Diagonal Dominance Design of Multivariable Feedback Control Systems using the Stability-equation Method

by TAIN-SOU TSAY *and* KUANG-WEI HAN†

Department of Electronics Engineering and Institute of Electronics, National Chiao Tung University, Taiwan, R.O.C.

ABSTRACT: *A systematic procedure is proposed for the design of multivariable feedback control systems. For an* N × N *multivariable control system, the desirable overall compensator is decomposed into* N *cascaded sub-compensators to be determined in each step. The stability-equation method is used to find the desirable sub-compensators. After step* N, *the overall closed-loop system can be designed diagonal dominant and have a desirable performance closely related to the characteristic roots of each subsystem. A* 4 × 4 *boiler furnace system is chosen as an example, and comparisons are made with methods in the current literature.*

I. Introduction

The concept of diagonal dominance plays a centrol role in the analysis and design of a multi-input multi-output system (**1**, **2**). In this paper, a systematic design procedure is proposed for the analysis and design of multivariable feedback control systems. The diagonal dominance of the overall system is achieved column by column (i.e. step by step), by closing only one loop of the considered system at each step. For an $N \times N$ multivariable feedback system, the desirable overall compensator matrix is decomposed into N cascaded subcompensator matrices to be determined in each step. The general configuration of the system is shown in Fig. 1.

In order to find the desirable sub-compensator in each step, the stability-equation method is used (**3**, **4**). The ratios of the parameters of the subcompensators can be determined by some criteria and the optimization procedure for the diagonal dominance of each column (**5**–**8**). The desirable values of parameters are found by inspecting the constant-ω curves and the stability boundaries generated by the stability-equation method. The relative differences among the constant-ω curves will show the relative damping characteristics (**3**, **4**).

† Kuang-Wei Han is also with Chung-Shan Institute, Taiwan.

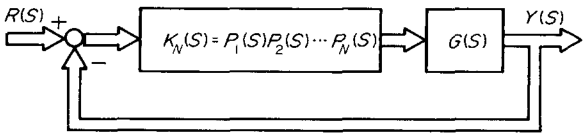

FIG. 1. System configuration of an $N \times N$ multivariable feedback control system.

Since the stability-equation method is highly capable of handling systems with adjustable parameters, the overall compensator for achieving a desirable performance can be easily designed.

II. Sequential Design for Diagonal Dominance and System Performance

The basic structure of the overall compensator for an $N \times N$ multivariable feedback control system is implemented by cascading the determined columns $P_{ik}(S)$ $(i = 1, 2, 3, \ldots, N)$ in each step. The general form of the compensator at step j (i.e. $k = j$) can be represented as

$$K_j(S) = P_1(S)P_2(S)\ldots P_j(S)$$

$$= \underset{\text{(step 1)}}{\begin{bmatrix} P_{11} & 0 & \ldots & 0 \\ P_{21} & 1 & \ldots & 0 \\ P_{31} & 0 & \ldots & 0 \\ \hdashline P_{N1} & 0 & \ldots & 1 \end{bmatrix}} \underset{\text{(step 2)}}{\begin{bmatrix} 1 & P_{12} & \ldots & 0 \\ 0 & P_{22} & \ldots & 0 \\ 0 & P_{32} & \ldots & 0 \\ \hdashline 0 & P_{N2} & \ldots & 1 \end{bmatrix}} \cdots \underset{\text{(step } j)}{\begin{bmatrix} 1 & 0 & \ldots & P_{1j} & \ldots & 0 \\ 0 & 1 & \ldots & P_{2j} & \ldots & 0 \\ 0 & 0 & \ldots & P_{3j} & \ldots & 0 \\ \hdashline 0 & 0 & \ldots & P_{Nj} & \ldots & 1 \end{bmatrix}} \quad (1)$$

where $P_{ik} = P_{ik}(S)$ $(i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, j)$ are called sub-compensators; $P_j(S)$ $(j = 1, 2, \ldots, N)$ are called sub-compensator matrices; $K_j(S)$ are called cascaded compensators and $K_N(S)$ is the overall compensator implemented in step N. Two typical block diagrams of control systems with this kind of structure are shown in Figs 2(a) and (b).

In step j, the loop-j with $P_{ij}(S)$ $(i = 1, 2, \ldots, N)$ and with subcompensators $P_{ik}(S)$ $(i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, j-1)$ found in step $j-1$, is closed; the remaining loops are open. The transfer function matrix $T^{(j)}(S)$ of this subsystem is

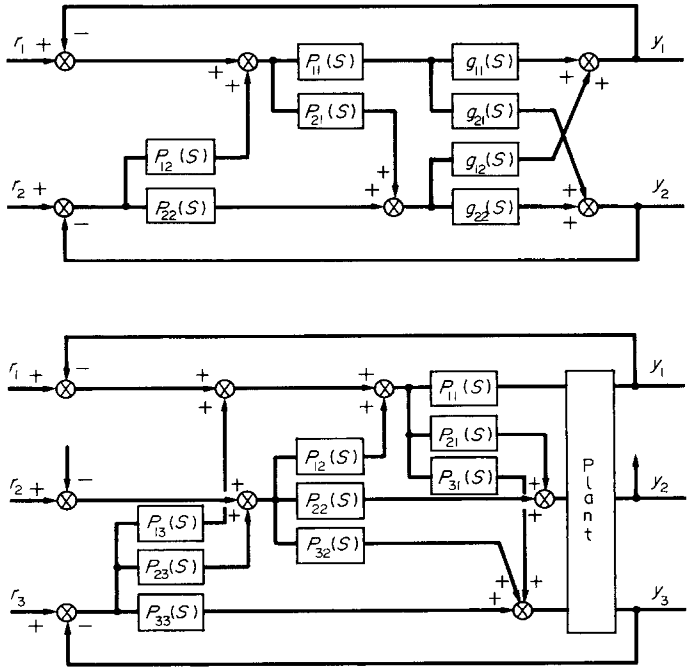

FIG. 2. Block diagram of (a) a 2×2 and (b) a 3×3 multivariable feedback control system.

