Corollary 4.3: Assume (4.1), (4.8), and (4.13) are satisfied and suppose K_1 and α_1 solve the nonzero set point problem with $K_1 \in \mathcal{S}_1^+$. Then there exist $n \times n$ $Q, P \geq 0$ such that

$$K_1 = -R_1^{-1}\mathcal{P}, \tag{4.14}$$

$$\alpha_1 = R_1^{-1}B_1^T\hat{A}_1^{-T}P\gamma - R_1^{-1}[B_1^T\hat{A}_1^{-T}(L^TR_0 - \mathcal{P}^TR_1^{-1}R_{01}^T) - R_{01}^T]\delta \tag{4.15}$$

and such that Q and P satisfy

$$0 = (A - B_1R_1^{-1}\mathcal{P})Q + Q(A - B_1R_1^{-1}\mathcal{P})^T + V_0, \tag{4.16}$$

$$0 = A^TP + PA + R_0 - \mathcal{P}^TR_1\mathcal{P}. \tag{4.17}$$

Finally, setting

$$\gamma = 0, \qquad R_{01} = 0, \qquad L = I_n \tag{4.18}$$

we obtain the result of [2].

Corollary 4.4: Assume (4.1), (4.8), (4.13), and (4.18) are satisfied and suppose K_1 and α_1 solve the nonzero set point problem with $K_1 \in \mathcal{S}_1^+$. Then there exists $n \times n$ $P \geq 0$ such that

$$K_1 = -R_1^{-1}B_1^TP, \tag{4.19}$$

$$\alpha_1 = -R_1^{-1}B_1^TA_P^{-T}R_0\delta \tag{4.20}$$

and such that P satisfies

$$0 = A^TP + PA + R_0 - P\Sigma P. \tag{4.21}$$

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Analysis of Time-Varying Scaled Systems Via General Orthogonal Polynomials

TSU TIAN LEE AND YIH FONG CHANG

***Abstract*—General orthogonal polynomials are introduced to analyze and approximate the solution of a class of scaled systems. Using the operational matrix of integration, together with the operational matrix of linear transformation, the dynamical equation of a scaled system is reduced to a set of simultaneous linear algebraic equations. The coefficient vectors of the general orthogonal polynomials can be determined recursively by the derived algorithm. An illustrative example is given to demonstrate the validity and applicability of the orthogonal polynomial approximations.**

I. INTRODUCTION

An investigation of the dynamics of an overhead current collection mechanism for an electric locomotive by Ockendon and Taylor [12] revealed that under certain conditions, the dynamics of the systems is characterized by a differential equation containing terms with a scaled argument of the form

$$\dot{X}(t) = AX(\lambda t) + BX(t)$$

$$X(0) = X_0$$

where $X(\lambda t)$ and $X(t)$ are n-vectors and A and B are $n \times n$ matrices and the constant $0 < \lambda < 1$. This type of differential equation also plays an important role in several chemical processes [3], [13]. This equation was first studied by Fox *et al.* [11] with the introduction of a finite difference method for $0 < \lambda < 1$. Recently, the solution of such a scaled system has been obtained by several different orthogonal functions, such as block-pulse functions [14], [2], [3], Walsh functions [1], delayed unit step functions [4], Laguerre polynomials [5], Chebyshev polynomials [6], [7], and Legender polynomials [15]. The common approach of these methods is the use of the operational matrix of integration together with the operational matrix of scaling to reduce the differential equation to a set of linear algebraic equations, which is more suitable for computer programming.

In this note we will employ the operational matrix of integration and product operational matrix of the general orthogonal polynomials, together with the operational matrix of linear transformation, which will be derived later, to obtain the solution of the scaled system. The operational matrix of linear transformation is derived based on the following properties, namely, the pure recurrence relation

$$\phi_{i+1}(z) = (a_iz + b_i)\phi_i(z) - c_i\phi_{i-1}(z) \tag{1}$$

with

$$\phi_0(z) = 1; \; \phi_1(z) = a_0z + b_0$$

and the differential recurrence relation

$$\phi_i(z) = A_i\dot{\phi}_{i+1}(z) + B_i\dot{\phi}_i(z) + C_i\dot{\phi}_{i-1}(z) \tag{2}$$

where recurrence coefficients a_i, b_i, c_i and differential recurrence coefficients A_i, B_i, and C_i, are specified by the particular orthogonal polynomials under consideration and some are listed in [9]. The aim of this paper is twofold: 1) to derive an operational matrix of linear transformation for general orthogonal polynomials so that the scaled

