Products of Nilpotent Matrices

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ABSTRACT

We show that any complex singular square matrix T is a product of two nilpotent matrices A and B with $\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} T$ except when T is a 2×2 nilpotent matrix of rank one.

An $n \times n$ complex matrix T is *nilpotent* if $T^n = 0$. It is easily seen that a product of finitely many nilpotent matrices must be singular. The purpose of this note is to prove the converse.

THEOREM. *Any complex singular square matrix T which is not 2×2 nilpotent is a product of two nilpotent matrices with ranks both equal to* $\operatorname{rank} T$.

Fong and Sourour [3] considered the product of two quasinilpotent operators on a complex separable Hilbert space. They proved that on an infinite-dimensional space every compact operator is a product of two compact quasinilpotent operators. The preceding theorem gives the finite-dimensional analogue.

We remark that singular square matrices can also be expressed as products of idempotent matrices (cf. [2], [4], and [1]).

To prove our theorem, we start with the following lemma, which was observed in [3].

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LEMMA 1. *The 2×2 matrix*

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is not the product of any two nilpotent matrices.

Proof. Since 2×2 nilpotent matrices must be of the following forms

$$a\begin{bmatrix} 1 & x \\ -1/x & -1 \end{bmatrix}, \quad b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

it is easily seen that the product of any two of them cannot be equal to

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \qquad \blacksquare$$

LEMMA 2. *For any $n \neq 2$, the $n \times n$ matrix*

$$J = \begin{bmatrix} 0 & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0 \end{bmatrix}$$

is the product of two nilpotent matrices with ranks equal to rank J.

Proof.

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

for odd n and

$$J = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

for even n are the required product. [To get some insight into the factorizations above, it is instructive to observe that for odd n the n-cycle $(123\cdots n)$ is the product of the two n-cycles $(135\cdots n246\cdots n-1)$ and $(1nn-1\cdots 32)$ and that the factorization of J in this case can be obtained by replacing appropriate ones by zeros in the corresponding permutation matrices.] The nilpotency of the factors can be verified by showing that their characteristic polynomials are all x^n. ■

LEMMA 3. *Any $n\times n$ $(n\neq 2)$ nilpotent matrix T is the product of two nilpotent matrices with ranks equal to* $\operatorname{rank} T$.

Proof. Since nilpotency is preserved under the similarity of matrices, we need only consider a nilpotent Jordan matrix. In view of Lemma 2 we may further reduce the problem to the factorizations of $J_k\oplus J_2$ $(k\geqslant 2)$ and $J_2\oplus J_2\oplus J_2$, where J_i denotes the nilpotent Jordan block of size i:

$$J_i=\begin{bmatrix} 0 & & & & 0\\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0\end{bmatrix}.$$

For the former, we have

$$\begin{bmatrix} J_k & 0\\ 0 & J_2\end{bmatrix}=\begin{cases} \left[\begin{array}{c|c} 0 & J_k\\ \hline \begin{matrix}0&0\\0&1\end{matrix} & 0\end{array}\right]\left[\begin{array}{c|c} 0 & J_2\\ \hline \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\\ 0 & & 0\end{matrix} & 0\end{array}\right] & \text{if } k \text{ is even},\\[3ex] \left[\begin{array}{c|c} 0 & \begin{matrix}0 & & & 0\\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{matrix}\\ \hline \begin{matrix}0&0\\0&1\end{matrix} & 0\end{array}\right]\begin{bmatrix} 0 & J_2\\ J_k & 0\end{bmatrix} & \text{if } k \text{ is odd},\end{cases}$$

and for the latter

$$\left[\begin{array}{c|c|c} J_2 & 0 & 0 \\ \hline 0 & J_2 & 0 \\ \hline 0 & 0 & J_2 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & 0 & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ \hline J_2 & 0 & 0 \\ \hline 0 & J_2 & 0 \end{array}\right] \left[\begin{array}{c|c|c} 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \\ \hline J_2 & 0 & 0 \end{array}\right]. \qquad \blacksquare$$

Finally, we are ready for the proof of our main theorem.

