Analysis of Discrete Dynamic Robot Models

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Abstract—**The discrete shift-transformation matrix of general orthogonal polynomials is introduced. The discrete shift-transformation matrix is employed to transform the difference equations, which describe the discrete dynamic robot model, into algebraic equations. Several lemmas are introduced which, together with the discrete shift-transformation matrix, solve for the joint positions and velocities of discrete dynamic robot models via discrete orthogonal polynomials approximations. The initial numerical experiment with a cylindrical coordinate robot shows the feasibility and applicability of discrete orthogonal polynomials approximations.**

I. Introduction

RECENTLY, discrete orthogonal polynomials have been applied to the analysis, parameter identification, model reduction, and optimal control of linear systems with some success. In particular, King and Paraskevopoulos [9] applied the discrete Laguerre orthogonal polynomials to solve the parametric identification problem. Hwang and Shih [6], [7] used the discrete Laguerre orthogonal polynomials and the discrete Chebyshev orthogonal polynomials, respectively, to solve the model reduction problem. Similarly, Horng and Ho [4], [5] applied the discrete Laguerre orthogonal polynomials and the discrete Chebyshev orthogonal polynomials, respectively, to solve the discrete optimal control problem.

Note that both the discrete Laguerre orthogonal polynomials and the discrete Chebyshev orthogonal polynomials possess the same recurrence relation. This fact motivates our attempts to start from the basic relation, the recurrence relation, to derive an algorithm for solving discrete-time control problems; hence the derived algorithm is so general that it is not only good for the discrete Laguerre orthogonal polynomials approach or the discrete Chebyshev orthogonal polynomials approach, but also, and most importantly, it is good for any other discrete orthogonal polynomials that possess the same recurrence relation. The basic idea of the paper is to use discrete orthogonal polynomials to approximate a discrete dynamic robot model, presented by Neuman and Tourassis [13], and then to use this approximation to solve for the joint positions and velocities.

This paper is organized as follows. In Section II, general discrete orthogonal polynomials are introduced, and the general discrete shift transformation matrix is derived. By means of the shifted transformation matrix, we can transform the difference equations describing the system into algebraic equations, regardless of whether the system is linear or nonlinear. The analysis of the resulting algebraic equations is then easily performed. In addition to the shift transformation matrix, several lemmas are developed so that the resulting algebraic equations will satisfy the specified initial and/or final conditions.

A discrete cylindrical robot model with highly coupled and nonlinear dynamics presented by Neuman and Tourassis [13] is outlined in Section III. This discrete robot model is analyzed via discrete general orthogonal polynomials in Section IV. It is shown that, by applying the shift transformation matrix of discrete general orthogonal polynomials, implicit nonlinear difference equations describing the discrete robot model can be transformed into explicit nonlinear algebraic equations. Based on these nonlinear algebraic equations, the approximate solutions for the joint positions and velocities can be obtained. These solutions are general enough that any of the discrete Chebyshev, discrete Laguerre, and other discrete orthogonal polynomials approximations that possess the recurrence relation can readily be obtained if required. Section V shows a few numerical results for the example presented by Neuman and Tourassis [13]. As results indicate, our approach yields comparable or greater accuracy. Section VI contains some concluding remarks.

II. General Discrete Orthogonal Polynomials and the Shift Transformation Matrix

The general discrete orthogonal polynomials $z_i(k)$ satisfy the orthogonal property of

$$\sum_{N-1}^{k=0} z_i(k)z_j(k)=\delta_{ij}, \qquad i, j=0, 1, \cdots, N-1 \tag{1}$$

where the orthogonal polynomials $z_i(k)$ have been normalized. These polynomials also satisfy the following recurrence relation:

$$z_{i+1}(k)=(a_ik+b_i)z_i(k)+c_iz_{i-1}(k), \qquad i=0, 1, \cdots, N-2, \quad k=0, 1, \cdots, N-1 \tag{2}$$

when $i < 0$, $z_i(k) = 0$.

The a_i, b_i, and c_i are the recurrence coefficients. Their values, along with $z_0(k)$, depend on the particular discrete polynomials under consideration (Appendix I).

Let $x(k)$, $k = 0, 1, \cdots, N-1$, be an arbitrary data sequence that can be expanded in terms of the general discrete orthogonal polynomials as

$$x(k)=\sum_{N-1}^{i=0} x_iz_i(k)=x^Tz(k) \tag{3}$$

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where the superscript T denotes transpose, $\boldsymbol{x}$ is called the discrete coefficient vector, and $z(k)$ is called the discrete polynomial vector. These two vectors are defined as

$$\boldsymbol{x} \triangleq (x_0 x_1 \cdots x_{n-1})^T \tag{4}$$

$$z(k) \triangleq (z_0(k) z_1(k) \cdots z_{N-1}(k))^T. \tag{5}$$

The coefficient x_i can be determined by using the property of (1), that is,

$$\sum_{N-1}^{k=0} x(k) z_i(k) = \sum_{N-1}^{k=0} \left(\sum_{N-1}^{j=0} x_j z_j(k) \right) z_i(k) = x_{ij} \tag{6}$$

hence

$$x_i = \sum_{N-1}^{k=0} x(k) z_i(k), \qquad i = 0, 1, \cdots, N-1. \tag{7}$$

Equations (3) and (7) can be viewed as a transform pair.

