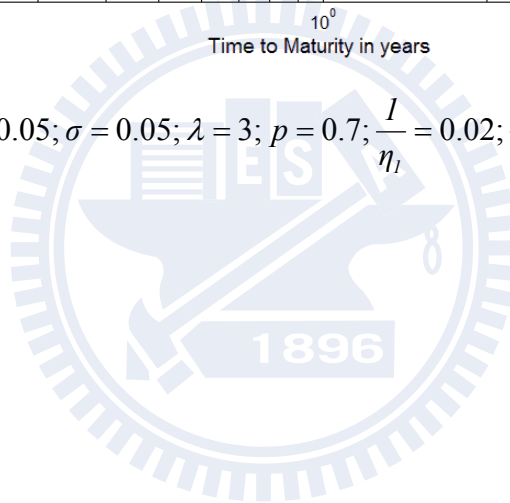

Parameter setting: $r = 0.05; \sigma = 0.05; \lambda = 3; p = 0.7; \dfrac{1}{\eta_1} = 0.02; \dfrac{1}{\eta_2} = 0.03; V_0 = 86.3; \hat{V}_t = 86.3$.

Appendix

Appendix A. Notations

<u>Notation 1</u>: (for equation (4))

Note that to simplify the notation, we shall drop the superscript $'$ in parameters, i.e., using θ, σ, p, q, η_1, η_2 rather than θ', σ', p', q', η_1', η_2'.

$$\pi_n = P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$L(\lambda,p,q,\eta_1,\eta_2) = \lim_{mn\to\infty} \frac{e^{\frac{(\sigma\eta_1)^2 \frac{t}{2}}{\sigma\sqrt{2\pi t}}}}{\sigma\sqrt{2\pi t}} \sum_{n=1}^{mn} \pi_n \sum_{k=1}^{n} P_{n,k} \left(\sigma\eta_1\sqrt{t}\right)^k I_{k-1}\left(x - \theta t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\sigma\eta_1\sqrt{t}\right)$$

$$M(\lambda,p,q,\eta_1,\eta_2) = \lim_{mn\to\infty} \frac{e^{\frac{(\sigma\eta_2)^2 \frac{t}{2}}{\sigma\sqrt{2\pi t}}}}{\sigma\sqrt{2\pi t}} \sum_{n=2}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} \left(\sigma\eta_2\sqrt{t}\right)^k I_{k-1}\left(x - \theta t; \eta_2, \frac{1}{\sigma\sqrt{t}}, -\sigma\eta_2\sqrt{t}\right)$$

$$P_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}, 1 \le k \le n-1$$

$$Q_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^i, 1 \le k \le n-1$$

$$P_{n,n} = p^n, Q_{n,n} = q^n, \binom{0}{0} = 1$$

$$I_n(c;\alpha,\beta,\delta) = \begin{cases} -\dfrac{e^{\alpha c}}{\alpha} \displaystyle\sum_{i=0}^{n} \left(\dfrac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) + \left(\dfrac{\beta}{\alpha}\right)^{n+1} \dfrac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}} \Phi\left(-\beta c + \delta + \dfrac{\alpha}{\beta}\right), \beta > 0, \alpha \ne 0, n \ge -1 \\ -\dfrac{e^{\alpha c}}{\alpha} \displaystyle\sum_{i=0}^{n} \left(\dfrac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) - \left(\dfrac{\beta}{\alpha}\right)^{n+1} \dfrac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}} \Phi\left(\beta c - \delta - \dfrac{\alpha}{\beta}\right), \beta < 0, \alpha < 0, n \ge -1 \end{cases}$$

$$Hh_n(x) = \int_x^{\infty} Hh_{n-1}(y)\,dy = \frac{1}{n!}\int_x^{\infty} (t-x)^n e^{-\frac{t^2}{2}}\,dt \ge 0, n = 0,1,2,\ldots.$$

Note that to compute the *Hh* function, the three-term recursion of *Hh* which is

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x), \quad n \ge 1 \quad \text{should be used.}$$