$$T^{(j)}(S) = \frac{\begin{bmatrix} 0 & 0 & \cdots & p_{oj}(S) \sum_{i=1}^{N} g_{1i}^{(j-1)}(S) P_{ij}(S) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p_{oj}(S) \sum_{i=1}^{N} g_{2i}^{(j-1)}(S) P_{ij}(S) & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & p_{oj}(S) \sum_{i=1}^{N} g_{Ni}^{(j-1)}(S) P_{ij}(S) & 0 & \cdots & 0 \end{bmatrix}}{p_{oj}(S) \left[1 + \sum_{i=1}^{N} g_{ji}^{(j-1)}(S) P_{ij}(S) \right]} \tag{2}$$

where $g_{ij}^{(j-1)}(S)$ is the (i, j) element of the compensated plant $G^{(j-1)}(S)$ with the sub-compensator matrices found in step $j-1$, and $p_{oj}(S)$ is the open-loop characteristic equation of the compensated plant $G^{(j-1)}(S)$ with the sub-compensators $P_{ij}(S)$ $(i = 1, 2, \ldots, N)$. Proper forms of $P_{ij}(S)$ $(i = 1, 2, \ldots, N)$ are chosen by the designer to satisfy the desirable performance and diagonal dominance of $T^{(j)}(S)$ (**1**, **2**, **5–8**). The compensated plant $G^{(j)}(S)$ with the subcompensators, is in the form of

$$G^{(j)}(S) = G(S)P_1(S)P_2(S)\ldots P_j(S)$$

$$= \frac{1}{p_{oj}(S)} \begin{bmatrix} p_{oj}(S)\, g_{11}^{(j-1)}(S) & \ldots & P_{oj}(S) \sum_{i=1}^{N} g_{1i}^{(j-1)}(S) P_{ij}(S) & \ldots & p_{oj}(S)\, g_{1N}^{(j-1)}(S) \\ p_{oj}(S)\, g_{21}^{(j-1)}(S) & \ldots & p_{oj}(S) \sum_{i=1}^{N} g_{2i}^{(j-1)}(S) P_{ij}(S) & \ldots & P_{oj}(S)\, g_{2N}^{(j-1)}(S) \\ \text{-----} & \text{-----} & \text{-----} & \text{-----} & \text{-----} \\ p_{oj}(S)\, g_{N1}^{(j-1)}(S) & \ldots & p_{oj}(S) \sum_{i=1}^{N} g_{Ni}^{(j-1)}(S) P_{ij}(S) & \ldots & P_{oj}(S)\, g_{NN}^{(j-1)}(S) \end{bmatrix} \tag{3a}$$

which can be written as

$$G^{(j)}(S) = \begin{bmatrix} g_{11}^{(j)}(S) & g_{12}^{(j)}(S) & \ldots & g_{1j}^{(j)}(S) & \ldots & g_{1N}^{(j)}(S) \\ g_{21}^{(j)}(S) & g_{22}^{(j)}(S) & \ldots & g_{2j}^{(j)}(S) & \ldots & g_{2N}^{(j)}(S) \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ g_{N1}^{(j)}(S) & g_{N2}^{(j)}(S) & \ldots & g_{Nj}^{(j)}(S) & \ldots & g_{NN}^{(j)}(S) \end{bmatrix}. \tag{3b}$$

From Eqs (2) and (3a), it can be seen that the numerators of the jth column of $T^{(j)}(S)$ are the same as those of the jth column of $G^{(j)}(S)$. Therefore, if the subsystem $T^{(j)}(S)$ is diagonal dominant then the jth column of $G^{(j)}(S)$ is also diagonal dominant, leaving the remaining columns unchanged. This property makes the compensated plant achieve the diagonal dominance column by column (i.e. step by step). In other words, in step j the columns denoted equal to or less than j are diagonal dominant while the remaining columns are unchanged.

The characteristic equation of the subsystem with transfer function matrix $T^{(j)}(S)$ is

$$F_{cj}(S) = p_{oj}(S)\left[1 + \sum_{i=1}^{N} g_{ji}^{(j-1)}(S) P_{ij}(S)\right]. \tag{4}$$

Since the compensated plant obtained at step N is diagonal dominant, the performance of each subsystem (i.e. each loop) after all loops are closed, will be closely related to the characteristic roots of $F_{cj}(S)$ ($j = 1, 2, 3, \ldots, N$) assigned at each corresponding step. Therefore, the desirable performance and diagonal dominance can be achieved simultaneously from step 1 to step N. This is the main approach of the paper.

III. Basic Method for Choosing the Values of Parameters in Compensators

Assume that the characteristic equation of a system is $F(S)$ which can be decomposed into two parts concerning even and odd exponents of S; i.e.

$$F(S) = F_e(S) + F_o(S) = 0. \tag{5}$$

Let $S = j\omega$ then the stability-equations are (**3, 4**)

$$f_e(\omega) = F_e(j\omega) \tag{6}$$

and

$$f_o(\omega) = F_o(j\omega)/j\omega. \tag{7}$$

From Ref. (3), one has the following stability criterion.

Stability Criterion: The system with the characteristic equation $F(S)$ is stable if the roots ω_{ei} and ω_{oj} $(i,\ j = 1, 2, \ldots)$ of the stability-equations $f_e(\omega) = 0$ and $f_o(\omega) = 0$, respectively, are all real and alternating in sequence.

For a system with two parameters (m_1 and m_2), the stability-equations can be written as

$$f_e(\omega) = \sum_{i=0}^{m} a_i \omega^{2i} = 0 \tag{8}$$

and

$$f_o(\omega) = \sum_{j=0}^{n} b_j \omega^{2j} = 0 \tag{9}$$

where the coefficients a_i's and b_j's are assumed in the form of

$$a_i = A_{ei} + B_{ei} m_1 + C_{ei} m_2 \tag{10}$$

and

$$b_j = A_{oj} + B_{oj} m_1 + C_{oj} m_2 \tag{11}$$

where A's, B's and C's are constants. By inserting Eqs (10) and (11) into Eqs (8) and (9), the result can be arranged as

$$\sum_{i=0}^{m} A_{ei}\omega^{2i} + m_1 \sum_{i=0}^{m} B_{ei}\omega^{2i} + m_2 \sum_{i=0}^{m} C_{ei}\omega^{2i} = 0 \tag{12}$$

for the even stability-equation, and

$$\sum_{j=0}^{n} A_{oj}\omega^{2j} + m_1 \sum_{j=0}^{n} B_{oj}\omega^{2j} + m_2 \sum_{j=0}^{n} C_{oj}\omega^{2j} = 0 \tag{13}$$

for the odd stability-equation. From these two equations the following two kinds of curves can be plotted.

(1) The Stability-boundary Curves: By solving Eqs (12) and (13) for a sufficient number of suitable values of ω, the simultaneous solutions of m_1 and m_2 can be used to sketch a number of curves in the m_1 vs m_2 plane. Then the curves for $\omega_{ei} = \omega_{oj}$ which constitute the stability-boundaries can be determined.

(2) The Constant-ω Curves: By assigning a sufficient number of values of ω to Eqs (12) and (13) the constant-ω curves for even and odd stability-equations can be plotted in the m_1 vs m_2 plane.

From Refs (3) and (4), it has been shown that the differences among the magnitudes of the real roots (ω_{ei} and ω_{oi}) can be used as indications of damping characteristics; therefore, the desirable values of parameters (m_1 and m_2) can be chosen by inspecting the relative differences among these constant-ω curves.

The design procedure and application of the stability-equation method in each step are explained along with the following numerical example.

