Poles-zeros placement and decoupling in dicrete LQG systems

Wen-June Wang, MSc
Tsu-Tian Lee, MSc, PhD

Indexing terms: Poles and zeros, Polynomials, Matrix algebra, Mathematical techniques

Abstract: A design algorithm is introduced to synthesise a discrete time LQG optimal controller subject to design constraints requiring (i) complete and arbitrary stable poles placement, (ii) some zeros assignment, and (iii) input-output decoupling. The zeros placement is partially used to deal with the deterministic reference tracking and disturbances rejection problems. In the paper, the Wiener-Hopf technique is employed and two weighting matrices are shaped by the inverse optimal control method, so that the controller is optimal with respect to the chosen weighting matrices and achieves the above three goals simultaneously.

1 Introduction

The design of continuous-time multivariable multipurpose controllers has received much attention [3, 4, 5, 12, 13, 14]. In particular, Wolovich [4] designed a multipurpose controller to achieve the following:

(i) input-output decoupling
(ii) complete and arbitrary closed-loop poles placement
(iii) disturbance rejection and reference signal tracking.

Recently, Chen and Wang [3] extended this problem to continuous-time LQG optimal systems. In their approach, an LQG optimal controller is synthesised by Wiener-Hopf's technique [5, 6], and two weighting matrices $\boldsymbol{Q}(s)$ and $\boldsymbol{R}(s)$ in the cost function J are shaped by the inverse optimal control method, so that the optimal controller will achieve the three purposes given above.

On the other hand, however, few results are known concerning the design of discrete-time multivariable multipurpose controllers, with a notable exception of the paper by Grimble [1], in which the solution to discrete-time linear quadratic stochastic optimal control, with cost function including both sensitivity measures and the usual LQG error and control quadratic terms, is obtained. However, the specification of a cost function (or weighting matrices) is one of the basic requirements for the formal analysis of quantitative decision processes. In fact, any mathematical criterion in a practical control problem cannot explicitly define the optimum system uniquely [15–19]. Consequently, we develop the selection of the cost function for economic stabilisation policy by solving the inverse optimal problem.

It is well known that the LQG optimal problem has a unique solution for prespecified weighting matrices. However, the solution of the inverse optimal control problem is not unique in general [15–19], but there is a relationship between two weighting matrices. In many cases, there may be a wide variety of performance indices that are equally suitable for determining optimal policies. This suggests that the design of physical and technological systems involves the inverse optimal control problem, instead of asking for a control policy corresponding to a given performance index.

In this paper, the inverse optimal control problem is considered, and the results of Grimble [1] and Chen and Wang [3] are modified and extended to solve the problem of discrete-time multivariable multipurpose controller synthesis for LQG based design. More specifically, we will present an algorithm for systematic design of discrete-time multivariable LQG optimal controllers which ensures:

(i) input-output decoupling
(ii) complete and arbitrary closed-loop poles placement
(iii) some zeros assignment.

Of course, stability of the system is ensured in our design work. In practical control design, the requirements of realisability and causality of the controller are necessary. These requirements have also been considered in this paper.

Our development will employ the Wiener Z-domain solution [2] for controller synthesis, and two weighting matrices $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ are shaped by the inverse optimal control method [15–19] to achieve the above design purposes in the LQG optimal systems. The feature of this paper is that the plant of the system is free of the constraints: (i) poles and/or zeros are all inside the unit circle and (ii) the plant matrix is strictly proper.

2 Problem formulation

The discrete-time multivariable linear time-invariant system to be controlled has known causal, square plant $\boldsymbol{P}(Z) \in \boldsymbol{R}^{n \times n}(Z)$, disturbance $\boldsymbol{W}_d(Z) \in \boldsymbol{R}^{n \times q}(Z)$, and measurement noise $\boldsymbol{W}_n(Z) \in \boldsymbol{R}^{n \times q}(Z)$ pulse transfer function matrices. The whole system is shown in Fig. 1, where $\boldsymbol{r}(t)$, $\boldsymbol{d}(t)$ and $\boldsymbol{n}(t)$ denote reference signal, disturbance and measurement noise, respectively, and $\tilde{\boldsymbol{r}}(t)$, $\tilde{\boldsymbol{d}}(t)$ and $\tilde{\boldsymbol{n}}(t)$ are zero-mean stationary white-noise signals with constant covariance matrices Θ_R, Θ_d, and Θ_n, respectively. Furthermore, we assume that $\tilde{r}(t)$, $\tilde{d}(t)$ and $\tilde{n}(t)$ are mutually uncorrelated.

Paper 5623D (C8), first received 3rd June and in revised form 6th November 1986

Wen-June Wang is with the Institute of Electronics, and Dr. Tsu-Tian Lee was formerly with the Institute of Control Engineering, National Chiao Tung University, Hsinchu, Taiwan

Dr. Lee is now with the Department of Electrical Engineering, University of Kentucky, Lexington, Kentucky 40506-0046, USA

The reference $r(t)$ is generated by the colouring filter

$$r(t) = W_r(Z)\tilde{r}(t) \tag{1}$$

the output $y(t) \in R^n$ is given by

$$y(t) = P(Z)u(t) + W_d(Z)\tilde{d}(t) \tag{2}$$

and the tracking error is defined as

$$e(t) = r(t) - y(t) \tag{3}$$

For the optimal control problem to have a solution, $P(Z)$ is assumed to be free of unstable hidden modes. Define the sensitivity matrix as

$$S(Z) = (I_n + P(Z)C(Z))^{-1} \tag{4}$$

then

$$e(t) = r(t) - y(t) = (I - S(Z))n(t) + S(Z)(r(t) - d(t)) \tag{5}$$

and

$$u(t) = C(Z)S(Z)(r(t) - n(t) - d(t)) \tag{6}$$

Let the cost function be

$$J \triangleq \frac{1}{2\pi j} \oint_{|Z|=1} t_r\{Q(Z)\phi_e(Z) + R(Z)\phi_u(Z)\} \frac{dZ}{Z} \tag{7}$$

where $Q(Z)$ and $R(Z)$ are two weighting matrices such that $Q(Z) = \bar{Q}(Z)\bar{Q}^*(Z)$ and $R(Z) = \bar{R}(Z)\bar{R}^*(Z)$ [9], where $\bar{Q}^*(Z) \triangleq \bar{Q}^T(Z^{-1})$, $\bar{R}^*(Z) \triangleq \bar{R}^T(Z^{-1})$, and $\bar{Q}(Z)$ and $\bar{R}(Z)$ are positive definite on $|Z| = 1$. Furthermore, $\phi_e(Z)$ and $\phi_u(Z)$ denote the power spectra of the signals $e(t)$ and $u(t)$, respectively, where

$$\phi_e(Z) = (I - S(Z))\phi_n(Z)(I - S(Z))^* + S(Z)(\phi_r(Z) + \phi_d(Z))S^*(Z) \tag{8}$$

and

$$\phi_u(Z) = C(Z)S(Z)(\phi_r(Z) + \phi_d(Z) + \phi_n(Z))S^*(Z)C^*(Z)$$

where $\phi_n(Z)$, $\phi_d(Z)$ and $\phi_r(Z)$ are the power spectral densities of measurement noise $n(t)$, disturbance $d(t)$ and reference signal $r(t)$, respectively.