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matrices derived by Hwang [4], Hwang and Shih [5], Chou and Horng [7], and Shih and Kung [15] can be obtained by the derived operational matrix as its special case; and 2) to present a general solution of scaled systems, whether it is time-varying or time-invariant, via orthogonal polynomials. This solution can certainly be reduced to different polynomials approximation solutions of the specific problem, including Chebyshev, Jacobi, Legender, ultraspherical, and any other orthogonal polynomials that possess recurrence relations (1) and (2).

II. General Orthogonal Polynomials on Finite Intervals

The orthogonal polynomials $\phi_i(z)$ with respect to the weight function $w(z)$ over the interval $a \le z \le b$ are defined as of degree precisely i in z and satisfy the condition [9]

$$\int_{a_i}^{b} w(z)\phi_i(z)\phi_j(z)\,dz = \begin{cases} r_i, & i=j \\ 0, & i\neq j \end{cases} \tag{3}$$

and the recurrence relation (1). The general shifted orthogonal polynomials may be obtained by letting $z = pt + q$ which transform domain $[a, b]$ into domain $[a', b']$, where $b' > a'$, a' and b' are both finite, $p = (a - b)/(a' - b')$ and $q = (a'b - ab')/(a' - b')$.

Thus, the shifted general orthogonal polynomial becomes

$$\phi^*_{i+1}(t) = [a_i^* t + b_i^*]\phi_i^*(t) - c_i^*\phi^*_{i-1}{}^*(t) \tag{4}$$

where $a_i^* = a_i p^*$, $b_i^* = b_i + a_i q$, $c_i^* = c_i$, for $i = 0, 1, \cdots$, with $\phi_0^*(t) = 1$; $\phi_1^*(t) = a_0^* t + b_0^*$. The new polynomials $\phi_i^*(t)$ with the recurrence relation (4) are orthogonal with respect to weight function $w^*(t) = w(pt + q)$ over the interval $[a', b']$. It has been shown that if $C(T)$ is an $n \times r$ matrix time function, $C(t)$ can be expanded by general orthogonal polynomials as [8]

$$C(t) = [C_0\ C_1\ \cdots\ C_{m-1}]\phi_r(t) = C^\tau\phi_r(t)$$

where C_i is an $n \times r$ coefficient matrix

$$\phi_r(t) = \begin{bmatrix} I_r \boxtimes \phi_0^*(t) \\ I_r \boxtimes \phi_1^*(t) \\ \vdots \\ I_r \boxtimes \phi^*_{m-1}(t) \end{bmatrix}, \text{ a } mr\times r \text{ matrix,}$$

is called the general orthogonal polynomial matrix, C^T, an $n \times mr$ matrix, is called the general orthogonal coefficients matrix, I_r, a $r \times r$ identity matrix, $\boxtimes$ denotes a Kronecker product, and $u(t)$ is an $r \times l$ time function,

$$u(t) = U\phi^*(t)$$

then

$$C(t)u(t) = [C_0UC_1U\ \cdots\ C_{m-1}U]L\phi^*(t) \tag{5}$$

where $L = [l_{0,0}l_{0,1}\cdots l_{0,m-1}: \cdots : l_{m-1,0}l_{m-1,1}\cdots l_{m-1,m-1}]^T$ is called the operational matrix of product, $l_{i,j}$ are the expansion coefficients vectors of the product of $\phi_i^*(t)$ and $\phi_j^*(t)$, for $i, j = 0, 1, \cdots, m - 1$.