IV. Example

Consider the boiler furnace control system with transfer function matrix (**2, 9**)

$$G(S) = \begin{bmatrix} \frac{1}{1+4S} & \frac{0.7}{1+5S} & \frac{0.3}{1+5S} & \frac{0.2}{1+5S} \\ \frac{0.6}{1+5S} & \frac{1}{1+4S} & \frac{0.4}{1+5S} & \frac{0.35}{1+5S} \\ \frac{0.35}{1+5S} & \frac{0.4}{1+5S} & \frac{1}{1+4S} & \frac{0.6}{1+5S} \\ \frac{0.2}{1+5S} & \frac{0.3}{1+5S} & \frac{0.7}{1+5S} & \frac{1}{1+4S} \end{bmatrix}$$

$$= \frac{1}{p_o(S)} \begin{bmatrix} 0.05+0.25S & 0.035+0.14S & 0.015+0.06S & 0.01+0.04S \\ 0.03+0.12S & 0.05+0.25S & 0.02+0.08S & 0.0175+0.07S \\ 0.0175+0.07S & 0.02+0.08S & 0.05+0.25S & 0.03+0.12S \\ 0.01+0.04S & 0.015+0.06S & 0.035+0.14S & 0.05+0.25S \end{bmatrix} \tag{14}$$

where $p_o(S) = 0.05+0.45S+S^2$ is the open-loop characteristic equation of $G(S)$. For the compensation of $G(S)$, the lead/lag sub-compensators $P_{ij}(S)$ ($i, j = 1, 2, 3, 4$) are chosen in the form of

$$K_4(S) = P_1(S)P_2(S)P_3(S)P_4(S)$$

$$= \underset{\text{(step 1)}}{\begin{bmatrix} \frac{S+b_{11}}{S+d_1}\bar{p}_{11} & 0 & 0 & 0 \\ \frac{S+b_{21}}{S+d_1}\bar{p}_{21} & 1 & 0 & 0 \\ \frac{S+b_{31}}{S+d_1}\bar{p}_{31} & 0 & 1 & 0 \\ \frac{S+b_{41}}{S+d_1}\bar{p}_{41} & 0 & 0 & 1 \end{bmatrix}} \underset{\text{(step 2)}}{\begin{bmatrix} 1 & \frac{S+b_{12}}{S+d_2}\bar{p}_{12} & 0 & 0 \\ 0 & \frac{S+b_{22}}{S+d_2}\bar{p}_{22} & 0 & 0 \\ 0 & \frac{S+b_{32}}{S+d_2}\bar{p}_{32} & 1 & 0 \\ 0 & \frac{S+b_{42}}{S+d_2}\bar{p}_{42} & 0 & 1 \end{bmatrix}}$$

$$\times \begin{bmatrix} 1 & 0 & \frac{S+b_{13}}{S+d_3}\bar{p}_{13} & 0 \\ 0 & 1 & \frac{S+b_{23}}{S+d_3}\bar{p}_{23} & 0 \\ 0 & 0 & \frac{S+b_{33}}{S+d_3}\bar{p}_{33} & 0 \\ 0 & 0 & \frac{S+b_{43}}{S+d_3}\bar{p}_{43} & 1 \end{bmatrix}_{\text{(step 3)}} \begin{bmatrix} 1 & 0 & 0 & \frac{S+b_{14}}{S+d_4}\bar{p}_{14} \\ 0 & 1 & 0 & \frac{S+b_{24}}{S+d_4}\bar{p}_{24} \\ 0 & 0 & 1 & \frac{S+b_{34}}{S+d_4}\bar{p}_{34} \\ 0 & 0 & 0 & \frac{S+b_{44}}{S+d_4}\bar{p}_{44} \end{bmatrix}_{\text{(step 4)}} \quad (15)$$

where d_j, b_{ij} and $\bar{p}_{ij}$ (i, $j = 1, 2, 3, 4$) are adjustable parameters. In order to determine these adjustable parameters, four steps are required; i.e. the parameters d_j, b_{ij} and $\bar{p}_{ij}$ ($i = 1, 2, 3, 4$) are determined in step j ($j = 1, 2, 3, 4$). The ratios of b_{ij} and $\bar{p}_{ij}s'$ ($i = 1, 2, 3, 4$) are first determined by consideration of diagonal dominance and then the desirable values of these parameters are found by inspecting the constant-ω curves generated by the stability-equation method. The details of each step are shown below.

Step 1. Assume that loop-1 with $P_{i1}(S)$ ($i = 1, 2, 3, 4$) is closed and the remaining loops are open. The transfer function matrix of this subsystem is

$$T^{(1)}(S) = \frac{\begin{bmatrix} p_o(S)\sum_{i=1}^{4}(S+b_{i1})g_{1i}(S)\bar{p}_{i1} & 0 & 0 & 0 \\ p_o(S)\sum_{i=1}^{4}(S+b_{i1})g_{2i}(S)\bar{p}_{i1} & 0 & 0 & 0 \\ p_o(S)\sum_{i=1}^{4}(S+b_{i1})g_{3i}(S)\bar{p}_{i1} & 0 & 0 & 0 \\ p_o(S)\sum_{i=1}^{4}(S+b_{i1})g_{4i}(S)\bar{p}_{i1} & 0 & 0 & 0 \end{bmatrix}}{p_o(S)\left[(S+d_1)+\sum_{i=1}^{4}(S+b_{i1})g_{1i}(S)\bar{p}_{i1}\right]} \quad (16)$$

where $p_o(S)$ is the open-loop characteristic equation of $G(s)$; and $g_{ij}(S)$ are the (i, j) elements of $G(S)$. In order to make the subsystem have low-interactions in both high and low frequencies, one may let the coefficients of the highest order and lowest order exponents of the off-diagonal terms of $T^{(1)}(S)$ approach zero.

For making the coefficients of the highest order exponents approach zero, one has

$$0.12\bar{p}_{11}+0.25\bar{p}_{21}+0.08\bar{p}_{31}+0.07\bar{p}_{41} = 0 \quad (17a)$$

$$0.07\bar{p}_{11}+0.08\bar{p}_{21}+0.25\bar{p}_{31}+0.12\bar{p}_{41} = 0 \quad (17b)$$

$$0.04\bar{p}_{11}+0.06\bar{p}_{21}+0.14\bar{p}_{31}+0.05\bar{p}_{41} = 0. \quad (17c)$$

For making the coefficients of the lowest order exponents approach zero, one has

$$0.03b_{11}\bar{p}_{11}+0.05b_{21}\bar{p}_{21}+0.02b_{31}\bar{p}_{31}+0.0175b_{41}\bar{p}_{41} = 0 \qquad (18a)$$

$$0.0175b_{11}\bar{p}_{11}+0.02b_{21}\bar{p}_{21}+0.05b_{31}\bar{p}_{31}+0.03b_{41}\bar{p}_{41} = 0 \qquad (18b)$$

$$0.01b_{11}\bar{p}_{11}+0.015b_{21}\bar{p}_{21}+0.035b_{31}\bar{p}_{31}+0.05b_{41}\bar{p}_{41} = 0. \qquad (18c)$$