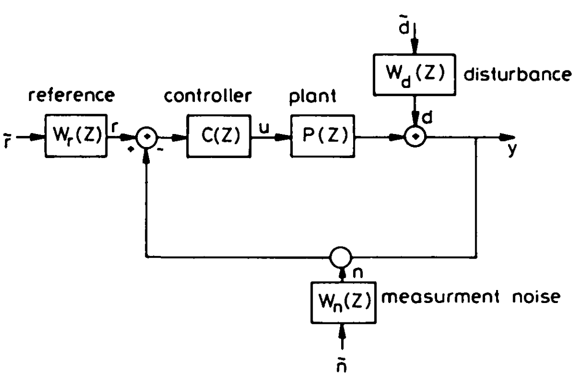

Fig. 1 *Closed-loop feedback control system*

Substituting eqn. 8 into eqn. 7 yields

$$J = \frac{1}{2\pi j} t_r \oint_{|Z|=1} \{Q(Z)[(I - S(Z))\phi_n(Z)(I - S(Z))^* + S(Z)(\phi_r(Z) + \phi_d(Z))S^*(Z)] + R(Z)[C(Z)S(Z)(\phi_r(Z) + \phi_d(Z) + \phi_n(Z))S^*(Z)C^*(Z)]\} \frac{dZ}{Z} \tag{9}$$

The problem considered in this paper is to develop a systematic design algorithm for an LQG optimal controller, and to shape weighting matrices $Q(Z)$ and $R(Z)$ so that the optimal controller achieves the three design purposes as mentioned in Section 1.

3 The optimal controller

In this Section, we will derive the optimal controller that achieves the three design purposes. Let us consider the optimal controller that minimises eqn. 9.

Introduce the spectral factors $\Delta_1(Z)$ and $\Delta_2(Z)$, which are free of poles and zeros in $|Z| > 1$, as [3, 6, 8]

$$\Delta_1^*(Z)\Delta_1(Z) = P^*(Z)Q(Z)P(Z) + R(Z)$$
$$\Delta_2(Z)\Delta_2^*(Z) = \phi_r(Z) + \phi_d(Z) + \phi_n(Z) \tag{10}$$

Then the optimal controller that minimises eqn. 9 can be stated as follows:

Theorem 1: The optimal controller $\hat{C}(Z)$ to minimise eqn. 9 for the system shown in Fig. 1, can be calculated from the optimal closed-loop transfer function

$$\hat{G}_c(Z) = \Delta_1^{-1}(Z)\{\hat{L}(Z) + [\Delta_1^{*-1}(Z)P^*(Z)Q(Z)(\phi_r(Z) + \phi_d(Z))\Delta_2^{*-1}(Z)p_i(Z)l(Z)]_+\}\Delta_2^{-1}(Z) \times p_i^{-1}(Z)l^{-1}(Z) \tag{11}$$

Using

$$\hat{C}(Z) = (I - \hat{G}_c(Z)P(Z))^{-1}\hat{G}_c(Z) \tag{12}$$

where $\hat{G}_c(Z) = P^{-1}(Z)(I - \hat{S}(Z))$, $\hat{L}(Z)$ is a polynomial matrix given by eqn. 49, $p_i(Z)$ is a polynomial which contains all poles of $P^*(Z)$ in $|Z| < 1$, and $l(Z)$ is a strict Hurwitz polynomial to be selected to ensure both $Q(Z)$ and $R(Z)$ are proper rational matrices. The proof is given in the Appendix.

Remarks:

(i) $\{f(Z)\}_+$ denotes the part associated with all the poles of $f(Z)$ in $|Z| < 1$. Similarly, $\{f(Z)\}_-$ denotes the part associated with all the poles of $f(Z)$ in $|Z| \geqslant 1$.

(ii) The degree of $\hat{L}(Z)$ and $l(Z)$ must be selected to ensure causality of $\hat{G}_c(Z)$ [5].

Thus, the problem considered in this paper may be restated as follows: Given a causal plant $P(Z)$, pulse transfer function matrices $W_d(Z)$, $W_n(Z)$ and $W_r(Z)$, which may be unstable, and power spectral densities $\phi_d(Z)$, $\phi_n(Z)$ and $\phi_r(Z)$, find the weighting matrices $Q(Z)$ and $R(Z)$ so that the optimal controller, given by eqn. 12, that minimises eqn. 9 and simultaneously achieves the three design purposes as mentioned earlier.

3.1 Realisability and causality

Before we discuss further the design of the multipupose controller, the realisability of the sensitivity matrix $S(Z)$ is discussed.

Definition: $S(Z)$ is realisable if, for some choice of controller $C(Z)$:

(i) $S(Z) = (I_n + P(Z)C(Z))^{-1}$

(ii) the closed-loop system of Fig. 1 is asymptotically stable.

Let [12]

$$P(X) = A^{-1}(Z)B(Z) = B_1(Z)A_1^{-1}(Z) \tag{13}$$

where the pairs $(A(Z), B(Z))$ and $(B_1(Z), A_1(Z))$ constitute any left-coprime and right-coprime polynomial decomposition of $\boldsymbol{P}(Z)$, respectively. The following lemma gives useful characterisation of realisable $\boldsymbol{S}(Z)$.

Lemma 1 [11]: Suppose det $(\boldsymbol{A}(Z))$ and det $(\boldsymbol{B}_1(Z))$ have no common zero in $|Z| \geqslant 1$. Then $\boldsymbol{S}(Z)$ is realisable if, and only if, det $(\boldsymbol{S}(Z)) \not\equiv 0$, and, for some appropriately dimensioned $(\boldsymbol{X}(Z), \boldsymbol{Y}(Z))$ having no unstable poles:

$$\boldsymbol{S}(Z) = \boldsymbol{Y}(Z)\boldsymbol{A}(Z)$$
$$\boldsymbol{I} - \boldsymbol{S}(Z) = \boldsymbol{B}_1(Z)\boldsymbol{X}(Z) \tag{14}$$

From lemma 1, we can see that $\boldsymbol{S}(Z)$ is realisable if, and only if, both the following conditions hold:

$$\boldsymbol{S}(Z)\boldsymbol{A}^{-1}(Z) \text{ has no poles in } |Z| \geqslant 1 \tag{14a}$$

$$\boldsymbol{B}_1^{-1}(Z)(\boldsymbol{I} - \boldsymbol{S}(Z)) \text{ has no poles in } |Z| \geqslant 1 \tag{14b}$$

Remark: There are no unstable hidden modes between $\boldsymbol{C}(Z)$ and $\boldsymbol{P}(Z)$ if both conditions 14*a* and 14*b* hold.

