mation regarding the lifter should read the references listed in the paper. The objective of the paper was to report the hydraulic equations that can be used for predicting the behavior of the lifter, and experiments to verify the equations.

Numerical Determination of Aquifer Constants[a]

Discussion by H. D. Yeh[2]

The author is to be commended for presenting a simple method that can be used to calculate the aquifer constants without using a graphical method. The author uses the finite difference formula to approximate ΔZ and chooses a single or average value for Z in Eq. 9 that results in Eqs. 10 and 13.

The writer does not agree with the conclusion that the author draws, i.e., that the accuracy of his method is of the same order as the other graphical methods. Giving two illustrative examples and some other data sets (not shown in the paper), the author uses his method to compute the values of T and S and claims that its results are within 10% deviation, compared to the results from the graphical methods. Actually, this statement is valid only for the case in which the time interval of pumping period $\Delta t = t_2 - t_1$ is very small. The logarithm of the pumping time $\log t$ versus the drawdown Z has a linear relationship as shown in figure 4.12 in Refs. 4 and 7 so that the term dt/t in Eq. 9 should be treated as $d \ln t$. Eq. 9 can be written as

$$f(u) = \frac{Z}{\dfrac{dZ}{d \ln t}} \qquad (23)$$

Its finite difference form is

$$f(u) = \frac{Z_2}{\dfrac{(Z_3 - Z_1)}{\ln\left(\dfrac{t_3}{t_1}\right)}} \qquad (24)$$

The drawdowns Z_1, Z_2, and Z_3 at pumping times t_1, t_2, and t_3 are arbitrarily selected from Table 2 and listed in Table 3. The values of T and S are calculated and shown in Table 4 by Eqs. 10–12, in Table 5 by Eqs. 13, 15, and 16, and in Table 6 by Eqs. 15, 16, and 24. It is obvious that

[a]July, 1985, Vol. 111, No. 7, by S. P. Rai (Paper 19826).

[2]Assoc., Prof., Dept. of Civ. Engrg., Natl. Chaio Tung Univ., Hsinchu, Taiwan, ROC.

J. Hydraul. Eng. 1987.113:948-951.

TABLE 3.—Selected Pumping Test Data from Table 2

Time, t_1 (min) (1)	Drawdown, Z_1 (ft) (2)	Time, t_2 (min) (3)	Drawdown, Z_2 (ft) (4)	Time, t_3 (min) (5)	Drawdown, Z_3 (ft) (6)
40	2.65	50	2.78	60	2.88
1	0.66	2	0.99	240	3.67
1	0.66	120	3.28	240	3.67
1	0.66	210	3.61	240	3.67
10	1.86	12	1.97	150	3.42
10	1.86	80	3.04	150	3.42
10	1.86	120	3.28	150	3.42
60	2.88	80	3.04	240	3.67
60	2.88	150	3.42	240	3.67
60	2.88	210	3.61	240	3.67

Note: 1 foot = 0.305 meter.

when Δt is large, Eqs. 10 and 13 give very poor approximations for the values of $f(u)$, hence the final results of the values of T and S shown in Tables 4 and 5 have large percentages of error. Using Eq. 24, the results in Table 6 show that the relative deviations [$100 \times (T - T')/T'$, $T' = 70$ gal/min/ft] are within 10% even in the extreme case when $t_1 = 1$ min and $t_3 = 240$ min.

Using the approximated values of $f(u)$ from Eq. 24, the values of u and $W(u)$ are found in Table 1 and Eq. 2 or figure 4.11 in Refs. 4 and 7, therefore, it may introduce errors in the values of u and $W(u)$ and have effects on the values of T and S. The writer proposes the use of Newton's method to find the values of u and $W(u)$ in Eq. 9. The basic algorithm for Newton's method (6) is set up as

$$U_{n+1} = U_n - \frac{F(u)}{F'(u)} \quad \text{(25)}$$

in which n is the number of iterations; $F(u) = W(u) \cdot e^u - f(u)$; and $F'(u) = dF(u)/du$. The well function $W(u)$ is expanded as a convergent series

TABLE 4.—Results by Eqs. 10–12

$f(u)$ (1)	$u \times 10^{-3}$ (2)	$W(u)$ (3)	T (4)	S (5)	$100 \times (T - T')/T'$ [a] (6)
4.6410	5.5899	4.6152	67.6359	0.000228	−3.38
1.6667	157.2075	1.4242	68.6880	0.000217	−1.87
1.4790	204.1553	1.2058	24.3550	0.004024	−65.21
1.4337	217.9257	1.1530	21.4876	0.006608	−69.30
3.1653	26.4173	3.0828	64.0519	0.000249	−8.50
3.2298	24.6266	3.1512	51.1763	0.000759	−26.89
3.0628	29.5545	2.9736	46.0380	0.001183	−34.23
5.2857	2.8950	5.2704	70.8460	0.000192	1.21
5.0000	3.8718	4.9807	62.9127	0.000342	−10.12
4.9391	4.1200	4.9188	60.3123	0.000449	−13.84

[a] $T' = 70$ gal/min/ft $= 0.8699$ m^3/min/m.

