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BOUNDARY ELEMENT TECHNIQUES IN MATHEMATICAL MODELLING

PROBABILISTIC MODELLING OF EXISTING STRUCTURES FOR DYNAMIC ANALYSIS

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Abstract. A probabilistic model for analyzing existing structures subjected to dynamic loads is presented. Modal parameters and their statistics are obtained from experimental modal tests. Structure stiffness matrix is constructed from measured modal parameters using the matrix of transfer functions in the Laplace domain. Finite element and finite difference methods are used for the dynamic analysis of structures with random stiffness. The application of the proposed model is demonstrated by a numerical example.

Keywords. Mechanical structures; random vibration; finite element analysis; experimental modal analysis.

INTRODUCTION

In recent years, many structures have been constructed and operated in hostile vibration environments. Although much of the early work centered around fatigue and life testing, the latest efforts have been directed towards analytical modelling and simulation of mechanical structures. The finite element method is generally used to model structures as networks of lumped mechanical elements, in an effort to predict failures more reliably and faster than is afforded by conventional life testing procedures. However, during the process of finite element idealization it is always difficult to determine the matrices of stiffness and damping of a structure. Due to the lack of knowledge about the actual behavior at joints and connections between structural components and the randomness of material properties, the dynamic behavior of the structure become uncertain and should be treated in a probabilistic way.

Experimental modal analysis is a widely used technique applied in vibration analysis to describe the dynamic behavior of mechanical structures. The information about structural dynamic behavior is gained by means of input-output measurements. Modal parameters such as natural frequencies and damping ratios can be evaluated from measured frequency response functions. Furthermore, the structure stiffness and damping matrices of lumped equivalent model can be constructed from the measured modal parameters [4]. Due to many uncertain factors involved in the measurements, test results always exhibit scattering and the so obtained modal parameters become uncertain. Therefore the experimental modal analysis of structures should be treated in a statistical way if any meaningful interpretation of the test results can be made.

In this paper a probabilistic method for the dynamic analysis of structures with random stiffness is presented. The experimental modal testing is used to evaluate the statistics of modal parameters which are then used to infer the statistics of structure stiffness through the second moment analysis. The response of the structure is then evaluated by utilizing the finite element and finite difference methods. A numerical example is given to illustrate the application of the proposed method.

STRUCTURE MATRICES FROM EXPERIMENTAL MODAL ANALYSIS

The equations of motion of a structure is assumed to be expressed as

$$M \ddot{X} + C \dot{X} + K X = f \tag{1}$$

where the dots denote differentiation with respect to time; $f = f(t)$ is the applied force vector, and $X = X(t)$ the displacement vector, while M, C, and K are the (nxn) mass, damping, and stiffness matrices, respectively.

Taking the Laplace transform of the above system equations gives

$$B(s)\ \hat{X}(s) = F(s) \tag{2}$$

where

$$B(s) = Ms^2 + C\ s + K \tag{3}$$

Here, s is the Laplace variable, and now F(s) is the applied force vector and $\hat{X}(s)$ is the displacement vector in the Laplace domain. B(s) is called the system matrix, and the transfer matrix H(s) is defined as

$$H(s) = B^{-1}(s) \tag{4}$$

It has been shown [4] that H(s) can be expressed in rational fractions in s,

$$H = \sum_{k=1}^{2n} \left(\frac{A_k}{s - s_k}\right) u_k\ u_k^t \tag{5}$$

where

$$s_k = \sigma_k + i\ \omega_{Dk} \quad k = 1, 2, \ldots, n \tag{6}$$

Here, s_k are the poles of H, σ_k the damping coefficients, ω_{Dk} the damped natural frequencies of the structure; A_k are modal participation factors and u_k are mode shape vectors.

The resonant frequencies are given by

$$\omega_k = \sqrt{\sigma_k^2 + \omega_{Dk}^2} \tag{7a}$$

and the damping ratios are

$$\xi_k = -\frac{\sigma_k}{\omega_k} \tag{7b}$$

If s is set to be $i\omega$ where ω is defined as frequency the transfer function obtained from Eq. 5 is called frequency response function.

There are many existing techniques [2,3] for estimating the modal parameters in Eq. 7 from measured frequency response functions. In general, a series of frequency response functions is measured and the satistics of the modal frequencies and damping ratios are extracted from the test data.