From Eqs (17a–c), the ratio of $P_{i1}(S)$ ($i = 1, 2, 3, 4$) can be found as

$$\bar{p}_{11}:\bar{p}_{21}:\bar{p}_{31}:\bar{p}_{41} = \bar{\bar{p}}_{11}:\bar{\bar{p}}_{21}:\bar{\bar{p}}_{31}:\bar{\bar{p}}_{41} = 31.089:-13.663:-4.813:-1$$

where $\bar{p}_{i1} = k_1\bar{\bar{p}}_{i1}$ ($i = 1, 2, 3, 4$). By use of the ratios, the ratios of b_{i1} ($i = 1, 2, 3, 4$) can be found from Eqs (18a–c); i.e.

$$b_{11}:b_{21}:b_{31}:b_{41} = \bar{\bar{b}}_{11}:\bar{\bar{b}}_{21}:\bar{\bar{b}}_{31}:\bar{\bar{b}}_{41} = 1:1.275:1.1887:3.0127$$

where $b_{i1} = k_2\bar{\bar{b}}_{i2}$ ($i = 1, 2, 3, 4$). Then the characteristic equation of $T^{(1)}(S)$ can be written as

$$F_{c1}(S) = (S+d_1)p_o(S)+p_o(S)\sum_{i=1}^{4}(S+\bar{\bar{b}}_{i1}k_2)g_{1i}(S)\bar{\bar{p}}_{i1}k_1 = 0$$

$$= (S+d_1)p_o(S)+\left[Sp_o(S)\sum_{i=1}^{4}g_{1i}(S)\bar{\bar{p}}_i\right]k_1$$

$$+\left[p_o(S)\sum_{i=1}^{4}g_{1i}(S)\bar{\bar{b}}_{i1}\bar{\bar{p}}_{i1}\right]k_1k_2 = 0 \qquad (19)$$

where k_1 and k_1k_2 are considered as two adjustable parameters to be analysed for a specified value of d_1.

Note that one may use other criteria and optimization procedures (**5–8**) to determine the ratios of $\bar{p}_{i1}$ and b_{i1} ($i = 1, 2, 3, 4$) for achieving diagonal dominance (**1,2**). It will be seen that the setting of the lowest and highest order exponents of the off-diagonal terms of each subsystem to approach zero is sufficient for making the considered system diagonal dominant.

Equation (19) can be decomposed into two stability-equations as shown in Section III, then the constant-ω curves can be plotted in the k_1 vs k_1k_2 plane. Figure 3 shows the results for $d_1 = 1$. The constant-X curves represent the negative sum of the characteristic roots. Generally, the larger the value of X, the better the damping characteristics of the system will be. By inspecting these curves, a suitable choice is made at Q_1 (0.8, 1.6), for which the roots of the stability-equations are at

$$\omega_{e1} = 0.498 \quad \text{and} \quad \omega_{o1} = 3.08033.$$

Since these roots are alternating in sequence and the differences among them are large, it can be predicted that the subsystem is stable and has adequate damping (**3, 4**).

Corresponding to the ratios $\bar{\bar{b}}_{i1}$ and $\bar{\bar{p}}_{i1}$ ($i = 1, 2, 3, 4$) found above and the choice of $(k_1, k_1k_2) = (0.8, 1.6)$, the sub-compensator matrix $P_1(s)$ is in the form of

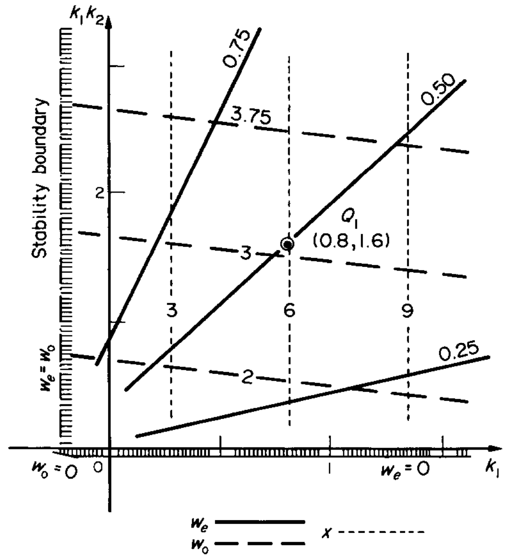

FIG. 3. Parameter plane analysis of the subsystem with the transfer function matrix $T^{(1)}(S)$.

$$P_1(S) = \begin{bmatrix} 24.871\dfrac{S+2}{S+1} & 0 & 0 & 0 \\ -10.93\dfrac{S+2.55}{S+1} & 1 & 0 & 0 \\ -3.85\dfrac{S+2.377}{S+1} & 0 & 1 & 0 \\ 0.8\dfrac{S+6.026}{S+1} & 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

The transfer function matrix is

$$T^{(1)}(S) = \frac{1}{F_{c1}(S)} \begin{bmatrix} 1.423+8.988S+4.488S^2 & 0 & 0 & 0 \\ -1.2569S & 0 & 0 & 0 \\ -0.4096S & 0 & 0 & 0 \\ 0.231S & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

where $F_{c1}(S) = S^3+5.9386S^2+9.4884S+1.4725$. The characteristic roots of $T^{(1)}(S)$ are found at

$$-0.1735, \; -2.8825 \pm j0.4235.$$

It can be seen that $T^{(1)}(S)$ is diagonal dominant for all frequencies.

With the sub-compensator matrix $P_1(S)$ found in Eq. (20), the open-loop transfer function matrix of the plant is

$$G^{(1)}(S) = G(S)P_1(S)$$

$$= \frac{.1}{p_{o1}(S)} \begin{bmatrix} 1.423+8.988S+4.4886S & (S+1)\times(0.035+0.14S) \\ -1.2569S & (S+1)\times(0.05+0.25S) \\ -0.4096S & (S+1)\times(0.02+0.08S) \\ 0.231S & (S+1)\times(0.015+0.06S) \end{bmatrix}$$

$$\times \begin{matrix} (S+1)\times(0.015+0.06S) & (S+1)\times(0.01+0.04S) \\ (S+1)\times(0.02+0.08S) & (S+1)\times(0.0175+0.07S) \\ (S+1)\times(0.05+0.25S) & (S+1)\times(0.03+0.12S) \\ (S+1)\times(0.015+0.06S) & (S+1)\times(0.05+0.25S) \end{matrix} \quad (22a)$$

$$= \begin{bmatrix} g_{11}^{(1)}(S) & g_{12}^{(1)}(S) & g_{13}^{(1)}(S) & g_{14}^{(1)}(S) \\ g_{21}^{(1)}(S) & g_{22}^{(1)}(S) & g_{23}^{(1)}(S) & g_{24}^{(1)}(S) \\ g_{31}^{(1)}(S) & g_{32}^{(1)}(S) & g_{33}^{(1)}(S) & g_{34}^{(1)}(S) \\ g_{41}^{(1)}(S) & g_{42}^{(1)}(S) & g_{43}^{(1)}(S) & g_{44}^{(1)}(S) \end{bmatrix} \quad (22b)$$

where $p_{o1}(S) = (S+1)p_o(S)$ which is the open-loop characteristic equation of $G^{(1)}(S)$. From Eqs (14), (21) and (22a), it can be seen that the sub-compensators make the first column diagonal dominant while the remaining columns (i.e. columns 2, 3, 4) are unchanged. Note that the numerators of the first column of $T^{(1)}(S)$ are the same as those of the first column of $G^{(1)}(S)$.