Definition: An arbitrary rational matrix

$$\boldsymbol{G}(Z) = \left\{\frac{G_{ij}(Z)}{F_{ij}(Z)}\right\}$$

where both $G_{ij}(Z)$ and $F_{ij}(Z)$ are polynomials of Z, i, $j = 1, \ldots, n$. Let the degree of a polynomial $f(Z)$ be denoted by $d(f(Z))$. Define $\bar{O}[\boldsymbol{G}(Z)] = \bar{g} = \max \{d(G_{ij}(Z)) - d(F_{ij}(Z))\}$ $\forall_{i,j}$, and $\underline{O}[\boldsymbol{G}(Z)] = \underline{g} = \min \{d(G_{ij}(Z)) - d(F_{ij}(Z))\}$ $\forall_{i,j}$. With these definitions, we can now state and establish the following useful result.

Lemma 2: Consider a feedback system of Fig. 1, where $\boldsymbol{P}(Z)$ is a causal plant and $\bar{O}[\boldsymbol{P}^{-1}(Z)] = \bar{q}$. If the sensitivity matrix $\boldsymbol{S}(Z)$ satisfying eqn. 14 has the following properties:

(i) $\lim_{Z\to\infty} \boldsymbol{S}(Z) = \text{diag}\,[s_1, s_2, \ldots, s_n]$, where s_i is a nonzero constant, $i = 1, \ldots, n$

(ii) $\bar{O}[\boldsymbol{I} - \boldsymbol{S}(Z)] = \bar{\delta}$

Then

$$\bar{\delta} \leqslant -\bar{q} \tag{15}$$

ensures that $\boldsymbol{C}(Z)$ is causal.

Proof: As $\boldsymbol{I} - \boldsymbol{S}(Z) = \boldsymbol{P}(Z)\boldsymbol{C}(Z)\boldsymbol{S}(Z)$, it follows that $\boldsymbol{P}^{-1}(Z)$ $(\boldsymbol{I} - \boldsymbol{S}(Z)) = \boldsymbol{C}(Z)\boldsymbol{S}(Z)$. Under the assumption that the plant $\boldsymbol{P}(Z)$ is nonsingular, therefore,

$$\bar{q} + \bar{\delta} \geqslant \bar{c} = \bar{O}[\boldsymbol{C}(Z)] \tag{16}$$

Eqn. 16 is derived based on the fact that

$$\bar{O}[\boldsymbol{S}(Z)] = \underline{O}[\boldsymbol{S}(Z)] = 0 \tag{17}$$

Hence, if $\bar{\delta} + \bar{q} \leqslant 0$, $\bar{c} \leqslant 0$, then $\boldsymbol{C}(Z)$ is causal.

3.2 Decoupling

The closed-loop transfer function of Fig. 1 is $\boldsymbol{P}(Z)\boldsymbol{C}(Z)(1 + \boldsymbol{P}(Z)\boldsymbol{C}(Z))^{-1} = \boldsymbol{I} - \boldsymbol{S}(Z)$. Under our decoupling constraint, $\boldsymbol{S}(Z)$ must have the diagonal form

$$\boldsymbol{S}(Z) = \begin{bmatrix} S_1(Z) & 0 & \cdots & 0 \\ 0 & S_2(Z) & \cdots & 0 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & \cdots & & S_n(Z) \end{bmatrix} \tag{18}$$

where $S_i(Z)$, $i = 1, \ldots, n$ are any rational functions of Z such that $\boldsymbol{S}(Z)$ satisfies lemmas 1 and 2. Then the controller is causal, reliable and input-output decoupled. The controller is given by

$$\boldsymbol{C}(Z) = \boldsymbol{P}^{-1}(Z)(\boldsymbol{I} - \boldsymbol{S}(Z))\boldsymbol{S}^{-1}(Z) \tag{19}$$

3.3 Reference signal tracking and disturbance rejection [7]

Consider the system of Fig. 1. Suppose $\bar{\boldsymbol{r}}(t)$, $\bar{\boldsymbol{d}}(t)$ and $\bar{\boldsymbol{n}}(t)$ are impulse functions $\delta(t)$, then $\boldsymbol{r}(t)$ and $\boldsymbol{n}(t)$ are deterministic signals generated by pulse transfer function matrices $\boldsymbol{W}_r(Z)$, $\boldsymbol{W}_d(Z)$ and $\boldsymbol{W}_n(Z)$, respectively. To distinguish the deterministic signals and stochastic signals, we denote $\bar{\boldsymbol{r}}(t)$, $\bar{\boldsymbol{d}}(t)$ and $\bar{\boldsymbol{n}}(t)$ to be deterministic signals instead of the stochastic notation $\boldsymbol{r}(t)$, $\boldsymbol{d}(t)$ and $\boldsymbol{n}(t)$. Similarly, the following $\bar{\boldsymbol{y}}(t)$ and $\bar{\boldsymbol{e}}(t)$ denote the deterministic signals derived by deterministic inputs $\bar{\boldsymbol{r}}(t)$, $\bar{\boldsymbol{d}}(t)$ and $\bar{\boldsymbol{n}}(t)$

$$\bar{\boldsymbol{y}}(Z) = (\boldsymbol{I} - \boldsymbol{S}(Z))\bar{\boldsymbol{r}}(Z) + \boldsymbol{S}(Z)\bar{\boldsymbol{d}}(Z) - (\boldsymbol{I} - \boldsymbol{S}(Z))\bar{\boldsymbol{n}}(Z) \tag{20}$$

and

$$\begin{aligned}\bar{\boldsymbol{e}}(Z) &\triangleq \bar{\boldsymbol{r}}(Z) - \bar{\boldsymbol{y}}(Z) \\ &= \boldsymbol{S}(Z)(\bar{\boldsymbol{r}}(Z) - \bar{\boldsymbol{d}}(Z)) + (\boldsymbol{I} - \boldsymbol{S}(Z))\bar{\boldsymbol{n}}(Z)\end{aligned} \tag{21}$$

where

$$\left.\begin{aligned}\bar{\boldsymbol{r}}(Z) &= \boldsymbol{W}_r(Z)\delta(Z) = \boldsymbol{W}_r(Z)[1 \ldots 1]^T \\ \bar{\boldsymbol{d}}(Z) &= \boldsymbol{W}_d(Z)\delta(Z) = \boldsymbol{W}_d(Z)[1 \ldots 1]^T \\ \bar{\boldsymbol{n}}(Z) &= \boldsymbol{W}_n(Z)\delta(Z) = \boldsymbol{W}_n(Z)[1 \ldots 1]^T\end{aligned}\right\} \tag{22}$$

hence

$$\left.\begin{aligned}\boldsymbol{S}(Z)\bar{\boldsymbol{r}}(Z) &= \boldsymbol{S}(Z)\boldsymbol{W}_r(Z)[1 \ldots 1]^T \\ \boldsymbol{S}(Z)\bar{\boldsymbol{d}}(Z) &= \boldsymbol{S}(Z)\boldsymbol{W}_d(Z)[1 \ldots 1]^T \\ (\boldsymbol{I} - \boldsymbol{S}(Z))\bar{\boldsymbol{n}}(Z) &= (\boldsymbol{I} - \boldsymbol{S}(Z))\boldsymbol{W}_n(Z)[1 \ldots 1]^T\end{aligned}\right\} \tag{23}$$