The shift transformation of a discrete orthogonal polynomial vector $z(k)$ is defined as

$$z(k+m) \triangleq G(m) z(k) \tag{8}$$

where $z(k)$ is an $N \times 1$ vector, and $G(m)$, an $N \times N$ matrix, is called the shift transformation matrix. By means of the shift transformation matrix $G(m)$, the discrete polynomial vector $z(k+m)$ can be expressed in terms of the original polynomial vector $z(k)$.

Let $g_{ij} = (G(m))_{ij}$. It can be shown that the general discrete shift transformation matrix is (Appendix II)

$$G(m) = g_{00} \begin{bmatrix} 1 & & & & 0 \\ g_{10} & 1 & & & \\ g_{20} & g_{21} & 1 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ g_{N-1,0} & \cdots & & g_{N-1,N-2} & 1 \end{bmatrix} \tag{9}$$

where g_{ij} satisfies the following recurrence relation:

$$\begin{aligned} g_{i+1,j} &= g_{ij}(ma_i + b_i - b_j a_i / a_j) + g_{i-1,j} c_i \\ &\quad + g_{i,j-1} a_i / a_{j-1} - g_{i,j+1} c_{j+1} a_i / a_{j+1}, \\ &\quad i = 0, 1, \cdots, N-2 \\ g_{ij} &= 0, \text{ for } j > i \text{ or } j < 0 \end{aligned} \tag{10}$$

and $g_{00} = 1$ for discrete Chebyshev polynomials, and $g_{00} = d^{m/2}$ for discrete Laguerre polynomials, where the real parameter $d \in (0, 1)$ is called the discount factor [5]. Let the difference equation describing the system be of the form

$$\begin{aligned} & c_m x(k+m) + c_{m-1} x(k+m-1) + \cdots + c_0 x(k) \\ & \quad = d_m y(k+m) + d_{m-1} y(k+m-1) + \cdots + d_0 y(k), \\ & \quad k = 0, 1, \cdots, N-1. \end{aligned} \tag{11}$$

By applying the shift transformation matrix, the previous equation becomes

$$\begin{aligned} & \boldsymbol{x}^T (c_m G(m) + c_{m-1} G(m-1) + \cdots + c_0 I) z(k) \\ & \quad = \boldsymbol{y}^T (d_m G(m) + d_{m-1} G(m-1) + \cdots + d_0 I) z(k), \\ & \quad k = 0, 1, \cdots, N-m+1, \end{aligned} \tag{12}$$

or equivalently,

$$\boldsymbol{a}^T z(k) = \boldsymbol{b}^T z(k), \qquad k = 0, 1, \cdots, N-m-1 \tag{13}$$

where $\boldsymbol{a}^T$ and $\boldsymbol{b}^T$ are $1 \times N$ vectors.

Thus the shift transformation matrix reduces mth-order difference equations with $k \in (0, 1, \cdots, N-1)$ to algebraic equations with $k \in (0, 1, \cdots, N-m-1)$. Since m equations are undetermined, no unique solution exists for $\boldsymbol{a}^T$ (or $\boldsymbol{b}^T$). For state-space representation $m = 1$, (13) becomes

$$\boldsymbol{a}^T z(k) = \boldsymbol{b}^T z(k), \qquad k = 0, 1, \cdots, N-2. \tag{14}$$

Again, no unique solution exists for $\boldsymbol{a}^T$ (or $\boldsymbol{b}^T$). In the case that, in addition to (14), there is a set of initial constraints, then $\boldsymbol{a}^T$ and $\boldsymbol{b}^T$ can be characterized by the following lemma.

Lemma 1: Given

$$\boldsymbol{a}^T z(k) = \boldsymbol{b}^T z(k), \qquad k = 0, 1, \cdots, N-2 \tag{15}$$

and

$$\boldsymbol{a}^T z(0) = a(0) \tag{16}$$

then

$$\boldsymbol{a}^T = \boldsymbol{b}^T + e \boldsymbol{W}^T \tag{17}$$

where

$$\boldsymbol{W}^T = \text{last row of } (z(0) \cdots z(N-1))^{-1} \tag{18}$$

and

$$e = (a(0) - \boldsymbol{b}^T z(0)) / \boldsymbol{W}^T z(0). \tag{19}$$

Proof: The proof is given in Appendix III.

If both initial and final conditions are to be satisfied, then $\boldsymbol{a}^T$ and $\boldsymbol{b}^T$ are characterized by Lemma 2.