Step 2. In this step, loop-2 with the sub-compensators $P_{i1}(S)$ ($i = 1, 2, 3, 4$) found in step 1 and with the sub-compensators $P_{i2}(S)$ ($i = 1, 2, 3, 4$) is closed. The remaining loops are open. The transfer function matrix of this subsystem is

$$T^{(2)}(S) = \frac{\begin{bmatrix} 0 & p_{o1}(S)\sum_{i=1}^{4}(S+b_{i2})g_{1i}^{(1)}(S)\bar{p}_{i2} & 0 & 0 \\ 0 & p_{o1}(S)\sum_{i=1}^{4}(S+b_{i2})g_{2i}^{(1)}(S)\bar{p}_{i2} & 0 & 0 \\ 0 & p_{o1}(S)\sum_{i=1}^{4}(S+b_{i2})g_{3i}^{(1)}(S)\bar{p}_{i2} & 0 & 0 \\ 0 & p_{o1}(S)\sum_{i=1}^{4}(S+b_{i2})g_{4i}^{(1)}(S)\bar{p}_{i2} & 0 & 0 \end{bmatrix}}{p_{o1}(S)\left[(S+d_1)+\sum_{i=1}^{4}(S+b_{i2})g_{2i}^{(1)}(S)\bar{p}_{i2}\right]}. \quad (23)$$

As in step 1, the ratios of $\bar{p}_{i2}$ and b_{i2} ($i = 1, 2, 3, 4$) are found as

$$\bar{p}_{12}:\bar{p}_{22}:\bar{p}_{32}:\bar{p}_{42} = \bar{\bar{p}}_{12}:\bar{\bar{p}}_{22}:\bar{\bar{p}}_{32}:\bar{\bar{p}}_{42} = 1 : -37.446 : 10.488 : 3.1136$$

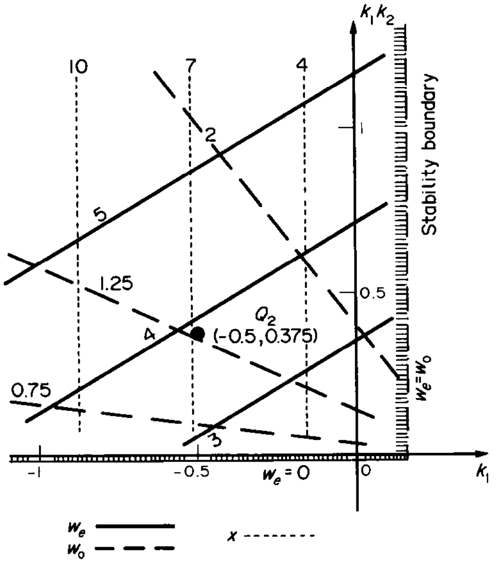

FIG. 4. Parameter plane analysis of the subsystem with the transfer function matrix $T^{(2)}(S)$.

and

$$b_{12}:b_{22}:b_{32}:b_{42}: =\bar{\bar{b}}_{12}:\bar{\bar{b}}_{22}:\bar{\bar{b}}_{32}:\bar{\bar{b}}_{42} = -1.8386: -2.4114: -3.2656: -1,$$

respectively. The characteristic equation of the subsystem with the transfer function matrix $T^{(2)}(S)$ can be written as

$$F_{c2}(S) = p_{o1}(S)\,(S+d_2)+\left[Sp_{o1}(S)\sum_{i=1}^{4} g_{2i}^{(1)}(S)\bar{\bar{p}}_{i2}\right]k_1 + \left[p_{o1}(S)\sum_{i=1}^{4} g_{2i}^{(1)}(S)\bar{\bar{b}}_{i2}\bar{\bar{p}}_{i2}\right]k_1k_2 = 0. \quad (24)$$

The constant-ω curves are plotted for $d_2 = 1.25$ as shown in Fig. 4. By inspecting these curves, a suitable choice is made at $Q_2(-0.5, 0.375)$, for which the roots of the stability-equations are at

$$\omega_{e1} = 0.3123, \quad \omega_{e2} = 3.893 \quad \text{and} \quad \omega_{o1} = 1.2738.$$

It can be seen that the differences among ω_{e1}, ω_{o1} and ω_{e2} are large enough for having adequate damping.

Corresponding to the ratios $\bar{\bar{b}}_{i2}$ and $\bar{\bar{p}}_{12}$ ($i = 1, 2, 3, 4$) found above and the choice of $(k_1, k_1k_2) = (-0.5, 0.375)$, the sub-compensator matrix $P_2(S)$ is in the form of

$$P_2(S) = \begin{bmatrix} 1 & -0.5\dfrac{S+1.379}{S+1.25} & 0 & 0 \\ 0 & 18.723\dfrac{S+1.809}{S+1.25} & 0 & 0 \\ 0 & -5.244\dfrac{S+2.449}{S+1.25} & 1 & 0 \\ 0 & -15.57\dfrac{S+0.75}{S+1.25} & 0 & 1 \end{bmatrix} \tag{25}$$

The transfer function matrix is

$$T^{(2)}(S) = \frac{1}{F_{c2}(S)} \begin{bmatrix} 0 & -1.44S-0.8605S^2 & 0 & 0 \\ 0 & \begin{matrix} 1.4157+10.442S \\ +12.941S^2+4.152S^3 \end{matrix} & 0 & 0 \\ 0 & -0.2943S-0.3719S^2 & 0 & 0 \\ 0 & -0.1982S-0.1544S^2 & 0 & 0 \end{bmatrix} \tag{26}$$

where $F_{c2}(S) = 1.4782+11.1173S+15.2532S^2+6.8522S^3+S^4$. The characteristic roots of the subsystem with the transfer function matrix $T^{(2)}(S)$ are found at

$$-0.1694, \quad -1.0829, \quad -2.8 \pm j0.4659.$$

It can be seen that $T^{(2)}(S)$ is diagonal dominant for all frequencies.