Therefore, if each entry $S_i(Z)$ of $\boldsymbol{S}(Z)$ is properly selected so that the matrices $\boldsymbol{S}(Z)\boldsymbol{W}_d(Z)$, $(\boldsymbol{I} - \boldsymbol{S}(Z))\boldsymbol{W}_n(Z)$ and $\boldsymbol{S}(Z)\boldsymbol{W}_r(Z)$ are stable, then the unstable disturbance and noise can be rejected and reference signal (which may be unstable) can be tracked. Let

$$S_k(Z) = \frac{\alpha_k(Z)\beta_k(Z)}{q_k(Z)} \qquad k = 1, 2, \ldots, n \tag{24}$$

where $q_k(Z)$ is a polynomial that contains the desired stable poles Z_{ki}, with multiplicity j_i, $i = 1, \ldots, m_k$ for each k, i.e. $q_k(Z) = \tilde{q}_k(Z)\bar{q}_k(Z)$, where

$$\tilde{q}_k(Z) = \prod_{i=1}^{m_k} (Z - Z_{ki})^{j_i} \tag{25}$$

and $\bar{q}_k(Z)$ are polynomials of Z to be determined to ensure $\bar{O}(\boldsymbol{S}(Z)) = 0$, and $\alpha_k(Z)$ is a polynomial selected for tracking reference $\bar{\boldsymbol{r}}(Z)$ and rejecting disturbance $\bar{\boldsymbol{d}}(Z)$. Hence,

$$\alpha_k(Z) = \alpha_{dk}(Z)\alpha_{rk}(Z)\alpha_{pk}(Z)$$

where $\alpha_{dk}(Z)$ and $\alpha_{rk}(Z)$ are polynomials which contain all the unstable poles of the kth row of $\boldsymbol{W}_d(Z)$ and the kth row of $\boldsymbol{W}_r(Z)$, respectively, so that $\boldsymbol{S}(Z)\boldsymbol{W}_d(Z)$ and $\boldsymbol{S}(Z)\boldsymbol{W}_r(Z)$ are stable. Furthermore, $\alpha_{pk}(Z)$ is a polynomial which contains all unstable poles of the kth row of $\boldsymbol{A}^{-1}(Z)$ (for condition 14*a*). In addition,

$$\beta_k(Z) = h_k \prod_{i=1}^{t_k} (Z - t_{ki}) \prod_{j=1}^{\delta_k'} (Z - \bar{Z}_{kj})$$

where $\delta_k' = \bar{q} - 1$, $t_k =$ (number of unstable poles of the kth row of $\boldsymbol{W}_n(Z)$) + (number of unstable poles of the kth column of $\boldsymbol{B}_1^{-1}(Z)$), t_{ki} for $i = 1, 2, \ldots, t_k$ are unknown constants, yet to be determined so that condition 14*b* is satisfied and measurement noise $\bar{\boldsymbol{n}}(Z)$ is rejected (i.e. $(\boldsymbol{I} - \boldsymbol{S}(Z))\boldsymbol{W}_n(Z)$ is stable), and constants h_k and $\bar{Z}_{kj}$, for $j = 1, \ldots, \delta_k'$, are chosen so that $\boldsymbol{I} - \boldsymbol{S}(Z)$ satisfies lemma 2.

As $\bar{O}(S(Z)) = \underline{O}(S(Z)) = 0$, it follows that $\bar{q}_k(Z)$ can be an arbitrary strict Hurwitz polynomial with degree $d(\bar{q}_k(Z)) = d(\alpha_k(Z)) + d(\beta_k(Z)) - d(\tilde{q}_k(k))$.

3.4 Determination of $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$

In general, in the inverse optimal control problem, the choices of $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ are not unique. For convenience, we choose $\boldsymbol{Q}(Z)$ such that the term $[.]_+$ in eqn. 11 is equal to zero; e.g. let

$$Q(Z) = h(\phi_r(Z) + \phi_d(Z))^{-1}/(l(Z)l^*(Z))$$

then $\Delta_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\phi_r(Z) + \phi_d(Z))\Delta_2^{*-1}(Z)p_i(Z)l(Z)$ has no pole in $|Z| < 1$. Note that h in $\boldsymbol{Q}(Z)$ is a small positive constant yet to be determined, to ensure $\boldsymbol{R}(Z)$ is positive-definite on $|Z| = 1$; $l(Z)$ is an arbitrary strict Hurwitz polynomial with enough degree to ensure that both $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ are proper (relisable). Substituting

$$Q(Z) = h(\phi_r(Z) + \phi_d(Z))^{-1}/(l(Z)l^*(Z)) \tag{26}$$

into eqn. 11, we obtain

$$\Delta_1(Z) = \hat{L}(Z)\Delta_2^{-1}(Z)p_i^{-1}(Z)\hat{G}_c^{-1}(Z)l^{-1}(Z) \tag{27}$$

where $\hat{L}(Z)$ is a polynomial of Z yet to be chosen such that $\Delta_1(Z)$ is free of poles and zeros in $|Z| > 1$. Hence, the corresponding weighting matrix $\boldsymbol{R}(Z)$ is

$$\boldsymbol{R}(Z) = \Delta_1^*(Z)\Delta_1(Z) - \boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)\boldsymbol{P}(Z) \tag{28}$$

4 The algorithm

In this Section, we present an algorithm to synthesise an optimal controller that achieves the goals described in Section 1:

Step 1: Perform the factorisation of $\boldsymbol{P}(Z)$ as eqn. 13 to obtain $\boldsymbol{A}(Z)$ ($\boldsymbol{A}_1(Z)$) and $\boldsymbol{B}(Z)(\boldsymbol{B}_1(Z))$.

Step 2: Choose $\boldsymbol{S}(Z)$ as eqns. 18 and 24. Assign $\alpha_k(Z)$, which contains the unstable poles of $\boldsymbol{W}_d(Z)$, $\boldsymbol{W}_r(Z)$ and $\boldsymbol{A}^{-1}(Z)$. Form polynomials $\beta_k(Z)$ with unknown coefficients, and set up $q_k(Z) = \tilde{q}_k(Z)\bar{q}_k(k)$ where $d(\bar{q}_k(Z)) = d(\alpha_k(Z)) + d(\beta_k(Z)) - d(\tilde{q}_k(Z))$. (Note: $\tilde{q}_k(Z)$ is assigned previously.)

Step 3: Choose the unknown coefficients of $\beta_k(Z)$ such that $\boldsymbol{S}(Z)$ satisfies eqns. 14 and 15, and rejects the unstable noise $\bar{\boldsymbol{n}}(Z)$.

Step 4: From eqn. 10, we obtain $\Delta_2(Z)$, which is free of poles and zeros in $|Z| > 1$.

Step 5: Choose $\boldsymbol{Q}(Z)$ as in eqn. 26 to let $[\Delta_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\phi_r(Z) + \phi_d(Z))l(Z)\Delta_2^{*-1}(Z)p_i(Z)]_+ = 0$ in eqn. 11, where $\boldsymbol{Q}(Z)$ is positive-definite for all $|Z| = 1$.