Lemma 2: Given

$$\boldsymbol{a}^T z(k) = \boldsymbol{b}^T z(k), \qquad k = 0, 1, \cdots, N-2 \tag{20}$$

and

$$\boldsymbol{a}^T z(0) = a(0) \tag{21}$$

$$\boldsymbol{a}^T z(N-1) = a(N-1), \tag{22}$$

then

$$\boldsymbol{a}^T = \boldsymbol{b}^T + e \boldsymbol{W}^T \tag{23}$$

where $\boldsymbol{W}^T$ is given by (18) and

$$\begin{aligned} e &= (a(0) - \boldsymbol{b}^T z(0)) / \boldsymbol{W}^T z(0) \\ &= (a(N-1) - \boldsymbol{b}^T z(N-1)) / \boldsymbol{W}^T z(N-1) \end{aligned} \tag{24}$$

and b^T is constrainted by

$$b^T\{z(0)z^T(N-1)-z(N-1)z^T(0)\}W$$
$$=\{a(0)z^T(N-1)-a(N-1)z^T(0)\}W. \quad (25)$$

Proof: The proof is given in the Appendix III.

III. The Discrete Robot Models

In 1985, Neuman and Tourassis [13] introduced an inherently discrete dynamic robot model which guarantees conservation of energy (and momentum, if appropriate). The model thus satisfies the fundamental principles of classical mechanics and advances nonlinear computational mechanics for robotic manipulators. The compact model, which incorporates a minimal set of coefficients, is particularly suitable for robot engineering applications.

The cylindrical robot, depicted schematically in (1), (11), (12), and (15), consists of three degrees of freedom (DOF's): a rotation θ, a vertical translation y, and a radial translation r. Let $(\theta(k)y(k)r(k))^T$ be the position vector; then $(w(k)\nu(k)\upsilon(k))^T$ is the velocity vector.

The discrete-time cylindrical robot model in state space is formulated in [13] using the following smoothing formula:

$$\theta(k+1)-\theta(k)=\frac{T}{2}[w(k+1)+w(k)] \quad (26)$$

$$y(k+1)-y(k)=\frac{T}{2}[\nu(k+1)+\nu(k)] \quad (27)$$

$$r(k+1)-r(k)=\frac{T}{2}[\upsilon(k+1)+\upsilon(k)] \quad (28)$$

to yield discrete dynamic equations

$$\{J+j[r(k+1)]\}w(k+1)$$
$$-\{J+j[r(k)]\}w(k)=TF_\theta(k) \quad (29)$$

$$M[\nu(k+1)-\nu(k)]+MgT=TF_y(k) \quad (30)$$

$$m[\upsilon(k+1)-\upsilon(k)]-(T/2)\{[j(r(k+1)]$$
$$-j[r(k)]\}/[r(k+1)-r(k)]$$
$$*w(k)w(k+1)=TF_r(k) \quad (31)$$

where J is the constant inertia of the vertical column; $j(r)$ is the variable inertia of the radial link and is a quadratic function of the radial displacement [1], [12] (i.e., $j(r) = mr^2 - m_pRr$); m is the mass of the radical link (including the mass of the payload m_P); M is the vertically translated mass (i.e., the sum of the masses of the vertical column and radical link); F_θ, F_y, and F_r are the external forces/torques that drive the θ, y, and r DOF, respectively, and T is the sampling period. For ease of presentation, we define the following state variables and inputs:

$$x_1(k)=\theta(k) \qquad x_2(k)=y(k) \qquad x_3(k)=r(k) \quad (32)$$

$$x_4(k)=w(k) \qquad x_5(k)=\nu(k) \qquad x_6(k)=\upsilon(k) \quad (33)$$

$$x_7(k)=j(r(k))w(k)=(mr^2(k)-m_PRr(k))w(k) \quad (34)$$

$$x_8(k)=\{\{j[r(k+1)]-j[r(k)]\}/[r(k+1)-R(k)]\}$$
$$w(k)w(k+1) \quad (35)$$

$$u_1(k)=F_\theta(k) \qquad u_2(k)=F_y(k)-Mg \qquad u_3(k)=F_r(k). \quad (36)$$

Thus (26)–(31) can be rewritten as follows:

$$x_1(k+1)-x_1(k)=\frac{T}{2}[x_4(k+1)+x_4(k)] \quad (37)$$

$$x_2(k+1)-x_2(k)=\frac{T}{2}[x_5(k+1)+x_5(k)] \quad (38)$$

$$x_3(k+1)-x_3(k)=\frac{T}{2}[x_6(k+1)+x_6(k)] \quad (39)$$

$$J[x_4(k+1)-x_4(k)]+[x_7(k+1)-x_7(k)]=Tu_1(k) \quad (40)$$

$$M[x_5(k+1)-x_5(k)]=Tu_2(k) \quad (41)$$

$$m[x_6(k+1)-x_6(k)]-\frac{T}{2}x_8(k)=Tu_3(k) \quad (42)$$

$$x_7(k)=[mx_3(k)-m_PR]x_3(k)x_4(k) \quad (43)$$

$$x_8(k)=[x_7(k+1)x_4(k)-x_7(k)x_4(k+1)]/$$
$$[x_3(k+1)-x_3(k)]. \quad (44)$$

To solve these equations, Neuman and Tourassis [13] presented a nested algorithm to eliminate indeterminancies of the coupled nonlinear difference equations for each sampling point, which led to a nested two-loop iterative algorithm consisting of an outer loop (which formats the system of $2N$ equations) and an inner loop (which solves the system of $2N$ equations) for each sampling point. In this paper, we will take advantage of the shift transformation matrix of general discrete orthogonal polynomials to transform the difference equations (37)–(44) into algebraic equations so that instead of using nested iterations for each sampling point as presented by Neuman and Tourassis [13], these equations can be solved for the whole range of sampling points simultaneously. Therefore, the number of iterations can be significantly reduced. In the next section, general discrete orthogonal polynomials are introduced to analyze the discrete robot model.

IV. Analysis of the Discrete Robot Model via General Discrete Orthogonal Polynomials

To establish a discrete robot model using an algebraic approach, we develop the state variable $x_i(k)$ ($i = 1, \cdots, 8$) in terms of an N-dimensional general discrete orthogonal polynomial vector $z(k)$ as follows:

$$x_i(k)=x_i^Tz(k), \qquad i=1,2,\cdots,8, \quad k=0,1,\cdots,N-1 \quad (45)$$

where $x_i = (x_{i0}x_{i1}\cdots x_{i,N-1})^T$ is the discrete coefficient vector of the state variables $x_1(k)$. Substituting (45) and (8) into (37), we obtain

$$x_1^T(G-I)z(k)=x_4^T(G+I)z(k)T/2, \quad k=0, 1, \cdots, N-2. \tag{46}$$

Here and in the sequel, the notation $G = G(1)$, an $N \times N$ matrix, shall be used.

The initial condition of $x_1(k)$ can be expressed as

$$x_1(0)=x_1^T z(0). \tag{47}$$

From Lemma 1, (46) can be reduced to

$$x_1^T(G-I)=\frac{T}{2}x_4^T(G+I)+e_1W^T. \tag{48}$$

Combining (47) and (48), we obtain

$$x_1^T[G-I+z(0)z^T(0)]=\frac{T}{2}x_4^T(G+I)+e_1W^T+x_1(0)z^T(0) \tag{49}$$

or

$$x_1^T=[x_4^T(G+I)\frac{T}{2}+e_1W^T+x_1(0)z^T(0)][G-I+z(0)^Tz(0)]^{-1}. \tag{50}$$

Simple manipulation yields

$$e_1=\frac{x_1(0)-\left[x_4^T(G+I)\dfrac{T}{2}+x_1(0)z^T(0)\right][G-I+z(0)z^T(0)]^{-1}z(0)}{W^T[G-I+z(0)z^T(0)]^{-1}z(0)}. \tag{51}$$

Let

$$H=[G-I+z(0)z^T(0)]^{-1} \tag{52}$$

$$h=[G-I+z(0)z^T(0)]^{-1}z(0) \tag{53}$$

$$h_0=W^T[G-I+z(0)z^T(0)]^{-1}z(0); \tag{54}$$

then

$$x_1^T=[x_4^T(G+I)\frac{T}{2}+e_1W^T+x_1(0)z^T(0)]H \tag{55}$$

and

$$e_1=\{x_1(0)-[x_4^T(G+I)\frac{T}{2}+x_1(0)z^T(0)]h\}/h_0. \tag{56}$$

Similarly, (38) to (42) can also be transformed to the following algebraic equations:

$$x_2^T=\left[x_5^T(G+I)\frac{T}{2}+e_2W^T+x_2(0)z^T(0)\right]H \tag{57}$$

$$x_3^T=\left[x_3^T(G+I)\frac{T}{2}+e_3W^T+x_3(0)z^T(0)\right]H \tag{58}$$

$$x_4^T=\frac{1}{J}[Tu_1^T-x_7(G-I)+e_4W^T+Jx_4(0)z^T(0)]H \tag{59}$$

$$x_5^T=\left[\left(\frac{T}{M}\right)u_2^T+e_5W^T+x_5(0)z^T(0)\right]H \tag{60}$$

$$x_6^T=\frac{1}{m}\left[Tu_3^T+\frac{T}{2}x_3^T+e_6W^T+mx_6(0)z^T(0)\right]H \tag{61}$$

where

$$e_2=\left\{x_2(0)-\left[\frac{T}{2}x_5^T(G+I)+x_2(0)z^T(0)\right]h\right\}/h_0 \tag{62}$$

$$e_3=\left\{x_3(0)-\left[\left(\frac{T}{2}\right)x_6^T(G+I)+x_3(0)z^T(0)\right]h\right\}/h_0 \tag{63}$$

$$e_4=\{Jx_4(0)-[Tu_1^T-x_7^T(G-I)+Jx_4(0)z^T(0)]h\}/h_0 \tag{64}$$

$$e_5=\left\{x_5(0)-\left[\left(\frac{T}{M}\right)u_2^T+x_5(0)z^T(0)\right]h\right\}/h_0 \tag{65}$$

and

$$e_6=\left\{mx_6(0)-\left[Tu_3^T+\left(\frac{T}{2}\right)x_8^T+mx_6(0)z^T(0)\right]h\right\}/h_0. \tag{66}$$