Furthermore, it has also been shown that [8]

$$\int_0^t \phi^*(t)\,dt = P\phi^*(t). \tag{6}$$

For m terms approximation, P is of the form

$$P = \frac{1}{p}\begin{bmatrix} B_o - q & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_1 + D_1 & B_1 & A_1 & 0 & \cdots & 0 & 0 & 0 \\ D_2 & C_2 & B_2 & A_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{m-2} & 0 & 0 & 0 & \cdots & C_{m-2} & B_{m-2} & A_{m-2} \\ D_{m-1} & 0 & 0 & 0 & \cdots & 0 & C_{m-1} & B_{m-1} \end{bmatrix} \tag{7}$$

where D_i for $i = 1, 2, \cdots, m - 1$ can be calculated from

$$D_i = -A_i\phi^*_{i+1}(0) - B_i\phi_i^*(0) - C_i\phi^*_{i-1}(0). \tag{8}$$

III. Operational Matrix of Linear Transformation

Define

$$\phi_n^*(t_s) = \phi_n^*(\alpha t - \beta) \equiv \sum_{i=0}^{n} d_{n,i}\phi_i^*(t) \tag{9}$$

that still satisfies the recurrence relation

$$\phi^*_{n+1}(t_s) \equiv (a_n t_s + b_n)\phi_n^*(t_s) - c_n\phi^*_{n-1}(t_s) \tag{10}$$

for any t_s even outside the interval $[a', b']$.

Substituting (9) into (10), we have

$$\sum_{i=0}^{n+1} d_{n+1,i}\phi_i^*(t) = \sum_{i=0}^{n} d_{n,i}[a_n(\alpha t - \beta) + b_n]\phi_i^*(t) - \sum_{i=0}^{n-1} c_n d_{n-1,i}\phi_i^*(t). \tag{11}$$

Since the first term of the right-hand side of (11), after simple manipulations, can be expanded as

$$\sum_{i=0}^{n} d_{n,i}[a_n(\alpha t - \beta) + b_n]\phi_i^*(t)$$

$$= \sum_{i=0}^{n+1} \frac{a_n\alpha}{a_{i-1}} d_{n,i-1}\phi_i^*(t) + \sum_{i=0}^{n-1} \frac{a_n\alpha c_{i+1}}{a_{i+1}} d_{n,i+1}\phi_i^*(t)$$

$$+ \sum_{i=0}^{n} \frac{a_i b_n - a_i a_n\beta - b_i a_n\alpha}{a_i} d_{n,i}\phi_i^*(t). \tag{12}$$

Substituting (12) into (11) and equating the like coefficients of the general orthogonal polynomials $\{\phi_i^*(t)\}$, $i = 0, 1, \cdots, n + 1$, we can obtain the recurrence relation of the form

$$d_{n+1,i} = \frac{a_n\alpha}{a_{i-1}} d_{n,i-1} + \frac{a_n\alpha c_{i+1}}{a_{i+1}} d_{n,i+1} + \frac{a_i b_n - a_i a_n\beta - b_i a_n\alpha}{a_i} d_{n,i} - c_n d_{n-1,i} \tag{13}$$

for $i = 0, 1, \cdots; n = 0, 1, \cdots$, with $d_{0,0} = 1$; $d_{1,0} = b_0(1 - \alpha) - a_0\beta$; $d_{1,1} = \alpha$; $d_{n,i} = 0$ ($i > n$ or $i < 0$).

Thus, we derive a matrix termed as the operational matrix of linear transformation T to relate general orthogonal polynomials to their transformed forms as

$$\phi^*(\alpha t - \beta) = T\phi^*(t) \tag{14}$$

and T is of the form

$$T = \begin{bmatrix} d_{0,0} & 0 & 0 & \cdots & 0 \\ d_{1,0} & d_{1,1} & 0 & \cdots & 0 \\ d_{2,0} & d_{2,1} & d_{2,2} & \cdots & 0 \\ & & \vdots & & \\ d_{m-1,0} & d_{m-1,1} & d_{m-1,2} & \cdots & d_{m-1,m-1} \end{bmatrix}. \tag{15}$$

When $\alpha = \lambda > 0$ and $\beta = 0$, then T becomes the scaled matrix S, i.e.,

$$\phi^*(\lambda t) = S\phi^*(t). \tag{16}$$

The scaled matrices proposed by Hwang [6], Hwang and Shih [5], Chou and Horng [7], and Shih and Kung [15], are the special cases of the present operational matrix. This operational matrix, together with the operational matrix of integration and product operation matrix, plays an important role in reducing the scaled systems to a set of algebraic equations.