Step 6: Calculate $\hat{\boldsymbol{G}}_c(Z) = \boldsymbol{P}^{-1}(Z)\ (\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))$. This $\hat{\boldsymbol{G}}_c(Z)$ ensures input-output decoupling, deterministic disturbance and noise rejection, and deterministic reference-signal tracking.

Step 7: Select suitable $\hat{L}(Z)$ such that $\Delta_1(Z)$ in eqn. 27 is free of poles and zeros in $|Z| > 1$.

Step 8: Determine a positive small constant h to ensure that $\boldsymbol{R}(Z)$ obtained from eqn. 28 is positive-definite.

Step 9: From eqn. 12, we obtain the optimal controller $\hat{\boldsymbol{C}}(Z)$ corresponding to the particular choice of $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ obtained from steps 5 and 8, respectively.

Remarks:

(i) Poles-zeros placement and decoupling are achieved in step 2.

(ii) The choice of $\boldsymbol{Q}(Z)$ in step 5 is not unique, and hence the corresponding $\boldsymbol{R}(Z)$ is also nonunique [15–19].

5 Illustrative example

Consider the system as shown in Fig. 1, where

$$\boldsymbol{P}(Z) = \begin{bmatrix} \dfrac{Z + 0.5}{Z(Z + 1.2)} & \dfrac{3(Z + 1.5)}{(Z + 1.2)(Z - 0.5)} \\ 0 & \dfrac{Z + 1.5}{Z(Z - 0.5)} \end{bmatrix}$$

$$\boldsymbol{W}_r(Z) = \boldsymbol{W}_d(Z) = \frac{Z}{\sqrt{2}(Z - 1)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\boldsymbol{W}_n(Z) = \frac{Z}{Z - 0.5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$\phi_r(Z) = \phi_d(Z) = \frac{1}{2(Z - 1)(Z^{-1} - 1)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\phi_n(Z) = \frac{1}{(Z - 0.5)(Z^{-1} - 0.5)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Furthermore, assume that $\bar{r}(t)$, $\bar{d}(t)$ and $\bar{n}(t)$ are deterministic signals generated by pulse generators $\boldsymbol{W}_r(Z)$, $\boldsymbol{W}_d(Z)$ and $\boldsymbol{W}_n(Z)$, respectively. The problem is to design a controller $\boldsymbol{C}(Z)$ so that the closed-loop system ensures (i) poles at -0.3 and 0.5 for each channel, (ii) input-output decoupling, and (iii) having zeros at certain locations for tracking $\bar{r}(t)$ as well as rejecting $\bar{d}(t)$ and $\bar{n}(t)$. Moreover, selecting two suitable weighting matrices $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ so that the controller is also optimal with respect to the chosen weighting matrices.

Solution: We solve the problem following our algorithm step by step as follows:

Step 1: From eqn. 13,

$$\boldsymbol{P}(Z) = \boldsymbol{A}^{-1}(Z)\boldsymbol{B}(Z) = \boldsymbol{B}_1(Z)\boldsymbol{A}_1^{-1}(Z) \tag{29}$$

Choose

$$\boldsymbol{A}(Z) = \begin{bmatrix} Z(Z + 1.2) & -3Z^2 \\ 0 & Z(Z - 0.5) \end{bmatrix}$$

$$\boldsymbol{A}_1(Z) = \begin{bmatrix} Z(Z + 1.2) & 1.2857Z \\ 0 & Z(Z - 0.5) \end{bmatrix}$$

$$\boldsymbol{B}(Z) = \begin{bmatrix} Z + 0.5 & 0 \\ 0 & Z + 1.5 \end{bmatrix}$$

$$\boldsymbol{B}_1(Z) = \begin{bmatrix} Z + 0.5 & 4.2857Z \\ 0 & Z + 1.5 \end{bmatrix} \tag{30}$$

Step 2: Let $\alpha_1(Z) = (Z - 1)(Z + 1.2)$, $\alpha_2(Z) = (Z - 1)$, and $\tilde{q}_1(Z) = \tilde{q}_2(Z) = (Z + 0.3)(Z - 0.5)$. Because $\bar{q} = 1$, $\delta'_1 = \delta'_2 = 0$, $t_1 = 0$ and $t_2 = 1$, therefore $\bar{q}_1(Z) = \bar{q}_2(Z) = 1$ and $\beta_1(Z) = h_1$, $\beta_2(Z) = h_2(Z - t_{21})$. Hence

$$\hat{\boldsymbol{S}}(Z) = \frac{1}{(Z + 0.3)(Z - 0.5)} \times \begin{bmatrix} h_1(Z + 1.2)(Z - 1) & 0 \\ 0 & h_2(Z - 1)(Z - t_{21}) \end{bmatrix} \tag{31}$$

Step 3: Lemma 2 requires $\bar{O}(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z)) \leqslant -1$, hence $h_1 = h_2 = 1$. To satisfy condition 14*b*, it is required that $t_{21} = -0.54$. Thus,

$$\boldsymbol{I} - \hat{\boldsymbol{S}}(Z) = \frac{1}{(Z+0.3)(Z-0.5)} \times \begin{bmatrix} 1.05 - 0.4Z & 0 \\ 0 & 0.26(Z+1.5) \end{bmatrix} \quad (32)$$

Step 4: From eqn. 10,

$$\Delta_2(Z)\Delta_2^*(Z) = \frac{3.25 - 1.5Y}{(2-Y)(1.25-0.5Y)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (33)$$

where $Y \triangleq Z + Z^{-1}$. Therefore,

$$\Delta_2(Z) = \frac{Z^{-1} - 1.5}{(Z^{-1}-1)(Z-0.5)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (34)$$

Step 5: In order to let

$$[\Delta_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\phi_r(Z) + \phi_d(Z))\Delta_2^{*-1}(Z) \times \boldsymbol{P}_i(Z)\boldsymbol{l}(Z)]_+ = 0$$

where $\boldsymbol{P}_i(Z) = 1 + 1.2Z$, we choose

$$\boldsymbol{Q}(Z) = \frac{h(\phi_r(Z) + \phi_d(Z))^{-1}}{\boldsymbol{l}(Z)\boldsymbol{l}^*(Z)} = h\frac{(2-Y)}{(1.25-0.5Y)^3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (35)$$

Where $\boldsymbol{l}(Z)$ is chosen as $\boldsymbol{l}(Z) = (Z - 0.5)^3$ to ensure that both $\boldsymbol{Q}(Z)$ and $\boldsymbol{R}(Z)$ are proper.