Substituting $x_3(k)$ and $x_4(k)$ directly into (43) and (44), we obtain $x_7(k)$ and $x_8(k)$. Next, via (7), the discrete coefficient vectors x_7 and x_8 can be computed:

$$x_7^T=\sum_{N-1}^{k=0}[mz^T(k)x_3-m_PR]x_3^Tz(k)x_4^Tz(k)z^T(k) \tag{67}$$

$$x_8^T=\sum_{N-1}^{k=0}\{z^T(k)[G^Tx_7x_4^T-x_7x_4^TG]z(k)z^T(k)/[x_3^T(G-I)z(k)]\}+e_8W^T \tag{68}$$

where

$$e_8=\left\{x_8(0)-\sum_{N-1}^{k=0}z^T(k)\{G^Tx_7x_4^T-x_7x_4^TG\}z(k)z^T(k)z(0)/\{x_3^T(G-I)z(k)\}\right\}/(W^Tz(0)). \tag{69}$$

Thus (55)–(69) form a set of algebraic equations from which the discrete orthogonal polynomials approximate solution of $x_i(k)$, $(i = 1, \cdots, 8)$ can be obtained. Unlike Neuman and Tourassis's method [13] in which nested two-loop iteration is required for every sampling point, the discrete orthogonal polynomials approach presented here allows us to consider all the sampling points within the time interval of interest as a whole, and the discrete coefficient vectors of all the state $x_i(k)$ for the whole time interval can be calculated directly via the presented algorithm. For a time interval consisting of N sampling points, the method of [13] requires the iteration of their algorithm N times, while the present method only requires the iteration of the proposed algorithm once.

Since the exact response cannot be computed analytically, two error criteria introduced by [13] are considered to compare the proposed algorithm and several existing methods. The totally applied energy and angular momentum of the system from $t = 0$ to $t = (N - 1)T$ are

$$E_1(N-1)=\sum_{N-1}^{k=0} \{\theta(k)F_\theta(k)+r(k)F_r(k)+y(k)F_y(k)\} \tag{70}$$

$$P_1(N-1)=\sum_{N-1}^{k=0} TF_\theta(k). \tag{71}$$

The energy and angular momentum increments of the system from $t = 0$ to $t = (N - 1)T$ are

$$\begin{aligned} E_2(N-1)=&\{J+j(r(N-1))\}\{w^2(N-1)-w^2(0)\}/2\\ &+\{v^2(N-1)-v^2(0)\}m/2\\ &+\{\nu^2(N-1)-\nu^2(0)\}M/2 \end{aligned} \tag{72}$$

$$P_2(N-1)=\{J+j(r(N-1))\}\{w(N-1)-w(0)\}. \tag{73}$$

Consequently, the normalized energy and angular momentum residuals [13] are

$$\left|\frac{\Delta E(N-1)}{E(N-1)}\right|=|[E_2(N-1)-E_1(N-1)]|/|E_1(N-1)| \tag{74}$$

$$\left|\frac{\Delta P_2(N-1)}{P_2(N-1)}\right|=|[P_2(N-1)-P_1(N-1)]|/|P_1(N-1)|. \tag{75}$$

These residuals have been used by Neuman and Tourassis [13] as a basis for the comparison of several numerical algorithms. Here we use the sample example of [13] to demonstrate the feasibility and applicability of analyzing the discrete robot model by discrete orthogonal polynomials. In the following, an algorithm incorporating the linear iteration technique [2], [3] is presented to solve the state response for given control inputs.

Step 1: Input the data: M, m, m_p, J, R, T, N, $x_i(0)$ ($i = 1, \cdots, 6$), and $u_i(k)$ ($i = 1, 2, 3$; $k = 0, 1, \cdots, N - 1$). Assume a sequence of $x_3(k)$ and $x_4(k)$, ($k = 1, \cdots, N - 1$) by linear interpolation between 0 and 1.

Step 2: Choose a specific set of discrete orthogonal polynomials (e.g., Chebyshev or Laguerre) and calculate $z_i(k)$ ($i = 0, 1, \cdots, N - 1$; $k = 0, 1, \cdots, N - 1$), G, W, H, h, h_0, etc., via (2), (9), (18), and (52)–(54). Consequently, the discrete coefficient vectors x_3, x_4, u_i ($i = 1, 2, 3$) can be obtained via (7).

Step 3: Via (62), (65), (57), and (60), we can obtain e_2, e_5, and the discrete coefficient vectors x_2 and x_5 in a straightforward and unique way.