<u>Notation 2</u>: (for equation (5))

$$A(\lambda, p, q, \eta_1, \eta_2, b) = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}$$

$$B(\lambda, p, q, \eta_1, \eta_2, b) = \frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} \left[e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}} \right]$$

$$C(\lambda, b, x) = \lim_{mn \to \infty} \sum_{n=0}^{mn} \frac{\lambda^n}{n!} H_0\left(-h, \gamma_\alpha, -\frac{\mu}{\sigma}; n \right)$$

$$D(\lambda, p, q, \eta_1, \eta_2, b, x) = \lim_{mn \to \infty} \sum_{n=1}^{mn} \sum_{j=1}^{n} \frac{\lambda^n}{n!} \left(A P_{n,j} + B \overline{P}_{n,j} \right) \left(\sum_{k=0}^{j-1} (\sigma\eta_1)^k H_k(h, \gamma_\alpha, c_+; n) \right)$$

$$E(\lambda, p, q, \eta_1, \eta_2, b, x) = B \lim_{mn \to \infty} \sum_{n=1}^{mn} \sum_{j=0}^{n} \frac{(\lambda p)^n}{n!} (\sigma\eta_1)^j H_j(h, \gamma_\alpha, c_+; n)$$

$$F(\lambda, p, q, \eta_1, \eta_2, b, x) = \lim_{mn \to \infty} \sum_{n=1}^{mn} \sum_{j=1}^{n} \frac{\lambda^n}{n!} \left(A Q_{n,j} + B \overline{Q}_{n,j} \right) \left(\sum_{k=0}^{j-1} (\sigma\eta_2)^k H_k(-h, \gamma_\alpha, c_-; n) \right)$$

$$P_{n,i} = \sum_{j=i}^{n-1} \binom{n-i-1}{j-i} \cdot \binom{n}{j} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-j} p^j q^{n-j}, 1 \le i \le n-1$$

$$Q_{n,k} = \sum_{j=i}^{n-1} \binom{n-i-1}{j-i} \cdot \binom{n}{j} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j} q^j p^{n-j}, 1 \le i \le n-1$$

while $P_{n,n} = p^n, Q_{n,n} = q^n$

$$\overline{P}_{n,1} = \sum_{i=1}^{n-1} Q_{n,i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^i, \qquad \overline{P}_{n,k} = P_{n,k-1}, \qquad 2 \le i \le n-1$$

$$\overline{Q}_{n,k} = \sum_{j=i}^{n-1} \binom{n}{j} q^j p^{n-j} \binom{n-i}{j-i} \cdot \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j+1}, \qquad 1 \le i \le n$$

$$c_+ = \sigma\eta_1 + \frac{\mu}{\sigma}, \quad c_- = \sigma\eta_2 - \frac{\mu}{\sigma}, \quad \gamma_\alpha = \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, \quad h = \frac{b-x}{\sigma}$$

$$H_i(a, b, c; n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\left(\frac{1}{2} c^2 - b \right) t} t^{n+\frac{i}{2}} Hh_i\left(c\sqrt{t} + \frac{a}{\sqrt{t}} \right) dt, \quad i \ge -1, \ n \ge 0$$

Note that to compute the H function, the three-term recursion of H which is

$$H_i(a,b,c;n) = \frac{1}{i} H_{i-2}(a,b,c;n+1) - \frac{c}{i} H_{i-1}(a,b,c;n+1) - \frac{a}{i} H_{i-1}(a,b,c;n) \quad \text{should be used..}$$

Appendix B. Derivation of the cumulative distribution of Z_t, $P\{Z_t \geq z\}$

Suppose $\{\xi_1, \xi_2, \ldots\}$ is a sequence of i.i.d. exponential random variables with rate $\eta > 0$, V

and U are random variables with normal distribution $N(0, \sigma^2)$ and $N(\bar{\mu}, a^2)$ respectively.