Step 6: From $\hat{\boldsymbol{G}}_c(Z) = \boldsymbol{P}^{-1}(Z)(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))$, it follows that

$$\hat{\boldsymbol{G}}_c(Z) = \frac{1}{Z+0.3} \times \begin{bmatrix} \dfrac{Z(Z+1.2)(1.05-0.4Z)}{(Z+0.5)(Z-0.5)} & \dfrac{-0.78(Z+1.5)Z^2}{(Z+0.5)(Z-0.5)} \\ 0 & 0.26Z \end{bmatrix} \quad (36)$$

Step 7: In order to let $\Delta_1(Z)$ be free of poles and zeros in $|Z| > 1$, we select

$$\hat{\boldsymbol{L}}(Z) = \begin{bmatrix} (Z+1.2)(1.05-0.4Z) & -1.56Z \\ 0 & 1 \end{bmatrix} \quad (37)$$

then

$$\Delta_1(Z) = \frac{(Z^{-1}-1)(Z+0.3)}{(1+1.2Z)(Z-0.5)(Z^{-1}-1.5)} \times \begin{bmatrix} \dfrac{Z+0.5}{Z} & 3 \\ 0 & \dfrac{1}{0.26Z(Z-0.5)} \end{bmatrix} \quad (38)$$

Step 8: $\boldsymbol{R}(Z)$ is obtained from eqn. 28

$$\boldsymbol{R}(Z) = \begin{bmatrix} R_{11}(Z) & R_{12}(Z) \\ R_{21}(Z) & R_{22}(Z) \end{bmatrix} \quad (39)$$

where

$$R_{11}(Z) = H(Z)(K_1(Z) - K_2(Z))$$

$$R_{12}(Z) = 3H(Z)\left[\frac{K_1(Z)}{Z^{-1}(Z+0.5)} - \frac{(Z+1.5)(Z^{-1}+0.5)}{Z^{-1}(Z-0.5)(1.25+0.5Y)} K_2(Z)\right]$$

$$R_{21}(Z) = 3H(Z)\left[\frac{K_1(Z)}{Z(Z^{-1}+0.5)} - \frac{(Z+0.5)(Z^{-1}+1.5)}{Z(Z^{-1}-0.5)(1.25+0.5Y)} K_2(Z)\right]$$

$$R_{22}(Z) = H(Z)\left[\frac{9}{1.25+0.5Y} K_1(Z) + \frac{14.8K_1(Z)}{(1.25-0.5Y)(1.25+0.5Y)} - \frac{9(3.25+1.5Y)}{(1.25+0.5Y)(1.25-0.5Y)^3}\right] - \frac{(2-Y)(3.25+1.5Y)}{(1.25-0.5Y)^4} h$$

in which $Y = Z^{-1} + Z = 2\cos\omega$, $Z = e^{j\omega}$;

$$K_1(Z) = \frac{1.09 + 0.3Y}{3.25 - 1.5Y}$$

$$H(Z) = \frac{(2-Y)(1.25+0.5Y)}{(2.44+1.2Y)(1.25-0.5Y)}$$

$$K_2(Z) = \frac{h}{(1.25-0.5Y)^2}$$

Here, $0 \leqslant h < 0.067$ guarantees $\boldsymbol{R}(Z)$ to be positive-definite.

Step 9: From eqn. 12, the optimal controller with respect to the weighting matrices $\boldsymbol{Q}(Z)$ in eqn. 35 and $\boldsymbol{R}(Z)$ in eqn. 36 is

$$\hat{\boldsymbol{C}}(Z) = \frac{1}{Z-1} \begin{bmatrix} \dfrac{Z(-0.4Z+1.05)}{Z+0.5} & \dfrac{-0.78Z^2(Z+1.5)}{(Z+0.5)(Z+0.54)} \\ 0 & \dfrac{0.26Z(Z-0.5)}{Z+0.54} \end{bmatrix} \quad (40)$$

Figs. 2–5 show the simulation results of this example.

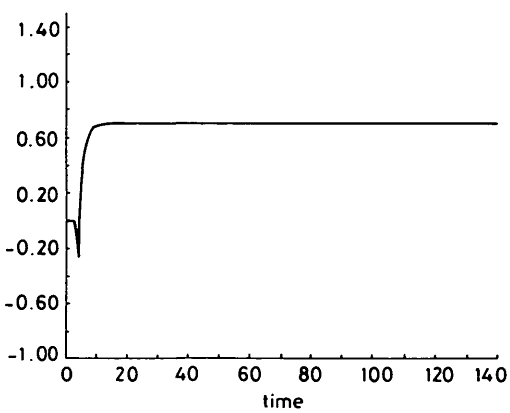

Fig. 2 *Deterministic response $\bar{y}_1(t)$ of the example with $\bar{d}(t) = \bar{n}(t) = 0$*

The responses $\bar{y}(t)$ of this example corresponding to

(i) $\bar{r}(t) = \frac{1}{\sqrt{2}} u_s(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

and

$\bar{n}(t) = \bar{d}(t) = 0$,

and

(ii) $\bar{r}(t) = \bar{d}(t) = \frac{1}{\sqrt{2}} u_s(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

and

$\bar{n}(t) = e^{-0.69t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

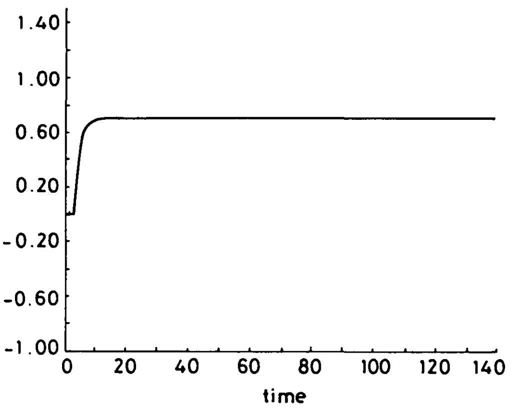

Fig. 3 *Deterministic response $\bar{y}_2(t)$ of the example with $\bar{d}(t) = \bar{n}(t) = 0$*

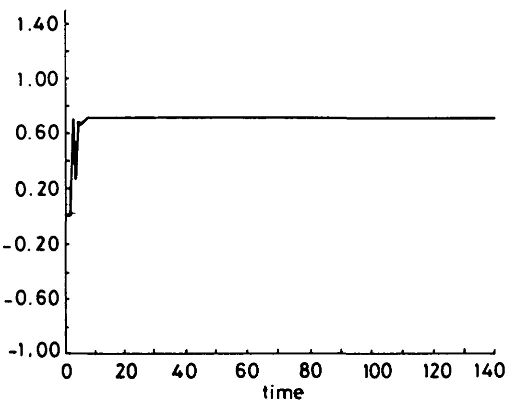

Fig. 4 *Deterministic response $\bar{y}_1(t)$ of the example with*

$\bar{d}(t) = \frac{1}{\sqrt{2}} u_s(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\bar{n}(t) = e^{-0.69t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

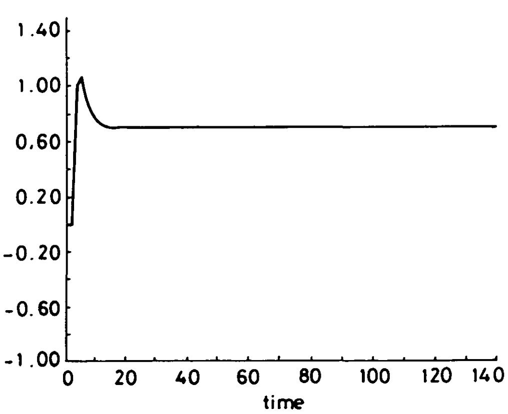

Fig. 5 *Deterministic response $\bar{y}_2(t)$ of the example with*

$\bar{d}(t) = \frac{1}{\sqrt{2}} u_s(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\bar{n}(t) = e^{-0.69t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

are shown in Figs. 2 and 3, respectively, where $u_s(t)$ denotes a unit step input. It is clear that the controller of eqn. 40 completely achieves these objectives: (i) tracking $\bar{r}(t)$ and (ii) rejecting $\bar{n}(t)$ and $\bar{d}(t)$.