Step 4: Via (67)–(69), find the e_8, x_7, and x_8.

Step 5: From (56), (63), (64), and (66), find e_1, e_3, e_4, and e_6.

Step 6: From (55), (58), (59), and (61), we determine x_1, x_3, x_4, and x_6.

Step 7: Let Δx_3 and Δx_4 be the deviations of x_3 and x_4 as follows:

$$\Delta x_3 = x_3\text{ (new)} - x_3\text{ (old)} \tag{76}$$

$$\Delta x_4 = x_4\text{ (new)} - x_4\text{ (old)}. \tag{77}$$

Then the error e can be defined as

$$e=\Delta x_3^T\Delta x_3+\Delta x_4^T\Delta x_4. \tag{78}$$

If the value of e is small enough (e.g., $e < 10^{-12}$), then go to Step 8; otherwise, go back to Step 4 for further iterations.

Step 8: Via (3), the states $x_i(k)$ ($i = 1, 2, \cdots 6$; $k = 0, 1, \cdots, N - 1$) can be obtained. Furthermore, via (74) and (75), we then obtain the normalized energy and angular momentum residuals, i.e.,

$$\left|\frac{\Delta E(N-1)}{E(N-1)}\right| \quad\text{and}\quad \left|\frac{\Delta P_2(N-1)}{P_2(N-1)}\right|.$$

Remarks: The sufficient condition of convergence of the algorithm can be expressed as [2], [3]

$$|\Delta x_i\text{ (new)}|<|\Delta x_i\text{ (old)}|, \qquad i=3, 4. \tag{79}$$

V. Numerical Example

Given a cylindrical robot with parameters as follow [13]:

$$J=10\text{ kg}\cdot\text{m}^2 \qquad M=20\text{ kg} \qquad m=7\text{ kg}$$

$$m_P=2\text{ kg} \qquad R=1\text{ m}$$

and the initial conditions

$$(\theta(0)z(0)r(0))^T=(0\text{ (rad) } 0\text{ (m) } 0\text{ (m)})^T$$

$$(w(0)\nu(0)v(0))^T=(0\text{ (rad/s) } 0\text{ (m/s) } 0\text{ (m/s)})^T,$$

find the step response of $(\theta(k)z(k)r(k))$ and $(w(k)\nu(k)v(k))$ when the input vector is

$$(F_\theta(k)F_y(k)F_r(k))^T=(5(N-m)Mg(N)1(N))^T.$$

In Table I, we listed the computed states $(\theta(k)r(k)w(k)v(k))^T$ of the baseline cylindrical robot and the magnitudes of the normalized residuals with $T = 0.01$, 0.1 (s) and $N = 6$, 11, respectively, for discrete orthogonal polynomials.

For comparison, we also listed the results presented by [13] in Table I. From Table I we observe that the discrete Chebyshev polynomials produce the smallest normalized error residuals no matter if the sampling time $T = 10$ ms or $T = 100$ ms.

Note that in this example results obtained via discrete Laguerre polynomials are inferior to those of the discrete

TABLE I
COMPARISON OF NUMERICAL RESULTS OF BASELINE CYLINDRICAL ROBOT AT t = 3 s

	Algorithm	T (ms)	θ (rad)	r (m)	w (rad/s)	v (m/s)	$\left\lvert\frac{\Delta E(N-1)}{E(N-1)}\right\rvert$	$\left\lvert\frac{\Delta P_\theta(N-1)}{P_\theta(N-1)}\right\rvert$
1	Forward Euler [13]	10	2.1533	0.8075	1.1606	0.9127	5.E-3	2.E-3
2	Backward Euler [13]	10	2.1457	0.8165	1.1452	0.9164	5.E-3	5.E-3
3	Trapezoidal [13]	10	2.1439	0.8071	1.1551	0.9091	2.E-4	3.E-3
4	Discrete mechanics [13]	10	2.1573	0.8145	1.1563	0.9168	4.E-3	3.E-3
5	Runge–Kutta (second order) [13]	10	2.1562	0.8156	1.1535	0.9184	2.E-3	2.E-3
6	Runge–Kutta (fourth order)	10	2.1538	0.8144	1.1521	0.9187	6.E-4	5.E-4
7	Adam [13]	adaptive	2.1548	0.8170	1.1504	0.9201	5.E-5	5.E-5
8	Gear [13]	adaptive	2.1550	0.8167	1.1507	0.9199	2.E-6	1.E-6
9	Runge–Kutta–Verner [13]	adaptive	2.1551	0.8167	1.1507	0.9200	5.E-8	5.E-8
10	Neuman and Tourassis [13]	10	2.1551	0.8168	1.1506	0.9201	8.E-7	4.E-6
11	Laguerre (N = 6, d = 0.1)	10	2.2408	0.8363	1.7746	1.0721	1.E 0	6.E-1
12	Laguerre (N = 6, d = 0.1)	100	2.1820	0.8252	1.2783	0.9235	2.E-1	1.E-1
13	Laguerre (N = 11, d = 0.1)	10	2.1952	0.8254	1.4732	0.9867	5.E-1	3.E-1
14	Laguerre (N = 11, d = 0.1)	100	2.1630	0.8195	1.2621	0.9082	1.E-1	1.E-1
15	Chebyshev (N = 6)	10	2.1551	0.8167	1.1507	0.9200	2.E-8	7.E-9
16	Chebyshev (N = 6)	100	2.1540	0.8179	1.1497	0.9204	2.E-8	2.E-8
17	Chebyshev (N = 11)	10	2.1551	0.8167	1.1507	0.9200	2.E-8	9.E-9
18	Chebyshev (N = 11)	100	2.1540	0.8179	1.1497	0.9204	2.E-8	2.E-8