The open-loop transfer function matrix of the plant with the sub-compensator matrices $P_1(S)$ and $P_2(S)$ is

$$G^{(2)}(S) = G(S)P_1(S)P_2(S)$$

$$= \frac{1}{p_{o2}} \times \begin{bmatrix} p_{o2}g_{11}^{(1)} & -1.44S-0.8605S2 & p_{o2}g_{13}^{(1)} & p_{o2}g_{14}^{(1)} \\ p_{o2}g_{21}^{(1)} & 1.4157+10.442S+12.941S2+4.152S^3 & p_{o2}g_{23}^{(1)} & p_{o2}g_{24}^{(1)} \\ p_{o2}g_{31}^{(1)} & -0.2943S-0.3719S^2 & p_{o2}g_{33}^{(1)} & p_{o2}g_{34}^{(1)} \\ p_{o2}g_{41}^{(1)} & -0.1982S-0.1544S^2 & p_{o2}g_{43}^{(1)} & p_{o2}g_{44}^{(1)} \end{bmatrix} \tag{27a}$$

$$= \begin{bmatrix} g_{11}^{(2)}(S) & g_{12}^{(2)}(S) & g_{13}^{(2)}(S) & g_{14}^{(2)}(S) \\ g_{21}^{(2)}(S) & g_{22}^{(2)}(S) & g_{23}^{(2)}(S) & g_{24}^{(2)}(S) \\ g_{31}^{(2)}(S) & g_{32}^{(2)}(S) & g_{33}^{(2)}(S) & g_{34}^{(2)}(S) \\ g_{41}^{(2)}(S) & g_{42}^{(2)}(S) & g_{43}^{(2)}(S) & g_{44}^{(2)}(S) \end{bmatrix} \tag{27b}$$

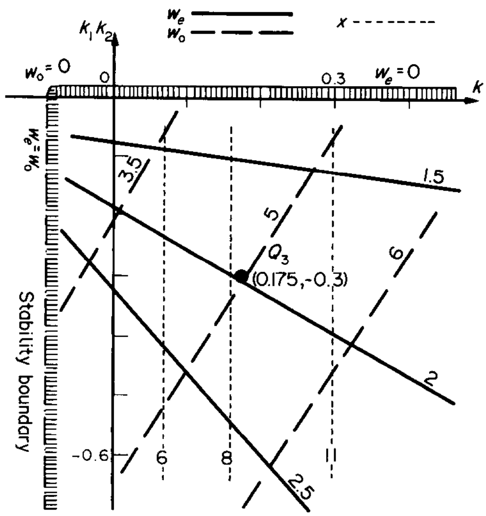

FIG. 5. Parameter plane analysis of the subsystem with the transfer function matrix $T^{(3)}(S)$.

where $p_{o2} = p_{o2}(S) = (S+1.25)p_{o1}(S)$, and $g_{ij}^{(1)} = g_{ij}^{(1)}(S)$ $(i, j = 1, 2, 3, 4)$ which are defined by Eq. (22b). From Eqs (22a) and (27a), it can be seen that the first and second columns are diagonal dominant while the remaining columns are unchanged.

The same design procedure is extended to the remaining steps. The details of steps 3 and 4 are omitted; only the main results are shown as follows:

Step 3. In this step, only loop-3 with $P_{ij}(S)$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3)$ is closed. From Eq. (27a) for diagonal dominance of the third column the ratios $\bar{p}_{i3}$ and b_{i3} $(i = 1, 2, 3, 4)$ are found as

$$\bar{p}_{13}:\bar{p}_{23}:\bar{p}_{33}:\bar{p}_{43} = \bar{\bar{p}}_{13}:\bar{\bar{p}}_{23}:\bar{\bar{p}}_{33}:\bar{\bar{p}}_{43} = -1:-1.173:119.377:-66.851$$

and

$$b_{13}:b_{23}:b_{33}:b_{43}: = \bar{\bar{b}}_{13}:\bar{\bar{b}}_{23}:\bar{\bar{b}}_{33}:\bar{\bar{b}}_{43} = -1:-1.0373:-1.49:-1.862,$$

respectively. Figure 5 shows the constant-ω curves for $d_3 = 1.75$. By inspecting these curves, a suitable choice is made at $Q_3(0.175, -0.3)$ for which the roots of the stability-equations are found at

$$\omega_{e1} = 0.251, \quad \omega_{e2} = 1.98 \quad \text{and} \quad \omega_{o1} = 0.8287, \quad \omega_{o2} = 4.9639.$$

Corresponding to the selection of $Q_3(0.175, -0.3)$ the resulted sub-compensator matrix is in the form of

$$P_3(S) = \begin{bmatrix} 1 & 0 & -0.175\dfrac{S+1.714}{S+1.75} & 0 \\ 0 & 1 & -0.2053\dfrac{S+1.778}{S+1.75} & 0 \\ 0 & 0 & 20.891\dfrac{S+2.554}{S+1.75} & 0 \\ 0 & 0 & -11.699\dfrac{S+3.192}{S+1.75} & 1 \end{bmatrix}. \quad (28)$$

The characteristic roots of this subsystem with transfer function matrix $T^{(3)}(S)$ are found at

$$-0.1701,\ -1,\ -1.273,\ -2.9126\pm j0.9739.$$

The open-loop transfer function of the plant with $P_j(S)$ ($j = 1, 2, 3$) is in the form of

$$G^{(3)}(S) = G(S)P_1(S)P_2(S)P_3(S) = \begin{bmatrix} g^{(3)}_{11}(S) & g^{(3)}_{12}(S) & g^{(3)}_{13}(S) & g^{(3)}_{14}(S) \\ g^{(3)}_{21}(S) & g^{(3)}_{22}(S) & g^{(3)}_{23}(S) & g^{(3)}_{24}(S) \\ g^{(3)}_{31}(S) & g^{(3)}_{32}(S) & g^{(3)}_{33}(S) & g^{(3)}_{34}(S) \\ g^{(3)}_{41}(S) & g^{(3)}_{42}(S) & g^{(3)}_{43}(S) & g^{(3)}_{44}(S) \end{bmatrix}. \quad (29)$$

Step 4. In this step, only loop-4 with $P_{ij}(S)$ ($i, j = 1, 2, 3, 4$) is closed. From Eq. (29) for diagonal dominance of the 4th column, the ratios of $\bar{p}_{i4}$ and b_{i4} ($i = 1, 2, 3, 4$) are found as

$$\bar{p}_{14}:\bar{p}_{24}:\bar{p}_{34}:\bar{p}_{44}: = \bar{\bar{p}}_{14}:\bar{\bar{p}}_{24}:\bar{\bar{p}}_{34}:\bar{\bar{p}}_{44}: = 1:1.8918:3.5261:112.214$$

and

$$b_{14}:b_{24}:b_{34}:b_{44}: = \bar{\bar{b}}_{14}:\bar{\bar{b}}_{24}:\bar{\bar{b}}_{34}:\bar{\bar{b}}_{44}: = -1:-1.619:-1.3691:-1.2677,$$

respectively. Figure 6 shows the constant-ω curves for $d_4 = 2.75$. A suitable choice is made at $Q_4(-0.15, 0.35)$, for which the roots of the stability-equations are

$$\omega_{e1} = 0.231,\ \omega_{e2} = 1.496,\ \omega_{e3} = 6.93 \text{ and } \omega_{o1} = 0.6983,\ \omega_{o2} = 2.976.$$

Corresponding to the selection of $Q_4(-0.15, 0.35)$, the sub-compensator matrix $P_4(S)$ is

$$P_4(S) = \begin{bmatrix} 1 & 0 & 0 & -0.15\dfrac{S+2.333}{S+2.75} \\ 0 & 1 & 0 & -0.284\dfrac{S+2.711}{S+2.75} \\ 0 & 0 & 1 & -0.529\dfrac{S+3.195}{S+2.75} \\ 0 & 0 & 0 & 16.832\dfrac{S+2.958}{S+2.75} \end{bmatrix}. \quad (30)$$