In Eq. 5, H can be written in matrix form as

$$H = \theta\, W^{-1}\, \theta^t \tag{8}$$

where the columns of θ comprise the u_k modal vectors

$$\theta = [u_1\ u_2\ \ldots\ u_{2n}]_{n\mathrm{x}2n} \tag{9}$$

and W^{-1} is a diagonal matrix containing all s dependence, i.e.,

$$W^{-1} = \begin{bmatrix} \frac{A_1}{s-s_1} & & \\ & \ddots & \\ & & \frac{A_{2n}}{s-s_n^*} \end{bmatrix}_{2n\mathrm{x}2n} \tag{10}$$

where s_i^* is the complex conjugate of s_i.

From Eqs. 3, 4 and 8, stiffness matrix can be obtained by setting s = 0,

$$K = (\theta\, W_o^{-1}\, \theta^t)^{-1} = \theta^{-t}\, W_o\, \theta^{-1} \tag{11}$$

where W_o is the inverse of W^{-1} for s = 0.

Since

$$HB = I \tag{12}$$

it follows that

$$HB' + H'B = 0 \tag{13}$$

where the prime denotes differentiation with respect to s.

Differentiating Eq. 3 with respect to s and observing the relation in Eq. 13, the damping matrix can be obtained by setting s equal to zero as

$$C = B'(0) = -K\, H'(0) K \tag{14}$$

Observing that

$$H'(0) = -\theta\, W_o^{-1}\, \delta\, W_o^{-1}\, \theta^t \tag{15}$$

and in view of Eq. 11, Eq. 14 can be rewritten as

$$C = \theta_o\, \delta\, \theta_o^t \tag{16a}$$

in which $\theta_o = \theta^{-t}$; and

$$\delta = \begin{bmatrix} \frac{1}{A_1} & & \\ & \ddots & \\ & & \frac{1}{A_{2n}} \end{bmatrix}_{2n\mathrm{x}2n} \tag{16b}$$

MOMENTS OF STRUCTURE STIFFNESS

The statistics of modal parameters can be obtained from test data. If the mean values and variances of modal frequencies, damping ratios and modal participation factors are known, the statistics of structure stiffness can be inferred from those of modal parameters by utilizing the second moment analysis [1]. Expansion of the stiffness matrix in truncated Taylor series about the mean values of the modal parameters gives

$$\tilde{K} = \bar{K} + \sum_{i=1}^{n} \left[\frac{\partial K}{\partial \bar{\omega}_i}(\tilde{\omega}_i - \bar{\omega}_i) + \frac{\partial K}{\partial \bar{\xi}_i}(\tilde{\xi}_i - \bar{\xi}_i) + \frac{\partial K}{\partial \bar{A}_i}(\tilde{A}_i - \bar{A}_i)\right] \tag{17}$$

where $(\tilde{\ })$ = random variables; $(\bar{\ })$ = mean values.

From the second moment analysis, the mean value and variance of $\tilde{K}$ are obtained from Eq. 17, with the assumption of independence of modal parameters, as

$$E[\tilde{K}] = \bar{K}(\bar{\omega}_i, \ldots, \bar{\omega}_n; \bar{\xi}_1, \ldots, \bar{\xi}_n) \tag{18a}$$

and

$$\mathrm{Var}[K] = \sum_{i=1}^{n} \left\{\left(\frac{\partial \bar{K}}{\partial \bar{\omega}_i}\right)^2 \mathrm{Var}[\tilde{\omega}_i] + \left(\frac{\partial \bar{K}}{\partial \bar{\xi}_i}\right)^2 \mathrm{Var}[\tilde{\xi}_i] + \left(\frac{\partial \bar{K}}{\partial \bar{A}_i}\right)^2 \mathrm{Var}[\tilde{A}_i]\right\} \tag{18b}$$

in which

$$\frac{\partial \bar{K}}{\partial \bar{\omega}_i} = \theta^{-t}\, \frac{\partial \bar{W}_o}{\partial \bar{\omega}_i}\, \theta^{-1} \tag{18c}$$

$$\frac{\partial \bar K}{\partial \bar\xi_i} = \theta^{-t}\,\frac{\partial \bar W_o}{\partial \bar\xi_i}\,\theta^{-1} \tag{18d}$$

and

$$\frac{\partial \bar K}{\partial \bar A_i} = \theta^{-t}\,\frac{\partial \bar W_o}{\partial \bar A_i}\,\theta^{-1} \tag{18e}$$