TABLE II
THE ENERGY AND MOMENTUM RESIDUALS OF DISCRETE CHEBYSHEV POLYNOMIALS APPROXIMATIONS FOR N = 3, 4, $\cdots$, 20 (SAMPLING TIME T = 10 ms)

N	$\left\lvert\frac{\Delta E(N-1)}{E(N-1)}\right\rvert$	$\left\lvert\frac{\Delta P_\theta(N-1)}{P_\theta(N-1)}\right\rvert$
3	2.46 E-8	1.65 E-8
4	2.57 E-14	9.54 E-9
5	1.23 E-8	1.28 E-9
6	2.46 E-8	6.97 E-9
7	2.70 E-8	1.52 E-8
8	2.66 E-13	1.91 E-8
9	6.16 E-9	7.00 E-9
10	1.23 E-8	1.89 E-9
11	2.46 E-8	8.93 E-9
12	2.46 E-8	1.33 E-8
13	3.08 E-8	9.57 E-9
14	3.70 E-8	3.11 E-8
15	4.31 E-8	2.09 E-8
16	2.29 E-14	3.18 E-8
17	3.08 E-9	3.53 E-8
18	6.16 E-9	2.93 E-8
19	6.16 E-9	3.24 E-8
20	1.23 E-9	4.32 E-8

Chebyshev polynomials. The reason is that discrete Chebyshev polynomials are defined over the interval $k = 0, 1, \cdots, N-1$, while the discrete Laguerre polynomials are defined over the infinite interval $k = 0, 1, \cdots$. Hence for a time sequence with N points, the discrete Chebyshev polynomials usually yield better approximate results than those of the discrete Laguerre polynomials.

Table II shows the energy and momentum residuals of discrete Chebyshev polynomials approximations for N = 3.4, $\cdots$, 20. From these results we may conclude that in this particular example discrete Chebyshev polynomials approximations produce comparable or smaller energy and momentum residues than those of [13].

VI. CONCLUSION

In this paper, we have introduced discrete general orthogonal polynomials to approximate the joint positions and velocities of a discrete robot model. A recursive algorithm for determining the entries of the shift transformation matrix has been developed. Based on the derived discrete shift transformation matrix of discrete general orthogonal polynomials, we are able to transform nonlinear difference equations describing a discrete robot model into nonlinear algebraic equations, thus simplifying the problem solution.

The proposed general discrete orthogonal polynomials include the discrete Laguerre polynomials, the discrete Chebyshev polynomials, and any other polynomial that possesses the recurrence relation of (2). The solutions are presented in a very general form. By specifying different values of a_i, b_i, and c_i ($i = 0, 1, \cdots, N-2$) in the recurrence relation, we can obtain the desired orthogonal polynomials approximation of a specific discrete dynamic model very easily.

The numerical example that has been used by Neuman and Tourassis [13] to confirm the feasibility and applicability of the discrete dynamic robot model is adopted here to show that the discrete Chebyshev orthogonal polynomials indeed provide smaller normalized energy and angular momentum residuals than those of [13]. Therefore, we may conclude that the present method provides a simple and straightforward algebraic approach for the analysis of discrete dynamic robot models.

Appendix I

The Recurrence Coefficients and $z_0(k)$

Some classical discrete orthogonal polynomials may be assigned specific a_i, b_i, c_i ($i = 0, 1, \cdots, N-2$), and $z_0(k)$ as follows.

Case 1: The Discrete Laguerre Polynomials [4], [6]: We have

$$a_i = d^{-1/2}(1-d)/(i+1) \tag{80}$$

$$b_i = -d^{-1/2}(i+(i+1)d)/(i+1) \tag{81}$$

$$c_i = -i/(i+1) \tag{82}$$

where the real parameter $d \in (0, 1)$ is termed the discount factor, and

$$z_{-1}(k) = 0 \tag{83}$$

$$z_0(k) = \{(1-d)d^k\}^{1/2}. \tag{84}$$

Case 2: The Discrete Chebyshev Polynomials [5], [7]: We have

$$a_i = -2B_{i+1}/(i+1) \tag{85}$$

$$b_i = (N-1)B_{i+1}/(i+1) \tag{86}$$

$$c_i = -(i/(i+1))(B_{i+1})/B_i) \tag{87}$$

$$\text{where } B_i = \{(2i-1)(2i+1)/(N^2-i^2)\}^{1/2} \tag{88}$$

$$\text{and } z_{-1}(k) = 0 \tag{89}$$

$$z_0(k) = N. \tag{90}$$

Appendix II

Derivation of the Discrete General Shift Transformation Matrix $G(m)$

The basic relationship of the shift transformation of a discrete orthogonal polynomial vector $z(k)$ is defined as