IV. Analysis of Scaled Systems

Consider the time-varying scaled system

$$\dot{x}(t)=A(t)x(\lambda t)+B(t)x(t)+C(t)u(t) \tag{17}$$

with $x(0)$ is given where $x(t)$ is an $n \times l$ state vector, $u(t)$ is a $r \times l$ input vector, and $A(t)$, $B(t)$, and $C(t)$ are time-varying matrices of appropriate dimensions.

Integrating (17) from $t' = 0$ to $t' = t$, we obtain

$$x(t)-x(0)=\int_0^t A(t')x(\lambda t')\,dt'+\int_0^t B(t')x(t')\,dt'+\int_0^t C(t')u(t')\,dt'. \tag{18}$$

Expanding $x(t)$, $u(t)$, $A(t)$, $B(t)$, and $C(t)$ by general orthogonal polynomials, we have

$$x(t)=[x_0\ \ x_1\ \ \cdots\ \ x_{m-1}]\phi^*(t)=X\phi^*(t) \tag{19}$$

$$u(t)=[u_0\ \ u_1\ \ \cdots\ \ u_{m-1}]\phi^*(t)=U\phi^*(t) \tag{20}$$

$$A(t)=[A_0\ \ A_1\ \ \cdots\ \ A_{m-1}]\phi_n(t) \tag{21}$$

$$B(t)=[B_0\ \ B_1\ \ \cdots\ \ B_{m-1}]\phi_n(t) \tag{22}$$

$$C(t)=[C_0\ \ C_1\ \ \cdots\ \ C_{m-1}]\phi_r(t). \tag{23}$$

Applying (16), $x(\lambda t)$ can be expanded as

$$x(\lambda t)=XS\phi^*(t). \tag{24}$$

Notice that initial condition $x(0)$ can be expanded into

$$x(0)=[x(0)\ \ 0\ \ \cdots 0]\phi^*(t)=X_0\phi^*(t). \tag{25}$$

Substituting (19)-(25) into (18) and using (5) and (6), we obtain

$$X\phi^*(t)-X_0\phi^*(t)=[A_0XS\ A_1KS\ \cdots\ A_{m-1}KS]LP\phi^*(t)+[B_0X\ B_1X\ \cdots\ B_{m-1}X]LP\phi^*(t)+[C_0U\ C_1U\ \cdots\ C_{m-1}U]LP\phi^*(t). \tag{26}$$

Equating the coefficient matrices of general orthogonal polynomials vector yields

$$X-X_0=[A_0XS\ A_1XS\ \cdots\ A_{m-1}XS]LP+[B_0X\ B_1X\ \cdots\ B_{m-1}X]LP+[C_0U\ C_1U\ \cdots\ C_{m-1}U]LP. \tag{27}$$

Letting $L = [L_0^T L_1^T \cdots L_{m-1}^T]^T$, where $L_i = [l_{i,0} l_{i,1} \cdots l_{i,m-1}]^T$, for $i = 0, 1, \cdots, m-1$, then (27) becomes

$$X-X_0=\sum_{i=0}^{m-1} A_iXSL_iP+\sum_{i=0}^{m-1} B_iXL_iP+\sum_{i=0}^{m-1} C_iUL_iP. \tag{28}$$

Letting $W_i = [w_{i1}w_{i2} \cdots w_{im-1}] = C_iUL_iP$, $X = [x_0x_1 \cdots x_{m-1}]$, $X_0 = [x(0)O \cdots O]$ and defining $\tilde{x} = [x_0^Tx_1^T \cdots x_{m-1}^T]^T$, $\tilde{x}_0 = [x^T(0)O^T \cdots O^T]^T$ and $\tilde{w}_i = [w_{i1}^Tw_{i2}^T \cdots w_{im-1}^T]^T$, then (28) can be solved by

$$\tilde{x}=\{I_{nm}-\sum_{i=0}^{m-1}[A_i \boxtimes (SL_iP)^\tau+B_i \boxtimes (L_iP)^\tau]\}^{-1}[\tilde{x}_0+\sum_{i=0}^{m-1}\tilde{w}_i]. \tag{29}$$

Note that if $A(t)$, $B(t)$ are constant matrices, and $C(t) = O$, which is the case considered by Hwang [5], the result of Hwang can be obtained by substituting $C(t) = O$ into (29). If $B(t) = 0$, and $A(t)$ and $C(t)$ are constant matrices, which is the case considered by Chen [2] and Chou and Horng [7], the approximate solutions can be obtained by substituting $B(t) = O$ into (29) which then will yield the results of Chou and Horng [7] and Chen [2]. If $A(t)$, $B(t)$, and $C(t)$ are all constant matrices, which is the case considered by Shih and Kung [15], the approximate solutions of Shih and Kung can be obtained by substituting $A(t) = A$, $B(t) = B$, and $C(t) = C$ into (29).