The characteristic roots of the subsystem with transfer function matrix $T^{(4)}(S)$ are found at

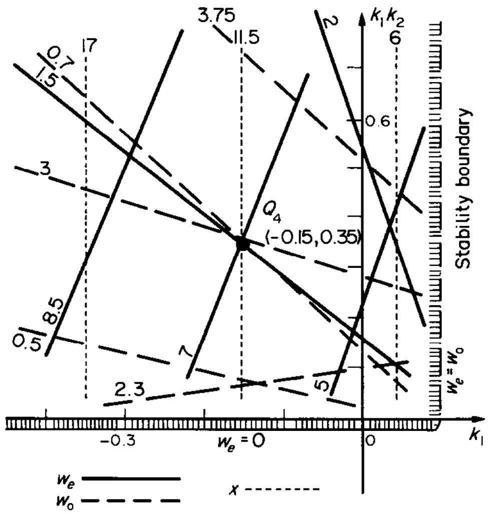

FIG. 6. Parameter plane analysis of the subsystem with the transfer function matrix $T^{(4)}(S)$.

$$-0.1737,\ -1.005,\ -1.2495,\ -2.2293,\ -3.375 \pm j0.6531.$$

The open-loop transfer function of the plant with $P_j(S)$ ($j = 1, 2, 3, 4$) is of the form

$$G^{(4)}(S) = G(S)P_1(S)P_2(S)P_3(S)P_4(S)$$

$$= \begin{bmatrix} g_{11}^{(4)}(S) & g_{12}^{(4)}(S) & g_{13}^{(4)}(S) & g_{14}^{(4)}(S) \\ g_{21}^{(4)}(S) & g_{22}^{(4)}(S) & g_{23}^{(4)}(S) & g_{24}^{(4)}(S) \\ g_{31}^{(4)}(S) & g_{32}^{(4)}(S) & g_{33}^{(4)}(S) & g_{34}^{(4)}(S) \\ g_{41}^{(4)}(S) & g_{42}^{(4)}(S) & g_{43}^{(4)}(S) & g_{44}^{(4)}(S) \end{bmatrix} \quad (31)$$

The overall compensator is

$$K_4(S) = P_1(S)P_2(S)P_3(S)P_4(S). \quad (32)$$

The step responses of the closed-loop system with $K_4(S)$ are shown in Fig. 7. It can be seen that the results are satisfactory for the considered system, and that the interactions among all the loops are very small. These result from the facts that the compensated system is diagonal dominant and the poles of each subsystem are well selected in each corresponding step.

To illustrate the dominance of the plant $G(S)$ and the compensated plant $G^{(4)}(S)$, the Gershgorin bands of $G(S)$ and $G^{(4)}(S)$ are plotted, as shown in Figs 8 and 9, respectively. It can be seen that the compensated plant $G^{(4)}(S)$ is diagonal dominant for all frequencies while the couplings of $G(S)$ are large. Note that the dominance is achieved column by column from plant $G(S)$.

V. Remarks

(1) For diagonal dominance of each column, the ratios of $\bar{p}_{ij}$ and b_{ij} are determined by setting the highest and lowest order exponents of the off-diagonal terms

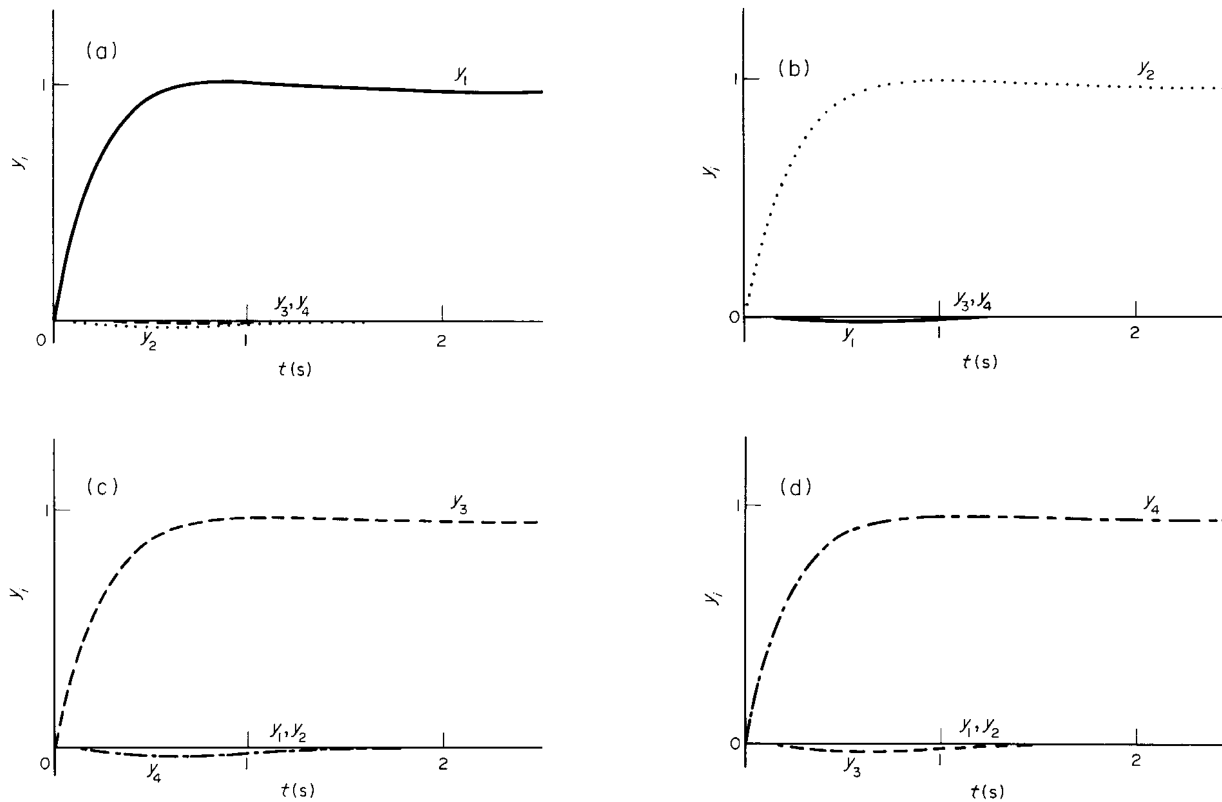

FIG. 7. (a) Step responses of the example for $r_1 = 1$ and $r_2 = r_3 = r_4 = 0$. (b) Step responses of the example for $r_2 = 1$ and $r_1 = r_3 = r_4 = 0$. (c) Step responses of the example for $r_3 = 1$ and $r_1 = r_2 = r_4 = 0$. (d) Step responses of the example for $r_4 = 1$ and $r_1 = r_2 = r_3 = 0$.