6 Conclusions

This paper presents an algorithm for synthesising a discrete multivariable optimal controller for input-output decoupling, complete and arbitrary closed-loop pole placement, and disturbance rejection and reference signal tracking. The weighting matrices $Q(Z)$ and $R(Z)$ corresponding to the optimal controller of the LQG systems are derived in a very simple and straightforward method. It should be noted that the choice of $Q(Z)$ is not unique, consequently the choice of $R(Z)$ is also nonunique. This is the characteristic of inverse optimal control problems. Moreover, as it is possible to select a suitable sensitivity matrix $S(Z)$ that possesses extra freedoms after achieving the three specified design goals, it is possible to design an optimal controller that not only achieves three design goals specified in this paper, but also ensures some other design objectives. This problem is currently under investigation.

7 Acknowledgment

W.J. Wang would like to thank Professor B.S. Chen, of the Tatung Institute of Technology, Taipei, Taiwan, for several helpful discussions on the subjects of inverse optimal problems.

8 References

1 GRIMBLE, M.J.: 'LQG design of discrete systems using a dual criterion', *IEE Proc. D, Control Theory & Appl.*, 1985, **132**, (2), pp. 61–68
2 GRIMBLE, M.J.: 'Solution of the discrete-time stochastic optimal control problem in the z-domain', *Int. J. Syst. Sci.*, 1979, **10**, pp. 1369–1390
3 CHEN, B.S., and WANG, W.J.: 'Multipurpose controller synthesis in LQG optimal systems', *IEE Proc. D, Control Theory & Appl.*, 1984, **131**, (5), pp. 182–188
4 WOLOVICH, W.A.: 'Multipurpose controllers for multivariable systems', *IEEE Trans.*, 1981, **AC-26**, pp. 162–170
5 YOULA, D.C., JABR, H.A., and BONGIORNO, J.J.: 'Modern Wiener Hopf design of optimal controllers, Part I and II', *ibid.*, 1976, **AC-21**, pp. 3–13, pp. 319–338
6 GRIMBLE, M.J.: 'Solution of the stochastic optimal control problem in the s-domain for systems with time delay', *Proc. IEE, Contr. & Sci.*, 1979, **126**, pp. 697–704
7 SAEKS, R., and MURRY, J.: 'Feedback system design: the tracking and disturbance rejection problems', *IEEE Trans.*, 1981, **AC-26**, pp. 203–217
8 SHAKED, U.: 'A general transfer function approach to linear stationary filtering and steady-state optimal control problems', *Int. J. Control*, 1976, **24**, pp. 741–770
9 KUCERA, V.: 'Discrete linear control — the polynomial equation approach' (Czechoslovak Academy of Sciences, Prague, 1979)
10 ASTROM, K. J., and WITTENMARK, B.: 'Computer controlled systems theory and design' (Prentice-Hall, Inc., Englewood Cliffs, 1984)
11 SAFONOV, M.G., and CHEN, B.S.: 'Multivariable stability-margin optimisation with decoupling and output regulation', *IEE Proc. D, Control Theory & Appl.*, 1982, **129**, (6), pp. 276–282
12 DESOR, C.A., LIU, R.W., MURRAY, J., and SAEKS, R.: 'Feedback system design: the fractional representation approach to analysis and synthesis', *IEEE Trans.*, 1980, **AC-25**, pp. 399–412
13 DESOR, C.A., and CHEN, M.J.: 'Design of multivariable feedback system with stable plant', *ibid.*, 1981, **AC-26**, pp. 408–414
14 DESOR, C.A., and CHEN, M.J.: 'Extention of design algorithm of "Design of multivariable feedback systems with stable plant" to the tracking problem', *ibid.*, 1981, **AC-26**, pp. 526–527
15 PARK, J.G., and LEE, K.Y.: 'An inverse optimal control problem and its application to the choice of performance index for economic stabilization policy', *ibid.*, 1975, **SMC-5**, pp. 64–76

16 KALMAN, R.E.: 'When is a linear control system optimal?' *J. Basic Eng.*, 1964, **86**, pp. 51–60
17 JAMESON, A., and KREINDLER, E.: 'Inverse problem of linear optimal control', *SIAM J. Control.*, 1973, **11**, pp. 1–19
18 YOKOYAMA, R., and KINNEN, E.: 'The inverse problem of the optimal regulator', *IEEE Trans.*, 1972, **AC-17**, pp. 499–504
19 CHEN, B.S.: 'The inverse problem of LQG control via frequency dependent cost/noise matrices'. Ph.D. Dissertation, University of Southern California, 1982

9 Appendix

Proof of theorem 1: From eqns. 7 and 8 we set $\boldsymbol{S}(Z) = \hat{\boldsymbol{S}}(Z) + \varepsilon \boldsymbol{T}(Z)$ (where the symbol ^ denotes 'optimal'). The necessary condition for minimising J is

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

Therefore

$$\begin{aligned}\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{2}{2\pi j}\, t_r \oint_{|Z|=1} \{\boldsymbol{Q}(Z)[(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))\boldsymbol{\phi}_n(Z)(-\boldsymbol{T}^*(Z)) \\ &\quad + \hat{\boldsymbol{S}}(Z)(\boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z))\boldsymbol{T}^*(Z)] \\ &\quad + \boldsymbol{R}(Z)[-\boldsymbol{P}^{-1}(Z)(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))\boldsymbol{\Sigma}(Z)\boldsymbol{T}^*(Z)]\} \frac{dZ}{Z} \\ &= \frac{2}{2\pi j}\, t_r \oint_{|Z|=1} \{\boldsymbol{Q}(Z)[\boldsymbol{S}(Z)\boldsymbol{\Sigma}(Z) - \boldsymbol{\phi}_n(Z)] \\ &\quad - \boldsymbol{P}^{-1*}(Z)[\boldsymbol{R}(Z)\boldsymbol{P}^{-1}(Z) \\ &\quad \times (\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))\boldsymbol{\Sigma}(Z)]\}\boldsymbol{T}^*(Z)\frac{dZ}{Z} \\ &= \frac{2}{2\pi j}\, t_r \oint_{|Z|=1} \{\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)[\hat{\boldsymbol{S}}(Z)\boldsymbol{\Sigma}(Z) - \boldsymbol{\phi}_n(Z)] \\ &\quad - [\boldsymbol{R}(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Sigma}(Z)]\}\boldsymbol{T}^*(Z)\frac{dZ}{Z} = 0 \end{aligned} \tag{41}$$

where

$$\begin{aligned}\boldsymbol{\Sigma}(Z) &= \boldsymbol{\phi}_n(Z) + \boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z) \\ \hat{\boldsymbol{G}}_c(Z) &= \boldsymbol{P}^{-1}(Z)(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z)) = \hat{\boldsymbol{C}}(Z)\hat{\boldsymbol{S}}(Z)\end{aligned} \tag{42}$$