Let $H_t = \sqrt{t}V + U \sim N(\bar{\mu}, \bar{\sigma}^2)$ and $\bar{\sigma}^2 = \sigma^2 t + a^2$. Then for every $n \geq 1$,

(1) The density functions are given by

$$f_{H_t + \sum_{i=1}^n \xi_i}(s) = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} + \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n e^{-s\eta} Hh_{n-1}\left(-\frac{s}{\bar{\sigma}} + \bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}} \right)$$

$$f_{H_t - \sum_{i=1}^n \xi_i}(s) = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n e^{s\eta} Hh_{n-1}\left(\frac{s}{\bar{\sigma}} + \bar{\sigma}\eta - \frac{\bar{\mu}}{\bar{\sigma}} \right)$$

(2) The tail probabilities are given by

$$P\{H_t + \sum_{i=1}^n \xi_i \geq x\} = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} + \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n I_{n-1}\left(x; -\eta, -\frac{1}{\bar{\sigma}}, -\bar{\sigma}\eta - \frac{\bar{\mu}}{\bar{\sigma}} \right)$$

$$P\{H_t - \sum_{i=1}^n \xi_i \geq x\} = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n I_{n-1}\left(x; \eta, \frac{1}{\bar{\sigma}}, -\bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}} \right)$$

Proof.

(1) The density functions of $H_t + \sum_{i=1}^n \xi_i$ and $H_t - \sum_{i=1}^n \xi_i$:

$$f_{H_t + \sum_{i=1}^n \xi_i}(s) = \int_{-\infty}^{\infty} f_{\sum_{i=1}^n \xi_i}(s-h) f_H(h) dh$$

$$= e^{-s\eta} \eta^n \int_{-\infty}^s \frac{e^{h\eta}(s-h)^{n-1}}{(n-1)!} \frac{1}{\bar{\sigma}\sqrt{2\pi}} \exp\left\{ -\frac{(h-\bar{\mu})^2}{2\bar{\sigma}^2} \right\} dh$$

$$= e^{-s\eta}\eta^n \int_{-\infty}^{s} \frac{(s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} \exp\left\{-\frac{1}{2\overline{\sigma}^2}\left(h^2 - 2(\overline{\sigma}^2\eta + \overline{\mu})h + \overline{\mu}^2\right)\right\}dh$$

$$= e^{-s\eta}\eta^n \int_{-\infty}^{s} \frac{(s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} \exp\left\{-\frac{1}{2\overline{\sigma}^2}\left[\left(h - \overline{\sigma}^2\eta - \overline{\mu}\right)^2 - \left(\overline{\sigma}^2\eta\right)^2 - 2\overline{\sigma}^2\eta\overline{\mu}\right]\right\}dh$$

$$= e^{-s\eta}\eta^n e^{\frac{(\overline{\sigma}\eta)^2}{2}+\eta\overline{\mu}} \int_{-\infty}^{s} \frac{(s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} \exp\left\{-\frac{1}{2\overline{\sigma}^2}\left(h - \overline{\sigma}^2\eta - \overline{\mu}\right)^2\right\}dh$$

Letting $y = \dfrac{\left(h - \overline{\sigma}^2\eta - \overline{\mu}\right)}{\overline{\sigma}}$, $h = y\overline{\sigma} + \overline{\sigma}^2\eta + \overline{\mu}$, $dh = \overline{\sigma}dy$, yields that

$$f_{H_t+\sum_{i=1}^{n}\xi_i}(s) = e^{-s\eta}\eta^n e^{\frac{(\overline{\sigma}\eta)^2}{2}+\eta\overline{\mu}} \int_{-\infty}^{\frac{s}{\overline{\sigma}}-\overline{\sigma}\eta-\frac{\overline{\mu}}{\overline{\sigma}}} \frac{\left(s - y\overline{\sigma} - \overline{\sigma}^2\eta - \overline{\mu}\right)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\}\overline{\sigma}dy$$

$$= e^{-s\eta}(\eta^n\overline{\sigma}^{n-1})e^{\frac{(\overline{\sigma}\eta)^2}{2}+\eta\overline{\mu}} \int_{-\infty}^{\frac{s}{\overline{\sigma}}-\overline{\sigma}\eta-\frac{\overline{\mu}}{\overline{\sigma}}} \frac{\left(\frac{s}{\overline{\sigma}} - \overline{\sigma}\eta - \frac{\overline{\mu}}{\overline{\sigma}} - y\right)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\}dy$$

$$= \frac{e^{\frac{(\overline{\sigma}\eta)^2}{2}+\eta\overline{\mu}}}{\overline{\sigma}\sqrt{2\pi}}(\eta\overline{\sigma})^n e^{-s\eta} Hh_{n-1}\left(-\frac{s}{\overline{\sigma}} + \overline{\sigma}\eta + \frac{\overline{\mu}}{\overline{\sigma}}\right)$$

here $\int_{-\infty}^{s} \dfrac{(s-y)^{n-1}}{(n-1)!} exp\left\{-\dfrac{y^2}{2}\right\}dy = Hh_{n-1}(-s)$.