Proof of Theorem. We need only consider Jordan matrices. Let J be an $n \times n$ singular Jordan matrix. We will prove by induction on n that if J is not 2×2 nilpotent, then j is the product of two nilpotent matrices $J = AB$ such that

(a) the ith row of A (jth column of B) is zero if and only if the ith row of J (jth column of J) is zero, $i, j = 1,2,\dots,n$, and

(b) the nonzero rows of A (nonzero columns of B) are independent.

In particular, $\operatorname{rank} A$ and $\operatorname{rank} B$ will both equal $\operatorname{rank} J$. Note that the factorizations for nilpotent J given in Lemmas 2 and 3 do satisfy these properties.

If $n = 2$, then

$$J = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (a \neq 0),$$

as asserted. For $n = 3$, a nonnilpotent J can be factored as one of the following:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1/a \\ 1 & -a & -1 \end{bmatrix} \begin{bmatrix} a & 0 & a \\ 2 & 0 & 1 \\ -a & 0 & -a \end{bmatrix} \qquad (a \neq 0),$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & c & b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (a, b \neq 0),$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (a \neq 0).$$

Now assume that our assertion holds for $n-1$ $(n \geqslant 4)$. Let $J = J_1 \oplus \cdots \oplus J_m$, where, for $k = 1, 2, \ldots, m$,

$$J_k = \begin{bmatrix} c_k & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & c_k \end{bmatrix}$$

is a Jordan block associated with the eigenvalue c_k. We may assume that $c_m \neq 0$. Let J' be the singular $(n-1)\times(n-1)$ matrix obtained from J by deleting its nth row and nth column. By the induction hypothesis, $J' = A'B'$, where A' and B' are nilpotent matrices with properties (a) and (b). Let a_i^T (b_j) denote the ith row of A' (jth column of B'), $i, j = 1, 2, \ldots, n-1$. To complete the proof, we need only determine a_n^T and b_n such that they are independent of $a_1^T, \ldots, a_{n-1}^T$ and $b_1, \ldots, b_{n-1}$, respectively, and satisfy

$$J = \left[\begin{array}{cccc|c} & & J' & & 0 \\ \hline 0 & \cdots & 0 & d & c_m \end{array}\right] = \left[\begin{array}{c|c} A' & 0 \\ \hline a_n^T & 0 \end{array}\right]\left[\begin{array}{c|c} B' & b_n \\ \hline 0 & 0 \end{array}\right] \equiv AB,$$

where d is either 0 or 1. In other words, a_n^T and b_n are to satisfy

$$a_n^T b_j = 0 \quad (1 \leqslant j \leqslant n-2), \qquad a_n^T b_{n-1} = d, \qquad a_n^T b_n = c_m, \tag{1}$$

and

$$a_i^T b_n = 0 \qquad (1 \leqslant i \leqslant n-1) \tag{2}$$

besides the independence property. Note that the A and B thus defined are indeed nilpotent with properties (a) and (b).

We first prove the existence of b_n which satisfies (2) and is independent of $b_1, \ldots, b_{n-1}$. Indeed, by the induction hypothesis we have rank $A' =$ rank $J' \leqslant n-2$. Hence there exists a nonzero b_n satisfying (2). If b_n is dependent on $b_1, \ldots, b_{n-1}$, say,

$$b_n = \sum_{j=1}^{n-1} x_j b_j$$

—where we may assume that $x_j = 0$ whenever $b_j = 0$, or, equivalently,

$$b_n = B'x,$$

where

$$x = [x_1 \quad \cdots \quad x_{n-1}]^T$$

—then

$$J'x = A'B'x = A'b_n = 0.$$

From the structure of the Jordan matrix J', it is easily seen that $x_j = 0$ for all j and therefore $b_n = 0$, contradicting our choice of b_n.

On the other hand, since $\operatorname{rank} B' = \operatorname{rank} J' \leqslant n-2$, the number of nonzero independent b_j's is at most $n-1$. Note that $c_m \neq 0$ implies that if $d=1$, then the size of J_m is at least 2, whence $b_{n-1} \neq 0$ by property (a). Hence there exists an a_n^T satisfying (1). If a_n^T is dependent on $a_1^T, \dots, a_{n-1}^T$, say

$$a_n^T = \sum_{i=1}^{n-1} y_i a_i^T,$$

then

$$c_m = a_n^T b_n = \sum_{i=1}^{n-1} y_i a_i^T b_n = 0,$$

a contradiction. This completes the proof. ■

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