After adding $\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_d(Z) + \boldsymbol{\phi}_r(Z))$ to eqn. 41 and subtracting $\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_d(Z) + \boldsymbol{\phi}_r(Z))$ from the resulting equation yields

$$\begin{aligned}\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{2}{2\pi j}\, t_r \oint_{|Z|=1} -\{\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{I} - \hat{\boldsymbol{S}}(Z))\boldsymbol{\Sigma}(Z) \\ &\quad - \boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z)) \\ &\quad + \boldsymbol{R}(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Sigma}(Z)\}\boldsymbol{T}^*(Z)\frac{dZ}{Z} \\ &= \frac{-1}{\pi j}\, t_r \oint_{|Z|=1} \{[\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)\boldsymbol{P}(Z) + \boldsymbol{R}(Z)] \\ &\quad \times \hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Sigma}(Z) - \boldsymbol{P}^*(Z)\boldsymbol{Q}(Z) \\ &\quad \times (\boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z))]\}\boldsymbol{T}^*(Z)\frac{dZ}{Z} = 0\end{aligned} \tag{43}$$

Perform the spectral factorisation:

$$\begin{aligned}\boldsymbol{\Delta}_1^*(Z)\boldsymbol{\Delta}_1(Z) &= \boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)\boldsymbol{P}(Z) + \boldsymbol{R}(Z) \\ \boldsymbol{\Delta}_2(Z)\boldsymbol{\Delta}_2^*(Z) &= \boldsymbol{\Sigma}(Z) = \boldsymbol{\phi}_n(Z) + \boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z)\end{aligned} \tag{44}$$

where $\boldsymbol{\Delta}_1(Z)$ and $\boldsymbol{\Delta}_2(Z)$ are free of poles and zeros in $|Z| \geqslant 1$. Hence, eqn. 43 can be written as

$$\begin{aligned}\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{1}{\pi j}\, t_r \oint_{|Z|=1} \{\boldsymbol{\Delta}_1^*(Z)\boldsymbol{\Delta}_1(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Delta}_2(Z)\boldsymbol{\Delta}_2^*(Z) \\ &\quad - \boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) + \boldsymbol{\phi}_d(Z))\}\boldsymbol{T}^*(Z)\frac{dZ}{Z} \\ &= \frac{1}{\pi j}\, t_r \oint_{|Z|=1} \boldsymbol{F}(Z)\boldsymbol{\Delta}_1^*(Z)\boldsymbol{\Delta}_2^*(Z)\boldsymbol{T}^*(Z)\,dZ = 0\end{aligned} \tag{45}$$

where

$$\begin{aligned}\boldsymbol{F}(Z) &= \{\boldsymbol{\Delta}_1(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Delta}_2(Z) - \boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\phi_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)\}Z^{-1}\end{aligned} \tag{46}$$

Because $\boldsymbol{\Delta}_1^*(Z)\boldsymbol{\Delta}_2^*(Z)\boldsymbol{T}^*(Z)$ is analytic in $|Z| < 1$, it follows that eqn. 45 is true provided that all the poles of $\boldsymbol{F}(Z)$ lie in $|Z| \geqslant 1$. Let

$$\boldsymbol{P}^*(Z) = \frac{1}{p_0(Z)p_i(Z)}\bar{\boldsymbol{P}}^*(Z)$$

$p_0(Z)p_i(Z)$ denotes the least common denominator (LCD) of $\boldsymbol{P}^*(Z)$, where $p_0(Z)$ absorbs all zeros of LCD in $|Z| \geqslant 1$, $p_i(Z)$ contains those zeros of LCD of $\boldsymbol{P}^*(Z)$ such that $|Z| < 1$. Hence, $p_i(Z)\boldsymbol{P}^*(z)$ is unstable.

Multiplying by $Zp_i(Z)\boldsymbol{l}(Z)$ to each term of eqn. 46 yields

$$\begin{aligned}Zp_i(Z)\boldsymbol{F}(Z)\boldsymbol{l}(Z) &= [\boldsymbol{\Delta}_1(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Delta}_2(Z)p_i(Z) \\ &\quad - \boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)p_i(Z)]\boldsymbol{l}(Z)\end{aligned} \tag{47}$$

Remark: The selection of $\boldsymbol{l}(Z)$, which is a polynomial with all zeros in $|Z| < 1$, is to ensure $\boldsymbol{Q}(Z)$ of eqn. 26 to be proper.

Note that the left-hand side of eqn. 47 is an unstable rational matrix, while the first term on the right-hand side of eqn. 47 is stable. Therefore, we perform the spectral factorisation on the second term of the right-hand side of eqn. 47 and rearrange the terms, to obtain

$$\begin{aligned}&Zp_i(Z)\boldsymbol{F}(Z)\boldsymbol{l}(Z) + [\boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)p_i(Z)\boldsymbol{l}(Z)]_- \\ &= \boldsymbol{\Delta}_1(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Delta}_2(Z)p_i(Z)\boldsymbol{l}(Z) \\ &\quad - [\boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)p_i(Z)\boldsymbol{l}(Z)]_+\end{aligned} \tag{48}$$

where all the terms on the left-hand side of eqn. 48 are analytic in $|Z| < 1$, while all the terms on the right-hand side of eqn. 48 are analytic in $|Z| \geqslant 1$.

Thus, eqn. 48 must be a polynomial matrix of Z, say $\hat{\boldsymbol{L}}(Z)$, so that

$$\begin{aligned}\hat{\boldsymbol{L}}(Z) &= \boldsymbol{\Delta}_1(Z)\hat{\boldsymbol{G}}_c(Z)\boldsymbol{\Delta}_2(Z)p_i(Z)\boldsymbol{l}(Z) \\ &\quad - [\boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)p_i(Z)\boldsymbol{l}(Z)]_+\end{aligned} \tag{49}$$

From eqn. 49, the optimal controller can be obtained:

$$\begin{aligned}\hat{\boldsymbol{G}}_c(Z) &= \boldsymbol{\Delta}_1^{-1}(Z)\{\hat{\boldsymbol{L}}(Z) + [\boldsymbol{\Delta}_1^{*-1}(Z)\boldsymbol{P}^*(Z)\boldsymbol{Q}(Z)(\boldsymbol{\phi}_r(Z) \\ &\quad + \boldsymbol{\phi}_d(Z))\boldsymbol{\Delta}_2^{*-1}(Z)\boldsymbol{l}(Z)p_i(Z)]_+\}\boldsymbol{\Delta}_2^{-1}(Z)p_i^{-1}(Z)\boldsymbol{l}^{-1}(Z)\end{aligned} \tag{50}$$

By simple calculation, it is easy to show that

$$\frac{\partial^2 J}{\partial \varepsilon^2} > 0 \qquad \text{when} \quad Z = e^{j\omega}$$

Therefore, $\hat{\boldsymbol{G}}(Z)$ given by eqn. 50 is the optimal controller corresponding to the cost functional given by eqn. 9.