TABLE 5.—Results by Eqs. 13, 15, and 16

$f(u)$ (1)	$u \times 10^{-3}$ (2)	$W(u)$ (3)	T (4)	S (5)	$100 \times (T - T')/T^{a}$ (6)
4.8348	4.5834	4.8127	68.8814	0.000211	−1.60
39.3040	—[b]	—	—	—	—
2.1703	82.5124	1.9984	24.2424	0.003211	−65.37
1.3650	241.1488	1.0725	11.8207	0.008007	−83.11
14.8280	—	—	—	—	—
3.4103	20.2642	3.3418	43.7395	0.000948	−37.52
2.4530	58.9043	2.3127	28.0544	0.002653	−59.92
8.6582	—	—	—	—	—
5.1949	3.1748	5.1785	60.2471	0.000384	−13.93
3.9168	11.8418	3.8707	42.6623	0.001419	−39.05

[a]$T' = 70$ gal/min/ft = 0.8699 m^3/min/m.
[b]Newton's method does not converge, so the solutions are not available.

in Eq. 2. Neglecting the high-order terms when the power of u is greater than 6, $F(u)$ and $F'(u)$ in Eq. 25 become

$$F(u) = e^{u} \cdot \left(-0.5772157 - \ln u + u - \frac{u^2}{4} + \frac{u^3}{18} - \frac{u^4}{96} + \frac{u^5}{600} - \frac{u^6}{4{,}320} \right) - f(u) \quad (26)$$

and $$F'(u) = e^{u} \cdot \left(0.4227843 - \ln u - \frac{1}{u} + \frac{u}{2} - \frac{u^2}{12} + \frac{u^3}{72} - \frac{u^4}{480} + \frac{u^5}{3{,}600} - \frac{u^6}{30{,}240} \right) \quad (27)$$

Beginning at $u_1 = 0.001$ and choosing the convergence criterion $|F(U_{n+1})| > 10^{-4}$, Newton's method never exceeds eight iterations to get the values of u as shown in Tables 4–6. The accuracy of u and $W(u)$ will

TABLE 6.—Results by Eqs. 15, 16, and 24

$f(u)$ (1)	$u \times 10^{-3}$ (2)	$W(u)$ (3)	T (4)	S (5)	$100 \times (T - T')/T'^{a}$ (6)
4.9008	4.2842	4.8799	69.8434	0.000200	−0.22
1.8026	131.1760	1.5810	63.5413	0.000223	−9.23
5.9723	1.4454	5.9636	72.3432	0.000168	3.35
6.5731	0.7893	6.5679	72.3906	0.000161	3.42
3.4418	19.5887	3.3751	68.1676	0.000214	−2.62
5.2772	2.9201	5.2618	68.8691	0.000215	−1.62
5.6939	1.9146	5.6830	68.9384	0.000212	−1.52
5.3346	2.7549	5.3199	69.6294	0.000205	−0.53
6.0014	1.4034	5.9930	69.7235	0.000196	−0.39
6.3348	1.0031	6.3285	69.7515	0.000197	−0.36

[a]$T' = 70$ gal/min/ft = 0.8699 m^3/min/m.

be up to four digits after decimal point. When the value of $f(u)$ is over 8.6, which is beyond the reasonable range of $f(u)$ as shown in figure 4.11 in Refs. 4 and 7, Newton's method has a convergence problem due to inappropriate initial guesses or nonlinearity of the equation.

Appendix.—References

6. Gerald, C. F., and Wheatley, P. O., *Applied Numerical Analysis*, 3rd ed., Addison-Wesley Publishing Co., Reading, Mass., 1984.
7. Todd, D. K., *Groundwater Hydrology*, 2nd ed., John Wiley and Sons, Inc., New York, N.Y., 1980.

Closure by S. P. Rai,[3] M. ASCE

The writer would like to thank Yeh for his interest in the paper and for presenting an alternate formula for $f(u)$ (Eq. 24).

The writer does not dispute the fact that the finite difference formula used, based on approximate values of Z will give large errors when the time interval is large (Table 4). This is obvious. However, the formulas (Eqs. 13–16) based on three successive values of the drawdown, such that the time intervals between the first and second values and between the second and third values are equal, will definitely give better results with smaller errors, even when the time intervals are large. The results presented in Table 5, showing large errors for data with large time intervals is not valid, because the data (Table 3) used for the calculation do not satisfy the criterion that the successive time intervals are equal, i.e., $(t_2 - t_1) = (t_3 - t_2)$, except for one or two cases; and in such cases, one may see that, the errors are on the order of 10% only.

Yeh's alternate formula for $f(u)$ (Eq. 24) may give marginally better results than Eq. 13 for larger time intervals only. However, Eq. 13 can be applied to the entire range of data, including the initial time $t = 0$; whereas, if Eq. 24 is used for the initial few data (say $t = 1$ to 5 min) this gives more than 100% error in the computation of T and S. It also fails completely if the initial point $t = 0$ is taken as one of the data points. Eq. 13, on the other hand, gives only about 10% error even when the first few data are considered. If a solution at large values of t is desired, one can numerically solve Cooper and Jacob's formula (2) without using the well function table, with

$$Z_1 - Z_2 = \frac{Q}{4\pi T} \ln\left(\frac{t_1}{t_2}\right) \qquad (28)$$

from which T and, hence, S can be determined. Thus, Eq. 24, which is also valid at large values of t only, has hardly any advantage over Eq. 26.

Since the purpose of the paper was to present a simple method for computation of aquifer constants either in the office or in situ, Eq. 13 stands in good stead in that respect. It is simple, involving arithmetic

[3]Sr. Lect., Civ. Engrg. Dept., Natl. Univ. of Singapore, Kent Ridge, Singapore 0511.

J. Hydraul. Eng. 1987.113:948-951.