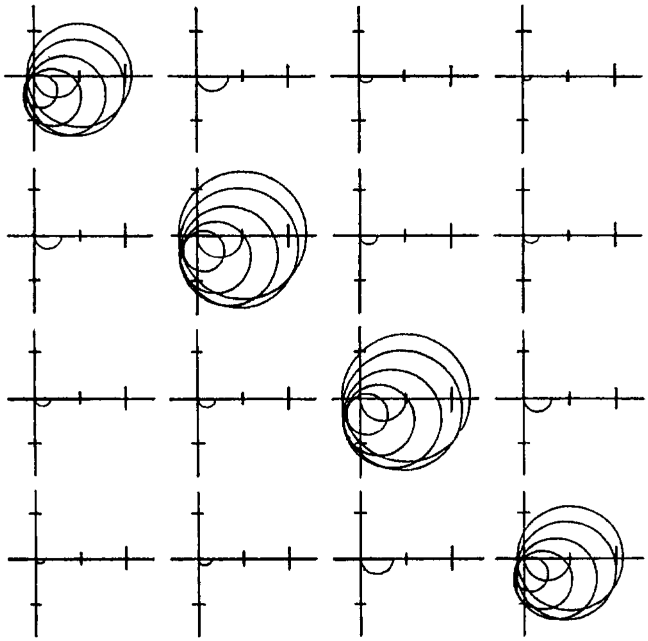

FIG. 8. Gershgorin bands of the plant $G(S)$.

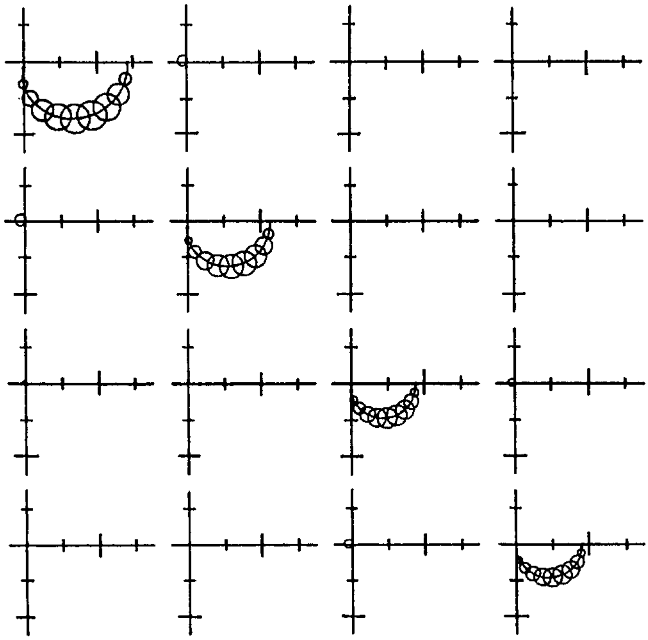

FIG. 9. Gershgorin bands of the plant with the overall compensator $K_4(S)$.

of each subsystem approach to zero. One may also use some other criteria or optimization procedures (**5–8**) for choosing the ratios of $\bar{p}_{ij}$ and b_{ij} to achieve diagonal dominance. Once the ratios are determined, the constant-ω curves and the stability boundaries can be plotted by use of the stability-equation method. Then a suitable choice of parameters can be made for the fine damping characteristics.

(2) To use the method proposed in this paper one can achieve high integrity against transducer failures. From the Gershgorin bands shown in Fig. 9, it can be seen that the compensated plant of the example is diagonal dominant prior to feedback loop closure. Therefore, high integrity is achieved.

(3) The same problem has been solved by Owens and Chotai (**9**) utilizing the diagonal approximated models of the precompensated plant to extend the inverse Nyquist array method (**1, 2**), and to apply an optimization procedure for the choice of the precompensators. The results obtained in this paper are better than those obtained by Owens and Chotai, while the compensators obtained by Owens and Chotai are simpler than those obtained by the proposed method. On the other hand, by use of the proposed method the compensators can be easily obtained, and a numerical overview of all the subsystems can also be obtained; i.e. one can select the locations of the poles by adjusting the parameters in each step.

(4) Mayne has also developed a sequential design procedure (**10–14**). Under his approach, a constant precompensating matrix is usually selected first to achieve the diagonal dominance, and then close the loops step by step with diagonal compensators in the diagonal elements. Compared to Mayne's approach, the method proposed in this paper uses a different structure for the overall compensator, and the compensators need to be in the diagonal elements in each step.

VI. Conclusions

A systematic design procedure for multivariable feedback control systems has been proposed and illustrated.

By use of the proposed design procedure together with the stability-equation method, it has been shown that the system characteristics, such as damping characteristics and diagonal dominance, can be achieved simultaneously and clearly in each step. In addition, high integrity against transducer failures can also be achieved.

From the example presented in this paper, it can be seen that the proposed method can be used to design a very complicated system, and that the design works in each step are easy and straightforward.

References

(**1**) H. H. Rosenbrock, "Design of multivariable control systems using the inverse Nyquist array", *Proc. IEE*, Vol. 116, pp. 1929–1936, 1969.

(**2**) H. H. Rosenbrock, "Computer Aided Control System Design", Academic Press, London, 1974.

(**3**) K. W. Han and G. J. Thaler, "High order system analysis and design using the root locus method", *J. Franklin Inst.*, Vol. 281, pp. 99–133, 1966.

(4) K. W. Han, "Nonlinear Control Systems: Some Practical Method", Academic Culture Company, 1977.
(5) D. J. Hawkins, "Pseudo-diagonalization and the inverse-Nyquist array method", *Proc. IEE*, Vol. 119, pp. 337–342, 1972.
(6) G. G. Leininger, "Diagonal dominance for multivariable Nyquist array method using function minimization", *Automatica*, Vol. 15, pp. 339–435, 1979.
(7) G. F. Bryant and L. F. Yeung, "Methods and concepts of dominance optimization", *Proc. IEE*, Vol. 130, pp. 72–82, 1983.
(8) D. J. N. Limebeer, "The application of generalized diagonal dominance to linear system stability theory", *Int. J. Control*, Vol. 36, pp. 185–212, 1982.
(9) D. H. Owens and A. Chotai, "Approximate models in multivariable process control: an inverse Nyquist array and robust tuning regulator interpretation", *Proc. IEE*, Vol. 133, pp. 1–12, 1986.
(10) S. C. Chuang, "Design of linear multivariable system by sequential return difference method", *Proc. IEE*, Vol. 121, pp. 745–747, 1974.
(11) D. Q. Mayne, "The design of linear multivariable systems", *Automatica*, Vol. 9, pp. 201–207, 1973.
(12) D. Q. Mayne and S. C. Chuang, "The sequential return difference method for designing linear multivariable systems", *Proc. Int. Conf. Computer-aided Control Syst.*, University of Cambridge, 1973.
(13) D. Q. Mayne, "The effect of feedback on linear multivariable systems", *Automatica*, Vol. 10, pp. 405–412, 1974.
(14) D. Q. Mayne, "Sequential design of linear multivariable systems", *Proc. IEE*, Vol. 126, pp. 568–572, 1979.

Vol. 325, No. 4, pp. 485–503, 1988
Printed in Great Britain