TABLE I
THE APPROXIMATION SOLUTION OF $x(t)$ FOR DIFFERENT ORTHOGONAL POLYNOMIAL EXPANSIONS

Polynomials Type \ Term	$\phi^*_0(t)$	$\phi^*_1(t)$	$\phi^*_2(t)$	$\phi^*_3(t)$	$\phi^*_4(t)$	$\phi^*_5(t)$
Jacobi# $P_n^{(1,3)}$	.91157	.08005	-.02407	-.00075	.00077	-.00004
Ultraspherical# $P_n^{(2)}$	.81617	.08018	-.01299	-.00086	.00031	-.00000
Chebyshev 1st# T_n	.76691	.31026	-.07309	-.00699	.00301	-.00004
Chebyshev 2nd# U_n	.80347	.15863	-.03805	-.00347	.00154	-.00002
Legendre# P_n	.79108	.31447	-.09974	-.01115	.00557	-.00009
Laguerre $L_n^{(0)}$	.56074	.33863	.18824	.09202	.03605	.00879
Hermite H_n	.78160	-.10226	-.05565	.00435	-.00217	.00230

All Jacobi type polynomials are shifted to domain [1, 1]; i.e., z = -2t + 1.

TABLE II
COMPARISON OF THE DIFFERENT POLYNOMIALS APPROXIMATION FOR $x(t)$

t	Jacobi (m=6)	Ultra. (m=6)	Chebyshev 1st (m=6)	Chebyshev 2nd (m=6)	Legendre (m=6)	Laguerre (m=6)	Hermite (m=6)	Runge Kutta
0.0	1.00009	1.00039	1.00006	1.00002	1.00014	1.22446	.86685	1.00000
0.2	.97752	.97752	.97754	.97752	.97752	1.00354	.86338	.97751
0.4	.90225	.90228	.90224	.90226	.90225	.81085	.83433	.90226
0.6	.77032	.77032	.77026	.77029	.77028	.64396	.76523	.77031
0.8	.59203	.59209	.59214	.59210	.59211	.50061	.64591	.59209
1.0	.39484	.39397	.39360	.39378	.39368	.37864	.47338	.39356

V. Illustrative Example

Example 1: Consider the scaled system

$$\dot{x}(t)=-tx(0.8t)-t^2x(t)$$

$$x(0)=1.$$

The expansion coefficients of $x(t)$ for $m = 6$ and $t_f = 1$ for different orthogonal polynomials are given in Table I. Some classical orthogonal polynomials approximation of $x(t)$ for $m = 6$ and $t_f = 1$, together with the solution obtained by the Runge-Kutta method is shown in Table II. It is clear that, in general, the agreement is very satisfactory. In particular, the Jacobi type polynomials approximation converges faster than the others. Noted the poor quality of results obtained via either the Laguerre or Hermite polynomial. This is due to the fact that the zeros of the Laguerre polynomial and the Hermite polynomial are widely spread over the interval of $[0, \infty]$ and $[-\infty, \infty]$, respectively. Hence, in general, these two polynomials require more terms than the Jacobi type polynomials in order to yield similar results as that of Jacobi type polynomials within a small interval. In this example, $m = 6$ is not large enough for these two polynomials.

VI. Conclusions

The operational matrix of linear transformation for general orthogonal polynomials is first introduced, and a systematic method is presented to analyze a class of time-varying scaled systems. The operational matrix of linear transformation, together with the operational matrix of integration, are applied to reduce the differential equation to a set of linear algebraic equations which is very convenient for digital computation. Illustrative example shows that only a small number of terms are required to obtain accurate approximations. Moreover, in general, the Jacobi type orthogonal polynomials solution converges faster than the Hermite polynomials solution and the Laguerre polynomials solution.