$$z(k+m) \triangleq G(m)z(k) \tag{91}$$

where $z(k)$ is an $N \times 1$ vector, and $G(m)$ is an $N \times N$ matrix. Equation (91) can be expressed as

$$Z_i(k+m) = \sum_{N-1}^{j=0} g_{ij} Z_j(k) \tag{92}$$

where $i = 0, 1, \cdots, N-1$, $k = 0, 1, \cdots, N-1$, and g_{ij} denotes the ijth element of $G(m)$. Note that $z_i(k+m)$ should still satisfy the recurrence relation, namely,

$$z_{i+1}(k+m) = (a_i(k+m)+b_i)z_i(k+m) + c_i z_i(k+m). \tag{93}$$

From (91) and (93) it is easy to show that

$$\sum_{i+1}^{j=0} g_{i+1,j} z_j(k) = (a_i(k+m)+b_i) \sum_{i}^{j=0} g_{ij} z_j(k) + c_i \sum_{i-1}^{j=0} g_{i-1,j} z_j(k). \tag{94}$$

Rearranging (2), one obtains

$$kz_j(k) = (z_{j+1}(k) - b_j z_j(k) - c_j z_{j-1}(k))/a_j. \tag{95}$$

Substituting (95) into (94) yields

$$\sum_{i}^{j=0} \{z_{j+1}(k) g_{ij} a_i/a_j + z_j(k)\{g_{ij}(ma_i+b_i) + g_{i-1,j}c_i - g_{ij}b_j a_i/a_j - g_{i+1,j}\} - z_{j-1}(k) g_{ij} c_j a_i/a_j\} = z_{i+1}(k) g_{i+1,i+1} \tag{96}$$

where $i = 0, 1, \cdots, N-2$.

Comparing the coefficients of $z_j(k)$, $j = 0, 1, \cdots, i$, one obtains

$$g_{i+1,j} = g_{ij}(ma_i + b_i - b_j a_i/a_j) + g_{i-1,j}c_i + g_{i,j-1}a_i/a_{j-1} - g_{i,j+1}c_{j+1}a_i/a_{j+1}, \quad i = 0, 1, \cdots, N-2. \tag{97}$$

Similarly, comparing the coefficients of $z_{i+1}(k)$ yields

$$g_{i+1,i+1} = g_{i,i}, \qquad i = 0, 1, \cdots, N-2; \tag{98}$$

hence

$$g_{00} = g_{11} = \cdots = g_{N-1,N-1}. \tag{99}$$

Thus the discrete general shift transformation matrix is

$$G(m) = g_{00} \begin{bmatrix} 1 & & & & & \\ g_{10} & 1 & & & 0 & \\ g_{20} & g_{21} & 1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ g_{N-1,0} & & \cdots & & g_{N-1,N-2} & 1 \end{bmatrix} \tag{100}$$

where $g_{00} = 1$ for discrete Chebyshev polynomials and $g_{00} = d^{m/2}$ for discrete Laguerre polynomials.

Appendix III

Proof of Lemmas 1 and 2

Given a set of algebraic equations describing the system

$$a^T z(k) = b^T z(k), \qquad k = 0, 1, \cdots, N-2 \tag{101}$$

subject to initial conditions

$$a^T z(0) = a(0), \tag{102}$$

let

$$a^T = b^T + v^T \tag{103}$$

where

$$v^T(z(0)z(1) \cdots z(N-1)) = (0\ 0 \cdots e). \tag{104}$$

Hence

$$v^T = (0\ 0 \cdots e)(z(0) \cdots z(N-1))^{-1} = eW^T \tag{105}$$

$$W^T = \text{last row of } (z(0) \cdots z(N-1))^{-1}. \tag{106}$$

Therefore,

$$a^T = b^T + eW^T. \tag{107}$$

Multiplying both sides of (107) by $z(0)$, we obtain

$$a(0) = b^T z(0) + eW^T z(0); \tag{108}$$

therefore,

$$e = (a(0) - b^T z(0))/W^T z(0). \tag{109}$$

Thus we have proved Lemma 1.

For the proof of the Lemma 2, (23) and (24) follow directly from the Lemma 1, while (25) can be derived by simple manipulation of (24). Hence the proof is omitted.

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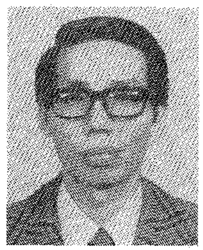

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