Similarly,

$$f_{H_t-\sum_{i=1}^{n}\xi_i}(s) = \int_{-\infty}^{\infty} f_{\sum_{i=1}^{n}\xi_i}(h-s)f_H(h)dh$$

$$= e^{s\eta}\eta^n \int_{s}^{\infty} \frac{e^{-h\eta}(h-s)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} exp\left\{-\frac{(h-\overline{\mu})^2}{2\overline{\sigma}^2}\right\}dh$$

Let $h=-h$,

$$f_{H_t-\sum_{i=1}^{n}\xi_i}(s) = e^{s\eta}\eta^n \int_{-\infty}^{-s} \frac{(-s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} exp\left\{-\frac{1}{2\overline{\sigma}^2}\left(h^2 + 2(-\overline{\sigma}^2\eta + \overline{\mu})h + \overline{\mu}^2\right)\right\}dh$$

$$= e^{s\eta}\eta^n \int_{-\infty}^{-s} \frac{(-s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} exp\left\{-\frac{1}{2\overline{\sigma}^2}\left[\left(h - \overline{\sigma}^2\eta + \overline{\mu}\right)^2 - \left(\overline{\sigma}^2\eta\right)^2 + 2\overline{\sigma}^2\eta\overline{\mu}\right]\right\}dh$$

$$= e^{s\eta}\eta^n e^{\frac{(\overline{\sigma}\eta)^2}{2}-\eta\overline{\mu}} \int_{-\infty}^{-s} \frac{(-s-h)^{n-1}}{(n-1)!} \frac{1}{\overline{\sigma}\sqrt{2\pi}} exp\left\{-\frac{1}{2\overline{\sigma}^2}\left(h - \overline{\sigma}^2\eta + \overline{\mu}\right)^2\right\}dh$$

Letting $y = \dfrac{\left(h - \overline{\sigma}^2\eta + \overline{\mu}\right)}{\overline{\sigma}}$, $h = y\overline{\sigma} + \overline{\sigma}^2\eta - \overline{\mu}$, $dh = \overline{\sigma}dy$ yields

$$f_{H_t - \sum_{i=1}^n \xi_i}(s) = e^{s\eta} \eta^n e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}} \int_{-\infty}^{\frac{s}{\bar{\sigma}} - \bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}}} \frac{\left(-s - y\bar{\sigma} - \bar{\sigma}^2\eta + \bar{\mu}\right)^{n-1}}{(n-1)!} \frac{1}{\bar{\sigma}\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} \bar{\sigma} dy$$

$$= e^{s\eta}(\eta^n \bar{\sigma}^{n-1}) e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}} \int_{-\infty}^{\frac{s}{\bar{\sigma}} - \bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}}} \frac{\left(-\frac{s}{\bar{\sigma}} - \bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}} - y\right)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{y^2}{2}\right\} dy$$

$$= \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n e^{s\eta} Hh_{n-1}\left(\frac{s}{\bar{\sigma}} + \bar{\sigma}\eta - \frac{\bar{\mu}}{\bar{\sigma}}\right)$$

(2) $P\{H_t + \sum_{i=1}^n \xi_i \geq x\}$ and $P\{H_t - \sum_{i=1}^n \xi_i \geq x\}$.

$$P\{H_t + \sum_{i=1}^n \xi_i \geq x\} = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} + \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n \int_x^\infty e^{-s\eta} Hh_{n-1}\left(-\frac{s}{\bar{\sigma}} + \bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}}\right) ds$$

$$= \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} + \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n I_{n-1}\left(x; -\eta, -\frac{1}{\bar{\sigma}}, -\bar{\sigma}\eta - \frac{\bar{\mu}}{\bar{\sigma}}\right)$$

where $I_n(c; \alpha, \beta, \delta) \equiv \int_c^\infty e^{\alpha x} Hh_n(\beta x - \delta) dx, \ n \geq 0$.