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The Operational Matrices of Integration and Differentiation for the Fourier Sine-Cosine and Exponential Series

P. N. PARASKEVOPOULOS

***Abstract*—For the Fourier sine-cosine series basis vector $\varphi(t)$ and the Fourier exponential series basis vector $\psi(t)$, a linear nonsingular transformation T is determined such that $\psi(t) = T\varphi(t)$. This result is then used to show that the operational matrices of integration P and Q for $\varphi(t)$ and $\psi(t)$, respectively, are related by the expression $TP = QT$. Analogous results are derived for the corresponding operational matrices of differentiation D and R. General expressions are derived for T, P, Q, D, and R.**

I. Introduction

Recently, orthogonal series have been used for studying various problems in system analysis and synthesis. The key idea involved is based on the integral expression

$$\int_0^t \varphi(\sigma)\, d\sigma \simeq P\varphi(t), \text{ where } \varphi(t)=[\varphi_0(t),\ \varphi_1(t),\ \cdots,\ \varphi_{r-1}(t)] \quad (1)$$

is the orthogonal basis vector and P is an $r \times r$ constant matrix called the operational matrix of integration. The matrix P has already been determined for many types of orthogonal series, such as Walsh [1], block-pulse [2], Laguerre [3], [4], Chebyshev [5], Legendre [6], [7], Hermite [8], Jacobi [9], Bessel [10], Fourier sine-cosine series [11], and the Haar functions [12]. Most of the problems that have been studied may be summarized as follows. State-space analysis [13]-[18], optimal control [19]-[22], identification [23]-[27], sensitivity analysis [28]-[30], observer design [31]-[33], model simplification [34]-[36], solution of integral and variational problems [37]-[42], etc.

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In this note the Fourier sine-cosine series are revisited and they are studied in conjunction with the Fourier exponential series. Let $\varphi(t)$ and $\psi(t)$ denote the orthogonal basis vectors for the sine-cosine series and the exponential series, respectively. Also let P and Q denote the respective operational matrices of integration. Then, it will be shown that a nonsingular transformation matrix T exists such that

$$\psi(t)=T\varphi(t) \quad (2)$$

$$TP=QT. \quad (3)$$

Since det $T \neq 0$, it follows that (2) and (3) are very useful since one may go from one set of orthogonal functions to another. In particular, (3) is used to derive Q knowing P. Similar results are derived for the operational matrices of differentiation.

II. The Operational Matrices of Integration

Consider the Fourier sine-cosine series basis vector $\varphi(t)$ having the form

$$\varphi(t)=[\varphi_0(t),\ \varphi_1(t),\ \cdots,\ \varphi_r(t),\ \varphi_1^*(t),\ \cdots,\ \varphi_r^*(t)]^T \quad (4)$$

where

$$\varphi_n(t)=\cos\frac{2n\pi t}{L}; \qquad n=0,\ 1,\ \cdots,\ r$$

$$\varphi_n^*(t)=\sin\frac{2n\pi t}{L}; \qquad n=1,\ 2,\ \cdots,\ r.$$

Also consider the Fourier exponential series basis vector $\psi(t)$ having the form

$$\psi(t)=[\psi_0(t),\ \psi_1(t),\ \cdots,\ \psi_r(t),\ \psi_1^*(t),\ \cdots,\ \psi_r^*(t)]^T \quad (5)$$

where

$$\psi_n(t)=e^{j2\pi nt/L}; \qquad n=0,\ 1,\ \cdots,\ r$$

$$\psi_n^*(t)=e^{-j2\pi nt/L}; \qquad n=1,\ 2,\ \cdots,\ r$$

where $j = \sqrt{-1}$.

Making use of the relation $e^{j\theta} = \cos\theta + j\sin\theta$ one may readily show that $\varphi(t)$ and $\psi(t)$ are related as follows:

$$\psi(t)=T\psi(t) \quad (6)$$

where T is an $2r + 1$ square nonsingular transformation matrix having the form

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & I_r & jI_r \\ 0 & I_r & -jI_r \end{bmatrix} \quad (7)$$

where I_r is the $r \times r$ unit matrix. Furthermore,

$$\varphi(t)=T^{-1}\psi(t) \quad (8)$$

where

$$T^{-1}=\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & I_r & I_r \\ 0 & -jI_r & jI_r \end{bmatrix}. \quad (9)$$

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