In view of Eqs. 6, 7, and 10, the derivatives of $\bar W_o$ with respect to $\bar\omega_i$, $\bar\xi_i$, and $\bar A_i$ can be written as

$$\frac{\partial \bar W_o}{\partial \bar\omega_i} = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && -\frac{1}{A_i}\frac{\partial \bar s_i}{\partial \bar\omega_i} &&&& \\ &&& 0 &&& \\ &&&& -\frac{1}{A_i}\frac{\partial s_i^*}{\partial \bar\omega_i} && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}_{2n\times 2n} \tag{19a}$$

$$\frac{\partial \bar W_o}{\partial \bar\xi_i} = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && -\frac{1}{A_i}\frac{\partial \bar s_i}{\partial \bar\omega_i} &&&& \\ &&& 0 &&& \\ &&&& -\frac{1}{A_i}\frac{\partial s_i^*}{\partial \bar\xi_i} && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}_{2n\times 2n} \tag{19b}$$

and

$$\frac{\partial \bar W_o}{\partial \bar A_i} = \begin{pmatrix} 0 &&&& \\ & \ddots &&& \\ && 0 && \\ &&& \frac{\bar s_i}{A_i^2} && \\ &&&& 0 \\ &&&&& \ddots \\ &&&&&& 0 \end{pmatrix} \tag{19c}$$

where

$$\frac{\partial \bar s_i}{\partial \bar\omega_i} = -\,\bar\xi_i + i\sqrt{1-\bar\xi_i^{\,2}} \tag{19d}$$

and

$$\frac{\partial \bar s_i}{\partial \bar\xi_i} = -\,\bar\omega_i - i\,\frac{\bar\omega_i\,\bar\xi_i}{\sqrt{1-\bar\xi_i^{\,2}}} \tag{19e}$$

STOCHASTIC RESPONSES OF RANDOM STRUCTURES

If a structure has random stiffness and is subjected to random loads, the equations of motion in Eq. 1 becomes

$$M\,\ddot{\tilde X} + C\,\dot{\tilde X} + \tilde K\,\tilde X = \tilde f \tag{20}$$

The structure is discretized into finite elements and masses are lumped at node points. Herein structure damping and stiffness matrices are obtained by the forementioned experimental modal analysis. Eq. 20 can be solved by using the finite difference method. The response of the ith degree of freedom at $t = t_n$ is $\tilde X_i^n$ and

$$\tilde X_i^{\,n} = \tilde Q_{ip}\,\tilde X_p^{\,n-1} - D_{ip}\tilde X_p^{\,n-2} + B_{ip}\tilde f_p^{\,n-1} \tag{21a}$$

where

$$\tilde A_{ip} = 2\{I_{ip} - \frac{\Delta t}{2} M_{i\ell}^{-1} C_{\ell p} - \frac{\Delta t^2}{2} M_{ir}^{-1}\tilde K_{rp}\} \tag{21b}$$

$$B_{ip} = \Delta t^2\, M_{ip}^{-1} \tag{21c}$$

and

$$D_{ip} = I_{ip} - \Delta t^2\, M_{i\ell}^{-1}\, C_{\ell p} \tag{21d}$$

In the above equation I_{ip} is 1 for $i = p$ and 0 for $i \neq p$.

Utilizing the second moment method, the mean value of $\tilde X_i^n$ is

$$E[\tilde X_i^{\,n}] = E[\tilde Q_{ip}]\,E[\tilde X_p^{\,n-1}] - D_{ip}E[\tilde X_p^{\,n-2}] + B_{ip}E[\tilde f_p^{\,n-1}] \tag{22a}$$

where

$$E[\tilde Q_{ip}] = 2\{I_{ip} - \frac{\Delta t}{2} M_{i\ell}^{-1} C_{\ell p} - \frac{\Delta t^2}{2} M_{ir}^{-1}\,E[\tilde K_{rp}]\} \tag{22b}$$

and the variance is

$$\mathrm{Var}[\tilde X_i^{\,n}] = \sum_{j=1}^{n}\{(\frac{\partial \bar X_i^{\,n}}{\partial \bar\omega_j})^2 \mathrm{Var}[\tilde\omega_j] + (\frac{\partial \bar X_i^{\,n}}{\partial \bar\xi_j})^2 \mathrm{Var}[\tilde\xi_j] + (\frac{\partial \bar X_i^{\,n}}{\partial \bar A_j})^2 \mathrm{Var}[\tilde A_j] + (\frac{\partial X_i^{\,n}}{\partial \bar f_j})^2 \mathrm{Var}[\tilde f_j]\} \tag{22c}$$