Similarly,

$$P\{H_t - \sum_{i=1}^n \xi_i \geq x\} = \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n \int_x^\infty e^{s\eta} Hh_{n-1}\left(\frac{s}{\bar{\sigma}} + \bar{\sigma}\eta - \frac{\bar{\mu}}{\bar{\sigma}}\right) ds$$

$$= \frac{e^{\frac{(\bar{\sigma}\eta)^2}{2} - \eta\bar{\mu}}}{\bar{\sigma}\sqrt{2\pi}} (\eta\bar{\sigma})^n I_{n-1}\left(x; \eta, \frac{1}{\bar{\sigma}}, -\bar{\sigma}\eta + \frac{\bar{\mu}}{\bar{\sigma}}\right)$$

Therefore,

$$P\{Z_t \geq z\} = \sum_{n=0}^\infty \pi_n P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + U_t + \sum_{j=1}^n Y_j \geq z\right)$$

$$= \sum_{n=0}^\infty \pi_n P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + H_t + \sum_{j=1}^n Y_j \geq z\right)$$

$$= \pi_0 P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + H_t \geq z\right)$$

$$+ \sum_{n=0}^\infty \pi_n \sum_{k=1}^n P_{n,k} P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + H_t + \sum_{j=1}^n \xi_j^+ \geq z\right)$$

$$+ \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + H_t - \sum_{j=1}^{n} \xi_j^- \geq z\right)$$

$$= \pi_0 \Phi\left(-\frac{z - \left(\mu - \frac{\sigma^2}{2}\right)t - \overline{\mu}}{\overline{\sigma}}\right)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_1)^2 + \overline{\mu}\eta_1}}{\overline{\sigma}\sqrt{2\pi}} \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} (\overline{\sigma}\eta_1)^k I_{k-1}\left(z - \left(\mu - \frac{\sigma^2}{2}\right)t; -\eta_1, -\frac{1}{\overline{\sigma}}, -\overline{\sigma}\eta_1 - \frac{\overline{\mu}}{\overline{\sigma}}\right)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_2)^2 - \overline{\mu}\eta_2}}{\overline{\sigma}\sqrt{2\pi}} \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} (\overline{\sigma}\eta_2)^k I_{k-1}\left(z - \left(\mu - \frac{\sigma^2}{2}\right)t; \eta_2, \frac{1}{\overline{\sigma}}, -\overline{\sigma}\eta_2 + \frac{\overline{\mu}}{\overline{\sigma}}\right)$$

Let

$$L'(\lambda, p, q, \eta_1, \eta_2) = \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} (\overline{\sigma}\eta_1)^k I_{k-1}\left(z - \left(\mu - \frac{\sigma^2}{2}\right)t; -\eta_1, -\frac{1}{\overline{\sigma}}, -\overline{\sigma}\eta_1 - \frac{\overline{\mu}}{\overline{\sigma}}\right)$$

$$M'(\lambda, p, q, \eta_1, \eta_2) = \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} (\overline{\sigma}\eta_2)^k I_{k-1}\left(z - \left(\mu - \frac{\sigma^2}{2}\right)t; \eta_2, \frac{1}{\overline{\sigma}}, -\overline{\sigma}\eta_2 + \frac{\overline{\mu}}{\overline{\sigma}}\right)$$

Then,

$$P\{Z_t \geq z\} = \sum_{n=0}^{\infty} \pi_n P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + U_t + \sum_{j=1}^{n} Y_j \geq z\right)$$

$$= \pi_0 \Phi\left(-\frac{z - \left(\mu - \frac{\sigma^2}{2}\right)t - \overline{\mu}}{\overline{\sigma}}\right)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_1)^2 + \overline{\mu}\eta_1}}{\overline{\sigma}\sqrt{2\pi}} L'(\lambda, p, q, \eta_1, \eta_2)$$

$$+ \frac{e^{\frac{1}{2}(\overline{\sigma}\eta_2)^2 - \overline{\mu}\eta_2}}{\overline{\sigma}\sqrt{2\pi}} M'(\lambda, p, q, \eta_1, \eta_2)$$