where

$$\frac{\partial \bar{X}_i^{\,n}}{\partial(\;)} = \bar{Q}_{ip}\frac{\partial \bar{X}_i^{\,n-1}}{\partial(\;)} + \frac{\partial \bar{Q}_{iq}}{\partial(\;)}\bar{X}_q^{\,n-1} - D_{ip}\frac{\partial \bar{X}^{n-2}}{\partial(\;)} \tag{22d}$$

$$\frac{\partial \bar{Q}_{iq}}{\partial(\;)} = -\,\Delta t^2\, M_{ip}^{\;-1}\,\frac{\partial \bar{K}_{pq}}{\partial(\;)} \tag{22e}$$

and

( ) = variables for $\bar{\omega}_j$, $\bar{\xi}_j$, $\bar{A}_j$ and $\bar{f}_j$.

In the above equations, independence of random variables is assumed. The derivatives of stiffness coefficients in Eq. 22 are evaluated from Eqs. 18 and 19.

NUMERICAL EXAMPLE

A two degree of freedom system subjected to a random load is shown in Fig. 1. The statistics of modal parameters obtained from test data are given in Table 1. The system properties and the statistics of applied load are tabulated in Table 2. Utilizing the foregoing proposed finite difference method, the statistics of response can be evaluated. Figs. 2 and 3 show, respectively, the mean and variance of X_1.

Table 1 Statistics of Modal Parameters

Frequencies:

$\bar{\omega}_1 = 200$, $\sigma_{\omega_1} = 5$

$\bar{\omega}_2 = 300$, $\sigma_{\omega_2} = 5$

Damping Ratios:

$\bar{\xi}_1 = 0.05$, $\sigma_{\xi_1} = 0.01$

$\bar{\xi}_2 = 0.1$, $\sigma_{\xi_2} = 0.05$

Table 2 System Properties and Statistics of Applied Load

Mass:	$m_1 = 1.15$,	$m_2 = 3.06$
Damping Coefficients:	$C_{11} = 68.1$,	$C_{12} = -10.9$
	$C_{21} = -10.9$,	$C_{22} = 55.2$
Load:	$\bar{f} = 100$ Kips,	$\sigma_f = 10$ Kips

CONCLUSIONS

A probabilistic method for modelling the dynamic behavior of existing structures is presented. Modal testing is used to evaluate system properties and their statistics. Finite element and finite difference methods are used to evaluate the responses of the structures. The statistics of response can be obtained from the statistics of modal parameters and applied loads by utilizing the second moment method.

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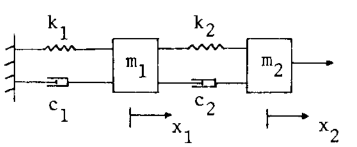

Fig.1 Mathematical Model of Two DOF System

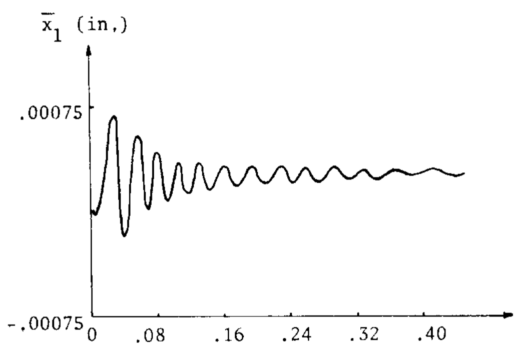

Fig. 2 Mean Value of x_1

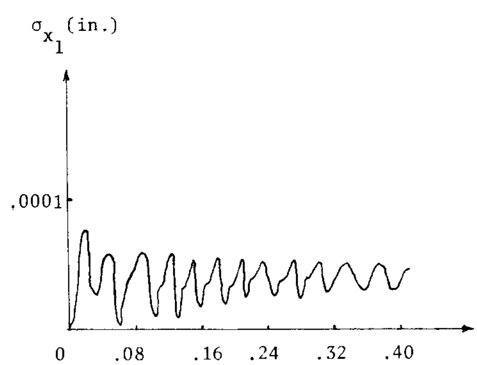

Fig. 3 Standard Deviation of x_1